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Stability of Delay Systems with Quadratic Nonlinearities

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Abstract: In this paper differential systems with delay and quadratic nonlinear terms are considered. Sufficient stability conditions, estimations of the stability domain and estimations of the convergence rate are derived.

Keywords: *Systems of differential equations with delay; stability of zero solution; estimates on the stability domain; rate of convergence.*

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1 Introduction

Two modifications are typically used when differential delay systems are studied by using the second Liapunov method [9 − 11]. The first one is the Liapunov–Krasovsky method. In this case, a segment of the trajectory is identified with a point in Banach space. Also, the main ideas of the Liapunov functions method are carried over to this case of functionals, and the stability theorems usually contain the necessary and sufficient conditions [9, 11]. The second modification uses the finite-dimensional Liapunov functions. In this case the derivative of the solution is estimated under the assumption that the solution remains inside the level surface of the Liapunov function. This assumption is called the Razumikhin condition [10].

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2 Preliminaries

In this paper we consider differential delay systems with a quadratic nonlinearity of the following form

$$\dot{x}(t) = Ax(t-\tau) + X^{\mathrm{T}}(t)Bx(t-\tau), \tag{1}$$

where $t \geq 0$, τ is a positive constant, $x(t) \in R^n$, A is a constant square matrix. The matrices $X^{\mathrm{T}}(t)$ and B are rectangular ones of the size $n \times n^2$ and $n^2 \times n$, respectively; $X^{\mathrm{T}}(t) = \left\{X_1^{\mathrm{T}}(t), X_2^{\mathrm{T}}(t), \dots, X_n^{\mathrm{T}}(t)\right\}$, $B^{\mathrm{T}} = \{B_1, B_2, \dots, B_n\}$. We suppose that square matrices B_i, $i = \overline{1,n}$, are constant and symmetric, and all elements of the square matrices $X_i^{\mathrm{T}}(t)$, $i = \overline{1,n}$, are zero except the i-th row, which equals to $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ [2,7].

Since the right hand side of system (1) does not contain the phase coordinate x at present time t the approach with the use of quadratic functionals encounters certain difficulties (see [10] for more details). Therefore, we shall study the stability of the zero solution $x(t) = 0$ and derive estimates on the stability region by using finite-dimensional Liapunov functions subject to the Razumikhin condition. For the Liapunov function we shall choose the following quadratic form

$$V(x,t) = e^{\gamma t} x^{\mathrm{T}} H x$$

with the positive definite matrix$H$ solving the Liapunov matrix equation [1,10]

$$A^{\mathrm{T}}H + HA = -C. \tag{2}$$

The exponential factor $e^{\gamma t}$, $\gamma > 0$, does not guarantee the existence of an infinitesimal limit of higher order for the function $V(x,t)$ [8,10,12]. It allows us however to obtain an estimate on the upper bound of decrease rate of solutions starting in the stability domain of zero solution.

In the case when matrix A is asymptotically stable the matrix equation (2) has a unique solution, positive definite matrix H, for every positive definite matrix C. We shall use the standard vector and matrix norms [6] as follows

$$|A| = \{\lambda_{\max}(A^{\mathrm{T}}A)\}^{1/2}, \quad |x(t)| = \left\{\sum_{i=1}^{n} x_i^2(t)\right\}^{1/2}, \quad \|x(t)\|_\tau = \max_{-\tau \leq s \leq 0}\{|x(t+s)|\}.$$

Here and in the sequel $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ stand for the smallest and the largest eigenvalues respectively for the symmetric positive definite matrices.

Let ∂V_α^γ be a level surface of the Liapunov function V and V_α^γ be the corresponding domain in the space $R^n \times R$, that is

$$\partial V_\alpha^\gamma = \{(x,t)\colon\ V(x,t) = \alpha\}, \quad V_\alpha^\gamma = \{(x,t)\colon\ V(x,t) < \alpha\}.$$

3 Main Results

3.1 Linear case

Consider the following linear system with delay

$$\dot{x}(t) = Ax(t-\tau). \tag{3}$$

Lemma 1 *Suppose solution $x(t)$ of system (3) satisfies $(x(t),t) \in V_\alpha^\gamma$, for $t > -\tau$. Then*

$$|x(t)| < \sqrt{\alpha/\lambda_{\min}(H)}e^{-\frac{1}{2}\gamma t}, \quad t \geq \tau. \tag{4}$$

Proof The Liapunov functions of quadratic type $X(x,T) = e^{\gamma t}x^{\mathrm{T}}Hx$ are known to satisfy the following two-sided inequality [3]

$$e^{\gamma t}\lambda_{\min}(H)|x(t)|^2 \leq V(x(t),\, t) \leq e^{\gamma t}\lambda_{\max}(H)|x(t)|^2. \tag{5}$$

Therefore, the assumptions of Lemma imply

$$\lambda_{\min}(H)|x(t)|^2 < \alpha.$$

From the latter inequality the estimate (4) follows.

Lemma 2 *Suppose there exist constants $\alpha > 0$, $\gamma > 0$ such that the solution $x(t)$ of system (3) satisfies $(x(t),t) \in V_\alpha^\gamma$, for all $T-2\tau \leq t < T$ and $(x(T),T) \in \partial V_\alpha^\gamma$. Then*

$$\begin{gathered} |x(T)-x(T-\tau)| < 2\,\frac{|A|}{\gamma}\,e^{\frac{1}{2}\gamma\tau}\sqrt{\varphi(H)}\big(e^{\frac{1}{2}\gamma\tau}-1\big)|x(T)|, \\ \varphi(H) = \lambda_{\max}(H)/\lambda_{\min}(H)\lambda_{\min}(H). \end{gathered} \tag{6}$$

Proof Solutions of system (3) can be represented in the following integral form

$$x(t) = x(t-\tau) + \int_{t-\tau}^{t} Ax(s-\tau)\,ds.$$

When $t = T$ the latter implies

$$|x(T)-x(T-\tau)| \leq \int_{T-\tau}^{T} |A||x(s-\tau)|\,ds.$$

From the assumptions of Lemma 2 and inequality (5) the following holds

$$\begin{aligned} e^{\gamma(s-\tau)}\lambda_{\min}(H)|x(s-\tau)|^2 &\leq V(x(s-\tau),\, s-\tau) \leq V(x(T),T) \\ &< e^{\gamma T}\lambda_{\max}(H)|x(T)|^2 \quad \text{for all} \quad T-\tau \leq s \leq T. \end{aligned}$$

Therefore

$$|x(s-\tau)| < e^{\frac{1}{2}\gamma(T-s+\tau)}\sqrt{\varphi(H)}\,|x(T)|, \quad \varphi(H) = \lambda_{\max}(H)/\lambda_{\min}(H). \tag{7}$$

By using the last inequality in the integral representation we derive the required estimate

$$\begin{aligned} |x(T)-x(T-\tau)| &< \int_{T-\tau}^{\mathrm{T}} |A|e^{\frac{1}{2}\gamma(T-s+\tau)}\sqrt{\varphi(H)}|x(T)|ds \\ &= 2\,\frac{|A|}{\gamma}\,e^{\frac{1}{2}\gamma\tau}\sqrt{\varphi(H)}\,\big[e^{\frac{1}{2}\gamma\tau}-1]\big|x(T)|. \end{aligned}$$

Lemma 3 *Every solution* $x(t)$ *of system (3) satisfies the inequality*

$$|x(t)| \le (1+|A|\tau)\|x(0)\|_\tau \tag{8}$$

on the time interval $0 \le t \le \tau$.

Proof Write system (3) in the integral form

$$x(t) = x(0) + \int_0^t Ax(s-\tau)\,ds.$$

Then

$$|x(t)| \le |x(0)| + \int_0^t |A||x(s-\tau)|\,ds \le |x(0)| + |A|\|x(0)\|_\tau \tau \le (1+|A|\tau)\|x(0)\|_\tau.$$

By using the above Lemmas the following Theorem on asymptotic stability of the system with pure delay (3) is derived.

Theorem 1 *Assume that matrix* A *is asymptotically stable. Then the system with pure delay (3) is also asymptotically stable for all* $\tau < \tau_0$, *where*

$$\tau_0 = \frac{\lambda_{\min}(C)}{2|A||HA|\sqrt{\varphi(H)}}. \tag{9}$$

Moreover, the solutions of the system satisfy the following exponential estimate on their rate of decrease

$$|x(t)| < (1+|A|\tau)\|x(0)\|_\tau \sqrt{\varphi(H)}\,\exp\left\{\frac{1}{2}\gamma t\right\}, \quad t \ge \tau, \tag{10}$$

where $0 < \gamma < \gamma^*$, γ^* *is the positive solution of the equation*

$$\gamma^*(\lambda_{\min}(C) - \gamma^*\lambda_{\max}(H)) = 4\sqrt{\varphi(H)}\,|HA||A|e^{\frac{1}{2}\gamma^*\tau}\Big(e^{\frac{1}{2}\gamma^*\tau} - 1\Big). \tag{11}$$

Proof Let $x(t)$ be any solution of system (3). Then, as it follows from Lemma 3, it satisfies the following inequality

$$|x(t)| \le (1+|A|\tau)\|x(0)\|_\tau,$$

for all $0 \le t \le \tau$. Also on the same time interval $x(t)$ satisfies $(x(t),t) \in V_\alpha^\gamma$, where $\gamma > 0$ is a constant to be determined later, and $\alpha > \lambda_{\max}(H)(1+|A|\tau)^2\|x(0)\|_\tau^2$.

We claim that also $(x(t),t) \in V_\alpha^\gamma$ for all $t > \tau$. Suppose not. Then there exists a time moment $T > \tau$, such that $(x(T),T) \in \partial V_\alpha^\gamma$. Evaluate now the total derivative of the Liapunov function V along the solutions of system (3):

$$\frac{d}{dt}V(x(t)) = e^{\gamma t}\gamma x^{\mathrm{T}}(t)Hx(t) + e^{\gamma t}\{x^{\mathrm{T}}(t)(A^{\mathrm{T}}H + HA)x(t) + 2x^{\mathrm{T}}(t)HA[x(t-\tau) - x(t)]\}.$$

If matrix A is asymptotically stable then, as it follows from the matrix Liapunov equation (2), for any positive define matrix C and matrix H solving the equation the total derivative of V satisfies

$$\frac{d}{dt}V(x(t)) \le e^{\gamma t}\{\gamma\lambda_{\max}(H) - \lambda_{\min}(C)\}|x(t)|^2 + 2e^{\gamma t}|HA||x(t)||x(t)-x(t-\tau)|.$$

As it follows from the assumptions of Theorem 1 and inequality (7) the last inequality at time $t = T$ reads

$$\frac{d}{dt}V(x(t)) \le -e^{\gamma T}\left\{\lambda_{\min}(C) - \gamma\lambda_{\max}(H) - 4|HA||A|\sqrt{\varphi(H)}\, e^{\frac{1}{2}\gamma\tau}\frac{e^{\frac{1}{2}\gamma\tau}-1}{\gamma}\right\}|x(t)|^2.$$

If in addition the inequality

$$\lambda_{\min}(C) - \gamma\lambda_{\max}(H) - 4|HA||A|\sqrt{\varphi(H)}\, e^{\frac{1}{2}\gamma\tau}\,\frac{e^{\frac{1}{2}\gamma\tau}-1}{\gamma} > 0 \tag{12}$$

holds, then the total derivative of the Liapunov function will be negative. This means that the velocity vector of the motion $x(t)$ is directed inside the domain at the moment $t = T$, and $(x(t), t) \in V_\alpha^\gamma$ for all $t > 0$. It follows from inequalities (4) and (8) that the following holds

$$|x(t)| < (1 + |A|\tau)\|x(0)\|_\tau\sqrt{\varphi(H)}\,\exp\left\{\frac{1}{2}\gamma t\right\}, \quad t \ge \tau,$$

that is, inequality (10) is true. Let us find the conditions for inequality (12) to be true. If $\gamma \to +0$ then inequality (11) has the form

$$\lambda_{\min}(C) - 2|HA||H|\sqrt{\varphi(H)}\,\tau > 0,$$

and if $\tau < \tau_0$, then

$$\tau_0 = \frac{\lambda_{\min}(C)}{2|HA||A|\sqrt{\varphi(H)}}.$$

That is, the maximum allowed delay τ_0 has the form given by (9). Let $\tau < \tau_0$. Then there is a threshold for the rate of exponential decrease of the solutions, which value is determined by the solution of equation (11).

Remark 1 In general it is not possible to represent the solution of equation (11) in an explicit analytic form. The value γ^* can be replaced by a smaller value $\tilde{\gamma}^*$, where

$$0 < \tilde{\gamma}^* = \gamma_0 - \frac{h(\gamma_0)}{\lambda_{\min}(C)}, \qquad \gamma_0 = \frac{\lambda_{\min}(C)}{\lambda_{\max}(H)},$$
$$h(\gamma_0) = 4|HA||A|\sqrt{\varphi(H)}\, e^{\frac{1}{2}\gamma_0\tau}\big(e^{\frac{1}{2}\gamma_0\tau}-1\big).$$

Proof The left-hand side of system (11) is the parabola

$$g(\gamma) = \gamma[\lambda_{\min}(C) - \gamma\lambda_{\max}(H)]$$

opening downward and having the following two zeros $\gamma_0 = \lambda_{\min}(C)/\lambda_{\max}(H)$, $\gamma_1 = 0$. The right-hand side of equality (11) is a parabola in the variable $e^{\frac{1}{2}\gamma\tau}$, where $\gamma \geq 0$

$$h(\gamma) = 4|HA||A|e^{\frac{1}{2}\gamma\tau}\left(e^{\frac{1}{2}\gamma\tau} - 1\right),$$

also opening downward. Since $g(0) = h(0) = 0$ and

$$g'(0) = \lambda_{\min}(C) > 2|HA||A|\sqrt{\varphi(H)}\,\tau = h'(0),$$

then a γ^* exists $(0 < \gamma^* < \gamma_0 = \lambda_{\min}(C)/\lambda_{\max}(H))$, such that $g(\gamma^*) = h(\gamma^*)$. The "parabola" $h(\gamma)$ is replaced by the line segment $\overline{h}(\gamma)$ passing through the origin and the point $(\gamma_0, h(\gamma_0))$ and having the form $\overline{h}(\gamma) = h(\gamma_0)\frac{\gamma}{\gamma_0}$. Point $\tilde{\gamma}^*$ is defined as the intersection of the parabola $g(\gamma)$ and the line $\overline{h}(\gamma)$. That is, as the positive solution of the equation

$$\gamma[\lambda_{\min}(C) - \gamma\lambda_{\max}(H)] = h(\gamma_0)\,\frac{\gamma}{\gamma_0}.$$

The latter gives the required value of $\tilde{\gamma}^*$.

Remark 2 Condition (9) is rather approximate but readily calculated one. For example, for the scalar equation

$$\dot{x}(t) = -ax(t-\tau), \quad a > 0$$

the stability condition is $\tau < \pi/2a$ (see [12]). By using the Liapunov function $V(x,t) = e^{\gamma t}x^2$ from inequality (9) we obtain the following stability condition $\tau < 1/a$.

3.2 Nonlinear case

Consider next systems of the form (1) with pure delay in the linear part.

Lemma 4 *Assume there exist constants $\alpha > 0$ and $\gamma > 0$ such that the solution $x(t)$ of system (1) satisfies $(x(T),T) \in \partial V_\alpha^\gamma$ for $t = T$, and $(x(t),t) \in V_\alpha^\gamma$ for $T - 2\tau \leq t < T$. Then the following inequality holds*

$$\begin{aligned} |x(T) - x(T-\tau)| < {} & \frac{2}{\gamma}\,e^{\frac{1}{2}\gamma\tau}\sqrt{\varphi(H)}\,|A|\left(e^{\frac{1}{2}\gamma\tau} - 1\right)|x(T)| \\ & + \frac{1}{\gamma}\,e^{\frac{1}{2}\gamma\tau}\varphi(H)|B|(e^{\gamma\tau} - 1)|x(T)|^2. \end{aligned} \tag{13}$$

Proof Write system (1) in the integral form

$$x(t) = x(t-\tau) + \int\limits_{t-\tau}^{t} [Ax(s-\tau) + X^{\mathrm{T}}(s)Bx(s-\tau)]\,ds.$$

At the time moment $t = T$ the latter inequality implies

$$|x(T) - x(T-\tau)| \leq \int\limits_{T-\tau}^{T} [|A||x(s-\tau)| + |X(s)||B||x(s-\tau)|]\,ds.$$

From the assumptions of Lemma 4 and estimate (7) it follows that the following inequality holds

$$|x(s-\tau)| < e^{\frac{1}{2}\gamma(T-s+\tau)}\sqrt{\varphi(H)}\,|x(T)|, \quad |x(s)| < e^{\frac{1}{2}\gamma(T-s)}\sqrt{\varphi(H)}\,|x(T)|,$$

for all $T-\tau \le s \le T$. By using the latter in the integral representation we derive

$$\begin{aligned}|x(T)-x(T-\tau)| < &\int\limits_{T-\tau}^{T} |A|e^{\frac{1}{2}\gamma(T-s+\tau)}\sqrt{\varphi(H)}\,|x(T)|\,ds\\ &+\int\limits_{T-\tau}^{T} e^{\frac{1}{2}(2T-2s+\tau)}\varphi(H)|B||x(T)|^2\,ds,\end{aligned}$$

or

$$|x(T)-x(T-\tau)| < \frac{2}{\gamma}\,e^{\frac{1}{2}\gamma\tau}\sqrt{\varphi(H)}\,|A|\Big(e^{\frac{1}{2}\gamma\tau}-1\Big)|x(T)|+\frac{1}{\gamma}\,e^{\frac{1}{2}\gamma\tau}\varphi(H)|B|(e^{\gamma\tau}-1)|x(T)|^2.$$

Lemma 5 *Every solution $x(t)$ of system (1) satisfies the following inequality*

$$|x(t)| \le (1+|A|\tau)\|x(0)\|_\tau e^{|B\|x(0)\||_\tau\tau t} \tag{14}$$

on the interval $0\le t\le\tau$.

Proof Write system (1) in the integral form

$$x(t) = x(0) + \int\limits_0^t [Ax(s-\tau)+X^{\mathrm{T}}(s)Bx(s-\tau)]\,ds.$$

Then

$$\begin{aligned}|x(t)| &\le |x(0)| + \int\limits_0^t [|A||x(s-\tau|+|X(s)|B||x(s-\tau||]\,ds\\ &\le (|x(0)|+|A|\|x(0)\|_\tau\tau)+|B|\|x(0)\|_\tau\int\limits_0^t |x(s)|\,ds\\ &\le (1+|A|\tau)\|x(0)\|_\tau e^{|B\|x(0)\||_\tau\tau t}.\end{aligned}$$

Lemma 6 *Suppose the derivative of the Liapunov function $V(x,t)=e^{\gamma t}x^{\mathrm{T}}Hx$ along solutions of system (1) satisfies the inequality*

$$\frac{d}{dt}V(x(t),t) \le -aV(x(t),t)+be^{-\frac{1}{2}\gamma t}V^{\frac{3}{2}}(x(t),t), \tag{15}$$

for all $t \geq 0$, *where* $a > 0$, $b > 0$, $\gamma > 0$. *Then all the solutions subjected to the initial condition*

$$\|x(0)\|_\tau < \frac{a+\gamma}{b\sqrt{\lambda_{\max}(H)}}$$

satisfy the inequality

$$|x(t)| \leq \frac{\sqrt{\varphi(H)}\,\|x(0)\|_\tau e^{-\frac{1}{2}at}}{1-\dfrac{b}{a+\gamma}\left(1-e^{-\frac{1}{2}(a+\gamma)t}\right)\sqrt{\lambda_{\max}(H)}\,\|x*(0)\|_\tau}. \tag{16}$$

Proof Inequality (15) is a Bernoulli type inequality. Since $V(x,t) > 0$, divide the inequality by $V^{3/2}(x,t)$. It follows

$$V^{-\frac{3}{2}}(x(t),t)\,\frac{d}{dt}V(x(t),t) \leq -aV^{-\frac{1}{2}}(x(t),t) + be^{-\frac{1}{2}\gamma t}.$$

By using the substitution $V^{-1/2}(x(t),t) = z(t)$, $z(0) > b/a$, we derive

$$\frac{d}{dt}\,z(t) \geq \frac{1}{2}\,az(t) - \frac{1}{2}\,be^{-\frac{1}{2}\gamma t}.$$

By solving the above differential inequality we obtain

$$z(t) \geq \left[z(0) - \frac{b}{a+\gamma}\right]e^{\frac{1}{2}at} + \frac{b}{a+\gamma}e^{-\frac{1}{2}\gamma t}, \quad z(0) \geq \frac{b}{a}.$$

Having returned to the original variables we have

$$\frac{1}{\sqrt{V(x(t),t)}} \geq \left[\frac{1}{\sqrt{V(x(0),0)}} - \frac{b}{a+\gamma}\right]e^{\frac{1}{2}at} + \frac{b}{a+\gamma}\,e^{-\frac{1}{2}\gamma t}$$

or

$$V(x(t),t) \leq \frac{1}{\left\{\left[\dfrac{1}{\sqrt{V(x(0),0)}} - \dfrac{b}{a+\gamma}\right]e^{\frac{1}{2}at} + \dfrac{b}{a+\gamma}e^{-\frac{1}{2}\gamma t}\right\}^2}.$$

Next we see that

$$V(x(t),t) \leq \frac{V(x(0),0)}{\left\{\left[1-\dfrac{b}{a+\gamma}\sqrt{V(x(0),0)}\right]e^{\frac{1}{2}at} + \dfrac{b}{a+b}e^{-\frac{1}{2}\gamma t}\sqrt{V(x(0),0)}\right\}^2}.$$

Finally by using the standard inequalities for quadratic forms we obtain inequality (16).

Theorem 2 *Assume that matrix A is asymptotically stable. Then for all $\tau < \tau_0$ where τ_0 is defined by (9), the zero solution of the differential system with delay (1) is also asymptotically stable. The stability domain contains the sphere U_δ, where the radius δ is found as the positive solution of the equation*

$$(1+|A|\tau)\delta e^{|B|\delta\tau} = \frac{a+\gamma}{b\sqrt{\lambda_{\max}(H)}}\,. \tag{17}$$

Moreover, for the solutions with the initial conditions inside the sphere U_δ the following estimate on the convergence rate holds

$$|x(t)| \le \frac{\sqrt{\varphi(H)}\,\|x(0)\|_\tau e^{-\frac{1}{2}(a+\gamma)t}}{1-\dfrac{b}{a+\gamma}\Big(1-e^{-\frac{1}{2}(a+\gamma)t}\Big)\sqrt{\lambda_{\max}(H)}\,\|x(0)\|_\tau}\,, \tag{18}$$

where

$$a = \frac{1}{\lambda_{\max}(H)}\Big\{\lambda_{\min}(C) - \gamma\lambda_{\max}(H) - 4|HA|\,\frac{|A|}{\gamma}\,E^{\frac{1}{2}\gamma\tau}(E^{\frac{1}{2}\gamma\tau}-1)\sqrt{\varphi(H)}\Big\},$$
$$b = \frac{2}{\lambda_{\min}(H)}|B|\sqrt{\varphi(H)}\,E^{\frac{1}{2}\gamma\tau}\Big\{|HA|\sqrt{\varphi(H)}\,\frac{1}{\gamma}\,(E^{\gamma\tau}-1)+\lambda_{\max}(H)\Big\}.$$

Proof Suppose the initial condition for the solution $x(t)$ of system (1) satisfies the assumption $\|x(0)\|_\tau < \delta$ where δ is defined by (17). Then inequality (14) of Lemma 5 implies that at the moment $t=\tau$ the following inequality

$$\|x(\tau)\|_\tau \le R, \quad R = (1+|A|t)\delta e^{|B|\delta t}$$

is true. On the time interval $-\tau \le t \le \tau$ the integral curve satisfies $(x(t),t) \in V_\alpha^\gamma$ where $\gamma > 0$ is a constant and $\alpha = e^{\gamma\tau}\lambda_{\max}(H)R$. We shall show that there exists a constant $\gamma^* > 0$ such that $(x(t),t) \in V_\alpha^{\gamma^*}$ for all $t > \tau$. Assume not. Then there exists $T > \tau$ such that $(x(T),T) \in \partial V_\alpha^\gamma$. We evaluate next the total derivative of the Liapunov function V along the solutions of system (1)

$$\frac{d}{dt}V(x(t)) = e^{\gamma t}\gamma x^{\mathrm{T}}(t)Hx(t) + e^{\gamma t}\{[Ax(t-\tau)+X^{\mathrm{T}}(t)Bx(t-\tau)]Hx(t)$$
$$+ x^{\mathrm{T}}(t)H[Ax(t-\tau)+X^{\mathrm{T}}(t)Bx(t-\tau)]\},$$

or

$$\frac{d}{dt}V(x(t)) = e^{\gamma t}x^{\mathrm{T}}(t)(\gamma H + A^{\mathrm{T}}H + HA)x(t)$$
$$+ 2e^{\gamma t}x^{\mathrm{T}}(t)HA[x(t-\tau)-x(t)] + 2e^{\gamma t}x^{\mathrm{T}}(t)HX^{\mathrm{T}}(t)Bx(t-\tau).$$

If matrix A is asymptotically stable then using the chosen matrix norm and the Liapunov equation (2) we obtain

$$\frac{d}{dt}V(x(t),t) \le -e^{\gamma t}\{\lambda_{\min}(C)-\gamma\lambda_{\max}(H)\}|x(t)|^2$$
$$+ 2e^{\gamma t}|HA||x(t)||x(t)-x(t-\tau)| + 2e^{\gamma t}\lambda_{\max}(H)|B||x(t)|^2|x(t-\tau)|.$$

Since $(x(T),T) \in \partial V_\alpha^\gamma$ by using inequalities (13) and (14) we obtain the following estimate for the derivative of the Liapunov function

$$
\begin{aligned}
&\frac{d}{dt}V(x(T),T)\\
&\quad\le -e^{\gamma T}\left\{\lambda_{\min}(C)-\gamma\lambda_{\max}(H)-4|HA|\,\frac{|A|}{\gamma}\,e^{\frac{1}{2}\gamma\tau}(e^{\frac{1}{2}\gamma\tau}-1)\sqrt{\varphi(H)}\right\}|x(T)|^2\\
&\qquad+2e^{\gamma T}|B|\sqrt{\varphi(H)}e^{\frac{1}{2}\gamma\tau}\left\{|HA|\sqrt{\varphi(H)}\,\frac{1}{\gamma}\,(e^{\gamma\tau}-1)+\lambda_{\max}(H)\right\}|x(T)|^3.
\end{aligned}
$$

By using the standard inequalities for quadratic forms we obtain

$$
\begin{aligned}
\frac{d}{dt}V(x(T),T) \le {}& -\frac{1}{\lambda_{\max}(H)}\left\{\lambda_{\min}(C)-\gamma\lambda_{\max}(H)\right.\\
&\left.-4|HA|\,\frac{|A|}{\gamma}\,e^{\frac{1}{2}\gamma\tau}\left(e^{\frac{1}{2}\gamma\tau}-1\right)\sqrt{\varphi(H)}\right\}V(x(T),T)\\
&+\frac{2}{\lambda_{\min}(H)}\,|B|\sqrt{\varphi(H)}\,e^{\frac{1}{2}\gamma\tau}\left\{|HA|\sqrt{\varphi(H)}\,\frac{1}{\gamma}(e^{\gamma\tau}-1)\right.\\
&\left.+\lambda_{\max}(H)\right\}e^{-\frac{1}{2}\gamma T}V^{3/2}(x(T),T).
\end{aligned}
\tag{19}
$$

Let $\tau < \tau_0$, where τ_0 is defined by (8) and let $0 < \gamma < \gamma^*$, where γ^* is the solution of equation (11). Define

$$
\begin{aligned}
a &= \frac{1}{\lambda_{\max}(H)}\left\{\lambda_{\min}(C)-\gamma\lambda_{\max}(H)-4|HA|\,\frac{|A|}{\gamma}\,e^{\frac{1}{2}\gamma\tau}\left(e^{\frac{1}{2}\gamma\tau}-1\right)\sqrt{\varphi(H)}\right\},\\
b &= \frac{2}{\lambda_{\min}(H)}\,|B|\sqrt{\varphi(H)}\,e^{\frac{1}{2}\gamma\tau}\left\{|HA|\sqrt{\varphi(H)}\,\frac{1}{\gamma}\,(e^{\gamma\tau}-1)+\lambda_{\max}(H)\right\}.
\end{aligned}
$$

Then $a > 0$, $b > 0$, and inequality (19) has the form (15)

$$
\frac{d}{dt}\,V(x(T),T) \le -aV(x(T),T)+bV^{3/2}(x(T),T).
$$

By using Lemma 6 we conclude that inequality (16) is true for the solutions $x(t)$ of system (1) satisfying the condition $\|x(\tau)\|_\tau \le R$, where δ is defined by (17). This completes the proof of the theorem.

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