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Observation for the descriptor systems with disturbances

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Abstract: In this paper, the observation problem for the descriptor systems with disturbances is studied. It is assumed that the disturbances and their first order derivatives are bounded, where the upper and lower bounds are unknown. First, the formulated descriptor system is decomposed into a dynamical system and an algebraic equation. The dynamical system is the relation among a part of the descriptor state, the input-output and the disturbance. The algebraic equation is the relation between the descriptor state variable and the disturbance. Second, the disturbances and one part of the descriptor state are estimated based on the obtained dynamical system. Finally, the other part of the descriptor state is estimated based on the obtained algebraic equation. Examples are presented to illustrate the proposed method.

Keywords: *Descriptor system, disturbance observer, state observer.*

Mathematics Subject Classification (2000): 93B07, 93B30, 93C15, 93C35.

1 Introduction

In the last years, considerable attention has been focused on the control synthesis problems of linear descriptor systems. Structures of such control systems were first studied in the frequency domain by Rosenbrock using matrix pencil theory [13]. Later, controllability, observability and feedback control problems have been investigated by many researchers [2, 5, 7, 8, 9, 15, 16, 19, 20]. However, little effort has been made to develop a theory of observers for descriptor systems. Based on singular-value decomposition,

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El-Tohami et al. have proposed the reduced-order observer for a class of descriptor systems satisfying a simple rank condition [9]. By using the concept of a matrix generalized inverse, a new method is given for constructing a minimal reduced-order observer under a certain observability condition on the constructed observer [14]. Then, a Luenberger-type observer is formulated by using the descriptor standard form [12]. It should be noted that these results are restricted to linear time-invariant descriptor systems with known parameters and without any additional uncertainties.

Recently, the problem of constructing a state observer for the input unknown systems has received some attention. Construction of a state variable observer is a very difficult task for the dynamical system with disturbances, not to say descriptor system with disturbances. For the dynamical systems with disturbances, one typical method is the disturbance decoupled observer by using an elegant geometric approach [3, 18]. Then, this method is applied to disturbance decoupling problems for descriptor systems [11, 10, 1]. However, the results are very complicated and far from complete. The index and stability of the resulting combined systems and the numerical computation of the desired observer have not been considered. As a matter of fact, these geometric solution methods are not suited for numerical computations. The need for reliable numerical method was pointed out in [18]. Later, the computation of the desired disturbance decoupling observer is effectively considered in [6] by using the orthogonal matrix transformation, where the descriptor systems under consideration must be regular and of index at most one.

For the input unknown dynamical systems, another typical effective method about the construction of the state observer is the VSS-type one [17]. However, this approach can only cope with the minimum phase dynamical systems with relative degree one, and the upper and lower bounds of the disturbances are required. It should be noted that this method cannot be applied to the state observation problem for input unknown descriptor systems.

In this paper, the observation problem for the descriptor systems with disturbances is studied by using a totally different approach, where both the descriptor state and the disturbances are estimated. The requirement that the descriptor system must be of index at most one is not needed. It is assumed that the disturbances and their first order derivatives are bounded in the open loop. However, the upper and lower bounds are unknown. The formulated descriptor system is decomposed into a dynamical system and an algebraic equation. The dynamical system is the relation among a part of the descriptor state, the input-output and the disturbance. The algebraic equation is the relation between the descriptor state variable and the disturbance. Based on the obtained dynamical system, the disturbances are first estimated, where the nonlinear method proposed by the authors in [4] for single disturbance single output (SDSO) systems is applied; then, one part of the descriptor state is estimated. Finally, the other part of the descriptor state is calculated based on the obtained algebraic equation.

This paper is organized as follows. Section 2 gives the problem formulation. In Section 3, the disturbance and the state variable are estimated for a special case, the dynamical system case, of the formulated descriptor system. In Section 4, the observation for the general descriptor system with disturbances is studied. In Section 5, design examples and computer simulation results are presented to illustrate the proposed method. Section 6 concludes this paper.

2 Problem formulation

Let us consider the following uncertain system

$$\begin{cases} E\dot{x}(t) = Fx(t) + Gu(t) + Kv(t), \\ y(t) = Hx(t) + Bu(t) + Dv(t), \end{cases} \tag{2.1}$$

where $u(t) \in R^q$, $y(t) \in R^r$ and $x(t) \in R^n$ are the input, output and the unknown descriptor state variable, respectively; $v(t) \in R^p$ represents the disturbance, which may include modeling errors, noise, higher order terms in linearization or just an unknown input to the system; $E \in R^{n\times n}$ is a known matrix which may not be nonsingular; $F \in R^{n\times n}$, $G \in R^{n\times q}$, $K \in R^{n\times p}$, $H \in R^{r\times n}$, $B \in R^{r\times q}$ and $D \in R^{r\times p}$ are known matrices.

About the system (2.1), the following assumptions are made.

Assumption 1 rank $\begin{bmatrix} E \\ H \end{bmatrix} = n$, rank $\begin{bmatrix} F - cE \\ H \end{bmatrix} =$ n for all $c \in C$, where C denotes the complex plane.

Assumption 2 For any $c \in C$ satisfying Re(c) ≥ 0 , $\begin{bmatrix} F - cE & K \\ H & D \end{bmatrix}$ is of full rank, i.e. the system (2.1) is in "minimum phase" with respect to the relation between the disturbance and the output.

Assumption 3 The signals $u(t)$, $y(t)$ and $v(t)$ are bounded. However, the upper bound of $\|v(t)\|_2$ is unknown.

Assumption 4 The disturbance $v(t)$ is continuous and piecewise differentiable. Furthermore, the derivative (at the undifferentiable points, we mean the right- and left-hand derivatives) is bounded.

Assumption 5 $r \geq p$, i.e. the number of the outputs is not smaller than that of the disturbances.

Remark 2.1 When the disturbance $v(t)$ is absent, Assumption 1 means that the system (2.1) is observable [12].

The purpose of this paper is to estimate the uncertain signal $v(t)$ and the descriptor state variable $x(t)$ by using the input-output information even though the matrix E may not be nonsingular.

In the following, we assume $B = 0$. Otherwise, we regard the signal $y(t) - Bu(t)$ as $y(t)$.

First, the observation problem is discussed for the case that E is nonsingular. Then, the observation problem is studied for the general descriptor system.

3 Observation for the system when E is nonsingular

Without loss of generality, we assume $E = I$. Otherwise, we pre-multiply the first equation of (2.1) with E^{-1}.

3.1 Observation for the system when D is of full rank

If D is of full rank, then the difference between the observed state $\hat{x}(t)$ and the genuine state $x(t)$ can be designed to decay to zero exponentially, and the disturbance can be asymptotically observed, where Assumption 4 about the disturbance $v(t)$ is not needed.

Theorem 3.1 *If D is of full rank, then the state observer of the system (2.1) with $E = I$ can be constructed as*

$$\begin{cases}\dot{\hat{x}}(t) = \big(F - KD_1^{-1}\Omega_1 H\big)\hat{x}(t) + Gu(t) + KD_1^{-1}\Omega_1 y(t) + \bar{L}\big(\Omega_2 y(t) - \hat{y}(t)\big), \ \hat{x}(t_0) = 0,\\ \hat{y}(t) = \Omega_2 H\hat{x}(t),\end{cases} \tag{3.1}$$

where $\hat{x}(t)$ is the estimated state, $\bar{L}$ is chosen such that $F - KD_1^{-1}\Omega_1 H - \bar{L}\Omega_2 H$ is a stable matrix, $\Omega = \begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix}$ is a $r \times r$ nonsingular matrix such that

$$\Omega D = \begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix} D = \begin{bmatrix}D_1\\ 0\end{bmatrix}, \tag{3.2}$$

in which D_1 is a $p \times p$ nonsingular matrix. Furthermore, the disturbance $v(t)$ can be observed by

$$\hat{v}(t) = D_1^{-1}\Omega_1 y(t) - D_1^{-1}\Omega_1 H\hat{x}(t), \tag{3.3}$$

where $\hat{x}(t)$ is the estimated state generated in (3.1). For the estimated state and the disturbance, we have

$$x(t) - \hat{x}(t) \to 0, \ v(t) - \hat{v}(t) \to 0 \tag{3.4}$$

as $t \to \infty$.

Proof Equation (2.1) gives

$$\begin{cases}\dot{x}(t) = Fx(t) + Gu(t) + Kv(t),\\ \Omega_1 y(t) = \Omega_1 Hx(t) + D_1 v(t),\\ \Omega_2 y(t) = \Omega_2 Hx(t).\end{cases} \tag{3.5}$$

From the second equation in (3.5), we have

$$v(t) = D_1^{-1}\Omega_1 y(t) - D_1^{-1}\Omega_1 Hx(t). \tag{3.6}$$

By substituting (3.6) into the first equation in (3.5), equation (3.5) yields

$$\begin{cases}\dot{x}(t) = \big(F - KD_1^{-1}\Omega_1 H\big)x(t) + Gu(t) + KD_1^{-1}\Omega_1 y(t),\\ \Omega_2 y(t) = \Omega_2 Hx(t).\end{cases} \tag{3.7}$$

Since

$$\begin{bmatrix}I & -KD_1^{-1} & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & \Omega_1\\ 0 & \Omega_2\end{bmatrix}\begin{bmatrix}F - cI & K\\ H & D\end{bmatrix} = \begin{bmatrix}F - KD_1^{-1}\Omega_1 H - cI & 0\\ \Omega_1 H & D_1\\ \Omega_2 H & 0\end{bmatrix}, \tag{3.8}$$

from Assumption 2, it can be seen that $\begin{bmatrix} F - KD_1^{-1}\Omega_1 H - cI \\ \Omega_2 H \end{bmatrix}$ is of full rank for all $c \in C$ satisfying Re(c) ≥ 0 by observing that $D_1 \in R^{p\times p}$ is a nonsingular matrix. Thus, the system (3.7) is detectable, i.e. the matrix $\bar{L}$ exists such that $F - KD_1^{-1}\Omega_1 H - \bar{L}\Omega_2 H$ is a stable matrix. If the observer is constructed as in (3.1), it yields

$$\frac{d}{dt}(x(t) - \hat{x}(t)) = \big(F - KD_1^{-1}\Omega_1 H - \bar{L}\Omega_2 H\big)(x(t) - \hat{x}(t)). \tag{3.9}$$

It can be easily seen that $x(t) - \hat{x}(t) \to 0$ as $t \to \infty$. From (3.6), it can be concluded that (3.3) is an observer of the disturbance $v(t)$ and $v(t) - \hat{v}(t) \to 0$ as $t \to \infty$.

3.2 Observation for the system when D is not of full rank

3.2.1 Some preliminaries

Let s denote the differential operator. Then, equation (2.1) can be written as

$$\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -Gu(t) \\ y(t) \end{bmatrix}. \tag{3.10}$$

Now, pre-multiplying (3.10) by $\left(adj\Big(\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}^T \begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}\Big)\right) \begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}^T$ yields

$$k(s)\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \left(adj\Big(\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}^T \begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}\Big)\right) \begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}^T \begin{bmatrix} -Gu(t) \\ y(t) \end{bmatrix}, \tag{3.11}$$

where $k(s)$ is defined as

$$k(s) = det\left(\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}^T \begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}\right) = k_0 s^{q_0} + \cdots + k_{q_0}, \quad k \neq 0. \tag{3.12}$$

By Assumption 2, it can be easily known that $k(s)$ is a Hurwitz polynomial.

By observing the calculation methods of the adjoint of a matrix and the multiplication of the matrices, equation (3.11) can be expressed as

$$\begin{cases} s^{l_{11}}(\beta_{11} y(t)) = \Phi_{11}(s)y(t) + \Psi_{11}(s)u(t) + k(s)x_1(t), \\ \vdots \\ s^{l_{1n}}(\beta_{1n} y(t)) = \Phi_{1n}(s)y(t) + \Psi_{1n}(s)u(t) + k(s)x_n(t), \\ s^{l_{21}}(\beta_{21} y(t)) = \Phi_{21}(s)y(t) + \Psi_{21}(s)u(t) + k(s)v_1(t), \\ \vdots \\ s^{l_{2p}}(\beta_{2p} y(t)) = \Phi_{1p}(s)y(t) + \Psi_{2p}(s)u(t) + k(s)v_p(t), \end{cases} \tag{3.13}$$

where $\beta_{ji} \neq 0$ are row vectors whose entries are constants, $\Phi_{ji}(s)$ are row vectors whose entries are at most $(l_{ji} - 1) - th$ order polynomials of s, $\Psi_{ji}(s)$ are row vectors whose entries are at most $(l_{ji} - 1) - th$ order polynomials of s.

Because F, G, K, H and D are known matrices, β_{ji} , $\Phi_{ji}(s)$, $\Psi_{ji}(s)$ and $k(s)$ can be calculated.

Remark 3.1 If $r = p$, i.e. the number of the outputs equals to that of the disturbances, then we can simply pre-multiply the both sides of (3.10) by $\begin{bmatrix} F - sI & K \\ H & D \end{bmatrix}$.

3.2.2 Observation of the disturbances

About the disturbance $v(t) = [v_1(t) \ \cdots \ v_p(t)]^T$, from (3.13), we have

$$\begin{cases} s^{l_{21}}(\beta_{21}y(t)) = \Phi_{21}(s)y(t) + \Psi_{21}(s)u(t) + k(s)v_1(t), \\ \qquad\qquad\qquad \vdots \\ s^{l_{2p}}(\beta_{2p}y(t)) = \Phi_{2p}(s)y(t) + \Psi_{2p}(s)u(t) + k(s)v_p(t). \end{cases} \tag{3.14}$$

For the $i-th$ equation in (3.14), $l_{2i} - q_0$ can be regarded as the "relative degree" with respect to the relation between the disturbance $v_i(t)$ and the "output" $\beta_{2i}y(t)$. It is easy to see that $l_{2i} \geq q_0$, otherwise, equation (3.13) contradicts with the original differential equation (2.1).

We start with equation (3.14) to estimate the disturbances.

For simplicity, let

$$\eta_i = l_{2i} - q_0. \tag{3.15}$$

To estimate the disturbances, the discussion is divided into the following two cases.

Case 1: $l_{2i} = q_0$

In this case, from (3.14), it gives

$$v_i(t) = \frac{s^{l_{2i}}}{k(s)}(\beta_{2i}y(t)) - \frac{\Phi_{2i}(s)}{k(s)}y(t) - \frac{\Psi_{2i}(s)}{k(s)}u(t), \tag{3.16}$$

i.e. the disturbance $v_i(t)$ can be expressed by the outputs and the filters of the inputs and outputs, where only the input and output information is employed. Thus,

$$w_{i,0} \triangleq \frac{s^{l_{2i}}}{k(s)}(\beta_{2i}y(t)) - \frac{\Phi_{2i}(s)}{k(s)}y(t) - \frac{\Psi_{2i}(s)}{k(s)}u(t) \tag{3.17}$$

can be regarded as the estimate of $v_i(t)$.

Remark 3.2 For a complex constant $\Gamma \in C$ satisfying $\mathrm{Re}(\Gamma) > 0$, $\frac{1}{s+\Gamma}y(t)$ is defined as the solution of the following differential equation

$$\dot{\xi}(t) + \Gamma\xi(t) = y(t), \ \xi(t_0) = 0, \tag{3.18}$$

where t_0 is the starting time. Thus, the filters in (3.17) and the upcoming ones can be analogously defined.

Case 2: $l_{2i} > q_0$

Introduce a monic $l_i - th$ order Hurwitz polynomial

$$g_i(s) = \frac{1}{k_0}k(s) \cdot (s + \lambda)^{\eta_i}, \tag{3.19}$$

where λ is a positive constant. Then, the $i-th$ equation in (3.14) can be rewritten as

$$\dot{z}_i(t) + \lambda z_i(t) = L_i(y(t), u(t)) + \frac{k_0}{(s+\lambda)^{\eta_i - 1}}v_i(t), \tag{3.20}$$

where $z_i(t)$ and $L_i(y(t),u(t))$ are respectively defined as

$$z_i(t) = \beta_{2i}y(t), \tag{3.21}$$

$$L_i(y(t),u(t)) = (s+\lambda)\left\{\frac{g_i(s)-s^{l_i}}{g_i(s)}\{\beta_{2i}y(t)\} + \frac{\Phi_{2i}(s)}{g_i(s)}y(t) + \frac{\Psi_{2i}(s)}{g_i(s)}u(t)\right\}. \tag{3.22}$$

Remark 3.3 It should be pointed out that $z_i(t)$ and $L_i(y(t),u(t))$ are computable signals.

Since $v_i(t)$ are bounded signals, it can be seen that, for a positive constant λ, signals $\left|\frac{1}{(s+\lambda)^{j_i}}v_i(t)\right|$ are also bounded for any positive integer j_i.

The next theorem gives a method to estimate $\frac{1}{(s+\lambda)^{\eta_i - j_i}}v_i(t)$, where the upper bounds of $\left|\frac{1}{(s+\lambda)^{\eta_i - j_i}}v_i(t)\right|$ are adaptively updated.

Theorem 3.2 *Construct the following differential equations*

$$\dot{\hat{z}}_i(t) + \lambda\hat{z}_i(t) = L_i(y(t),u(t)) + k_0 w_{i,1}(t),\ \ \hat{z}_i(t_0) = z_i(t_0), \tag{3.23}$$

$$\dot{\hat{w}}_{i,\mu_i-1}(t) + \lambda\hat{w}_{i,\mu_i-1}(t) = w_{i,\mu_i}(t),\ \ \hat{w}_{i,\mu_i-1}(t_0) = 0, \tag{3.24}$$

where $\hat{z}_i(t)$ *and* $\hat{w}_{i,\mu_i-1}(t)$ $(1<\mu_i\le\eta_i)$ *are the variables which can be obtained by respectively solving (3.23) and (3.24);* $w_{i,1}(t)$ *and* $w_{i,\mu_i}(t)$ *are the inputs described respectively by*

$$w_{i,1}(t) = \hat{\omega}_{i,1}(t)\frac{k_0\{z_i(t)-\hat{z}_i(t)\}}{\left|k_0\{z_i(t)-\hat{z}_i(t)\}\right| + \delta_{i,1}} \tag{3.25}$$

and

$$w_{i,\mu_i}(t) = \hat{\omega}_{i,\mu_i}(t)\frac{w_{i,\mu_i-1}(t)-\hat{w}_{i,\mu_i-1}(t)}{\left|w_{i,\mu_i-1}(t)-\hat{w}_{i,\mu_i-1}(t)\right| + \delta_{i,\mu_i}},\ \ (1<\mu_i\le\eta_i) \tag{3.26}$$

$\delta_{i,j_i} > 0$ $(i=1,\cdots,p;\ j_i=1,\cdots,\eta_i)$ *are design parameters which are usually chosen to be very small;* $\hat{\omega}_{i,\mu_i}(t)$ $(1\le\mu_i\le\eta_i)$ *are updated by the following adaptive algorithms*

$$\dot{\hat{\omega}}_{i,1}(t) = \begin{cases} 2\alpha_{i,1}|z_i(t)-\hat{z}_i(t)| & if\ |k_0\{z_i(t)-\hat{z}_i(t)\}| > \delta_{i,1} \\ 0 & otherwise \end{cases}, \tag{3.27}$$

$$\dot{\hat{\omega}}_{i,\mu_i}(t) = \begin{cases} 2\alpha_{i,\mu_i}|w_{i,\mu_i-1}(t)-\hat{w}_{i,\mu_i-1}(t)| & if\ |w_{i,\mu_i-1}(t)-\hat{w}_{i,\mu_i-1}(t)| > \delta_{i,\mu_i} \\ 0 & otherwise \end{cases} \tag{3.28}$$

for $1<\mu\le\eta_i$, $\hat{\omega}_{i,\mu_i}(t_0)$ *can be chosen as any positive constants,* α_{i,μ_i} *are positive constants for* $i=1,\cdots,p$, $1\le\mu_i\le\eta_i$. *It can be concluded that* $w_{i,\mu_i}(t)$ *are the corresponding approximate estimates of* $\frac{1}{(s+\lambda)^{\eta_i-\mu_i}}v_i(t)$ *for* $1\le\mu_i\le\eta_i$ *as* t *is large* *enough, i.e. there exist $T_{i,\mu_i} \geq t_0$ and functions $\epsilon_{i,\mu_i}(\nu_1, \cdots, \nu_{\mu_i}) > 0$ with the property $\lim_{\sum_{j=1}^{\mu_i} |\nu_i| \to 0} \epsilon_{i,\mu_i}(\nu, \cdots, \nu_{\mu_i}) = 0$ such that*

$$\left| \frac{1}{(s+\lambda)^{\eta_i - \mu_i}} v_i(t) - w_{i,\mu_i(t)} \right| < \epsilon_{i,\mu_i}(\delta_{i,1}, \cdots, \delta_{i,\mu_i}) \tag{3.29}$$

for all $t \geq T_{i,\mu_i}$

Proof This theorem can be proved by a similar procedure as in [4], where Assumptions 3 and 4 are employed.

Remark 3.4 The design parameters $\delta_{i,j_i} > 0$ $(1 \leq j_i \leq \eta_i)$ and $\lambda > 0$ determine the estimating precision and the estimating speed. The parameters $\alpha_{i,j_i} > 0$ should be chosen large enough to adjust the estimated upper bounds $\hat{\omega}_{i,j_i}(t)$ rapidly for $1 \leq j_i \leq \eta_i$. The estimation error for the disturbances can be designed to be arbitrarily small by choosing the design parameters. The influence of the measurement noises in the output can be similarly discussed as in [4].

Remark 3.5 For $i \neq j$, it can be seen that the estimation of $v_i(t)$ is independent of the estimation of $v_j(t)$.

3.2.3 Observation of the state

About the state $x(t) = [x_1(t) \ \cdots \ x_n(t)]^T$, from (3.13), we have

$$\begin{cases} s^{l_{11}}(\beta_{11} y(t)) = \Phi_{11}(s) y(t) + \Psi_{11}(s) u(t) + k(s) x_1(t), \\ \qquad \vdots \\ s^{l_{1n}}(\beta_{1n} y(t)) = \Phi_{1n}(s) y(t) + \Psi_{1n}(s) u(t) + k(s) x_n(t). \end{cases} \tag{3.30}$$

To estimate the state, the discussion is divided into the following two cases.
Case 1: $l_{1i} \leq q_0$

In this case, from (3.30), it gives

$$x_i(t) = \frac{s^{l_{1i}}}{k(s)}(\beta_{1i} y(t)) - \frac{\Phi_{1i}(s)}{k(s)} y(t) - \frac{\Psi_{1i}(s)}{k(s)} u(t), \tag{3.31}$$

i.e. the partial state $x_i(t)$ can be expressed by the outputs and the filters of the inputs and outputs, where only the input and output information is employed. Thus,

$$\hat{x}_i(t) \triangleq \frac{s^{l_{1i}}}{k(s)}(\beta_{1i} y(t)) - \frac{\Phi_{1i}(s)}{k(s)} y(t) - \frac{\Psi_{1i}(s)}{k(s)} u(t) \tag{3.32}$$

can be regarded as the estimate of $x_i(t)$.

Remark 3.6 If $l_{1i} \leq q_0$ for all $i = 1, \cdots, n$, then there is no steady error between the estimated state and the genuine state $x(t)$.

Case 2: $l_{2i} > q_0$

In this case, the partial state $x_i(t)$ can be similarly estimated by the method proposed for estimating the disturbances in Section 3.2.2 if the partial state $x_i(t)$ is bounded.

However, the computation for all such partial states $x_i(t)$ satisfying $l_{2i} > q_0$ will become very complicated, and the partial state $x_i(t)$ may not be bounded.

One simple method of estimating the partial state in this case is to construct a Luenberger-type state observer for the full state $x(t)$ by using the estimates of the disturbances obtained in Section 3.2.2, and then extract the partial states $x_i(t)$ satisfying $l_{2i} > q_0$. We have the following theorem to approximately construct the full state observer. The estimation error is controlled by the design parameters.

Theorem 3.3 *The state observer of the system (2.1) with $E = I$ can be considered as*

$$\begin{cases} \dot{\hat{x}}(t) = F\hat{x}(t) + Gu(t) + Kw(t) + L(y(t) - \hat{y}(t)), \ \hat{x}(t_0) = 0, \\ \hat{y}(t) = H\hat{x}(t) + Dw(t), \end{cases} \tag{3.33}$$

where $\hat{x}(t)$ is the estimated state, $w(t) = [w_{1,\eta_1} \ \cdots \ w_{p,\eta_p}]^T$ is the estimate of the disturbance $v(t)$ obtained in Section 3.2.2, the design matrix L is chosen such that the matrix $F - LH$ is stable. Then, there exists a function $\epsilon(\nu_{i,j_i}\big|i \in S;\ j_i = 1, \cdots, \eta_i) > 0$ with the property $\lim_{\sum_{i=1}^{p}\sum_{j_i=1}^{\eta_i}|\nu_{i,j_i}|\to 0} \epsilon(\nu_{i,j_i}\big|i \in S;\ j_i = 1, \cdots, \eta_i) \to 0$ such that

$$\|x(t) - \hat{x}(t)\|_2 \le \epsilon(\delta_{i,j_i}\big|i \in S;\ j = 1, \cdots, \eta_i), \tag{3.34}$$

as $t \to \infty$, where S is the subset of $\{1, \cdots, p\}$ satisfying the condition: if $i \in S$, then $\eta_i > 0$.

Proof It can be seen from Assumption 1 that there exists a matrix L such that $F - LH$ is stable. From (2.1) and (3.33), it gives

$$\dot{e}(t) = (F - LH)e(t) - (K + LD)\{v(t) - w(t)\}, \tag{3.35}$$

where $e(t)$ is defined as $e(t) = x(t) - \hat{x}(t)$. As $w(t)$ is the estimate of $v(t)$, by employing Theorem 3.1 and the stability of matrix $F - LH$, the result can be easily proved.

3.3 The numerical observation algorithm for the case that E is nonsingular

Suppose $E = I$. Otherwise, pre-multiply the first equation of (2.1) with E^{-1}.

S1 If D is of full rank, then the disturbance $v(t)$ and the state $x(t)$ are asymptotically identified by Theorem 3.1. Otherwise, go to S2.

S2 Derive the system (3.13) based directly on (2.1).

S3 Identify the disturbance $v_i(t)$ by (3.17) or Theorem 3.2.

S4 Identify the state $x_i(t)$ by using (3.32) or extracting from the constructed Luenberger-type state observer formulated in Theorem 3.3.

4 Observation for the general descriptor system

4.1 Some preparations

Suppose the matrix E is of rank l ($l < n$). Since E is known, we can find nonsingular matrices $P, Q \in R^{n\times n}$ such that

$$PEQ^{-1} = \begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.1}$$

Thus, by taking the transformation

$$\bar{x}(t) = Qx(t), \tag{4.2}$$

the system (2.1) can be rewritten as

$$\begin{cases} \begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} \dot{\bar{x}}(t) = PFQ^{-1}\bar{x}(t) + PGu(t) + PKv(t), \\ y(t) = HQ^{-1}\bar{x}(t) + Dv(t). \end{cases} \tag{4.3}$$

Lemma 4.1 *For the system (4.3), we have*

$$\text{rank} \begin{bmatrix} \begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} \\ HQ^{-1} \end{bmatrix} = n, \tag{4.4}$$

$$\text{rank} \begin{bmatrix} PFQ^{-1} - c\begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} \\ HQ^{-1} \end{bmatrix} = n \ \textit{for all}\ c \in C, \tag{4.5}$$

and $\begin{bmatrix} PFQ^{-1} - c\begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} & PK \\ HQ^{-1} & D \end{bmatrix}$ *is of full rank for any* $c \in C$ *satisfying* $Re(c) \geq 0$.

Proof The lemma can be easily proved by observing the following facts

$$\begin{bmatrix} \begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} \\ HQ^{-1} \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E \\ H \end{bmatrix} Q^{-1}, \tag{4.6}$$

$$\begin{bmatrix} PFQ^{-1} - c\begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} \\ HQ^{-1} \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F - cE \\ H \end{bmatrix} Q^{-1}, \tag{4.7}$$

$$\begin{bmatrix} PFQ^{-1} - c\begin{bmatrix} I_{l\times l} & 0 \\ 0 & 0 \end{bmatrix} & PK \\ HQ^{-1} & D \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F - cE & K \\ H & D \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix}, \tag{4.8}$$

From now on, we will start with the system (4.3) to estimate the disturbance and the state. Now rewrite the system (4.3) as

$$\begin{cases} \dot{\bar{x}}_1(t) = F_{11}\bar{x}_1(t) + F_{12}\bar{x}_2(t) + G_1u(t) + K_1v(t), \\ 0 = F_{21}\bar{x}_1(t) + F_{22}\bar{x}_2(t) + G_2u(t) + K_2v(t), \\ y(t) = H_1\bar{x}_1(t) + H_2\bar{x}_2(t) + Dv(t), \end{cases} \tag{4.9}$$

where

$$\bar{x}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix},\ PFQ^{-1} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},\ PG = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix},\ HQ^{-1} = [H_1\ H_2],\ PK = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \tag{4.10}$$

Lemma 4.2 *$H_2 \in R^{r\times(n-l)}$ is of full rank and r satisfies $r \geq n-l$.*

Proof From Lemma 4.1 and the assumptions, the lemma is obvious.

Since the matrix $H_2 \in R^{r\times(n-l)}$ is of full rank, there exists a nonsingular matrix $O = \begin{bmatrix} O_1 \\ O_2 \end{bmatrix} \in R^{r\times r}$ such that

$$OH_2 = \begin{bmatrix} H_{21} \\ 0 \end{bmatrix}, \tag{4.11}$$

where $H_{21} \in R^{(n-l)\times(n-l)}$ is a nonsingular matrix.

Therefore, by pre-multiplying the third equation in (4.9) with O, equation (4.9) yields

$$\begin{cases} \dot{\bar{x}}_1(t) = F_{11}\bar{x}_1(t) + F_{12}\bar{x}_2(t) + G_1u(t) + K_1v(t), \\ 0 = F_{21}\bar{x}_1(t) + F_{22}\bar{x}_2(t) + G_2u(t) + K_2v(t), \\ O_1y(t) = O_1H_1\bar{x}_1(t) + H_{21}\bar{x}_2(t) + O_1Dv(t), \\ O_2y(t) = O_2H_1\bar{x}_1(t) + O_2Dv(t). \end{cases} \tag{4.12}$$

By the third equation in (4.12), $\bar{x}_2(t)$ can be expressed as

$$\bar{x}_2(t) = H_{21}^{-1}O_1\big(y(t) - H_1\bar{x}_1(t) - Dv(t)\big). \tag{4.13}$$

By substituting (4.13) into the first two equations in (4.12), equation (4.12) yields

$$\begin{cases} \dot{\bar{x}}_1(t) = (F_{11} - F_{12}H_{21}^{-1}O_1H_1)\bar{x}_1(t) + F_{12}H_{21}^{-1}O_1y(t) + G_1u(t) + (K_1 - F_{12}H_{21}^{-1}O_1D)v(t), \\ \begin{bmatrix} F_{22}H_{21}^{-1}O_1 \\ O_2 \end{bmatrix} y(t) = \begin{bmatrix} F_{21} - F_{22}H_{21}^{-1}O_1H_1 \\ O_2H_1 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} G_2 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} K_2 - F_{22}H_{21}^{-1}O_1D \\ O_2D \end{bmatrix} v(t). \end{cases} \tag{4.14}$$

Now, for simplicity, we rewrite the system (4.14) in the following compact form

$$\begin{cases} \dot{\bar{x}}_1(t) = \bar{F}\bar{x}_1(t) + \bar{u}(t) + \bar{K}v(t), \\ \bar{y}(t) = \bar{H}\bar{x}_1(t) + \bar{D}v(t), \end{cases} \tag{4.15}$$

where the matrices $\bar{F},\ \bar{K},\ \bar{H},\ \bar{D}$ are defined as

$$\bar{F} = F_{11} - F_{12}H_{21}^{-1}O_1H_1,\ \bar{K} = K_1 - F_{12}H_{21}^{-1}O_1D, \tag{4.16}$$

$$\bar{H} = \begin{bmatrix} F_{21} - F_{22}H_{21}^{-1}O_1H_1 \\ O_2H_1 \end{bmatrix},\ \bar{D} = \begin{bmatrix} K_2 - F_{22}H_{21}^{-1}O_1D \\ O_2D \end{bmatrix}, \tag{4.17}$$

$\bar{u}(t)$ and $\bar{y}(t)$ are represented by

$$\bar{u}(t) = F_{12}H_{21}^{-1}O_1y(t) + G_1u(t),\ \bar{y}(t) = \begin{bmatrix} F_{22}H_{21}^{-1}O_1y(t) - G_2u(t) \\ O_2y(t) \end{bmatrix}. \tag{4.18}$$

Remark 4.1 The matrices $\bar{F} \in R^{l\times l},\ \bar{K} \in R^{l\times p},\ \bar{H} \in R^{r\times l},\ \bar{D} \in R^{r\times p}$ are available because they can be computed out by using the known matrices $E,\ F,\ G,\ K,\ H$, and D.

Remark 4.2 $\bar{u}(t) \in R^l$ and $\bar{y}(t) \in R^r$ are available signals. Since $r \geq p$, the number of the outputs of the system (4.15) is not smaller than that of the disturbances.

Lemma 4.3 *The system (4.15) is observable in the absence of the disturbance $v(t)$.*

Proof Since

$$\left[\begin{array}{c} PFQ^{-1} - c\left[\begin{array}{cc} I_{l\times l} & 0 \\ 0 & 0 \end{array}\right] \\ HQ^{-1} \end{array}\right] = \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & O^{-1} \end{array}\right] \left[\begin{array}{cc} F_{11} - cI & F_{12} \\ F_{21} & F_{22} \\ \hline O_1H_1 & H_{21} \\ O_2H_1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & O^{-1} \end{array}\right] \left[\begin{array}{cc|cc} I & 0 & F_{12}H_{21}^{-1} & 0 \\ 0 & I & F_{22}H_{21}^{-1} & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{cc} F_{11} - F_{12}H_{21}^{-1}O_1H_1 - cI & 0 \\ F_{21} - F_{22}H_{21}^{-1}O_1H_1 & 0 \\ \hline O_1H_1 & H_{21} \\ O_2H_1 & 0 \end{array}\right], \tag{4.19}$$

we obtain by Lemma 4.1 that $\left[\begin{array}{c} F_{11} - F_{12}H_{21}^{-1}O_1H_1 - cI \\ \hline F_{21} - F_{22}H_{21}^{-1}O_1H_1 \\ O_2H_1 \end{array}\right]$, i.e. $\left[\begin{array}{c} \bar{F} - cI \\ \bar{H} \end{array}\right]$, is of full rank for all $c \in C$ by using the fact that $H_{21} \in R^{(n-l)\times(n-l)}$ is a nonsingular matrix. Thus, the observability of the system (4.15) is verified.

Lemma 4.4 *The system (4.15) is in minimum phase with respect to the relation between the disturbance $v(t)$ and the "output" $\bar{y}(t)$.*

Proof Since

$$\left[\begin{array}{cc} PFQ^{-1}-c\left[\begin{array}{cc} I_{l\times l} & 0 \\ 0 & 0 \end{array}\right] & PK \\ HQ^{-1} & D \end{array}\right] = \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & O^{-1} \end{array}\right] \left[\begin{array}{cc|c} F_{11}-cI & F_{12} & K_1 \\ F_{21} & F_{22} & K_2 \\ \hline O_1H_1 & H_{21} & O_1D \\ O_2H_1 & 0 & O_2D \end{array}\right]$$

$$= \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & O^{-1} \end{array}\right] \left[\begin{array}{cc|cc} I & 0 & F_{12}H_{21}^{-1} & 0 \\ 0 & I & F_{22}H_{21}^{-1} & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right] \times$$

$$\left[\begin{array}{cc|c} F_{11}-F_{12}H_{21}^{-1}O_1H_1-cI & 0 & K_1 - F_{12}H_{21}^{-1}O_1D \\ F_{21}-F_{22}H_{21}^{-1}O_1H_1 & 0 & K_2-F_{22}H_{21}^{-1}O_1D \\ \hline O_1H_1 & H_{21} & O_1D \\ O_2H_1 & 0 & O_2D \end{array}\right], \tag{4.20}$$

we obtain from Lemma 4.1 that $\left[\begin{array}{c|c} F_{11} - F_{12}H_{21}^{-1}O_1H_1 - cI & K_1 - F_{12}H_{21}^{-1}O_1D \\ \hline F_{21} - F_{22}H_{21}^{-1}O_1H_1 & K_2 - F_{22}H_{21}^{-1}O_1D \\ O_2H_1 & O_2D \end{array}\right]$, i.e. $\left[\begin{array}{cc} \bar{F} - cI & \bar{K} \\ \bar{H} & \bar{D} \end{array}\right]$, is of full rank for any $c \in C$ satisfying $\mathrm{Re}(c) \geq 0$ by observing that $H_{21} \in R^{(n-l)\times(n-l)}$ is a nonsingular matrix. Thus, the lemma is proved.

From Lemmas 4.3 and 4.4, the system (4.15) can be used to estimate the disturbances and to observe the partial state $\bar{x}_1(t)$ by the algorithm proposed in Section 3.3. Furthermore, from (4.13), the partial state $\bar{x}_2(t)$ can be estimated as

$$\hat{\bar{x}}_2(t) = H_{21}^{-1} O_1(y(t) - H_1 \hat{\bar{x}}_1(t) - Dw(t)), \tag{4.21}$$

where $\hat{\bar{x}}_1(t)$ is the estimate of the partial state $\bar{x}_1(t)$, $w(t)$ is the estimate of the disturbance $v(t)$. Therefore, the state $x(t)$ can be estimated by using the transformation

$$\hat{x}(t) = Q^{-1} \begin{bmatrix} \hat{\bar{x}}_1(t) \\ \hat{\bar{x}}_2(t) \end{bmatrix}. \tag{4.22}$$

4.2 The numerical observation algorithm for the general descriptor systems with disturbances

Step1 If E is nonsingular, then the algorithm is given in Section 3.3. Otherwise, go to step 2.

Step2 Determine the nonsingular matrices P and Q satisfying (4.1), derive the system (4.9), and consider the state observer and the disturbance observer for the system (4.9).

Step3 For the matrix H_2 in the system (4.9), determine the nonsingular matrix O satisfying (4.11). The system (4.10) is rearranged as the dynamical system (4.15) and relation (4.13). For the dynamical system (4.15), the algorithm presented in Section 3.3 can be used to estimate the disturbance $v(t)$ and the partial state $\bar{x}_1(t)$.

Step4 Construct the observer $\hat{\bar{x}}_2(t)$ for the partial descriptor state $\bar{x}_2(t)$ by (4.21).

Step5 The descriptor state $x(t)$ is estimated by (4.22).

5 Design examples and simulation results

Example 5.1 Consider the descriptor system

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \tag{5.1}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{5.2}$$

where the input $u(t)$ is assumed as zero, the disturbances are governed by

$$v_1(t) = \phi(t) + \psi(t), \ v_2(t) = 1 + \psi(t) \tag{5.3}$$

with $\phi(t) = \begin{cases} t & 0 \le t \le 3 \\ 3 & t > 3 \end{cases}$ and $\psi(t) = \begin{cases} t & 0 \le t \le 6 \\ 4 & t > 6 \end{cases}$.

It can be easily checked that the assumptions in Section 2 are all satisfied. Furthermore, it can be checked that $\deg(\det(sE-F)) = rank(E)$, i.e. this descriptor system is of index at most one.

In the following, the disturbance observer and the descriptor state observer will be formulated by following the algorithm summarized in Section 4.2.

Step1 Since E is singular with $rank(E) = 2$, go to step 2.

Step2 The nonsingular matrices P and Q satisfying (4.1) are determined as

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let $\bar{x}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{12} \\ \hline \bar{x}_{21} \end{bmatrix} = Qx(t)$. Then, corresponding to (4.9), it yields

$$\begin{cases} \dot{\bar{x}}_1(t) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \bar{x}_2(t) + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} v(t), \\ 0 = [-1 \quad 2]\bar{x}_1(t) + \bar{x}_2(t) + [0 \quad -1]v(t), \\ y(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{x}_2(t) + \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} v(t). \end{cases}$$

Step3 For the matrix $H_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the nonsingular matrix O satisfying (4.11) can be determined as

$$O = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Thus, corresponding to (4.12), it gives

$$\begin{cases} \dot{\bar{x}}_1(t) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \bar{x}_2(t) + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} v(t), \\ 0 = [-1 \ 2]\bar{x}_1(t) + \bar{x}_2(t) + [0 \ -1]v(t), \\ [1 \ 0]\, y(t) = [0 \ 0]\bar{x}_1(t) + \bar{x}_2(t) + [0 \ 1]v(t), \\ [-1 \ 1]\, y(t) = [0 \ 1]\bar{x}_1(t) + [0 \ 1]v(t). \end{cases} \tag{5.4}$$

Then, from the third equation in (5.4), $\bar{x}_2(t)$ can be expressed as

$$\bar{x}_2(t) = [1 \quad 0]y(t) - [0 \quad 1]v(t). \tag{5.5}$$

By substituting the expression of $\bar{x}_2(t)$ into the other equations in (5.4), the equation corresponding to (4.15) is given by

$$\begin{cases} \dot{\bar{x}}_1(t) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} y(t) + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} v(t), \\ \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} y(t) = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} v(t). \end{cases} \tag{5.6}$$

Now, based on (5.6), the disturbance $v(t)$ and the variable $\bar{x}_1(t)$ will be estimated by following the algorithm summarized in Section 3.3.

S1 Since $\bar{D} = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$ is not of full rank, go to S2.

S2 Rewrite (5.6) as

$$\left[\begin{array}{cc|cc} 2-s & 0 & 1 & 1 \\ 0 & -2-s & 0 & 1 \\ \hline -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right] \begin{bmatrix} \bar{x}_1(t) \\ v(t) \end{bmatrix} = \left[\begin{array}{cc} 2 & 0 \\ -1 & 0 \\ \hline -1 & 0 \\ -1 & 1 \end{array}\right] y(t). \tag{5.7}$$

By pre-multiplying the both sides of (5.7) with

$$adj \left[\begin{array}{cc|cc} 2-s & 0 & 1 & 1 \\ 0 & -2-s & 0 & 1 \\ \hline -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} 0 & 4 & s+3 & 2s+2 \\ 0 & 1 & 0 & -1 \\ -s-3 & 4s-7 & s^2+s-6 & 2s^2-s-2 \\ 0 & -1 & 0 & -s-2 \end{bmatrix}$$

the system corresponding to (3.13) is derived as

$$\begin{cases} s(3y_1 - 2y_2) + 9y_1 - 2y_2 = (s+3)\bar{x}_{11}, \\ y_2 = (s+3)\bar{x}_{12}, \\ s^2(3y_1 - 2y_2) = -s(6y_1 + y_2) + 9y_1 - 2y_2 + (s+3)v_1, \\ s(-y_1 + y_2) = 3y_1 - 2y_2 + (s+3)v_2. \end{cases} \tag{5.8}$$

S3 Based on (5.8), the disturbance $v_2(t)$ can be simply estimated by

$$w_{2,0} \triangleq \frac{s}{s+3}(-y_1 + y_2) - \frac{1}{s+3}(3y_1 - 2y_2).$$

The disturbance $v_1(t)$ is estimated as follows.
Introduce the Hurwitz polynomial

$$g_1(s) = (s+3)(s+2),$$

where λ is chosen as $\lambda = 2$.
Define $z_1 = 3y_1 - 2y_2$. Corresponding to (3.10), the third equation in (5.8) can be rewritten as

$$\dot{z}_1(t) + 2z_1(t) = \frac{s}{s+3}(9y_1 - 11y_2) + \frac{1}{s+3}(27y_1 - 14y_2) + v_1(t).$$

By Theorem 3.2, construct the following differential equation

$$\dot{\hat{z}}_1(t) + 2\hat{z}_1(t) = \frac{s}{s+3}(9y_1 - 11y_2) + \frac{1}{s+3}(27y_1 - 14y_2) + w_{1,1}(t),$$
$$\hat{z}_1(0) = z_1(0),$$
$$w_{1,1}(t) = \hat{\omega}_{1,1}(t)\frac{z_1(t) - \hat{z}_1(t)}{|z_1(t) - \hat{z}_1(t)| + \delta_{1,1}},$$

$\hat{\omega}_{1,1}(t)$ is updated by the following adaptive algorithms

$$\dot{\hat{\omega}}_{1,1}(t) = \begin{cases} 1200|z_1(t) - \hat{z}_1(t)| & \text{if } |z_1(t) - \hat{z}_1(t)| > \delta_{1,1} \\ 0 & \text{otherwise} \end{cases}, \quad \dot{\hat{\omega}}_{1,1}(0) = 5\,.$$

Then, $w_{1,1}(t)$ can be regarded as an estimate of $v_1(t)$.

S4 By the theory in Section 3.2.3, the variables $\bar{x}_{11}(t)$ and $\bar{x}_{12}(t)$ can be respectively estimated by $\hat{\bar{x}}_{11}(t)$ and $\hat{\bar{x}}_{12}(t)$ defined by

$$\hat{\bar{x}}_{11} = \frac{s}{s+3}(3y_1 - 2y_2) + \frac{1}{s+3}(9y_1 - 2y_2),$$

$$\hat{\bar{x}}_{12} = \frac{1}{s+3}y_2.$$

Step4 Construct the observer $\hat{\bar{x}}_2(t)$ for the partial descriptor state $\bar{x}_2(t)$ by (5.5).

$$\hat{\bar{x}}_2(t) = [1 \quad 0]y(t) - [0 \quad 1]\begin{bmatrix} w_{1,1}(t) \\ w_{2,0}(t) \end{bmatrix} = y_1(t) - w_{2,0}(t).$$

Step5 The descriptor state $x(t)$ is estimated by

$$\hat{x}(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\bar{x}}_{11}(t) \\ \hat{\bar{x}}_{12}(t) \\ \hat{\bar{x}}_2(t) \end{bmatrix}.$$

It can be seen that some steady error exists in the estimation of $v_1(t)$, and the error depends on the design parameter δ_{11}. Furthermore, there are no steady errors existing in the estimation of the disturbance $v_2(t)$ and the descriptor state $x(t)$.

Computer simulation results show that the disturbance $v_2(t)$ and the descriptor state $x(t)$ can be perfectly identified. The figures are omitted. The estimation error of the disturbance $v_1(t)$ is shown in Figure 5.1, where the parameter δ_{11} is chosen as $\delta_{11} = 0.0001$.

It should be noted $v_1(t)$ is not differentiable at $t = 3$ and $t = 6$ and is not continuous at $t = 6$. Simulation results show that the disturbance observer works well at the continuous points and has a transient error at the discontinuous points. This is because that the proposed method is trying to identify the unknown signals by using a differentiable approach. It is considered that the new method can be applied to practical problems with piecewise differentiable disturbances. For the sake of strictness, the disturbances are assumed to be continuous and piecewise differentiable.

Example 5.2 Consider the descriptor system

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad (5.9)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad (5.10)$$

where the input $u(t)$ is assumed as zero, $v_1(t)$ and $v_2(t)$ are the disturbances.

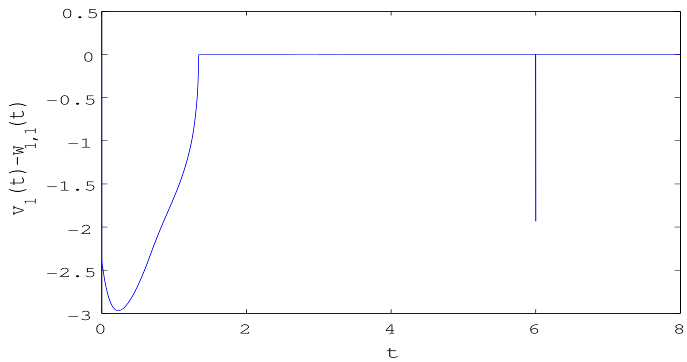

Figure 5.1: The difference between the disturbance $v_1(t)$ and its estimate

It can be easily checked that the assumptions in Section 2 are all satisfied. Furthermore, it can be checked that $\deg(\det(sE - F)) \neq rank(E)$, i.e. this descriptor system is not of index at most one.

Since E is singular with $rank(E) = 2$, the nonsingular matrices P and Q satisfying (3.32) are determined as

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}, \qquad Q = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let $\bar{x}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{12} \\ \hline \bar{x}_{21} \end{bmatrix} = Qx(t)$. Then, by a computation similar to that in Example 5.1, the relations corresponding to (5.5) and (5.8) are derived as

$$\bar{x}_2(t) = [1 \quad 0]y(t) - [0 \quad 1]v(t), \tag{5.11}$$

$$\begin{cases} s(-y_2) + 15y_1 - 4y_2 = (2s+3)\bar{x}_{11}, \\ 6y_1 - y_2 = (2s+3)\bar{x}_{12}, \\ s^2(-y_2) = -s(15y_1 - y_2) + 33y_1 - 10y_2 + (2s+3)v_1, \\ s(y_2) = 3y_1 - 2y_2 + (2s+3)v_2. \end{cases} \tag{5.12}$$

Similar to Example 5.1, the disturbances and the descriptor state can be identified. It can be seen that the proposed method can also deal with the descriptor systems which are not of index at most one.

6 Conclusions

In this paper, the observation problem for the descriptor systems with disturbances is studied. It is assumed that the disturbances and their first order derivatives are bounded in the open loop. However, the upper and lower bounds are unknown. The formulated descriptor system can be decomposed into a dynamical system and an algebraic equation. Based on this obtained dynamical system, first, the disturbances are estimated; then, one part of the descriptor state is observed. Finally, the other part of the descriptor state is estimated based on the obtained algebraic equation.

If D (if E is nonsingular; or $\bar{D}$ if E is singular) is of full rank, then the estimation errors of the full state and all the disturbances decay to zero exponentially. For the cases $l_{ji} < q_0$, the estimation errors of the corresponding partial states and disturbances still remain, and they can be controlled to be as small as necessary by choosing the design parameters. For the cases $l_{ji} \geq q_0$, no steady errors exist in the estimates of the corresponding partial states and disturbances. After the disturbance and the descriptor state are estimated, the controller can be designed by referring to the results in [8, 16, 20].

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