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Synchronization of Discrete-Time Hyperchaotic Systems Through Extended Kalman Filtering

A. Y. Aguilar-Bustos [1] and C. Cruz-Hernández [2*]

[1] *Electronics & Telecommunications Department*
[2] *Telematics Direction*
Scientific Research and Advanced Studies of Ensenada (CICESE)
Km. 107 Carretera Tijuana-Ensenada, 22860 Ensenada, B.C., México

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Abstract: In this paper, we use an extended Kalman filter (EKF) to sinchronize discrete-time hyperchaotic systems. In particular, we consider unidirectionally coupled maps corrupted by noise. Approximate synchronization is obtained between master and slave maps in case that the slave is designed as an EKF which is driven by a noisy drive signal from a noisy master dynamics. Two numerical examples are provided to illustrate the efficiency of the proposed approach.

Keywords: *Synchronization; hyperchaotic maps; discrete-time systems; extended Kalman filter; Lyapunov stability; convergence analysis.*

Mathematics Subject Classification (2000): 34C15, 34C28, 37B25, 37D45, 60G35, 93C55, 93E11.

1 Introduction

Synchronization of chaotic oscillations has attracted in recent decades much attention. Different approaches have been reported in the literature see, e.g. [1-14] and references therein. This phenomenon is supposed to have interesting applications in secure communications, see for example [14-23]. However, it has been shown [24, 25] that masking information signals by means of comparatively simple chaos with only one positive Lyapunov exponent does not ensure a sufficient level of security. In some cases, extracting of the information can be performed using common signal processing techniques. For **higher security** the **hyperchaotic systems** characterized by more than one positive Lyapunov exponent **are advantageous** over "simple" chaotic systems. Two factors of primordial importance in security considerations related to chaotic communication systems are:

* Corresponding author: ccruz@cicese.mx

i) *the dimensionality of the chaotic attractor, and*

ii) *the effort required to obtain the necessary parameters for the matching of a slave dynamics.*

One way to enhance the level of encryption security of the communication system consists in applying proper cryptographic techniques to the information signal in combination with chaos [26, 27]. Another way to solve this security problem is to encrypt the information by using high dimensional chaotic attractors, or hyperchaotic attractors, which take advantage of the increased randomness and unpredictabilily of the higher dimensional dynamics. In such option, one generally encounters multiple positive Lyapunov exponents. However, *hyperchaotic synchronization is a much more difficult problem*, see for example [28-30] and [11] for discrete-time context. Other alternative of synchronize hyperchaotic dynamics is using delay differential (or difference) equations, such systems have an infinite-dimensional state space and produce hyperchaos with an arbitrary large number of positive Lyapunov exponents [22, 23].

On the other hand, most of the previous work done on chaos synchronization has been concentrated on continuous-time chaotic systems. Discrete-time systems used for chaos synchronization though, having potential in applications of discrete communications, have not been thoroughly discussed. While a lot of work is available in the control of chaotic mappings, only a few works face the problem of synchronization of discrete-time chaotic systems.

Moreover, parameter uncertainty or unstructured uncertainty in the master dynamics and coupling signal, **noise** may appear due to measurement noise or uncertainties in the dynamics. In this case, synchronization becomes a more difficult problem, certainly no exact state reconstruction will be possible. Nevertheless, a **filtering** approach may be very suitable in this case, see [31] and in the discrete-time context [6, 7].

On the basis of these considerations, the objective of this paper is to extend the approach developed in [6, 7] to the synchronization of hyperchaotic noisy maps with noisy coupling signal. Our goal is achieved by designing an EKF as a slave. In this work, we show that synchronization of discrete-time hyperchaotic systems is indeed suitable from this viewpoint and, moreover, we proceed to apply this approach to synchronize two noisy maps as illustrative examples.

The paper is organized as follows. In Section 2 we state the problem under consideration, the noisy synchronization of discrete-time systems. In Section 3, based on Lyapunov theory, we present an analysis of asymptotic convergence of the EKF. To illustrate the proposed approach, we use in Section 4 an EKF as a slave to synchronize two noisy hyperchaotic maps. Finally, some conclusions are drawn in Section 5.

2 Problem Statement

We consider **noisy master dynamics** given by the state equation

$$x(k+1) = f(x(k)) + w(k), \qquad x(0) = x_0, \tag{1}$$

with coupling signal

$$y(k) = h(x(k)) + v(k). \tag{2}$$

In system (1), $w(k)$ represents the noise in the dynamics of the master system, which is assumed to be a zero mean noise process with $E\left[w(k)\, w^T(l)\right] = Q\delta_{kl} > 0$, with δ_{kl} the Kronecker delta function. Also $v(k)$ is a zero mean noise process with $E\left[v(k)\, v(l)\right] = R\delta_{kl} > 0$; the processes $v(k)$ and $w(k)$ are assumed to be independent.

The EKF that we use as **slave dynamics** for (1) with noisy coupling signal (2) is described as follows [32]:
Measurement update equations:

$$\hat{x}(k) = \hat{x}(k/k-1) + K_{\hat{x}}(k)\left[y(k) - h\left(\hat{x}(k/k-1)\right)\right], \tag{3}$$

where the vector $\hat{x}(k)$ is referred as the *filtered* estimate for the master state vector $x(k)$ at time k. The covariance of the error in $\hat{x}(k)$ is given by

$$P_{\hat{x}}(k) = \left[I - K_{\hat{x}}(k)\, H_{\hat{x}}(k)\right] P_{\hat{x}}(k/k-1). \tag{4}$$

Time update equations:
The (one-step ahead) *predictor* of $\hat{x}(k+1)$ is given by

$$\hat{x}(k+1/k) = f\left(\hat{x}(k)\right), \tag{5}$$

the covariance matrix of the prediction error is

$$P_{\hat{x}}(k+1/k) = F_{\hat{x}}(k)\, P_{\hat{x}}(k)\, F_{\hat{x}}^T(k) + Q, \tag{6}$$

where

$$K_{\hat{x}}(k) = P_{\hat{x}}(k/k-1)\, H_{\hat{x}}^T(k)\left[H_{\hat{x}}(k)\, P_{\hat{x}}(k/k-1)\, H_{\hat{x}}^T(k) + R\right]^{-1} \tag{7}$$

is the *Kalman gain matrix*, and

$$\begin{aligned} F_{\hat{x}}(k) &= \left.\frac{\partial\, f\left(x(k)\right)}{\partial\, x(k)}\right|_{x(k)=\ \hat{x}(k)}, \\ H_{\hat{x}}(k) &= \left.\frac{\partial\, h\left(x(k)\right)}{\partial\, x(k)}\right|_{x(k)=\ \hat{x}(k/k-1)}. \end{aligned} \tag{8}$$

In this paper, our main objective is: Given a noisy master system, and a noisy coupling signal; we want to design a suitable EKF for synchronization in the master-slave framework, such that the following problem is solved.

Definition (*Noisy synchronization*). *The slave dynamics (3)-(8)* ***synchronizes*** *with the noisy master dynamics (1) with noisy drive signal (2), if*

$$\|x(k) - \hat{x}(k)\| \le \rho, \qquad \forall k \ge \tau, \tag{9}$$

where ρ should be related to Q and R and is a constant of the synchronization/estimation error. If for a given ρ there exists a time instant τ (to be called the ***synchronization time****) such that condition (9) is fulfilled, then the noisy master (1) and the EKF slave (3)-(8) are* ***approximately synchronized*** *with level ρ.*

One might also consider as an adequate condition for approximate (or noisy) synchronization, if there exists a positive constant τ such that

$$E\left\{\|(x(k) - \hat{x}(k))\|^2\right\} \le \rho, \qquad \forall k \ge \tau.$$

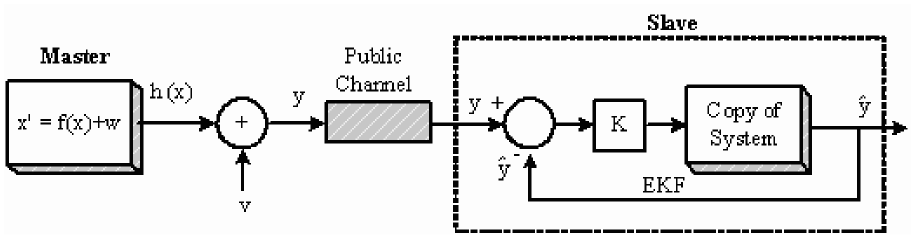

Figure 2.1: Master-slave coupling scheme for noisy hyperchaotic maps: where $x' = x(k+1)$ is the state master, w and v are independent noise processes, y the coupling signal, and $\hat{y}$ the output of EKF.

In particular, this may be a more relevant requirement if $w(k)$ is not necessarily bounded. Since we will assume that $v(k)$ and $w(k)$ are bounded, then is sufficient to take condition (9). Figure 2.1 shows the master-slave coupling scheme for approximate (or noisy) synchronization of maps (1) and (3)-(8) when the noisy drive signal (2) is used. Also, in all subsequent simulations we check the condition (9) over a long but finite time interval $[0, t_f]$.

3 Convergence Analysis of the EKF for Synchronization

In this section, based on Lyapunov theory, we make an analysis of the convergence of the synchronization error. Define the *estimation (synchronization) error* as

$$e(k) = x(k) - \hat{x}(k), \tag{10}$$

and the error between the state and the prediction of the estimation as

$$e(k/k-1) = x(k) - \hat{x}(k/k-1).$$

If we assume that f and h are C^1 functions, then f can be expanded (using Taylor's Theorem) as follows,

$$f(x) = f(\hat{x}) + F(k)[x(k) - \hat{x}(k)] + \varphi(x(k), \hat{x}(k)), \tag{11}$$

where $f(\hat{x})$ represents the copy of the system $f(x)$, $F = \partial f(x)/\partial x$ the first derivative of $f(x)$, and $\varphi(x, \hat{x})$ the remainder after the first order expansion of $f(x)$.

The dynamics of the error between the state of the master and the prediction are given by the equation,

$$\begin{aligned} e(k+1/k) &= x(k+1) - \hat{x}(k+1/k) \\ &= f(x(k)) + w(k) - f(\hat{x}(k)) \\ &= F(k)e(k) + \varphi(x(k), \hat{x}(k)) + w(k) \end{aligned}$$

and the dynamics of the state estimation (synchronization) error system are governed by the following equation

$$\begin{aligned} e(k+1) &= x(k+1) - \hat{x}(k+1) \\ &= f(x(k)) + w(k) - f(\hat{x}(k)) - K(k+1)[y(k+1) - H\hat{x}(k+1/k)] \\ &= [I - K(k+1)H]F(k)e(k) + [I - K(k+1)H]\varphi(x(k), \hat{x}(k)) \\ &\quad + [I - K(k+1)H]w(k) - K(k+1)v(k+1), \\ e(k+1) &= [I - K(k+1)H]F(k)e(k) + r(k) + s(k), \end{aligned} \tag{12}$$

where

$$\begin{aligned} r(k) &= [I - K(k+1)H]\varphi(x(k), \hat{x}(k)), \\ s(k) &= [I - K(k+1)H]w(k) - K(k+1)v(k+1). \end{aligned}$$

Before going to analyze the stability of the error system (12) we make the following assumptions:

(A1) *There exist positive constants $\bar{f}$, $\bar{h}$, p_1, and p_2 such that the following bounds hold for all $k \geq 0$:*

$$\|F(k)\| \leq \bar{f}, \tag{13}$$
$$\|H(k)\| \leq \bar{h}, \tag{14}$$
$$p_1 I \leq P(k) \leq p_2 I, \tag{15}$$
$$qI \leq Q, \tag{16}$$
$$rI \leq R. \tag{17}$$

(A2) *$F(k)$ is nonsingular for all $k \geq 0$.*

(A3) *There exist positive constants ϵ and κ such that the function $\varphi(x(k), \hat{x}(k))$ in (11) is bounded by*

$$\|\varphi(x(k), \hat{x}(k))\| \leq \kappa \|x(k) - \hat{x}(k)\|^2,$$

for $x(k), \hat{x}(k) \in \mathbb{R}^n$ with $\|x(k) - \hat{x}(k)\| \leq \epsilon$.

In addition to this, we demonstrate the following lemmas to be required to establish the necessary conditions on stability of the estimation (synchronization) error given by the EKF.

Lemma 3.1 *Under the boundedness conditions (13)-(17) there exists a real number $0 < \alpha < 1$ such that $P^{-1}(k)$ satisfies the inequality*

$$F^T(k)[I - K(k+1)H]^T P^{-1}(k+1)[I - K(k+1)H]F(k) \leq (1-\alpha)P^{-1}(k)$$

for all $k \geq 0$.

Proof The term $P(k+1) = [I - K(k+1)H]\left[F(k)P(k)F^T(k) + Q\right]$ can be rewritten as follows

$$\begin{aligned} P(k+1) &= [I - K(k+1)H]F(k)P(k)F^T(k)[I - K(k+1)H]^T \\ &\quad + [I - K(k+1)H]Q[I - K(k+1)H]^T \\ &\quad + [I - K(k+1)H]\left[Q + F(k)P(k)F^T(k)\right]H^T K^T(k+1), \end{aligned} \tag{18}$$

where $[I-K(k+1)H]\left[Q+F(k)P(k)F^T(k)\right]$ is a symmetric matrix. Making use of the matrix inversion Lemma, we obtain

$$\begin{aligned}[I-K(k+1)H]\left[Q+F(k)P(k)F^T(k)\right] &= \\ [\left(Q+F(k)P(k)F^T(k)\right)^{-1}+H^TR^{-1}H]^{-1} &> 0,\end{aligned} \tag{19}$$

from Eq. (19) it follows that

$$[I-K(k+1)H]\left[Q+F(k)P(k)F^T(k)\right]H^TK^T(k+1)\geq 0, \tag{20}$$

using the condition (20) and eliminating that term of (18), the following inequality holds

$$\begin{aligned}P(k+1) \geq\ & [I-K(k+1)H]F(k)P(k)F^T(k)[I-K(k+1)H]^T \\ & +[I-K(k+1)H]Q[I-K(k+1)H]^T.\end{aligned}$$

Now, the above inequality can be rewritten as follows

$$P(k+1)\geq[I-K(k+1)H]F(k)\left[P(k)+F^{-1}(k)QF^{-T}(k)\right]F^T(k)[I-K(k+1)H]^T.$$

Using the conditions (13), (15), and (16), we have that

$$P(k+1)\geq[I-K(k+1)H]F(k)\left(I+\frac{qI}{\bar{f}^2p_2}\right)P(k)F^T(k)[I-K(k+1)H]^T \tag{21}$$

and taking the inverse in both sides of inequality (21) and multiplying by $F^T(k)[I-K(k+1)H]^T$ and $[I-K(k+1)H]F(k)$, we have that

$$F^T(k)[I-K(k+1)H]^TP^{-1}(k+1)[I-K(k+1)H]F(k)\leq\left(1+\frac{q}{p_2\bar{f}^2}\right)^{-1}P^{-1}(k)$$

with $(1-\alpha)=\left(1+\frac{q}{p_2\bar{f}^2}\right)^{-1}$.

Lemma 3.2 *Since conditions (13)-(17) hold. Then, there exist positive constants* ϵ *and* k_{nom} *such that*

$$r^T(k)P^{-1}(k)\left[2[I-K(k+1)H]F(k)e(k)+r(k)\right]\leq k_{nom}\|e(k)\|^3$$

holds for all $\|e(k)\|\leq\epsilon$.

Proof Since $r(k)=[I-K(k+1)H]\varphi(x(k),\hat{x}(k))$ and by Assumption (A3), we have that $\|\varphi(x(k),\hat{x}(k))\|\leq\kappa\|e(k)\|^2$ in addition, considering $Q\leq\delta_1I$ it follows that

$$\begin{aligned}\|K(k+1)\| &\leq \left\|\left[F(k)P(k)F^T(k)+Q\right]H^T\left[H\left[F(k)P(k)F^T(k)+Q\right]H^T+R\right]^{-1}\right\| \\ &\leq \left\|\left[F(k)P(k)F^T(k)+Q\right]\right\|\left\|H^T\right\|\left\|\left[H\left[F(k)P(k)F^T(k)+Q\right]H^T+R\right]^{-1}\right\| \\ &\leq \left\|\left[F(k)P(k)F^T(k)+Q\right]\right\|\left\|H^T\right\|\left\|\left[H\left[F(k)P(k)F^T(k)+Q\right]H^T+R\right]^{-1}\right\| \\ &\leq \left(\bar{f}^2p_2+\delta_1\right)\frac{\bar{h}}{r}.\end{aligned}$$

The term $\|K(k+1)\|$ can be expressed as $\|K(k+1)\| \le \rho_1 + \rho_2\delta_1$ with $\rho_1 = \frac{\bar{f}^2 p_2 \bar{h}}{r}$ and $\rho_2 = \frac{\bar{h}}{r}$. Therefore, we obtain

$$\begin{aligned}\|r(k)\| &\le \|I - K(k+1)H\| \, \|\varphi(x(k), \hat{x}(k))\| \\ &\le \|I - K(k+1)H\| \, \kappa \|e(k)\|^2 \\ &\le [\|I\| + \|K(k+1)H\|] \, \kappa \|e(k)\|^2 \\ &\le [1 + \|K(k+1)\| \, \|H\|] \, \kappa \|e(k)\|^2 \\ &\le \left(1 + \rho_1\bar{h} + \rho_2\bar{h}\delta_1\right) \kappa \|e(k)\|^2 \end{aligned} \tag{22}$$

and making use of inequality (22), we have that

$$\begin{aligned} & r^T(k) P^{-1}(k) \left[2[I - K(k+1)H]F(k) e(k) + r(k) \right] \\ \le\ & \left\| r^T(k) P^{-1}(k) \left[2[I - K(k+1)H]F(k) e(k) + r(k) \right] \right\| \\ \le\ & \left\| r^T(k) \right\| \left\| P^{-1}(k) \right\| \left\| \left[2[I - K(k+1)H]F(k) e(k) + r(k) \right] \right\| \\ \le\ & \left(1 + \rho_1\bar{h} + \rho_2\bar{h}\delta_1\right) \kappa \|e(k)\|^2 \left(\frac{1}{p_1}\right) \\ & \times \left[2\left(1 + \rho_1\bar{h} + \rho_2\bar{h}\delta_1\right) \bar{f} \|e(k)\| + \left(1 + \rho_1\bar{h} + \rho_2\bar{h}\delta_1\right) \kappa \|e(k)\|^2 \right] \\ \le\ & \left(1 + \rho_1\bar{h} + \rho_2\bar{h}\delta_1\right)^2 \kappa \left(\frac{1}{p_1}\right) \left(2\bar{f} + \kappa\epsilon\right) \|e(k)\|^3 \\ \le\ & k_{nom} \|e(k)\|^3 \end{aligned}$$

with

$$k_{nom} = \left(1 + \rho_1\bar{h} + \rho_2\bar{h}\delta\right)^2 \kappa \left(\frac{1}{p_1}\right) \left(2\bar{f} + \kappa\epsilon\right) \quad \text{and} \quad \delta = \delta_1.$$

Lemma 3.3 *Under the boundedness conditions (13)-(17). There exist positive real numbers ρ_3, ρ_4, and ρ_5 independent of δ, such that*

$$E\left\{s^T(k) P^{-1}(k+1) s(k)\right\} \le \rho_3\delta^3 + \rho_4\delta^2 + \rho_5\delta$$

holds for some constant $\delta > 0$.

Proof Firstly, we have that

$$\begin{aligned} s^T(k) P^{-1}(k+1) s(k) = {} & w^T(k)[I - K(k+1)H]^T P^{-1}(k+1)[I - K(k+1)H] w(k) \\ & - w^T(k)[I - K(k+1)H] P^{-1}(k+1) K(k+1) v(k) \\ & - v^T(k) K^T(k+1) P^{-1}(k+1)[I - K(k+1)H] w(k) \\ & + v^T(k) K^T(k+1) P^{-1}(k+1) K(k+1) v(k) \end{aligned} \tag{23}$$

since $w(k)$ and $v(k)$ are uncorrelated signals, the expression (23) becomes,

$$\begin{aligned} s^T(k) P^{-1}(k+1) s(k) = {} & w^T(k)[I - K(k+1)H]^T P^{-1}(k+1)[I - K(k+1)H] w(k) \\ & + v^T(k) K^T(k+1) P^{-1}(k+1) K(k+1) v(k). \end{aligned}$$

From Lemma 3.2, we have obtained that $\|K(k+1)\| \le \rho_1 + \rho_2\delta_1$ with $\rho_1 = \frac{\bar{f}^2 p_2 \bar{h}}{r}$ and $\rho_2 = \frac{\bar{h}}{r}$ and, considering again that $Q \le \delta_1 I$ and $R \le \delta_2 I$, we have

$$\begin{aligned} s^T(k) P^{-1}(k+1) s(k) &\le \left(1 + \rho_1\bar{h} + \rho_2\bar{h}\delta_1\right)^2 \frac{1}{p_1} w^T(k) w(k) + (\rho_1 + \rho_2\delta_1)^2 \frac{1}{p_1} v^T(k) v(k) \\ &\le \left(1 + \rho_1\bar{h} + \rho_2\bar{h}\delta_1\right)^2 \frac{1}{p_1}\delta_2 + (\rho_1 + \rho_2\delta_1)^2 \frac{1}{p_1}\delta_1 \end{aligned} \tag{24}$$

considering $\delta_1 = \delta_2 = \delta$ the above inequality can be rewritten as follows,

$$s^T(k)\,P^{-1}(k+1)\,s(k) \leq \rho_3\delta^3 + \rho_4\delta^2 + \rho_5\delta$$

with

$$\rho_3 = \frac{\rho_2^2\left(1+\bar{h}^2\right)}{p_1}, \quad \rho_4 = \frac{2\rho_2\left(\bar{h}+\rho_1\bar{h}^2+\rho_1\right)}{p_1}, \quad \rho_5 = \frac{1+2\rho_1\bar{h}+\rho_1^2\bar{h}^2+\rho_1^2}{p_1}.$$

Lemma 3.4 ([35]) *Suppose that $V(e(k))$ is a stochastic process and that exist real numbers v_1, v_2, $\mu > 0$, and $0 < \alpha' \leq 1$ such that:*

$$v_1\left\|e(k)\right\|^2 \leq V(e(k)) \leq v_2\left\|e(k)\right\|^2,$$

$$E\left\{V(e(k+1))/e(k)\right\} - V(e(k)) \leq \mu - \alpha' V(e(k))$$

hold for all solution of Eq. (12). Then, the stochastic process is exponentially bounded as follows

$$E\left\{\left\|e(k)\right\|^2\right\} \leq \frac{v_2}{v_1}E\left\{\left\|e(0)\right\|^2\right\}(1-\alpha)^k + \frac{\mu}{v_1\alpha'}.$$

In order to prove stability of the estimation (synchronization) error (10), we propose the following function as a Lyapunov function candidate

$$V(e(k)) = e^T(k)\,P^{-1}(k)\,e(k) \tag{25}$$

since $P(k)$ is a positive definite matrix, then $P^{-1}(k)$ is another positive definite matrix, and therefore $V(e(k))$ is positive definite, hence Lyapunov function candidate. From (15), we can obtain

$$\frac{1}{p_2}\left\|e(k)\right\|^2 \leq V(e(k)) \leq \frac{1}{p_1}\left\|e(k)\right\|^2,$$

iterating both sides of (25), we have

$$\begin{aligned} V(e(k+1)) &= e(k+1)^T P^{-1}(k+1)\,e(k+1) \\ &= e^T(k)\,F^T(k)\,[I-K(k+1)\,H]^T P^{-1}(k+1)\,[I-K(k+1)\,H]F(k)\,e(k) \\ &\quad + r^T(k)\,P^{-1}(k+1)\,[2[I-K(k+1)\,H]F(k)\,e(k)+r(k)] \\ &\quad + 2s^T(k)\,P^{-1}(k+1)\,[[I-K(k+1)\,H]F(k)\,e(k)+r(k)] \\ &\quad + s^T(k)\,P^{-1}(k+1)\,s(k). \end{aligned}$$

Using the Lemma 3.1, we obtain

$$\begin{aligned} V(e(k+1)) \;\leq\; & (1-\alpha)\,V(e(k)) \\ & + r^T(k)\,P^{-1}(k+1)\,[2[I-K(k+1)\,H]F(k)\,e(k)+r(k)] \\ & + 2s^T(k)\,P^{-1}(k+1)\,[[I-K(k+1)\,H]F(k)\,e(k)+r(k)] \\ & + s^T(k)\,P^{-1}(k+1)\,s(k). \end{aligned}$$

Taking the conditional expectation $E\left\{V(e(k+1))/e(k)\right\}$ and considering the properties of the white Gaussian random process, it is not difficult to see that the term

$$E\left\{2s^T(k)\,P^{-1}(k+1)\,[[I-K(k+1)\,H]F(k)\,e(k)+r(k)]\,|e(k)\right\}$$

vanishes. Thus, we have that

$$\begin{aligned} E\left\{V\left(e\left(k+1\right)\right)/e\left(k\right)\right\} \quad &\leq \quad \left(1-\alpha\right)V\left(e\left(k\right)\right)+s^{T}\left(k\right)P^{-1}\left(k+1\right)s\left(k\right) \\ &\quad +r^{T}\left(k\right)P^{-1}\left(k+1\right)\left[2[I-K\left(k+1\right)H]F\left(k\right)e\left(k\right)+r\left(k\right)\right]. \end{aligned}$$

Now, invoking to Lemmas 3.2 and 3.3, we obtain that

$$E\left\{V\left(e\left(k+1\right)\right)/e\left(k\right)\right\} \leq V\left(e\left(k\right)\right)-\alpha V\left(e\left(k\right)\right)+\rho_3\delta^3+\rho_4\delta^2+\rho_5\delta+\ k_{nom}\left\|e\left(k\right)\right\|^3 \quad (26)$$

or equivalently,

$$E\left\{V\left(e\left(k+1\right)\right)/e\left(k\right)\right\}-V\left(e\left(k\right)\right) \leq -\frac{\alpha}{p_2}\left\|e\left(k\right)\right\|^2+\rho_3\delta^3+\rho_4\delta^2+\rho_5\delta+\ k_{nom}\left\|e\left(k\right)\right\|^3. \quad (27)$$

The function (27) will be negative semidefinite if satisfies:

$$k_{nom}\left\|e\left(k\right)\right\| \leq \frac{\alpha}{2p_2}, \quad (28)$$

$$\rho_3\delta^3+\rho_4\delta^2+\rho_5\delta \leq \frac{\alpha}{2p_2}\left\|e\left(k\right)\right\|^2, \quad (29)$$

the expression (29) can be replaced by

$$\left(1+\rho_1\bar{h}+\rho_2\bar{h}\delta_1\right)^2\frac{1}{p_1}\delta_2+\left(\rho_1+\rho_2\delta_1\right)^2\frac{1}{p_1}\delta_1 \leq \frac{\alpha}{2p_2}\left\|e\left(k\right)\right\|^2,$$

if we do not consider $\delta_1=\delta_2=\delta$. Defining $\varepsilon=\min\left(\epsilon,\frac{\alpha}{2p_2k_{nom}}\right)$ and using it in (26), the following inequality

$$E\left\{V\left(e\left(k+1\right)\right)/e\left(k\right)\right\}-V\left(e\left(k\right)\right) \leq -\frac{\alpha}{2}V\left(e\left(k\right)\right)+\rho_3\delta^3+\rho_4\delta^2+\rho_5\delta$$

holds for all $\left\|e\left(k\right)\right\| \leq \varepsilon$. Now, invoking to Lemma 3.4 with $\left\|e\left(0\right)\right\| \leq \varepsilon$, $v_1=\frac{1}{p_2}$, $v_2=\frac{1}{p_1}$, $\alpha'=\frac{\alpha}{2}$, and $\mu=\rho_3\delta^3+\rho_4\delta^2+\rho_5\delta$ to quantify the estimation/synchronization error $e\left(k\right)$.

The previous results can be combined to obtain the main result of this paper on the stability of the estimation (synchronization) error given by EKF, when it is applied to hyperchaotic synchronization of stochastic discrete-time systems, which is established in the following theorem.

Theorem 3.1 *Consider a stochastic nonlineal system defined by (1) with noisy coupling signal (2). In addition, consider an extended Kalman filter described by (3)-(8). Assume that Assumptions (A1)-(A3) hold. Then, the estimation (synchronization) error $e\left(k\right)$ given by (10) is exponentially bounded, if the initial error satisfies*

$$\left\|e\left(0\right)\right\| \leq \varepsilon,$$

and the covariance matrices of the noise terms are bounded by

$$\begin{aligned} Q &\leq \delta_1 I, \\ R &\leq \delta_2 I \end{aligned}$$

for some constants δ_1, δ_2, $\varepsilon>0$.

The above result shows that the stability of the estimation (synchronization) error depends on the nature of the nonlinearities and of the size of the noise in the processes, thus as of the boundedness of the initial estimation error. Therefore, this result can be used to design nonlinear filters (EKF) with stability to approximate synchronize noisy hyperchaotic maps, as will be shown in the next section. In addition to this, we mention that other bounds on the error dynamics of the EKF can be obtained with a prescribed degree of stability from [33–35].

4 Synchronization of Noisy Hyperchaotic Maps

Example 1. Consider the following discrete-time system

$$\begin{aligned} x_1(k+1) &= x_2(k) + ax_1(k), \\ x_2(k+1) &= x_1(k)^2 + b \end{aligned} \tag{30}$$

with parameter values $a = -0.1$ and $b = -1.7$, the map (30) exhibits hyperchaotic dynamics [12]. Figure 4.1 shows the hyperchaotic attractor generates for the map (30). In the sequel, based on this mapping, we show approximate synchronization, by using an EKF as slave dynamics, which will try to estimate the master dynamics (30) corrupted by noise, described by

$$\begin{aligned} x_1(k+1) &= x_2(k) + ax_1(k) + w_1(k), \\ x_2(k+1) &= x_1(k)^2 + b + w_2(k), \end{aligned} \tag{31}$$

with output corrupted by noise defined by

$$y(k) = x_1(k) + v(k).$$

The EKF will generate the estate estimates $\hat{x}_i(k)$, $i = 1, 2$ for the master states $x_i(k)$. The state equations of EKF as *slave,* are described by

$$\begin{aligned} \hat{x}_1(k) &= \hat{x}_1(k/k-1) + K_1(k)\left[y(k) - \hat{x}_1(k/k-1)\right], \\ \hat{x}_2(k) &= \hat{x}_2(k/k-1) + K_2(k)\left[y(k) - \hat{x}_1(k/k-1)\right], \end{aligned} \tag{32}$$

where the Kalman gain $(K_1(k), K_2(k))^T$ is given by (7).

For the noisy map (31) with the above parameter values, we obtain that: $\bar{h} = 1$, $\bar{f} = 4$, $p_1 = 4.52 \times 10^{-6}$, $p_2 = 5.52 \times 10^{-6}$, and $\kappa = 2$. By computer simulations, we take $\delta_1 = 0.0005$ such that the system remains with hyperchaotic dynamics. In addition, we propose the values for q and r as $q = r = \frac{\delta_1}{100}$. With previous data and by using conditions (28) and (29), we obtain the values $\delta_2 = 0.0001$ and $\|e(0)\| \le 0.02$ which satisfy the mentioned conditions. In the sequel, we show some computer simulations.

We take $x(0) = (0.1, 0.1)$, $P_0 = \text{diag}\{p_{0i}\}$, $p_{0_i} = 5 \times 10^{-6}$. Figure 4.2 shows the time evolution of synchronization errors $e_1(k)$ and $e_2(k)$ for $\hat{x}(0) = (0.13, 0.13)$ for one realization of the noise, where approximate synchronization is achieved for $\tau = 0$ when the level of synchronization $\rho = 0.06$ was considered. While, in Figure 4.3 we can see the time evolution of synchronization errors $e_1(k)$ and $e_2(k)$ for one realization of the noise, starting at $\hat{x}(0) = (0.31, 0.31)$, in this case approximate synchronization is achieved for $\tau = 7$ when $\rho = 0.06$ was considered.

To evaluate the performance of EKF from the point of view of sensitivity to initial error and noise, twenty different Monte Carlo runs were taken in order to obtain root-mean-square error statistics. The results are summarized in Table 4.1, where SSE_i is the *sum of square errors* for each realization of the noise given by

$$SSE_i = \sum_{k=0}^{N} \left(x_i\left(k\right) - \hat{x}_i\left(k\right) \right)^2, \qquad i = 1, 2, ..., n$$

where $x_i\left(k\right)$ and $\hat{x}_i\left(k\right)$ are the true and estimated states, respectively, and N the number of time steps. So, t*he mean-square error (MSE$_i$)* is obtained as $\frac{1}{N+1}\left(SSE_i\right)$. Therefore, the *Monte Carlo sum of square errors* $\left(SSE_i\right)_{MC}$ is given by

$$\left(SSE_i\right)_{MC} = \frac{1}{20} \sum_{j=1}^{20} \left(SSE_i\right)_j, \qquad i = 1, 2, ..., n.$$

With the purpose of knowing the same statistics, when the transient has died out we define the *truncated mean-square error* $\left(TMSE_i\right)$ for each realization of the noise as

$$TMSE_i = \frac{1}{N + 1 - \tau} \sum_{k=\tau}^{N} \left(x_i\left(k\right) - \hat{x}_i\left(k\right) \right)^2, \qquad i = 1, 2, ..., n,$$

so, the *Monte Carlo truncated mean-square error* is obtained as

$$\left(TMSE_i\right)_{MC} = \frac{1}{20} \sum_{j=1}^{20} \left(TMSE_i\right)_j, \qquad i = 1, 2, ..., n,$$

and the *Monte Carlo synchronization time* τ_{MC} by

$$\tau_{MC} = \frac{1}{20} \sum_{j=1}^{20} \max\left(\tau_i\left(\rho\right)\right)_j, \qquad i = 1, 2, ..., n.$$

From Table 4.1, it possible to appreciate the suitable performance of the EKF as slave for the estimation/synchronization of the noisy master (31), when the conditions $e\left(0\right) < \varepsilon$ and $R < \delta_1$ and $Q < \delta_2$ are satisfied. Note that last three lines in Table 4.1, we take the initial error values $e\left(0\right) > \varepsilon$, nevertheless the EKF achieves approximate synchronization, due to the bounds used are conservatives.

Example 2. Consider the hyperchaotic Rössler map

$$\begin{aligned}
x_1\left(k+1\right) &= \alpha x_1\left(k\right)\left(1 - x_1\left(k\right)\right) - \beta\left(x_3\left(k\right) + \gamma\right)\left(1 - 2x_2\left(k\right)\right), \\
x_2\left(k+1\right) &= \delta x_2\left(k\right)\left(1 - x_2\left(k\right)\right) + \zeta x_3\left(k\right), \\
x_3\left(k+1\right) &= \eta\left(\left(x_3\left(k\right) + \gamma\right)\left(1 - 2x_2\left(k\right)\right) - 1\right)\left(1 - \theta x_1\left(k\right)\right),
\end{aligned} \tag{33}$$

with the set of parameter values: $\alpha = 3.8$, $\beta = 0.05$, $\gamma = 0.35$, $\delta = 3.78$, $\zeta = 0.2$, $\eta = 0.1$, and $\theta = 1.9$ the Rössler map (33) exhibits hyperchaotic dynamics [12]. Figure 4.4 shows

Table 4.1: Monte Carlo sum of square errors $(SSE_i)_{MC}$, Monte Carlo truncated mean-square error $(TMSE_i)_{MC}$, and synchronization time (τ_{MC}) for Example 1 with $p_{0_i} = 5\times10^{-6}$, $i = 1, 2$, $Q = R = 5 \times 10^{-5}$, $\rho = 0.06$, and $N = 100$.

$e(0)$	$(SSE_1)_{MC}$	$(SSE_2)_{MC}$	$(TMSE_1)_{MC}$	$(TMSE_2)_{MC}$	(τ_{MC})
$(0.2, 0.2)$	0.0043	0.0595	0.0035	0.0203	2
$(0.05, 0.05)$	0.0065	0.0235	0.0064	0.0228	0
$(0.01, 0.01)$	0.0040	0.0219	0.0042	0.0210	1
$(-0.01, -0.01)$	0.0038	0.0204	0.0042	0.0213	0
$(-0.05, -0.05)$	0.0065	0.0243	0.0064	0.0262	0
$(-0.1, -0.1)$	0.0141	0.0319	0.0055	0.0216	2
$(-0.2, -0.2)$	0.0450	0.0721	0.0042	0.0229	4
$(-0.5, -0.5)$	0.2676	0.4756	0.0040	0.0214	6
$(-1, -1)$	1.1183	3.09	0.0052	0.0230	7
$(-5, -5)$	30.37	544.12	0.0038	0.0194	10

several hyperchaotic attractors generate for the Rössler map (33). Consider the noisy master map

$$\begin{aligned} x_1(k+1) &= \alpha x_1(k)(1-x_1(k)) - \beta(x_3(k)+\gamma)(1-2x_2(k)) + w_1(k), \\ x_2(k+1) &= \delta x_2(k)(1-x_2(k)) + \zeta x_3(k) + w_2(k), \\ x_3(k+1) &= \eta((x_3(k)+\gamma)(1-2x_2(k))-1)(1-\theta x_1(k)) + w_3(k), \end{aligned} \tag{34}$$

and the noisy drive signal

$$y(k) = x_1(k) + v(k). \tag{35}$$

The covariance Q and variance R were fixed at $R = Q = 1 \times 10^{-6}$. The slave system (EKF) will generate the state estimates $\hat{x}_i(k)$, $i = 1, 2, 3$ for each $x_i(k)$, which is designed as

$$\begin{aligned} \hat{x}_1(k+1) &= \alpha\hat{x}_1(k)(1-\hat{x}_1(k)) - \beta(\hat{x}_3(k)+\gamma)(1-2\hat{x}_2(k)) + k_1(k)(y(k)-\hat{x}_1(k)), \\ \hat{x}_2(k+1) &= \delta\hat{x}_2(k)(1-\hat{x}_2(k)) + \zeta\hat{x}_3(k) + k_2(k)(y(k)-\hat{x}_1(k)), \\ \hat{x}_3(k+1) &= \eta((\hat{x}_3(k)+\gamma)(1-2\hat{x}_2(k))-1)(1-\theta\hat{x}_1(k)) + k_3(k)(y(k)-\hat{x}_1(k)), \end{aligned} \tag{36}$$

where $(k_1(k), k_2(k), k_3(k))^T$ is given by (7).

For noisy Rössler map (34), we obtain that: $\bar{h} = 1$, $\bar{f} = 3.84$, $p_1 = 5.5 \times 10^{-3}$, $p_2 = 248$, and $\kappa = 7.6$. By computer simulations, we take $\delta_1 = 0.00005$ such that the mapping remains with hyperchaotic dynamics. We propose $q = r = \delta_1/1000$ and using (28) and (29), we have that $\delta_2 = 1 \times 10^{-7}$ and $\|e(0)\| \leq 0.03$ which satisfy these conditions. In the following simulations we take $x(0) = (0.95, 0.9, 0)$, $P_0 = \mathrm{diag}\{p_{0_i}\}$, $p_{0_i} = 500$, $i = 1, 2, 3$. Figure 4.5 shows the synchronization error evolution between (34) and (36) for $\hat{x}(0) = (0.9, 0.95, 0.05)$ for one realization of the noise. We can see, after some transient behavior, that approximate synchronization is achieved at time $\tau = 3$ when $\rho = 0.05$ was considered. Tables 4.2 and 4.3 show the suitable behavior of EKF as a estimator of the state vector of noisy hyperchaotic map (34).

Table 4.2: Monte Carlo sum of square errors $(SSE_i)_{MC}$ and synchronization time (τ_{MC}) for Example 2 with $p_{0_i} = 500$, $i = 1, 2, 3$, $Q = R = 1 \times 10^{-6}$, $\rho = 0.05$, and $N = 100$.

$e(0)$	$(SSE_1)_{MC}$	$(SSE_2)_{MC}$	$(SSE_3)_{MC}$	τ_{MC}
$(0.1, 0.1, 0.1)$	0.0100	0.1066	0.0110	6
$(0.05, 0.05, 0.05)$	0.0025	0.0532	0.0035	5
$(0.01, 0.01, 0.01)$	1.0095×10^{-4}	0.0142	0.0010	4
$(-0.01, -0.01, -0.01)$	1.0100×10^{-4}	0.0176	0.0011	3

Table 4.3: Monte Carlo truncated mean-square error and synchronization time (τ_{MC}) for Example 2 with $p_{0_i} = 500$, $i = 1, 2, 3$, $Q = R = 1 \times 10^{-6}$, $\rho = 0.05$, and $N = 100$.

$e(0)$	$(TMSE_1)_{MC}$	$(TMSE_2)_{MC}$	$(TMSE_3)_{MC}$	τ_{MC}
$(0.1, 0.1, 0.1)$	1.0205×10^{-6}	0.0131	9.7194×10^{-4}	6
$(0.05, 0.05, 0.05)$	9.6606×10^{-7}	0.0128	9.7529×10^{-4}	5
$(0.01, 0.01, 0.01)$	8.5910×10^{-5}	0.0117	9.8750×10^{-4}	4
$(-0.01, -0.01, -0.01)$	9.5976×10^{-5}	0.0148	0.0011	3

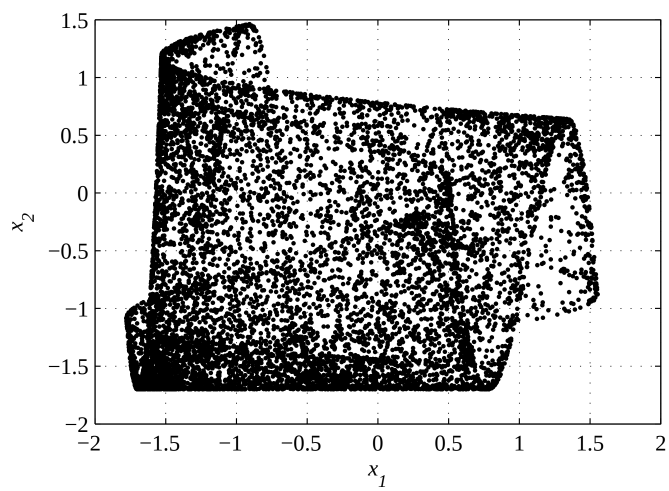

Figure 4.1: Hyperchaotic attractor of discrete-time system (30).

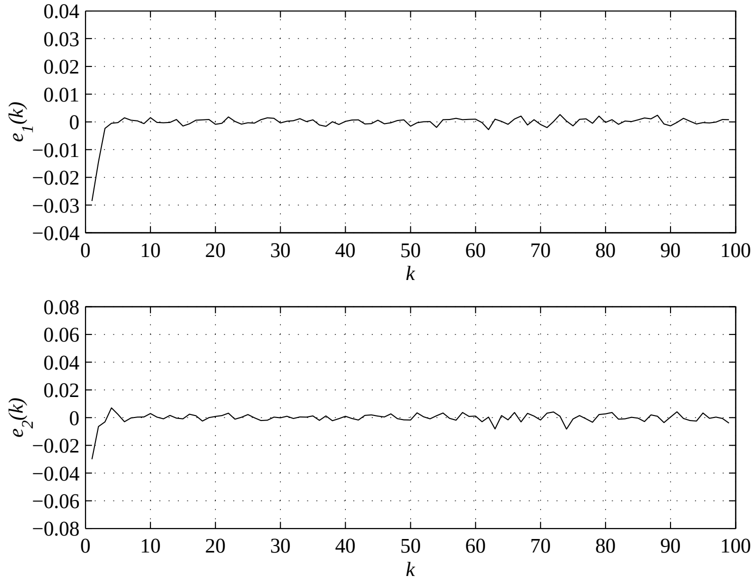

Figure 4.2: Time evolution of synchronization errors $e_1\,(k)$ and $e_2\,(k)$ for $\hat{x}\,(0) = (0.13, 0.13)$; $\tau = 0$ when $\rho = 0.06$ was considered (for one realization of the noise).

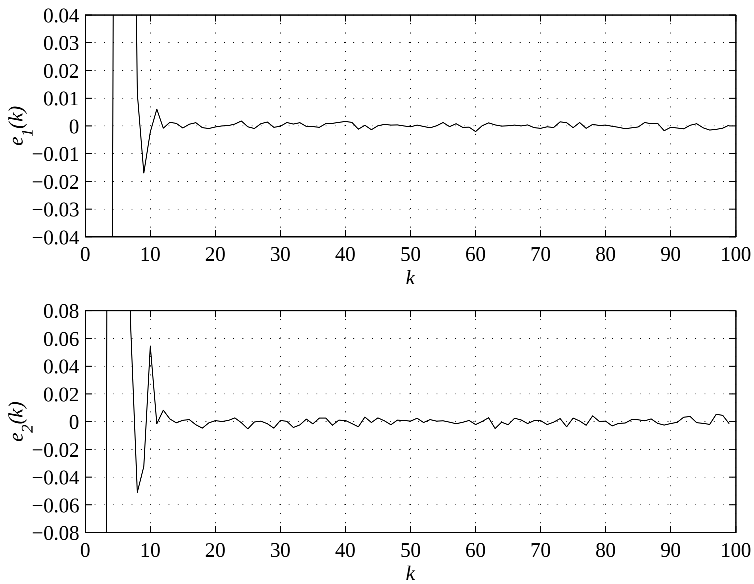

Figure 4.3: Time evolution of synchronization errors $e_1\,(k)$ and $e_2\,(k)$ for $\hat{x}\,(0) = (0.31, 0.31)$; $\tau = 7$ when $\rho = 0.06$ was considered (for one realization of the noise).

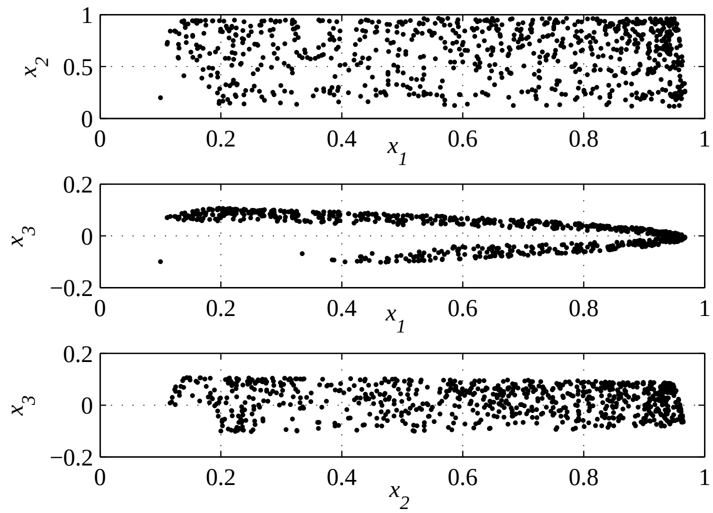

Figure 4.4: Hyperchaotic attractors of Rössler map (33).

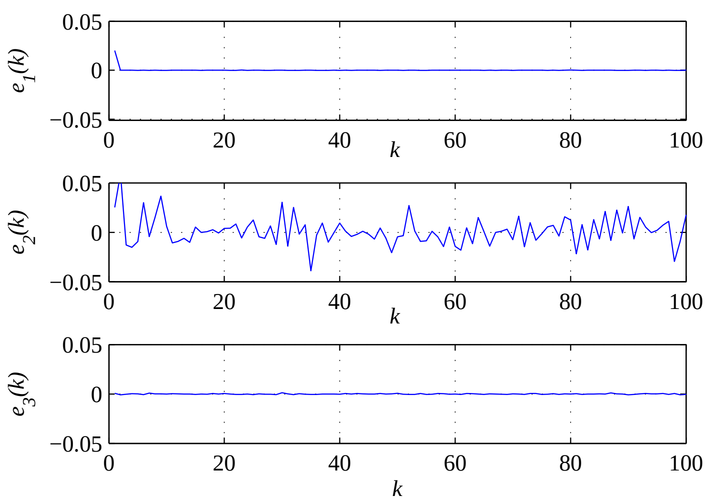

Figure 4.5: Time evolution of the estimation errors $e_i\,(k)$, $i = 1, 2, 3$ between (34) and (36) for $\hat{x}\,(0) = (0.9, 0.95, 0.05)$ (for one realization of the noise).

5 Conclusions

In this paper, we have approached the problem of synchronization of discrete-time hyperchaotic systems from the perspective of an extended Kalman filter (EKF) designed as slave. Approximate synchronization was obtained between a noisy master and slave dynamics when the slave was driven by a noisy drive signal from the master. Based on Lyapunov theory, we have demonstrated stability of the estimation/synchronization error, this result provides necessary conditions to achieve approximate synchronization. By extensive computer simulations, we have shown that the filter/slave is indeed suitable for synchronization of noisy hyperchaotic maps, it was illustrated by means of two numerical examples. The adopted approach shows great potential for actual communication systems in which the encoding is required to be secure. In a forthcoming article we will be concerned with the application to secure communication, and with the quantization of the degree of safety of the proposal in actual communication systems. Finally, we comment that this type of approximate synchronization method can be applied to secure chaotic communication, by using similar idea developed in [6, 7] and for continuous case in [31].

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References

[1] Pecora, L.M. and Carroll, T.L. Synchronization in chaotic systems. *Phys. Rev. Lett.* **64** (1990) 821–824.

[2] Feldmann, U., Hasler, M. and Schwarz, W. Communication by chaotic signals: the inverse system approach. *Int. J. Circuits Theory and Applications* **24** (1996) 551–579.

[3] Nijmeijer H. and Mareels I.M.Y. An observer looks at synchronization. *IEEE Trans. Circuits Syst. I* **44** (10) (1997) 882–890.

[4] Special Issue on Chaos synchronization and control: Theory and applications. *IEEE Trans. Circuits Syst. I* **44** (10) (1997).

[5] Fradkov, A.L. and Pogromsky, A. Yu. *Introduction to control of oscillations and chaos*, World Scientific Publishing, Singapore, 1998.

[6] Cruz-Hernández, C. and Nijmeijer, H. Synchronization through extended Kalman filtering. In: *New Trends in Nonlinear Observer Design* (Eds: H. Nijmeijer, T.I. Fossen), Lecture Notes in Control and Information Sciences 244 Springer-Verlag, 1999, 469–490.

[7] Cruz-Hernández, C. and Nijmeijer, H. Synchronization through filtering. *Int. J. Bifurc. Chaos* **10** (4) (2000) 763–775.

[8] Special Issue on Control and synchronization of chaos. *Int. J. Bifurc. Chaos* **10** (3-4) (2000).

[9] Sira-Ramírez H. and Cruz-Hernández C. Synchronization of chaotic systems: A generalized Hamiltonian systems approach. *Int. J. Bifurc. Chaos* **11** (5) (2001) 1381-1395. And In: *Procs. of American Control Conference (ACC'2000)*, Chicago, USA, P. 769–773.

[10] Pikovsky, A., Rosenblum, M. and Kurths, J. *Synchronization: A universal concept in nonlinear sciences*, Cambridge University Press, 2001.

[11] Aguilar, A.Y. and Cruz-Hernández, C. Synchronization of two hyperchaotic Rössler systems: Model-matching approach. *WSEAS Transactions on Systems* **1** (2) (2002) 198–203.

[12] Itoh, M., Yang, T. and Chua, L.O. Conditions for impulsive synchronization of chaotic and hyperchaotic systems. *Int. J. Bifurc. Chaos* **11** (2) (2001) 551–560.

[13] López-Mancilla, D. and Cruz-Hernández C. An analysis of robustness on the synchronization of chaotic systems under nonvanishing perturbations using sliding modes. *WSEAS Transactions on Mathematics* **3** (2) (2004) 364–369.

[14] López-Mancilla, D. and Cruz-Hernández C. Output synchronization of chaotic systems: Model-matching approach with application to secure communication. *Nonlinear Dynamics and Systems Theory* **5** (2) (2005) 141–156.

[15] Parlitz, U., Chua, L.O., Kocarev, Lj., Halle, K.S. and Shang, A. Transmission of digital signals by chaotic synchronization. *Int. J. Bifurc. Chaos* **2** (4) (1992) 973–977.

[16] Cuomo, K.M., Oppenheim, A.V. and Strogratz, S.H. Synchronization of Lorenz-based chaotic circuits with applications to communications *IEEE Trans. Circuits Syst. II* **40** (10) (1993) 626–633.

[17] Dedieu, H., Kennedy, M.P., and Hasler, M. Chaotic shift keying: Modulation and demodulation of a chaotic carrier using self-synchronizing Chua's circuits. *IEEE Trans. Circuits Syst. II* **40** (10) (1993) 634–642.

[18] Yang, T., and Chua, L.O. Secure communication via chaotic parameter modulation. *IEEE Trans. Circuits Syst. I* **43** (9) (1996) 817–819.

[19] López-Mancilla, D. and Cruz-Hernández, C. A note on chaos-based communication schemes. *Revista Mexicana de Física* **51** (3) (2005) 265–269.

[20] López-Mancilla, D., Cruz-Hernández, C. and Posadas-Castillo, C. A modified chaos-based communication scheme using Hamiltonian forms and observer. *Journal of Physics: Conference Series* 23 (2005) 267–275.

[21] Cruz-Hernández, C., López-Mancilla, D., García, V., Serrano, H. and Núñez, R. Experimental realization of binary signal transmission using chaos *J. Circuits, Syst. Computers* **14** (3) (2005) 453–468.

[22] Cruz-Hernández, C. Synchronization of time-delay Chua's oscillator with application to secure communication. *Nonlinear Dynamics and Systems Theory* **4** (1) (2004) 1–13.

[23] Cruz-Hernández, C. and Romero-Haros, N. Communicating via synchronized time-delay Chua's circuits. To be published in: *Communications in Nonlinear Science and Numerical Simulation* (2006).

[24] Short, K.M. Steps towards unmasking chaotic communication. *Int. J. Bifurc. Chaos* **4** (4) (1994) 959–977.

[25] Pérez, G. and Cerdeira, H.A. Extracting messages masked by chaos. *Phys. Rev. Lett.* **74** (1995) 1970–1973.

[26] Yang, T, Wu, C.W. and Chua, L.O. Cryptography based on chaotic systems. *IEEE Trans. Circuits Syst. I* **44** (5) (1997) 469–472.

[27] Cruz-Hernández, C. and Serrano-Guerrero, H. Cryptosystems based on synchronized Chua's circuits. In *Procs. of the 16th IFAC World Congress*, July 3–8 (2005) Prague, Czech Republic.

[28] Brucoli, M., Cafagna, D. and Carnimeo, L. Synchronization of hyperchaotic circuits via continuous feedback control with application to secure communications. *Int. J. Bifurc. Chaos* **8** (10) (1998) 2031–2040.

[29] Peng, J.H., Ding, E.J., Ding, M. and Yang, W. Synchronizing hyperchaos with a scalar transmitted signal. *Phys. Rev. Lett.* **76** (6) (1996) 904–907.

[30] Cruz-Hernández, C., Posadas, C. and Sira-Ramírez, H. Synchronization of two hyperchaotic Chua circuits: A generalized Hamiltonian systems approach. *Procs. of the 15th IFAC World Congress*, July 21–26 (2002), Barcelona, Spain.

[31] Sobiski, D.J. and Thorp, J.S. PDMA-1: Chaotic communication via the extended Kalman filter. *IEEE Trans. Circuit Syst. I* **45** (2) (1998) 194–197.

[32] Anderson, B.D.O. and Moore, J.B. *Optimal Filtering*. Prentice-Hall, Englewood Cliffs, NJ, 1979.

[33] La Scala, B.F., Bitmead, R.R. and James, M.R. Conditions for stability of the extended Kalman filter and their application to the frequency tracking problem. *Math. Control Signals Systems* **8** (1) (1995) 1–26.

[34] Reif, K., Sonnemann, F. and Unbehauen, R. An EKF-based nonlinear observer with a prescribed degree of stability. *Automatica* **34** (9) (1998) 1119–1123.

[35] Reif, K., Günther, S. and Yaz, E. Stochastic stability of the discrete-time extended Kalman filter. *IEEE Trans. Automat. Contr.* **44** (4) (1999) 714–728.