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On a Class of Strongly Nonlinear Impulsive Differential Equation with Time Delay

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Abstract: In this paper, we prove the existence of solutions for a class of strongly nonlinear impulsive differential equations with time delay in infinite dimensional Banach spaces by means of a fixed point theorem due to Leray-Schauder.

Keywords: *Monotone operator; impulsive differential equation; delay; existence of solutions.*

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1 Introduction

In recent years there has been intensive research on systems governed by impulsive differential equations and impulsive functional equations in Banach spaces (see [1], [2], [3], [6], and the references therein). This is probably due to the fact that though a vast majority of physical systems are described by differential or difference equations, a more realistic model of a physical system can be constructed using differential equations with time delay and impulsive effects in describing the evolution and discrete events occurring in the system. In fact, many evolution processes in nature are characterized by the fact that there are inherently time delays and at certain moments of time experience an abrupt change of state. Most papers in the literature dealt with ordinary differential systems and semilinear differential equations. Their emphasis and advantage lie in the fact that solutions of these systems are being represented by means of integration formula via appropriate semigroup of operators. It seems that only a few papers discuss the strongly nonlinear impulsive functional differential system, which cover quasilinear partial differential equations with time delay.

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Let $D = \{t_1 < t_2 < \cdots < t_m\}$ be fixed impulsive points in $(0, T)$. In this paper we study a class of strongly nonlinear impulsive differential equations with time delay in the form

$$\begin{cases} \dot{x}(t) + A(t,x) = f(t,x_t), \quad t \in [0,T] \setminus D, \\ x(t) = \phi(t), \quad t \in [-r,0], \\ \triangle x(t_i) = G_i(x(t_i)), \quad i = 1,2,\cdots,m. \end{cases} \tag{1.1}$$

Here A is a nonlinear monotone operator and f is a nonlinear nonmonotone perturbation, and G_i denotes the jump operator defined as

$$G_i(x(t_i)) = x(t_i+) - x(t_i-) = x(t_i+) - x(t_i),$$

$\phi \in PF$ and $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$. The space PF will be introduced in Section 2.

We present here sufficient conditions for the existence of solution to this particular class of nonlinear impulsive functional equations in an appropriate infinite dimensional Banach space. The results are obtained by using the theory of nonlinear functional analysis and a fixed point theorem due to Leray-Schauder.

The rest of the paper is organized as follows. In Section 2, we introduce some basic notations. In Section 3, we prove the existence of solutions for a class of nonimpulsive delay differential equations in Banach spaces. In Section 4, we establish the new existence result for a class of nonlinear impulsive functional differential equation in Banach spaces. Finally, we conclude with an example to illustrate our results in Section 5.

2 Preliminaries

Let H be a separable Hilbert space and V be a dense subspace of H having the structure of a reflexive Banach space with continuous embedding, so that $V \hookrightarrow H \hookrightarrow V^*$ forms a Gelfand triple. We assume the injection $V \hookrightarrow H$ is continuous and compact. The system model considered here is based on this Gelfand triple (see [1] or Chapter 23 of [9]).

Let $I \equiv [0,T]$, $r > 0$, and $m > 0$ be given. The norm in any Banach space X will be denoted by $\|\cdot\|_X$. Let $PF(X) = \{\psi : [-r,0] \to X;\ \ \psi$ is continuous everywhere except for a finite number of points $\widetilde{t}$ at which $\psi(\widetilde{t}-)$ and $\psi(\widetilde{t}+)$ exist and $\psi(\widetilde{t}-) = \psi(\widetilde{t})\}$ be endowed with the norm

$$\|\psi\|_{PF(X)} = sup\{\|\psi(\theta)\|_H, \theta \in [-r,0]\}.$$

For any continuous function x defined on $[-r,T] \setminus D$ and $t \in [0,T]$, we denote by x_t the element of $PF \equiv PF(H)$ defined by

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-r,0].$$

Let $1 < q \le p < +\infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. The space $W_{pq}(I) \equiv W_{pq}$ is defined as follows:

$$W_{pq}(I) = \{x | x \in L_p(I,V), \dot{x} \in L_q(I,V^*)\}$$

with the norm

$$\|x\|^2_{W_{pq}} = \|x\|^2_{L_p(I,V)} + \|\dot{x}\|^2_{L_q(I,V^*)},$$

where $\dot{x}$ denotes the derivative of x in the generalized sense. $\{W_{pq}, \|\cdot\|_{W_{pq}}\}$ is a Banach space and the embedding $W_{pq} \hookrightarrow C(I,H)$ is continuous. If the embedding $V \hookrightarrow H$ is compact, the embedding $W_{pq} \hookrightarrow L_p(I,H)$ is also compact (see [9] and [1]). Similarly, we can define $W_{pq}([s,u])$ for $0 \leq s < t < u \leq T$. Furnished with the norm $\| \cdot \|_{W_{pq}([s,u])}$, the space $(W_{pq}([s,u]), \| \cdot \|_{W_{pq}([s,u])})$ becomes a Banach space which is clearly reflexive and separable. Moreover, the embedding $W_{pq}([s,u]) \hookrightarrow C([s,u],H)$ is continuous and the embedding $W_{pq}([s,u]) \hookrightarrow L_p((s,u),H)$ is also compact.

Set $PC(I,H) = \{x : x$ is a map from I into H such that $x(t)$ is continuous at $t \in I \backslash D$ and $x(t)$ is left continuous at $t \in D$ and the right limit $x(t_i+)$ exists for $i = 1,2,\cdots m\}$, and

$$PW_{pq}(I) = \{x : x \mid_{[t_i,t_{i+1}]} \in W_{pq}([t_i,t_{i+1}]) \text{ for } i = 0,1,\cdots,m\},$$

where $t_0 = 0$ and $t_{m+1} = T$. For $x \in PW_{pq}(I) \cap PC(I,H) \triangleq PWC$, define

$$\|x\|_{PWC} = \sum_{i=0}^{m} \|x\|_{W_{pq}[t_i,t_{i+1}]} + \sum_{i=1}^{m} \|x(t_i+) - x(t_i-)\|_H.$$

It is easy to show that PWC is a Banach space.

Let us consider the following nonlinear impulsive differential equation with time delay

$$\begin{cases} \dot{x}(t) + A(t,x) = f(t,x_t), \quad t \in [0,T] \setminus D, \\ x(t) = \phi(t), \quad t \in [-r,0], \\ \triangle_l x(t_i) = G_i(x(t_i)) \quad i = 1,2,\cdots,m, \end{cases} \tag{2.1}$$

where A is a nonlinear monotone operator, f is a nonlinear nonmonotone perturbation, $G_i(i = 1,2\cdots m)$ are nonlinear maps. Here $\phi \in PF$, and $\triangle_l x(t_i) = x(t_i+) - x(t_i-) = x(t_i+) - x(t_i)$, which represents the jump in the state x at time t_i with G_i determining the size of the jump at time t_i.

We will impose the following hypotheses on problem (2.1).

(A1) $A : I \times V \to V^*$ is an operator such that

(i) $t \to A(t,x)$ is measurable.

(ii) $x \to A(t,x)$ is monotone and hemicontinuous; i.e., $\forall t \in I$,

$$\langle A(t,x_1) - A(t,x_2), x_1 - x_2 \rangle \geq 0 \qquad \forall x_1,\, x_2 \in V, \quad t \in I;$$
$$A(t,x+sy) \xrightarrow{W} A(t,x) \qquad \text{in } V^* \quad \text{as } s \to 0 \quad \forall x,y \in V.$$

(iii) There exist positive constants c_1, c_2, c_3 and a nonnegative function $c_4(\cdot) \in L_q(I)$ such that $\forall t \in I$,

$$\langle A(t,x), x \rangle \geq c_1 \| x \|_V^p - c_2, \qquad \text{for all } x \in V,$$
$$\|A(t,x)\|_{V^*} \leq c_4(t) + c_3 \|x\|_V^{p-1} \text{ for all } x \in V.$$

(A2) $f : I \times PF \to H$ is an operator such that

(i) $t \to f(t,\xi)$ is measurable, and
$\xi \to f(t,\xi)$ is continuous.

(ii) There exist a constant $\alpha \geq 0$ and a nonnegative function $h(\cdot) \in L_2(I)$ such that

$$\|f(t,\xi)\|_H \leq h(t) + \alpha \|\xi\|_{PF}^{\frac{2}{q}}, \forall t \in I, \xi \in PF.$$

(A3) For $i = 1,2,\cdots,m$, $G_i : H \to H$ is a bounded map (i.e., G_i maps a bounded set to a bounded set).

To arrive at the main results of the paper, we need the following fixed point theorem due to Leray and Schauder [5].

Theorem 2.1 *Let B be a convex subset of a normed linear space E and $0 \in B$. Let $P : B \to B$ be a completely continuous operator and let*

$$\xi(P) = \{x \in B : x = \sigma P(x) \text{ for some } 0 < \sigma < 1\}.$$

Then either the set $\xi(P)$ is unbounded, or P has a fixed point.

3 Existence of solutions of functional differential equation

In this section we consider the following functional differential equation without impulsive effects:

$$\begin{cases} \dot{x}(t) + A(t,x) = f(t,x_t), & t \in [0,T], \\ x(t) = \phi(t), & t \in [-r,0], \end{cases} \tag{3.1}$$

Definition 3.1 *A function x is called a weak solution of (3.1) if $x|_{[0,T]} \in W_{pq}$ satisfies the equation in a weak sense and $x(t) = \phi(t)$ $\forall t \in [-r,0]$.*

Theorem 3.1 *Under assumptions (A1) and (A2), problem (3.1) has a solution in W_{pq}.*

Proof

Step 1: The proof will be given first for the case where $\phi(0) = 0$.

(1) Set

$$B = \{y | y \in C([0,T],H), y(0) = 0\}.$$

Obviously, B is a Banach space with the supremum norm. For any $x \in B$, we define $F : B \to L_2(I,H)$ by $F(x)(t) = f(t,\widehat{x}_t)$ with

$$\widehat{x}_t(s) = \begin{cases} \phi(t+s) & \text{for } t+s \in [-r,0), \\ x(t+s) & \text{for } t+s \in [0,T]. \end{cases}$$

The operator P is defined on B by letting $y = Px$ be the corresponding solution of the following Cauchy problem

$$\begin{cases} \dot{y}(t) + A(t,y(t)) = F(x)(t), & t \in I, \\ y(0) = 0. \end{cases}$$

Indeed, by assumption $(A2)$ and $1 < q \le p < +\infty$, $F(x)(t) = f(t,\widehat{x}_t)$ is measurable and

$$F(x)(\cdot) \in L_2(I,H) \subset L_q(I,V^*).$$

Thus, the above Cauchy problem has a unique solution $y \in W_{pq} \hookrightarrow C(I,H)$ (see Theorem 30.A of [9]). Hence P maps B into itself.

(2) $P : B \to B$ is continuous.

Suppose $x_n \longrightarrow x$ in B as $n \longrightarrow \infty$. This means

$$\sup_{0 \le t \le T} \|x_n(t) - x(t)\|_H \longrightarrow 0,$$

as $n \to +\infty$. Hence, there exists a constant $M > 0$ such that

$$\|\widehat{x}_n\|_{PC([-r,T],H)} \leq M \quad \text{and} \quad \|\widehat{x}\|_{PC([-r,T],H)} \leq M.$$

By virtue of assumption (A2), we have for $t \in I$,

$$F(x_n)(t) \longrightarrow F(x)(t) \text{ in } H$$

as $n \to \infty$ and there exists a constant $M_1 > 0$ such that

$$\|F(x_n)(t)\|_H \leq h(t) + M_1 \quad \text{and} \quad \|F(x)(t)\|_H \leq h(t) + M_1.$$

It follows from the majorized convergence principle that

$$F(x_n) \longrightarrow F(x) \text{ in } L_2(I,H)$$

as $n \to \infty$.

Let $y_n = Fx_n$ and $y = Fx$ satisfy the following equations respectively. For $t \in I$,

$$\begin{aligned} \dot{y}_n(t) + A(t, y_n(t)) &= F(x_n)(t), \quad y_n(0) = 0, \\ \dot{y}(t) + A(t, y(t)) &= F(x)(t), \quad y(0) = 0. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2}\|y_n(t) - y(t)\|_H^2 &\leq \|F(x_n) - F(x)\|_{L_2(0,t;H)}\|y_n - y\|_{L_2(0,t;H)} \\ &\leq \frac{1}{2}\|F(x_n) - F(x)\|^2_{L_2(I,H)} + \frac{1}{2}\int_0^t \|y_n(\tau) - y(\tau)\|_H^2 d\tau. \end{aligned}$$

Thanks to Gronwall's lemma, it is easy to show that

$$y_n \longrightarrow y \quad \text{in } B \text{ as } n \to \infty.$$

(3) P is a compact operator on B.

Let $\{x_n\}$ be a bounded sequence in B. That is, there is a constant $M_2 > 0$ such that

$$\|x_n\|_{C(I,H)} \leq M_2.$$

Again, by assumption $(A2)$, there exist constants $M_3, M_4 > 0$ such that

$$\|F(x_n)(t)\|_H \leq h(t) + M_3 \quad \text{and} \quad \|F(x_n)\|_{L_2(I,H)} \leq M_4.$$

Let $y_n = Px_n$ be a solution of the following equation

$$\begin{cases} \dot{y}_n(t) + A(t, y_n(t)) = F(x_n)(t). \\ y_n(0) = 0. \end{cases} \tag{3.2}$$

Integrating by parts in (3.2) and using assumption (A1), one can obtain

$$\frac{1}{2}\|y_n(t)\|_H^2 + C_1\|y_n\|^p_{L_p(0,t;V)} \leq \|F(x_n)\|_{L_2(0,t;H)}\|y_n\|_{L_2(0,t;H)} + C_2.$$

It follows from the Cauchy inequality that there exist constants $\gamma > 0$ and $K > 0$ such that

$$\frac{1}{2}\|y_n(t)\|_H^2 + \gamma\|y_n\|^p_{L_p(0,t;V)} \leq K\|F(x_n)\|^q_{L_2(I,H)} + C_2.$$

Hence $\{y_n\}$ is bounded in $C(I,H)\cap L_p(I,V)$. It follows from Eq.(3.2) that $\{\dot{y}_n\}$ is bounded in $L_q(I,V^*)$ and therefore $\{y_n\}$ is bounded in W_{pq}.

Since $W_{pq}\hookrightarrow L_p(I,H)$ is compact, there exists a subsequence, relabelled $\{y_n\}$, such that

$$y_n \longrightarrow y \quad \text{in } L_p(I,H) \quad \text{as } n\to\infty.$$

So $\{y_n\}$ is a Cauchy sequence in $L_p(I,H)$. Hence there exists a constant $M_5>0$ such that

$$\begin{aligned}\frac{1}{2}\|y_n(t)-y_m(t)\|_H^2 &\leq \|F(x_n)-F(x_m)\|_{L_q(I,H)}\|y_n-y_m\|_{L_p(I,H)}\\ &\leq M_5\|y_n-y_m\|_{L_p(I,H)}.\end{aligned}$$

This inequality implies that $\{y_n\}$ is a Cauchy sequence in B. Since B is closed, the sequence $\{y_n\}$ has a limit in B. This proves that P is compact.

(4) Boundedness of the set $\xi(P)$.

We will show that the set $\xi(P)$ is bounded. To this end, suppose $x\in B$ and $x=\sigma Px$ where $\sigma\in(0,1)$. This implies that x satisfies the following Cauchy problem:

$$\begin{cases}\frac{1}{\sigma}\dot{x}(t)+A(t,\frac{1}{\sigma}x(t))=g(t,\widehat{x}_t), \quad t\in I,\\ x(0)=0.\end{cases} \tag{3.3}$$

We will show that there exists a $Q>0$ such that

$$\|x\|_{C(I,H)}\leq Q.$$

Using the same arguments and assumptions $(A1)$ and $(A2)$, we have

$$\begin{aligned}&\frac{1}{2\sigma}\|x(t)\|_H^2+\frac{C_1}{\sigma^{p-1}}\|x\|^p_{L_p(0,t;V)}\leq\int_0^t\langle f(\tau,\widehat{x}_\tau),x(\tau)\rangle d\tau+C_2\\ \leq\ &(\int_0^t\|f(\tau,\widehat{x}_\tau)\|_H^q d\tau)^{1/q}(\int_0^t\|x(\tau)\|_H^p d\tau)^{1/p}+C_2\\ \leq\ &\frac{1}{q\varepsilon^q}\int_0^t\|f(\tau,\widehat{x}_\tau)\|_H^q d\tau+\frac{\varepsilon^p}{p}\|x\|^p_{L_p(0,t;H)}+C_2,\end{aligned}$$

for any constant $\varepsilon>0$ and some constants $C_1\geq0$ and $C_2\geq0$. Hence

$$\frac{\sigma^{p-2}}{2}\|x(t)\|_H^2+C_1\|x\|^p_{L_p(0,t;V)}\leq C_2\sigma^{p-1}+b_1\varepsilon^p\sigma^{p-1}\|x\|^p_{L_p(0,t;V)}+\frac{d_1\sigma^{p-1}}{q\varepsilon^q}\int_0^t\|\widehat{x}_\tau\|^2_{PF}d\tau$$

where b_1 and d_1 are positive constants. So, we can choose $\varepsilon>0$ small enough such that

$$\frac{\sigma^{p-2}}{2}\|x(t)\|_H^2\leq a_1\sigma^{p-1}+b_2\sigma^{p-1}\int_0^t\|\widehat{x}_\tau\|^2_{PF}d\tau$$

where a_1 and b_2 are positive constants. It follows from $0<\sigma<1$ that

$$\|x(t)\|_H^2\leq a_2+d_1\int_0^t\|\widehat{x}_\tau\|^2_{PF}d\tau.$$

We denote

$$k(t)=a_2+d_1\int_0^t\|\widehat{x}_\tau\|^2_{PF}d\tau.$$

It is obvious that $k(t)$ is an increasing function. So,

$$\sup_{0\le\theta\le t}\|x(\theta)\|_H^2 \le a_2 + d_1 r\|\phi\|_{PF}^2 + \int_0^t \sup_{0\le\theta\le\tau}\|x(\theta)\|_H^2 d\tau \quad \text{for all } t\in[0,T]$$

Let $\omega(t) = \sup_{0\le\theta\le t}\|x(\theta)\|_H^2$. Then $\omega(t)$ is continuous and increasing since $x(t)$ is continuous. An application of the Gronwall lemma implies that

$$\|x\|_{C([0,T],H)} \le Q, \tag{3.4}$$

and so $\xi(P)$ is bounded.

By the Leray-Schauder fixed point theorem (Theorem 2.1), P has a fixed point x^* in B. Then x^* is a corresponding solution of (3.1).

Step 2: For the proof of the theorem, in general case where $\phi(0)\neq 0$, at first we assume that $\phi(0)\in V$, we use the transformation

$$y = x - \phi(0)$$

to reduce the problem (3.1) into the following problem:

$$\begin{cases} \dot{y}(t) + A(t, y+\phi(0)) = f(t, y_t + \phi(0)), & t\in I,\\ y(t) = \phi(t) - \phi(0), & t\in[-r,0], \end{cases} \tag{3.5}$$

We set $\hat{A}(t,y) = A(t, y+\phi(0))$ and $\hat{f}(t,y_t) = f(t, y_t+\phi(0))$. Then it is easy to see that $\hat{A}$ satisfies assumptions (A1)(i) and (ii). It follows from assumption (A1) (iii) that

$$\begin{aligned} \|\hat{A}(t,y)\|_{V^*} &= \|A(t,y+\phi(0))\|_{V^*} \le c_4(t) + c_3\|y+\phi(0)\|_V^{p-1} \\ &\le c_4(t) + c_3 2^{p-1}\|y\|_V^{p-1} + c_3 2^{p-1}\|\phi(0)\|_V^{p-1}. \end{aligned} \tag{3.6}$$

Let $m_4 = c_4(t) + c_3 2^{p-1}\|\phi(0)\|_V^{p-1}$ and $m_3 = c_3 2^{p-1}$. Then

$$\|\hat{A}(t,y)\|_{V^*} \le m_4(t) + m_3\|y\|_V^{p-1}$$

for all $y\in V$ and $t\in I$.

By assumption (A1)(iii), one can get

$$\begin{aligned} \langle \hat{A}(t,y), y\rangle &= \langle A(t,y+\phi(0)), y+\phi(0)\rangle - \langle A(t,y+\phi(0)), \phi(0)\rangle \\ &\ge c_1\|y+\phi(0)\|_V^p - c_2 - \|A(t,y+\phi(0))\|_{V^*}\cdot\|\phi(0)\|_V \\ &\ge c_1\|y+\phi(0)\|_V^p - c_2 - \frac{1}{p\epsilon^p}\|\phi(0)\|_V^p - \frac{\epsilon^q}{q}\|A(t,y+\phi(0))\|_{V^*}^q \end{aligned} \tag{3.7}$$

for any constant $\epsilon > 0$. Then, by (3.6), one can reduce (3.7) into

$$\langle \hat{A}(t,y), y\rangle \ge \left(c_1 - \frac{c_3 2^{q-1}}{q}\epsilon^q\right)\|y+\phi(0)\|_V^p - c_2 - \frac{1}{p\epsilon^p}\|\phi(0)\|_V^p - \frac{2^{q-1}c_4^q(t)}{q}\epsilon^q.$$

We can choose ϵ small enough such that $m_1 \equiv c_1 - \frac{c_3 2^{q-1}}{q}\epsilon^q > 0$ and note that the following inequality

$$\begin{aligned} \|y+\phi(0)\|_V^p + \|\phi(0)\|_V^p &\ge |\|y\|_V - \|\phi(0)\|_V|^p + \|\phi(0)\|_V \\ &\ge C\left(\|y\|_V - \|\phi(0)\|_V + \|\phi(0)\|_V\right)^p = C\|y\|_V^p \end{aligned}$$

holds for some constant $C > 0$.

One can obtain

$$\langle \hat{A}(t,y), y\rangle \geq m_1 C\|y\|_V^p - m_2$$

where $m_2 = c_2 - \frac{1}{p\epsilon^p}\|\phi(0)\|_V p - c_1\|\phi(0)\|_V^p - \frac{2^{q-1}c_4^q(t)}{q}\epsilon^q$. That is, $\hat{A}(t,y)$ satisfies assumption (A1).

For $\hat{f}(t,y) = f(t, y+\phi(0))$, one can easily verify that $\hat{f}(t,y)$ satisfying assumption (A2). Then the problem (3.5) has a solution from Step 1.

If $\phi(0) \in H$, there exists a sequence $\{\xi_n\} \subset V$, such that $\xi_n \to \phi(0)$ in H. Set

$$\phi_n(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0), \\ \xi_n & \text{for } t = 0. \end{cases}$$

Then there exists $x_n \in W_{pq}$ such that

$$\begin{cases} \dot{x}_n(t) + A(t, x_n(t)) = f(t,(x_n)_t), & t \in I, \\ x_n(t) = \phi_n(t), & t \in [-r,0]. \end{cases} \tag{3.8}$$

We define $\hat{A}(x)(t) = A(t,x(t))$ for $x \in L_p(I,V)$ and $t \in I$. Then $\hat{A} : L_p(I,V) \to L_q(I,V^*)$ is bounded, monotone, hemicontinuous, and coercive (see Theorem 30.A of [9]). It follows from (3.4) and assumption (A1) that

$$\|x_n\|_{W_{pq}} \leq M \qquad \text{and} \qquad \|A(x_n)\|_{L_q(I,V^*)} \leq M$$

for some constant $M > 0$. Then there exists a subsequence of $\{x_n\}$, denoted $\{x_n\}$ again, such that

$$\begin{aligned} x_n &\xrightarrow{W} x && \text{in} \quad L_p(I,V), \\ \dot{x}_n &\xrightarrow{W} \dot{x} && \text{in } L_q(I,V^*), \\ \hat{A}(x_n) &\xrightarrow{W} w && \text{in } L_q(I,V^*), \end{aligned}$$

as $n \to +\infty$. Since $W_{pq} \hookrightarrow L_p(I,H)$ is compact, we know that

$$\begin{aligned} x_n &\xrightarrow{W} x && \text{in } W_{pq}, \\ x_n &\xrightarrow{S} x && \text{in } L_p(I,H), \\ x_n(t) &\xrightarrow{S} x(t) && \text{a.e. on } I \text{ in } H. \end{aligned}$$

By assumption (A2) and using the similar method as in the proof of Lemma 1 of [8], it follows that

$$F(x_n) \xrightarrow{S} F(x) \qquad \text{in } L_q(I,H).$$

Hence

$$\begin{cases} \dot{x} + w = F(x), & t \in I, \\ x(t) = \phi(t), & t \in [-r,0]. \end{cases} \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$\langle \dot{x}_n - \dot{x}, x_n - x\rangle + \langle \hat{A}x_n - w, x_n - x\rangle = \langle F(x_n) - F(x), x_n - x\rangle.$$

Hence

$$\begin{aligned}\langle \hat{A}(x_n) - w, x_n \rangle &= \frac{1}{2}\|x_n(0) - x(0)\|_H^2 - \frac{1}{2}\|x_n(T) - x(T))\|_H^2 \\ &+ \langle \hat{A}(x_n) - w, x \rangle + \langle F(x_n) - F(x), x_n - x \rangle.\end{aligned}$$

So,

$$\limsup_{n\to+\infty} \langle Ax_n, x_n \rangle \leq \langle w, x \rangle.$$

Note that $\hat{A} : L_p(I, V) \to L_q(I, V^*)$ is monotone, hemicontinuous, and so $\hat{A}$ satisfies the condition (M) (see p. 538 of [9]). We deduce that

$$w = \hat{A}(x).$$

Thus,

$$\begin{cases} \dot{x} + \hat{A}(x) = F(x), & t \in I, \\ x(t) = \phi(t), & t \in [-r, 0], \end{cases} \tag{3.10}$$

That is, x is a solution of (4.1).

The theorem is proved.

Remark 3.1 *It follows from the proof of Theorem 3.2 that if x is a solution of (3.1) then x is bounded in W_{pq}.*

Theorem 3.2 guarantees the existence of solutions for (3.1), but not the uniqueness of solutions. In order to obtain uniqueness, we have to impose a somewhat stronger assumption on f. Assume that

(A4) f is locally Lipschitz continuous with respect to ξ, i.e., for any $\rho > 0$, there exists a constant $L(\rho)$, such that

$$\|f(t, \xi_1) - f(t, \xi_2)\|_H \leq L(\rho)(\|\xi_1 - \xi_2\|_{PF}), \quad \forall t \in I$$

and for all $\xi_1, \xi_2 \in PF(H)$ satisfying $\|\xi_1\|_{PF} \leq \rho, \|\xi_2\|_{PF} \leq \rho$.

Theorem 3.2 *(Uniqueness of solution) If assumption (A4) holds, then the problem (3.1) has at most one solution.*

Proof Let x_1 and x_2 be two solutions of problem (3.1). Then

$$\begin{aligned}\frac{1}{2}\|x_1(t) - x_2(t)\|_H^2 &\leq \|F(x_1) - F(x_2)\|_{L_2(0,t;H)} \|x_1 - x_2\|_{L_2(0,t;H)} \\ &\leq \frac{1}{2}\|F(x_1) - F(x_2)\|_{L_2(0,t;H)}^2 + \frac{1}{2}\|x_1 - x_2\|_{L_2(0,t;H)}^2.\end{aligned}$$

By assumption (A4), there exist constants $C_1^* > 0$ and $C_2^* > 0$ such that

$$\|x_1(t) - x_2(t)\|_H^2 \leq C_1^* \int_0^t \|x_1(\tau) - x_2(\tau)\|_H^2 d\tau + C_2^* \int_0^t \|(x_1)_\tau - (x_2)_\tau\|_{PF}^2 d\tau.$$

Because $x_1(t) = x_2(t) = \phi(t), t \in [-r, 0]$ and the solution of (3.1) is continuous in $[0, T]$, one can modify x_1 and x_2 by setting $x_1(t) = x_2(t) \equiv \xi, \forall t \in [-r, 0]$. Then $x_1, x_2 \in C([-r, T]; H)$ such that

$$\|x_1(t) - x_2(t)\|_H^2 \leq C_1^* \int_0^t \|x_1(\tau) - x_2(\tau)\|_H^2 d\tau + C_2^* \int_0^t \|(x_1)_\tau - (x_2)_\tau\|_C^2 d\tau \quad \forall t \in [0, T]$$

where $C = C([-r,0],H)$ denotes all continuous maps from $[-r,0]$ into H with the usual supremum norm. Thanks to Gronwall's lemma, it implies

$$x_1(t) = x_2(t) \text{ for all } t \in [0,T].$$

That is,

$$x_1 = x_2.$$

4 Existence of solutions for impulsive delay differential equations

In this section, we deal with the nonlinear impulsive differential equation (2.1) with time delay in Banach Space.

Definition 4.1 A function $x \in PWC$ is called a PWC solution of (2.1) if it satisfies the equation in a weak sense on every interval $[t_i, t_{i+1}]$ $(i = 0, 1, \cdots, m)$, $x(t) = \phi(t), \quad t \in [-r, 0]$, and the state jump at t_i $(i = 1, 2, \cdots, m)$.

Theorem 4.1 *Suppose assumptions (A1), (A2), and (A3) hold. Then, for each $\phi \in PF(H)$, the problem (2.1) has a solution $x \in PWC$. Moreover, there is a constant $M > 0$ such that*

$$\|x\|_{PWC} \leq M \quad \text{and} \quad \|x\|_{PC} \leq M.$$

Proof Define $I_i \equiv (t_i, t_{i+1}], i = 0, 1, \cdots, m$ with $t_0 = 0, t_{m+1} = T$. It follows from assumptions (A1), (A2), Theorem 3.2, and Theorem 3.4 that for each $\phi \in PF$ and $\phi(0) \in V$, the equation (2.1) has a unique solution $x^{(1)}$ where $x^{(1)} |_{I_0} \in W_{pq}(I_0) \bigcap C(I_0, H)$ and $x^{(1)} |_{[-r,0]} = \phi$. By assumption (A3), $x(t_1 + 0)$ is well defined and it is given by

$$x(t_1 + 0) = G_1(x^{(1)}(t_1)) + x^{(1)}(t_1) \equiv \xi^1.$$

Consider the following problem

$$\begin{cases} \dot{x}(t) + A(t,x) = f(t, x_t), \quad t \in I_1, \\ x(t) = x^{(1)}(t), \quad t \in [t_1 - r, t_1], \\ x(t_1) = \xi^1. \end{cases} \tag{4.1}$$

Using the same argument as in the proof of Theorem 3.2 and the fact $x^{(1)} \in PF([t_1 - r, t_1], H)$, one obtains that there is a unique solution $x^{(2)}$ in I_1.

We continue this process taking into account that $x^{(m+1)} := x |_{I_m}$ is a solution to the problem

$$\begin{cases} \dot{x}(t) + A(t,x) = f(t, x_t), \quad t \in (t_m, T], \\ x(t) = \phi(t), \quad t \in [t_m - r, t_m], \\ x(t_m + 0) = x^{(m)}(t_m) + G_m(x^{(m)}(t_m)). \end{cases} \tag{4.2}$$

The solution x of the problem (2.1) is defined by

$$x(t) = \begin{cases} x^{(1)}(t), & \text{if } t \in [-r, t_1], \\ x^{(2)}(t), & \text{if } t \in (t_1, t_2], \\ \vdots & \\ x^{(m+1)}(t), & \text{if } t \in (t_m, T]. \end{cases}$$

And it follows from Remark 3.3 that for $k = 0, 1, \cdots, m$

$$\|x^{(k+1)}\|_{W_{pq}(I_k)} \le M$$

for some constant $M > 0$. Hence x is a PWC solution of (2.1) and

$$\|x\|_{PWC} \le M_1 \quad \text{and} \quad \|x\|_{PC} \le M_1.$$

for some constant $M_1 > 0$.

5 Examples

In this section we present an example of delay evolution equations with impulse to which our general theory applies.

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$, $0 < t_1 < \cdots < t_k < T$ are given fixed points and $D \equiv \{t_1, t_2, \cdots, t_k\}$, $Q_T = (0,T) \setminus D \times \Omega$, $0 < T < \infty$. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index with nonnegative integers $\{\alpha_i\}$, $i = 1, \ldots, n$, and $\|\alpha\| = \sum_{i=1}^n \alpha_i$. Let $p \ge 2$ and $q = p/(p-1)$and let $m > 0$ be an integer. $W^{m,p}(\Omega)$ denotes the standard Sobolev space with the usual norm:

$$\|\varphi\|_{W^{m,p}} = (\sum_{|\alpha| \le m} \|D^\alpha \varphi\|^p_{L^p(\Omega)})^{1/p}.$$

Let $W_0^{m,p}(\Omega) = \{\varphi \in W^{m,p} | D^\beta \varphi|_{\partial\Omega} = 0, \quad |\beta| \le m-1\}$. It is well known that $C_0^\infty(\Omega) \hookrightarrow W_0^{m,p}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-m,p}(\Omega)$ and the embedding $W_0^{m,p}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Denote $V \equiv W_0^{m,p}(\Omega)$, $H \equiv L_2(\Omega)$, then $V^* \equiv W^{-m,q}(\Omega)$.

We consider the following initial-boundary impulsive value problem of $2m$-order quasilinear delay parabolic equation:

$$\begin{cases} \frac{\partial}{\partial t} y(t,x) + \sum\limits_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(t,x,\eta(y)(t,x)) = g(t,x,y(t-r,x)) \quad \text{on } Q_T, \\ D^\beta y(t,x) = 0 \quad \text{on } [0,T] \times \partial\Omega, \qquad \text{for all } \beta \text{ satisfying } |\beta| \le m-1, \\ y(s,x) = \phi(s,x) \quad \text{for } x \in \Omega \text{ and } -r \le s \le 0, \\ y(t_i+) = -y(t_i-), \quad i = 1, 2, \cdots, k, \end{cases} \tag{5.1}$$

where $\eta(y) \equiv \{(D^\gamma y), \quad |\gamma| \le m\}, \phi(t,x)$ is a given function, $\phi \in C([-r,0], L_2(\Omega))$, $\phi(0) \in W_0^{m,p}(\Omega)$, and $M = \frac{(n+m)!}{n!m!}$.

For $y_1, y_2 \in W_0^{m,p}(\Omega)$ and $t \in I$, we set

$$a(t, y_1, y_2) = \int_\Omega \sum_{|\alpha| \le m} A_\alpha(t,x,\eta(y_1)(t,x)) D^\alpha y_2 dx$$

and assume that for all α with $|\alpha| \le m$, the function $A_\alpha : Q_T \times R^M \to R$ satisfies the following properties.

(H1) $(1)(t,x) \to A_\alpha(t,x,\eta)$ is measurable on Q_T for $\eta \in R^M$, $\eta \to A_\alpha(t,x,\eta)$ is continuous on R^M for a.e. $(t,x) \in Q_T$;

(2) For $\eta = (\eta_\alpha) \in R^M, \widetilde{\eta} = (\widetilde{\eta}_\alpha) \in R^M$, there exist positive constants c, c_1, c_2, c_3, and c_4 such that

$$\sum_{|\alpha|\leq m} (A_\alpha(t,x,\eta) - A_\alpha(t,x,\widetilde{\eta}))(\eta_\alpha - \widetilde{\eta}_\alpha) \geq 0,$$
$$\sum_{|\alpha|\leq m} A_\alpha(t,x,\eta)\eta_\alpha \geq c_1 \sum_{|\gamma|\leq m} |\eta_\gamma|^p - c_2,$$
$$|A_\alpha(t,x,\eta)| \leq c_4 + c_3 \sum_{|\gamma|\leq m} |\eta_\gamma|^{p-1}.$$

It is not difficult to verify that under the above assumption, for each $y_1 \in V$ and $t \in [0,T]$, $y_2 \to a(t, y_1, y_2)$ is a continuous linear form on V. Hence there exists an operator $A : I \times V \to V^*$ such that

$$\langle A(t,y_1),\ y_2\rangle_{V^*,V} = a(t,y_1,y_2).$$

Under the given assumption (H1), it is easy to see that A satisfies our assumption (A1) of Section 3.

Assume the function $f : Q_T \times R \to R$ satisfies the following properties.

(H2) (1) $(t,x) \to f(t,x,\eta)$ is measurable on Q_T for all $\eta \in R$;

(2) $\eta \to f(t,x,\eta)$ is continuous on R for almost all $(t,x) \in Q_T$;

(3) there exist constants $b_1 > 0$ and $b_2 > 0$ such that

$$|f(t,x,\eta)| \leq b_1|\eta|^{2/q} + b_2(t,x)$$

for almost all $(t,x) \in Q_T$.

For $\phi_1 \in H$ and $t \in I$, set

$$b(t,\phi_1,\psi) = \int_\Omega f(t,x,\phi_1)\psi dx.$$

Then $\psi \to b(t,\phi_1)$ is a continuous linear form on H. Hence there exists an operator $F : [0,T] \times H \to H$ such that

$$b(t,\phi_1,\psi) = (F(t,\phi_1),\psi).$$

Noting that $y_t(\theta) = y_t(r)$ for all $-r \leq \theta \leq 0$ and using (H2), one can verify that F satisfies assumption (A2) of Section 3.

With the operators A and F as defined above, problem (5.1) can be written as the abstract evolution equation

$$\begin{cases} \dot{y}(t) + A(t,y(t)) = F(t,y(t-r)), \quad t \in [0,T] \setminus D, \\ y(t) = \phi(t), \quad t \in (-r,0), \\ y(t_i) = -y(t_i), \quad 0 < t_1 < t_2 < \cdots < t_k < T. \end{cases}$$

Hence our result can be applied to this model to assert the existence of its solutions.

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