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Cause Effect Nonlinear Relations in Continuous Orbital Transfers under Superposed Pitch and Yaw Deviations

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Abstract: The thrust direction deviations effects in orbital transfers maneuvers cause linear and angular misalignments that displace the vehicle with respect to its nominal directions. Corrections maneuvers are realized, but these deviations are increasing during the vehicle life time due to the propulsion system consuming. The main of the corrections maneuvers are not reached during this period. The vehicle is lost due to the dissipatives forces and the thrusters systems deviations. The understanding of these deviations effects through the final orbit is very important to the mission control under pitch and yaw deviations. In this paper, we show the algebraic relations between these deviations and the keplerian elements of the vehicle final orbit. This analysis allowed to found the theoretical and exact nonlinear cause effect relation between the media values of the keplerian elements (final semi-major axis) and the superposed burn-direction deviations. The dissipatives forces effects were not considered with respect to these thrust deviations during the transfers maneuvers.

Keywords: *Pitch; yaw; thrust deviations; superposed; nonlinear relation.*

Mathematics Subject Classification (2000): 70M20.

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1 Introduction

The optimum orbital transfer problem of the space vehicle was studied initially by Goddard [1] in 1919, with his pioneer paper about the maximization of the final altitude of the rocket under the gravitation field and atmosphere drag. Hohmann [2] in 1925 found the minimum fuel solution for the bi-impulsive transfer problem between two space vehicle circular and coplanar orbits. This solution was considered the final solution for this problem until 1959. In this year Hoelker and Silber [3] published the minimum fuel condition for the Hohmann transfer limited to 11.94. This value is the ratio between the final orbit and initial orbit radius to the bi-impulsive maneuver. In the non-impulsives maneuvers is it very important to know one of the thrust burn point to avoid the misalignment's thrust. Lawden [4] in 1955 found optimum directions to thrust application and showed that the thrust direction would be tangente to the trajectory. There are others maneuvers of tri- and multi-impulsives under minimum fuel consumption condition and under others in-orbital restrictions, studied by many authors through several methods. Many authors used the propulsion system as control system to reach many purposes. See e.g., Kluever and Tanck [5] (1997), Javorsek II and Longuski [6] (1999), Vassar and Sherwood [7] (1985), Ulybyshev [8] (1998), etc. The orbital and rotational motions coupling effects through the transfer maneuvers were studied, e.g., Duboshin [9] (1958), Barkin [10] (1985), Beletski [11] (1990), Wang et alli [12] (1991), Wang et alii [13] (1992) and Maciejewski [14] (1995), etc.

Study of the superposition thrust deviations effects for the space vehicle trajectories is important, due to their technological importance and space missions feasibility to vehicle under thrusters burns. The poorly modeled maneuvers and/or non optimized to turn aside from nominal orbits and the correction or vehicle capture maneuvers can be unworkable, due to the operational expense and the fuel availability on board. The satellite orbits under non-ideal propulsion system are affected due to the non-superposed directions thrust. This results was verified by Jesus [15] in (1999) to orbital transfers planar maneuvers. The numerical analysis of the maneuvers under superposed and correlated thrust deviations were published in 2004 by Jesus and Santos [16] and the mission feasibility analysis under thrust direction deviations by Jesus et al. [17]. In this paper we realized the algebraic demonstration of the cause/effect relation between the semi-major axis deviations and the thrust superposed pitch and yaw deviations to orbital transfers maneuvers. Our results were found without restrictions on the kind of maneuvers with respect to the their altitude, power thrusters, etc.

2 Mathematical Formulation and Coordinate System

The mathematical problem is to find the motion equations of the space vehicle and show the cause/effect algebraic relation between superposed "pitch" α and "yaw" β, thrust direction deviations and the semi-major axis deviations of the orbital transfer trajectory. Besides this, the maneuvers were considered optimum as minimum fuel consumption condition. The Figure 2.1, shows the reference system where we wrote the Newton laws. The optimum problem associated with the space vehicle orbital dynamic is:

1) Globally minimize the performance index: $J = m(t_0) - m(t_f)$;

2) With respect to $\alpha : [t_0, t_f] \to R$ ("pitch" angle) and $\beta : [t_0, t_f] \to R$ ("yaw" angle) with α , $\beta \in C^{-1}$ in $[t_0, t_f]$;

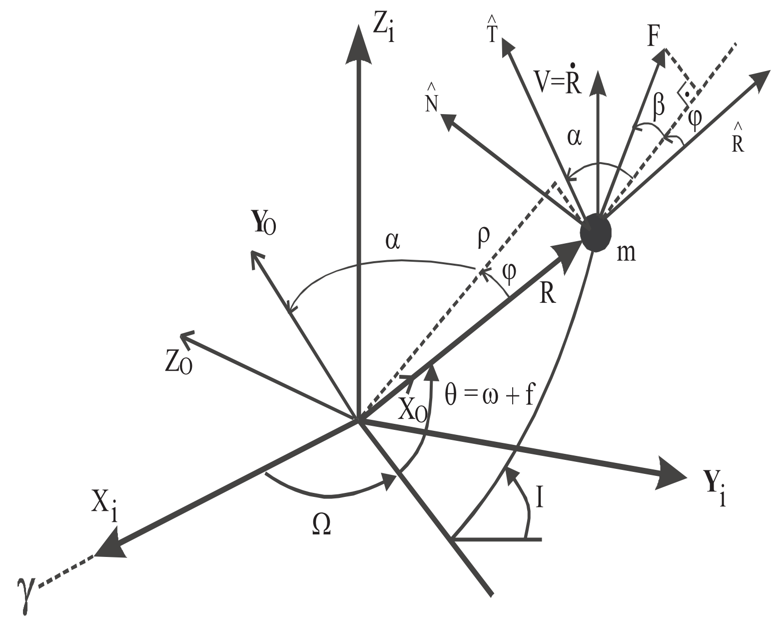

Figure 2.1: Reference systems used in this work.

3) Consideration of the dynamics in inertial coordinates X_i, Y_i, Z_i (see Figure 2.1) for $\forall t \in [t_0, t_f]$

$$m\frac{d^2X}{dt^2} = -\mu m\frac{X}{R^3} + F_x, \tag{1}$$

$$m\frac{d^2Y}{dt^2} = -\mu m\frac{Y}{R^3} + F_y, \tag{2}$$

$$m\frac{d^2Z}{dt^2} = -\mu m\frac{Z}{R^3} + F_z, \tag{3}$$

$$F_x = F\left[\cos\beta\sin\alpha\left(\cos\Omega\cos\theta - \sin\Omega\cos I\sin\theta\right) + \sin\beta\sin\Omega\sin I - \cos\beta\cos\alpha\left(\cos\Omega\sin\theta + \sin\Omega\cos I\cos\theta\right)\right], \tag{4}$$

$$F_y = F\left[\cos\beta\sin\alpha\left(\sin\Omega\cos\theta + \cos\Omega\cos I\sin\theta\right) - \sin\beta\cos\Omega\sin I - \cos\beta\cos\alpha\left(\sin\Omega\sin\theta - \cos\Omega\cos I\cos\theta\right)\right], \tag{5}$$

$$F_z = F\left(\cos\beta\sin\alpha\sin I\sin\theta + \cos\beta\cos\alpha\sin I\cos\theta + \sin\beta\cos I\right). \tag{6}$$

The V_N, V_T and V_R are the normal, transversal and radial velocity components, respectively. Their accelerations are a_N, a_T and a_R. These equations in orbital coordinates (radial R, transverse T, and binormal N) of Figure 2.1 are:

$$
\begin{aligned}
ma_R(t) &= F\cos(\beta(t)+\Delta\beta(t))\sin(\alpha(t)+\Delta\alpha(t)) - \frac{\mu m}{R^2(t)}, && (7)\\
ma_T(t) &= F\cos(\beta(t)+\Delta\beta(t))\cos(\alpha(t)+\Delta\alpha(t)), && (8)\\
ma_N(t) &= F\sin(\beta(t)+\Delta\beta(t)), && (9)\\
a_R(t) &= \dot{V}_R - \frac{V_T^2}{R} - \frac{V_N^2}{R}, && (10)\\
a_T(t) &= \dot{V}_T + \frac{V_R V_T}{R} - V_N \dot{I}\cos\theta - V_N\dot{\Omega}\sin I\sin\theta, && (11)\\
a_N(t) &= \dot{V}_N + \frac{V_R V_N}{R} + V_T \dot{I}\cos\theta + V_T\dot{\Omega}\sin I\sin\theta, && (12)\\
V_R &= \dot{R}, && (13)\\
V_T &= R(\dot{\Omega}\cos I + \dot{\theta}), && (14)\\
V_N &= R(-\dot{\Omega}\sin I\cos\theta + \dot{I}\sin\theta), && (15)\\
\theta &= \omega + f, && (16)
\end{aligned}
$$

where $\Delta\alpha$ and $\Delta\beta$ are the errors in the "pitch" and in the "yaw" angles, respectively. In this way, for each implementation of the orbital transfer arc, values of α and β are chosen, whose errors are inside the range, that produce the direction for the overall minimum fuel consumption. If we consider mass time-variable, for example, linear variation, so,

$$m(t) = m(t_o) + \dot{m}(t - t_o) \tag{17}$$

with $\dot{m} < 0$ and

$$F \cong \mid \dot{m} \mid c. \tag{18}$$

The motion Equations (1),(2),(3) etc. must be modified to include the force associated to mass variation.

4) Given the initial and final orbits, and the parameters of the problem $m(t_o), c, \ldots$ these equations were obtained by: 1) writing in coordinates of the dexterous rectangular reference system with inertial directions $OX_iY_iZ_i$ the Newton's laws for the motion of a satellite S with mass m, with respect to this reference system, centered in the Earth's center of mass O with X_i axis toward the Vernal point, X_iY_i plane coincident with Earth's Equator,and Z_i axis toward the Polar Star approximately; 2) rewriting them in coordinates of the dexterous rectangular reference system with radial, transverse, binormal directions $SRTN$, centered in the satellite center of mass S; helped by 3) a parallel system with $OX_oY_oZ_o$ directions, centered in the Earth's center of mass O, X_o axis toward the satellite S, X_oY_o plane coincident with the plane established by the position $\vec{R}$ and velocity $\vec{V}$ vectors of the satellite, and Z_o axis perpendicular to this plane; and helped by 4) the instantaneous Keplerian coordinates $(\Omega, I, \omega, f, a, e)$. These equations were later rewritten and simulated by using 5) 9 state variables, defined and used by Biggs [19, 20] and Prado [21].

3 Orbital Continuous Transfer under Thrust Deviations – General Equations

The cause/effect relation between the thrust vector directions deviations and the semi-major axis of the final orbit can be found if we consider the mechanical energy of the space vehicle under thrusters burns. This dynamics is under action of two forces: natural force (gravity) and nonnatural force (due to the thrusters). Our algebraic approach for the semi-major axis deviations is done through the rate of change of the space vehicle mechanical energy with respect to the time, which is equal to the power applied by forces components in the transverse, radial and normal directions. We considered the Earth mass and the space vehicle mass as punctual. Also we considered the thrust to be nonideal in direction, transfering the deviations effects to the vehicle dynamics. Their energy rate of change are:

$$\frac{dE_M(t)}{dt} = F\cos\alpha(t)\cos\beta(t)v_T(t) + F\sin\alpha(t)\cos\beta(t)v_R(t) + F\sin\beta(t)v_N(t). \tag{19}$$

In the Equation (19) the powers are included applied by forces components in the transverse, radial and normal directions, without the thrust deviations. During the time interval Δt, we integrated and found the change of the mechanical energy,

$$\begin{aligned}\Delta E_M(t_1,t_2) = E_M(t_2) - E_M(t_1) = \\ \int_{t_1}^{t_2} F[\cos\alpha(t)\cos\beta(t)v_T(t) + \sin\alpha(t)\cos\beta(t)v_R(t) + \sin\beta(t)v_N(t)]dt = \\ \frac{-\mu m}{2a(t_2)} + \frac{\mu m}{2a(t_1)}\end{aligned} \tag{20}$$

with $a(t_i)$ the semi-major axis of the space vehicle orbit in the instant i. This mechanical energy change, Equation (20), can be computed for one transfer under "pitch", $\Delta\alpha(t)$ and "yaw", $\Delta\beta(t)$ deviations. So,

$$\begin{aligned}\Delta E'_M(t_1,t_2) = E'_M(t_2) - E'_M(t_1) = \\ \int_{t_1}^{t_2} F[\cos(\alpha(t)+\Delta\alpha(t))\cos(\beta(t)+\Delta\beta(t))v'_T(t)]dt + \int_{t_1}^{t_2} F\sin(\beta(t) + \\ \Delta\beta(t))v'_N(t)dt + \int_{t_1}^{t_2} F[\cos(\beta(t)+\Delta\beta(t))\sin(\alpha(t)+\Delta\alpha(t))v'_R(t)]dt = \\ \frac{-\mu m}{2a'(t_2)} + \frac{\mu m}{2a'(t_1)}.\end{aligned} \tag{21}$$

The terms in ($'$) denotes functions under thrust deviations influence. We define $\Delta_2 E_M$ as change of the mechanical energy change, that is, the difference between its values with and without thrust deviations. So, taking the difference between Equations (20) and (21) and using a small mathematical manipulation,

$$\Delta_2 E_M(t_1,t_2) \equiv \Delta E'_M(t_1,t_2) - \Delta E_M(t_1,t_2) = \frac{-\mu m}{2a'(t_2)} + \frac{\mu m}{2a'(t_1)} + \frac{\mu m}{2a(t_2)} + \frac{-\mu m}{2a(t_1)} =$$

$$\int_{t_1}^{t_2} F[\cos(\beta(t)+\Delta\beta(t))\sin(\alpha(t)+\Delta\alpha(t))v'_R(t) - \sin\alpha(t)\cos\beta(t)v_R(t)]dt + \int_{t_1}^{t_2} F[\sin(\beta(t)+\Delta\beta(t))v'_N(t) - \sin\beta(t)v_N(t)]dt +$$

$$\int_{t_1}^{t_2} F[\cos(\alpha(t)+\Delta\alpha(t))\cos(\beta(t)+\Delta\beta(t))v'_T(t) - \cos\alpha(t)\cos\beta(t)]v_T(t)dt. \tag{22}$$

4 $\Delta\alpha(t)$ and $\Delta\beta(t)$ Not Correlated with the Transverse, Radial and Normal Velocities (Uniform Deviations)

Equation (22) is general, but we need to realize the integrations to realistic space missions conditions. In this way, we consider that the direction deviations are not correlated (in the time) with the transversal, radial and normal velocities components. That is, during the burn time, the thrust vector deviations have not functional dependence with space vehicle velocity. Besides this, we consider that the semi-major axis in the initial instant to the initial and final orbits are equal. This condition is physically reasonable, because during the initial instant there is no time to the deviations affect the semi-major axis.

To find the cause/effect relation, we apply the expectation operator, $\mathcal{E}$, (or the first moment, or mean) over the Equation (22). In this way, we select the mean of the functions inside the integrations. We consider the probabilistic approach, where the mean over the physical functions is very good to represent the dynamic phenomena under deviations, define through probabilistic errors function (Gaussian, Uniform, etc.). This approach is applicable in the space technology, because the direction deviations are due to the several unpredictable reasons such as: vehicle mass center displacement, due to the fuel consumption or movable parts as solar panels, antennas, booms, pendulums, etc., and their angular deviations. These deviations and others in the thrust magnitude cause resultant nonideal force, which do not pass through the vehicle mass center. So, the linear and/or angular misalignments displace the vehicle with respect to its nominal directions. The technological implementations has shown that these deviations can be modeled through the uniform and gaussian probability function. We assume that stochastic processes are ergotic. So, the expectation operator (mean in the ensemble) commutes with the integral operator (in time). Besides this, the function F and the trigonometric functions are deterministic in time.

The non-correlation of the deviations with the velocities allows us to decompose the expectation operator as one product of the individual expectatins for the product of the functions. Therefore, taking the expectation, $\mathcal{E}$, and doing some algebraic manipulation, we have

$$\mathcal{E}[\Delta_2 E_M(t_1,t_2)] = [Q_{11}+Q_{22}][\mathcal{E}[\cos\Delta\beta(t)\cos\Delta\alpha(t)] - 1] + [Q_{12} + Q_{21}][\mathcal{E}[\cos\Delta\beta(t)\sin\Delta\alpha(t)] - 1] + [Q_{31}+Q_{42}][\mathcal{E}[\sin\Delta\beta(t)\cos\Delta\alpha(t)] - 1] + [Q_{32}+Q_{41}][\mathcal{E}[\sin\Delta\beta(t)\sin\Delta\alpha(t)] - 1] + Q_{93}[\mathcal{E}[\cos\Delta\beta(t)] - 1] + Q_{10}[\mathcal{E}[\sin\Delta\beta(t)] - 1] + [Q_{51}+Q_{52}+Q_{61}+Q_{71}+Q_{72}+Q_{82}], \tag{23}$$

where Q_{ij} are quadratures in sines and cosines. Besides this, we consider that the velocities effects inside the interval $[-\Delta\alpha_{max}, \Delta\alpha_{max}]$ in the same time are,practically, balanced, because the deviations occur between values maxima and minima inside them. That is, the velocities with and without deviations have, in mean, very close values. So,

$$\mathcal{E}[v'_{R,T,N}(t)] = v_{R,T,N}(t_1). \tag{24}$$

We consider the important approach of the $\Delta\alpha(t)$ and $\Delta\beta(t)$ are random-bias deviations with uniform distribution inside the interval $[-\Delta\alpha_{max}, \alpha_{max}]$, that is, $\Delta\alpha(t) = \Delta\alpha(t_1) = \Delta\alpha$ and $\Delta\beta(t) = \Delta\beta(t_1) = \Delta\beta$ (systematic deviations) through the orbital transfers. Besides this, we consider that the pitch and yaw deviations are not correlated with each other (it occurs in the practice) and that their values in anterior instant (due to thrusters action) are not correlated with their values in posterior instant (due to thrusters action again). This last effect was analized in 2004 by Jesus and Santos [16] in numerical approach. They modeled the consuming of the thrusters through it. Hier, we do not consider this effect. Therefore, applying the expectation operator over the first term (not correlated) of the Equation (23), for example,

$$\begin{aligned}\mathcal{E}\{\cos\Delta\alpha(t_1)\} = \frac{1}{2\Delta\alpha_{max}}\int_{\Delta\alpha_{max}}^{\Delta\alpha_{max}} \cos\Delta\alpha d(\Delta\alpha) = \\ \frac{1}{2\Delta\alpha_{max}}\sin[\Delta\alpha]_{\Delta\alpha_{max}}^{\Delta\alpha_{max}} = \frac{\sin\Delta\alpha_{max}}{\Delta\alpha_{max}}\end{aligned} \tag{25}$$

and

$$\mathcal{E}[\sin\Delta\beta(t_1)] = \mathcal{E}[\sin\Delta\alpha(t_1)] = 0. \tag{26}$$

If we compute the expectation over all the terms of the Equation (23), we obtain

$$\begin{aligned}\Delta_2 E_M(t_1,t_2) = C_1[[\frac{\sin\Delta\alpha_{max}}{\Delta\alpha_{max}}][\frac{\sin\Delta\beta_{max}}{\Delta\beta_{max}}] - 1] + C_2[[\frac{\sin\Delta\beta_{max}}{\Delta\beta_{max}} - 1] + \\ Q_{T1} - Q_{10} + Q_{T2}\end{aligned} \tag{27}$$

or, writting in Taylor expansion to the $\sin\Delta\alpha$, $\sin\Delta\beta$, we obtain

$$\begin{aligned}\mathcal{E}\{\Delta_2 E_M(t_1,t_2)\} = C_1\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\Delta\alpha_{max}^{2n}.\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\Delta\beta_{max}^{2n} + \\ C_2\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\Delta\beta_{max}^{2n} + Q_T,\end{aligned} \tag{28}$$

where Q_T, C_1 and C_2 are quadratures. The expectation over the left side of the Equation (23) provide

$$\mathcal{E}\{\Delta_2 E_M(t_1,t_2)\} = \mathcal{E}\{\frac{\mu m}{2a(t_2)} - \frac{\mu m}{2a'(t_2)}\} = \frac{\mu m}{2}\frac{1}{a(t_2)}\mathcal{E}\{\frac{\Delta a(t_2)}{a'(t_2)}\}. \tag{29}$$

If we expand Equation (29) about the rate $\frac{\Delta a(t_2)}{a(t_2)}$, we get

$$\mathcal{E}\{\Delta_2 E_M(t_1,t_2)\} = \frac{\mu m}{2}[\frac{1}{a^2(t_2)}\mathcal{E}\{\Delta a(t_2)\} - \frac{1}{a^3(t_2)}\mathcal{E}\{\Delta^2 a(t_2)\} + \frac{1}{a^4(t_2)2!}\mathcal{E}\{\Delta^3 a(t_2)\} - \frac{1}{a^5(t_2)3!}\mathcal{E}\{\Delta^4 a(t_2)\} + \ldots] = \frac{\mu m}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)(n-1)!}\mathcal{E}\{\Delta^n a(t_2)\}, \tag{30}$$

where, $\Delta a(t_2)$=$a'(t_2) - a(t_1)$. So,

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)(n-1)!}\mathcal{E}\{\Delta^n a(t_2)\} = [C_4 + C_3\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\Delta\alpha_{max}^{2n}].\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\Delta\beta_{max}^{2n} + C_5. \tag{31}$$

Equation (31) is the cause/effect relation between thrust vector pitch and yaw direction deviations and semi-major axis deviation of the final orbit. It shows that the direction deviations affect directly the transfer maneuvers. It is nonlinear relation in even power of the maxima deviations terms

$$C_3 = \frac{2C_1}{\mu m}; C_5 = \frac{2C_2}{\mu m}; C_5 = \frac{2Q_T}{\mu m}. \tag{32}$$

Equations (30) and (31) can be expandied, that is,

$$\mathcal{E}\{\Delta_2 E_M(t_1,t_2)\} = C_7 - \frac{C_3}{3!}(\Delta\alpha_{max}^2 + \Delta\beta_{max}^2) + \frac{1}{5!}(C_3\Delta\alpha_{max}^4 + C_6\Delta\beta_{max}^4) + \frac{1}{7!}(C_3\Delta\alpha_{max}^6 + C_6\Delta\beta_{max}^6) + \frac{1}{(3!)^2}(\Delta\alpha_{max}\Delta\beta_{max})^2 + \frac{1}{(5!)^2}(\Delta\alpha_{max}^2\Delta\beta_{max}^2)^2 + \frac{1}{(7!)^2}(\Delta\alpha_{max}^3\Delta\beta_{max}^3)^2 - \frac{C_3}{(3!5!)}(\Delta\alpha_{max}^2\Delta\beta_{max}^4 + \Delta\alpha_{max}^4\Delta\beta_{max}^2) + \frac{C_3}{(3!7!)}(\Delta\alpha_{max}^2\Delta\beta_{max}^6 + \Delta\alpha_{max}^6\Delta\beta_{max}^2) + \frac{C_3}{(5!7!)}(\Delta\alpha_{max}^4\Delta\beta_{max}^6 + \Delta\alpha_{max}^6\Delta\beta_{max}^4) + \ldots \tag{33}$$

The space missions conditions request direction deviations inside the practical interest range, that is, maximum two degree. So, for small deviations we can neglect high power terms. In this condition we can choose $n = 0$ for the expansion

$$\mathcal{E}\{\Delta a(t_2)\} = K_1 - K_2\Delta\alpha_{max}^2 - K_2\Delta\beta_{max}^2. \tag{34}$$

This is the first order nonlinear cause/effect relation between thrust direction deviations and semi-major axis. It is one revolution paraboloid. Our paper [18] in 2004 showed numerical simulation results of these superposed direction deviations case and found a revolution paraboloid deformed inside general deviations pitch and yaw range and revolution paraboloid not deformed inside the space missions practical interest range. This algebraic results confirm it (Figures 4.1, 4.2, 4.3, 4.4).The deviations in modulus thrust (DES1),in pitch direction (DES2) and in yaw direction (DES3). These deviations are modeled as operational (white-noise) and systematic (random-bias).

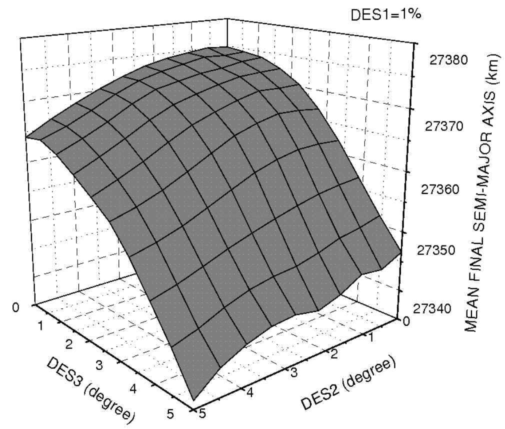

Figure 4.1: Noncoplanar Transfer under Operational Deviations.

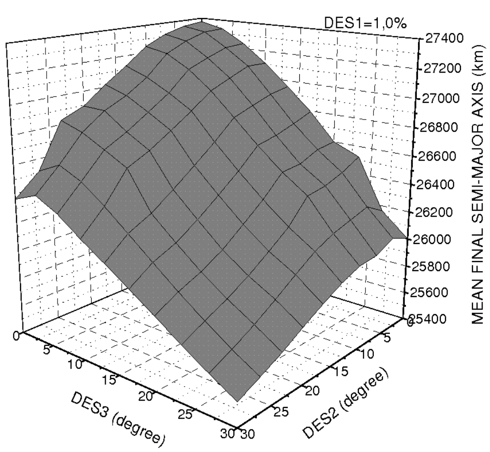

Figure 4.2: Noncoplanar Transfer under Systematic Deviations.

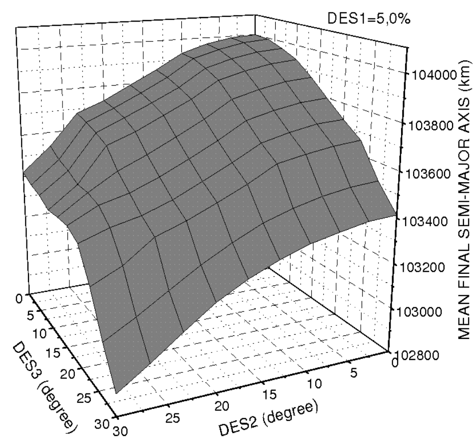

Figure 4.3: Coplanar Transfer under Operational Deviations.

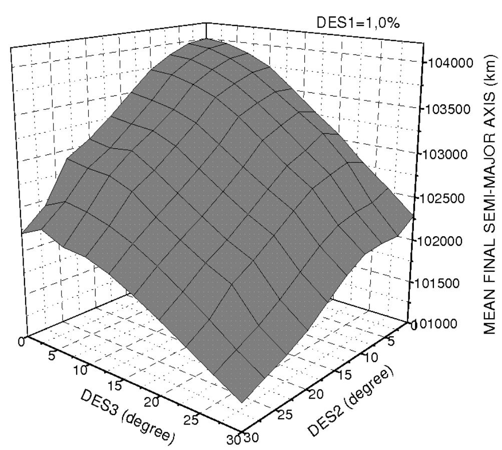

Figure 4.4: Coplanar Transfer under Systematic Deviations.

5 $\Delta\alpha(t)$ and $\Delta\beta(t)$ Not Correlated with the Transverse, Radial and Normal Velocities (Gaussian Deviations)

The procedures for the $\Delta\alpha(t)$ and $\Delta\beta(t)$ with Gaussian distribution inside the interval $[-\Delta\alpha_{max}, \Delta\alpha_{max}]$ and $[\Delta\alpha_{max}, \Delta\alpha_{max}]$ are the same for the uniform distribution. Therefore, applying the expectation operator over the first term (not correlated) of the Equation (23) for the Gaussian distribution, for example, we have

$$\mathcal{E}\{\cos\Delta\alpha(t_1)\} = \int_{-\infty}^{\infty} \cos\Delta\alpha \frac{\exp^{-\frac{(\Delta\alpha)}{2\sigma_\alpha}}}{\sqrt{2\pi}\sigma_\alpha} d(\Delta\alpha) = \exp^{\frac{-\sigma_\alpha{}^2}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{\sigma_\alpha^{2n}}{2^n n!}. \tag{35}$$

The expectation of the sinos terms are zero. So, we can obtain the nonlinear cause e effect relation between thrust direction gaussian deviations and the semi-major axis,

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{a^{n+1}(t_2)(n-1)!} \mathcal{E}\{\Delta^n a(t_2)\} =$$
$$[C_4 + C_3 \sum_{n=0}^{\infty} (-1)^n \frac{\sigma_\alpha^{2n}}{(2^n)n!}].\sum_{n=0}^{\infty} (-1)^n \frac{\sigma_\beta^{2n}}{(2^n)n!} + C_5, \tag{36}$$

where σ_α and σ_β are standard pitch and yaw deviations, respectively. Equation (36) is similar to Equation (31), that is, the nonlinear cause/effect relation do have dependence with the probability deviations function.

So, for small deviations we can neglect high power terms. In this condition we can choose $n = 0$ for the expansion

$$\mathcal{E}\{\Delta a(t_2)\} = K_3 - K_4 \frac{\sigma_\alpha^2}{2} - K_5 \frac{\sigma_\beta^2}{2}. \tag{37}$$

6 $\Delta\alpha(t)$ Correlated with Transverse, Radial and Normal Velocities

In this case, we cannot decompose the expectation operator as a product of the individual expectations for the trigonometric functions of the $\Delta\alpha(t)$ and $\Delta\beta(t)$ and the velocities components, because now they are correlated. The procedures are the same done until this point, except that we must evaluate the expectation of the products of the different variables, without decomposing them. Besides this, we consider the $\Delta\alpha(t)$ and $\Delta\beta(t)$ random-bias deviations, that is, $\Delta\alpha(t)$=constant=$\Delta\alpha(t_1)$=$\Delta\alpha$ and $\Delta\beta(t)$=constant=$\Delta\beta(t_1)$=$\Delta\beta$. After many mathematical manipulations we found the following equation, for both cases, uniform and Gaussian distribution,

$$I_{t,r,n} = \int_{t_2}^{t_1} \mathcal{E}\{(\cos\Delta\alpha)(\cos\Delta\beta) v'_{t,r,n}(t) \dot{f}'_{t,r,n}(t)\} dt. \tag{38}$$

We know that the integral of the odd functions for symmetrical distributions is null. But Equation (38) has even product of the functions. The odd functions inside the product are not known, but we can modeled its product as one even function, for example, $\cos\Delta\alpha$. Besides this, we consider that the I, Ω, θ, $\dot{I}$ and $\dot{\Omega}$ effects inside the $[-\Delta\alpha_{max}, \Delta\alpha_{max}]$ and $[-\Delta\beta_{max}, \Delta\beta_{max}]$ intervals in the same time are, practically, balanced, because the deviations occur between values maxima and minima inside them.

That is, these instantaneous Keplerian coordinates values with and without deviations are, in mean, very close values. So,

$$\mathcal{E}\{\dot{I}'(t)\cos\theta'(t)\} = \dot{I}(t_1)\cos\theta(t_1), \tag{39}$$

$$\mathcal{E}\{\dot{\Omega}'(t)\sin I'(t)\sin\theta'(t)\} = \dot{\Omega}(t_1)\sin I(t_1)\sin\theta(t_1). \tag{40}$$

Other important approach in this way is to consider for Equations (9) and (11) that the expectation of the product is equal to the product of the expectations of the functions, so that

$$\begin{aligned}\mathcal{E}\{\frac{(\cos\Delta\alpha)(\cos\Delta\beta)}{(r')^2(t)}\} = \mathcal{E}\{(\cos(\Delta\alpha)(\cos\Delta\beta)\frac{1}{(r')^2(t)}\} \cong \\ \mathcal{E}\{\cos\Delta\alpha\}\mathcal{E}\{\cos\Delta\beta\}\mathcal{E}\{\frac{1}{(r')^2(t)}\} = \frac{\mathcal{E}\{\cos\Delta\alpha\}\mathcal{E}\{\cos\Delta\beta\}}{r^2(t)}.\end{aligned} \tag{41}$$

The final forms are:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}\mathcal{E}\{\Delta^n a(t_2)\} = \lambda_1 + \lambda_2\Delta\alpha_{max} + \lambda_3\Delta\beta_{max} + \\ \lambda_4\Delta\alpha_{max}\Delta\beta_{max} - \lambda_5\Delta\alpha_{max}^2 - \lambda_6\Delta\alpha_{max}^2\Delta\beta_{max} - \lambda_7\Delta\beta_{max}^2\Delta\alpha_{max} + \\ \lambda_8\Delta\alpha_{max}^2\Delta\beta_{max}^2 + \lambda_9\Delta\alpha_{max}^4 + \lambda_{10}\Delta\alpha_{max}^5 - \\ \lambda_{11}\Delta\alpha_{max}\Delta\beta_{max}^4 - \lambda_{12}\Delta\alpha_{max}^2\Delta\beta_{max}^4 + \lambda_{13}\Delta\alpha_{max}^4\Delta\beta_{max}^2 + \\ \lambda_{14}\Delta\alpha_{max}^4\Delta\beta_{max}^4 + \ldots\end{aligned} \tag{42}$$

for the uniform deviations and

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}\mathcal{E}\{\Delta^n a(t_2)\} = f_1 + f_2\sigma_\alpha + f_3\sigma_\beta + f_4\sigma_\alpha\sigma_\beta - f_5\sigma_\alpha^2 \\ -f_6\sigma_\alpha^2\sigma_\beta - f_7\sigma_\beta^2\sigma_\alpha + f_8\sigma_\alpha^2\sigma_\beta^2 + f_9\sigma_\alpha^4 + f_{10}\sigma_\alpha^5 - f_{11}\sigma_\alpha\sigma_\beta^4 - f_{12}\sigma_\alpha^2\sigma_\beta^4 \\ +f_{13}\sigma_\alpha^4\sigma_\beta^2 + f_{14}\sigma_\alpha^4\sigma_\beta^4 + \ldots\end{aligned} \tag{43}$$

for the Gaussian deviations.

The coefficients λ_i, λ_{ij}, f_i and f_{ij} are mathematical operations (sums, products and sums of the products) between quadratures in sines and cosines of the pitch and yaw angles.

If we compute the first order terms, Equations (42) and (43), and consider deviations inside the practical range for the space missions, we obtain, for the both cases,

$$\mathcal{E}\{\Delta\alpha(t_2)\} = C_1 + C_2\Delta\alpha_{max} + C_3\Delta\beta_{max} + C_4\Delta\alpha_{max}\Delta\beta_{max} - C_5\Delta\alpha_{max}^2 \tag{44}$$

for the uniform deviations and

$$\mathcal{E}\{\Delta\alpha(t_2)\} = C_6 + C_7\sigma_\alpha + C_8\sigma_\beta + C_9\sigma_\alpha\sigma_\beta - C_{10}\sigma_\alpha^2 \tag{45}$$

for the Gaussian deviations.

These results show once again the nonlinear relationship between cause and effect since n=1 of the expansions.

We modeled in the Equation (38) the product of the not known odd functions as an even function equal to the $\cos\Delta\alpha$. If we choose it as the $\cos\Delta\beta$ the results are different. So, with the same previous algebraic proceedings, we have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}\mathcal{E}\{\Delta^n a(t_2)\} = \lambda'_1 + \lambda'_2\Delta\alpha_{max} + \lambda'_3\Delta\beta_{max} + \lambda'_4\Delta\alpha_{max}\Delta\beta_{max} - \lambda'_5\Delta\alpha^2_{max} - \lambda'_6\Delta\beta^2_{max} - \lambda'_7\Delta\alpha^2_{max}\Delta\beta_{max} - \lambda'_8\Delta\beta^3_{max} + \lambda'_9\Delta\alpha^4_{max} + \lambda'_{10}\Delta\beta^4_{max} + \lambda'_{11}\Delta\alpha^2_{max}\Delta\beta^2_{max} - \lambda'_{12}\Delta\alpha^2_{max}\Delta\beta^3_{max} - \lambda'_{13}\Delta\alpha^2_{max}\Delta\beta^4_{max} - \lambda'_{14}\Delta\beta^6_{max} + \lambda'_{15}\Delta\alpha^2_{max}\Delta\beta^6_{max} + \dots \tag{46}$$

for the uniform deviations, and

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}\mathcal{E}\{\Delta^n a(t_2)\} = f'_1 + f'_2\sigma_\alpha + f'_3\sigma_\beta + f'_4\sigma_\alpha\sigma_\beta - f'_5\sigma^2_\alpha - f'_6\sigma^2_\beta - f'_7\sigma^2_\alpha\sigma_\beta - f'_8\sigma^3_\beta + f'_9\sigma^4_\alpha + f'_{10}\sigma^4_\beta + f'_{11}\sigma^2_\alpha\sigma^2_\beta - f'_{12}\sigma^2_\alpha\sigma^3_\beta - f'_{13}\sigma^2_\alpha\sigma^4_\beta - f_{14}\sigma^6_\beta + f'_{15}\sigma^2_\alpha\sigma^6_\beta + \dots \tag{47}$$

for the Gaussian deviations.

These results for the space missions interest deviations range are

$$\mathcal{E}\{\Delta\alpha(t_2)\} = C_{11} + C_{12}\Delta\alpha_{max} + C_{13}\Delta\beta_{max} + C_{14}\Delta\alpha_{max}\Delta\beta_{max} - C_{15}\Delta\alpha^2_{max} - C_{16}\Delta\beta^2_{max}, \tag{48}$$

$$\mathcal{E}\{\Delta\alpha(t_2)\} = C_{17} + C_{18}\sigma_\alpha + C_{19}\sigma_\beta + C_{20}\sigma_\alpha\sigma_\beta - C_{21}\sigma^2_\alpha - C_{11}\sigma^2_\beta. \tag{49}$$

These results show, again, the nonlinear relation between the thrust deviations and the mean semi-major axis uncertainess. The difference between this case, correlated with cossine of the yaw, is that, in the practical interest range, it occurs the $-\cos\Delta\beta^2_{max}$ contribution. It is the out-plane angle deviation and the nonlinear relation in the space missions interest must avoid it, because the in-plain maneuvers are fuel-optimal.

7 Conclusions

Our results show the nonlinear relations between thrust superposed pitch and yaw direction deviations and the final mean semi-major axis. We analysed the correlated and not correlated deviations with the satellite velocity. In all the cases, the relation shows a progressive deformation of the trajectory due these deviations. This dependence is presented as a revolution paraboloid in the space mission practical interest in the range deviations and the deformed revolution paraboloid in general case. In the space mission interest the relation is dominated by $(\Delta\alpha_{max})^2$ term for the α-correlation and $(\Delta\beta_{max})^2$ and $(\Delta\alpha_{max})^2$ terms for the β-correlation. We suggest the first correlation for the transfers maneuvers under fuel consumption optimal. Besides this, these algebraic results confirm the exhaustive numerical simulation realized before in all the deviations ranges. These results do not depend of the trajectory.

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