Nonlinear Dynamics and Systems Theory, **6**(2) (2006) 111–127

Minimal Representations, Controllability and Free Energies in a Heat Conductor with Memory

A.M. Caucci*

Department of Mathematics, University of Bologna,
Piazza di Porta San Donato 5, 40126 Bologna, Italy

Received: July 8, 2005; Revised: June 7, 2006

Abstract: A rigid linear heat conductor with memory effects is considered. Some results about state-space representation, minimality and controllability of heat conductors with memory kernel of exponential type are presented. In such a context, the asymptotic behavior and the existence of a bounded absorbing set for solutions of the energy equation are studied by means of a suitable class of quadratic free energies.

Keywords: *Heat conduction; free energy; absorbing set.*

Mathematics Subject Classification (2000): 80A20, 74D05, 35K05.

1 Introduction

In this paper we consider a rigid linear heat conductor with memory effects — within the framework proposed by Gurtin and Pipkin [10] — when the memory kernel is finite sum of exponentials, namely

$$\dot{K}(s) = \sum_{i=1}^{n} b_i\, e^{-a_i\, s},$$

where n is a positive integer, $a_i, b_i \in R,\ a_i > 0,\ i = 1, 2, \dots, n$.

On the basis of Coleman's results concerning materials with memory [3], a non-linear model for a rigid heat conductor was developed by Gurtin and Pipkin in [10]. Moreover, they considered the linearization of their theory appropriate to infinitesimal temperature gradients, which for isotropic materials yields a constitutive equation for the heat flux $\boldsymbol{q}$ expressed in terms of the history of the temperature gradient $\boldsymbol{g}$; this linear theory is important because the obtained constitutive equation for $\boldsymbol{q}$ is a generalization of the

*Corresponding author: caucci@dm.unibo.it

so-called Cattaneo–Maxwell equation [2], which follows from it as a special case. Subsequently, many authors considered this linearized equation to study problems connected with heat propagation. Among all the results so obtained, we remember, in particular, those derived in [6], where an approximate theory of thermodynamics is developed for Gurtin and Pipkin's model and maximal free energy and maximal free enthalpy functions are explicitly constructed and used to prove stability and domain of dependence results. We recall that in [6], following [11], the thermodynamic states and processes are connected with the integrated history of the temperature gradient and the temperature gradient, respectively.

In this work, the linear theory introduced in [10] is taken into account in Section 2.

In Section 3, following the lines of [4] and [5], where analogous problems are studied for viscoelastic solids of exponential type, we prove that the minimal representation of the state space is a finite dimensional vector space and each minimal state element represents an equivalence class of integrated histories; the full controllability of the minimal state space is also verified.

In the following Section 4, an explicit representation of a class of quadratic free energies is taken into consideration with respect to some minimal, finite-dimensional state space. Finally, the last part of the paper is devoted to study, by means of uniform energy estimate, the asymptotic behavior of solutions of the evolutive (semilinear) equation, obtained by substituting the constitutive equations for the internal energy e and for the heat flux $\boldsymbol{q}$ into the energy equation for rigid heat conductors.

2 Preliminary Notions and Setting of the Problem

Within the linear theory of thermodynamics developed in [10], the *internal energy* e is assumed of the form

$$e(\boldsymbol{x},t) = \alpha_0\, \theta(\boldsymbol{x},t), \tag{2.1}$$

where α_0 is here assumed to be constant, $\boldsymbol{x} \in R^3$ denotes the position within the conductor[1], $t \in R^+$ denotes the time variable[2] and $\theta = (\Theta - \Theta_0)$ is the temperature difference with respect to a fixed reference absolute temperature $\Theta_0 > 0$, uniform in R^3. The *heat flux* $\boldsymbol{q} \in R^3$ is assumed to satisfy the constitutive equation

$$\boldsymbol{q}(\boldsymbol{x},t) = -\int_0^\infty K(\tau)\nabla\theta(\boldsymbol{x},\, t-\tau)\, d\tau, \tag{2.2}$$

where $K(\tau)$ is the *heat flux relaxation function*, given by

$$K(t) = K_0 + \int_0^t \dot{K}(s)\, ds; \tag{2.3}$$

[1] More precisely, it should be require that $\boldsymbol{x} \in \mathcal{B} \subset R^3$, where $\mathcal{B}$ denotes the bounded closed set in R^3 which represents the configuration domain of the conductor, here not specified since of no interest in the present study.

[2] Throughout the whole paper, $R^+ = [0,\infty)$ and $R^{++} = (0,\infty)$.

K_0 represents the initial (positive) value of the flux relaxation function, thus termed *initial heat flux relaxation coefficient*. It is further required that

$$\dot{K} \in L^1(R^+) \cap L^2(R^+) \quad \text{and} \quad K \in L^1(R^+), \tag{2.4}$$

which implies

$$K_\infty = \lim_{t\to\infty} K(t) = 0.$$

The latter can be physically interpreted recalling that there is no heat flux when, at infinity, the thermal equilibrium is reached.

In the sequel, we will focus our attention on a material element of the conductor; thus we will omit to show explicit dependence on the position $\boldsymbol{x}$ in the conductor and all the quantities introduced will be represented by functions of the time variable alone.

When the integral kernel satisfies both the requirements (2.3) and (2.4), (2.2) is equivalent to the following

$$\boldsymbol{q}(t) = \int_0^\infty \dot{K}(t)\, \overline{\boldsymbol{g}}^t(\tau)\, d\tau, \tag{2.5}$$

where $\boldsymbol{g} = \nabla\theta$ denotes the temperature-gradient and

$$\overline{\boldsymbol{g}}^t(\tau) = \int_{t-\tau}^{t} \boldsymbol{g}(s)\, ds$$

represents the *integrated history of the temperature-gradient*.

To specify those thermodynamical phenomena to study, the following vectorial space can be introduced

$$\Gamma = \left\{ \overline{\boldsymbol{g}}^t\colon\ R^+ \to R^3\colon\ \left| \int_0^\infty \dot{K}(s+\tau)\overline{\boldsymbol{g}}^t(s)\, ds \right| < \infty, \quad \forall \tau \ge 0 \right\}. \tag{2.6}$$

Following the theory proposed by Noll, Coleman and Owen in the seventies, we introduce some basic definitions.

The *thermodynamic state* of the conductor is chosen to be

$$\sigma(t) = (\theta(t), \overline{\boldsymbol{g}}^t), \quad \forall\, t \ge 0,$$

where $\theta(t) > 0$ and $\overline{\boldsymbol{g}}^t$ belongs to Γ. Such a definition implies that, the thermodynamic state function is known as soon as the temperature and the integrated history of the temperature-gradient are given. The (metric) space, Σ, of all admissible states (*state space*) is the set comprising all those states σ which correspond to a finite heat flux; Σ may be written as

$$\Sigma = R^{++} \times \Gamma$$

where Γ is given by (2.6).

We define *thermal process of duration* $T > 0$ as a map

$$P\colon\ [0, T) \to R \times R^3,$$

piecewise continuous on the time interval $[0,T)$ and such that

$$P(\tau) = (\dot{\theta}_P(\tau),\, \boldsymbol{g}_P(\tau)), \quad \forall \tau \in [0,T).$$

Let Π be the set of all admissible thermal processes P, that is the set of piecewise continuous functions $P\colon\ [0,T) \to R\times R^3,\ T>0,$ which satisfies the following properties:

(1) if $P\in\Pi,$ then its restriction $P_{[t_1,t_2)}$ to the interval $[t_1,t_2)\subseteq[0,T)$ belongs to Π;

(2) if $P_1,P_2\in\Pi,$ then the composition $P_1 * P_2,$ defined as

$$(P_1 * P_2)(t) = \begin{cases} P_1(t) & \text{if} \quad t\in[0,T_1), \\ P_2(t-T_1) & \text{if} \quad t\in[T_1,\,T_1+T_2), \end{cases}$$

belongs to Π.

To any given rigid heat conductor are associated two maps:

(i) $\rho\colon\ \Sigma\times\Pi\to\Sigma$ called *evolution* (or *state-transition*) *function*, which transforms the state σ_1 under the process P into $\sigma_2=\rho(\sigma_1,P)$. The map ρ obeys the semi-group property. If $(\sigma_0,P)\in\Sigma\times\Pi,$ where $\sigma_0=\sigma(0)=(\theta_\star(0),\overline{\boldsymbol{g}}^0_\star)$ $(\theta_\star(0)$ denotes the temperature and $\overline{\boldsymbol{g}}^0_\star$ the integrated history of the temperature-gradient at time $t=0)$ and $P=(\dot{\theta}_P,\boldsymbol{g}_P),$ then, through the map ρ, it is possible to define the state function

$$\sigma(t)=\left(\theta(t),\overline{\boldsymbol{g}}^t\right)=\rho\left(\sigma_0,P_{[0,t)}\right), \quad t\in[0,T)$$

in the following manner

$$\theta(t)=\theta_\star(0)+\int_0^t \dot{\theta}_P(\zeta)\,d\zeta,$$

$$\overline{\boldsymbol{g}}^t(s)=\begin{cases} \displaystyle\int_{t-s}^t \boldsymbol{g}_P(\zeta)\,d\zeta, & 0\le s<t, \\ \displaystyle\int_0^t \boldsymbol{g}_P(\zeta)\,d\zeta+\overline{\boldsymbol{g}}^0_\star(s-t), & s\ge t. \end{cases}$$

The particular nature of the state space Σ and the properties of the state-transition function ρ provide all the thermal properties of the system and enable it to model physical phenomena. We say that a state $\sigma^f\in\Sigma$ is *attainable* from a state $\sigma^i\in\Sigma$ if there exists a process $P\in\Pi$ such that

$$\rho(\sigma^i,P)=\sigma^f.$$

The state space Σ is

* *attainable* from a state σ_0 if, for every final state $\bar{\sigma}\in\Sigma,$ there exists a process $P\in\Pi$ such that

$$\rho(\sigma_0,P)=\bar{\sigma};$$

* *controllable* in a state σ_0 if σ_0 is attainable from any state of Σ;
* *completely controllable*, if, for any pair $\sigma_1,\ \sigma_2 \in \Sigma$, there exists *at least* a process $P \in \Pi$ such that

$$\rho(\sigma_1, P) = \sigma_2.$$

(ii) $\mathcal{Q}$ called *response function* which maps the pair $(\sigma(t), P(t))$ into the pair $(e(t), \boldsymbol{q}(t))$ at time t, namely

$$(e(t), \boldsymbol{q}(t)) = \mathcal{Q}(\sigma(t), P(t)), \quad t \in [0, T).$$

The notion of *equivalence between material states* is introduced to associate together all those different thermal histories which correspond to the same heat flux.

Definition 2.1 Two states $\sigma_1, \sigma_2 \in \Sigma$ are said to be *equivalent* $(\sigma_1 \sim \sigma_2)$ if

$$\mathcal{Q}(\sigma_1, P) = \mathcal{Q}(\sigma_2, P), \quad \forall\, P \in \Pi.$$

For rigid heat conductors described by constitutive equations (2.1), (2.2), the thermodynamic states

$$\sigma_1(t) = (\theta_1(t), \overline{\boldsymbol{g}}_1^t), \quad \sigma_2(t) = (\theta_2(t), \overline{\boldsymbol{g}}_2^t)$$

are equivalent in the sense of Definition 2.1 if and only if $\theta_1(t) = \theta_2(t), \forall\, t \geq 0$ and

$$\int_0^\infty \dot{K}(s+\tau)\overline{\boldsymbol{g}}_1(s)\, ds = \int_0^\infty \dot{K}(s+\tau)\overline{\boldsymbol{g}}_2(s)\, ds, \quad \forall\, \tau \geq 0. \tag{2.7}$$

An equivalent way to rephrase relationship (2.7) can be found in [1].

Remark 2.1 Definition 2.1 introduces an *equivalence relation* between states; the quotient space

$$\Sigma_R = \Sigma/_\sim$$

is the *minimal representation* of the state space.

3 Minimal Representation and Controllability: The Exponential Case

For linear heat conductors with relaxation function of exponential type $(n \geq 1)$, the explicit form of the relaxation function $K(s)$ is given by

$$K(s) = \sum_{i=1}^n k_i\, e^{-a_i\, s},$$

$$K_0 = \sum_{i=1}^n k_i, \quad K_\infty = \lim_{s\to\infty} K(s) = 0,$$

where a_i and $k_i,\ i = 1, \dots, n,$ are assumed to be strictly positive; moreover, it is reasonable to assume $a_i \neq a_j, \forall\, i \neq j$, and $a_i < a_j,\ i < j$. Then

$$\dot{K}(s) = \sum_{i=1}^n b_i e^{-a_i s}, \quad b_i = -a_i k_i < 0, \tag{3.1}$$

and, substituting (3.1) into (2.5), the heat flux $\boldsymbol{q}(t)$ becomes

$$\boldsymbol{q}(t) = \sum_{i=1}^{n} b_i \int_0^\infty e^{-a_i s} \overline{\boldsymbol{g}}^t(s)\, ds. \tag{3.2}$$

Recalling (2.7), two different integrated histories of the temperature-gradient, $\overline{\boldsymbol{g}}_1^t$, $\overline{\boldsymbol{g}}_2^t$, are equivalent if

$$\sum_{i=1}^{n} b_i e^{-a_i \tau} \int_0^\infty e^{-a_i s} (\overline{\boldsymbol{g}}_1^t(s) - \overline{\boldsymbol{g}}_2^t(s))\, ds = 0, \quad \tau \geq 0,$$

which in turn implies

$$\int_0^\infty e^{-a_i s} (\overline{\boldsymbol{g}}_1^t(s) - \overline{\boldsymbol{g}}_2^t(s))\, ds = 0, \quad i = 1, \dots, n.$$

This means that two thermodynamic states

$$\sigma_1(t) = (\theta_1(t), \overline{\boldsymbol{g}}_1^t), \quad \sigma_2(t) = (\theta_2(t), \overline{\boldsymbol{g}}_2^t)$$

are equivalent if and only if $\theta_1(t) = \theta_2(t)$, $\forall\, t \geq 0$ and

$$\boldsymbol{g}_{1,a_i} = \boldsymbol{g}_{2,a_i}, \quad i = 1, \dots, n, \tag{3.3}$$

where

$$\boldsymbol{g}_{a_i}(t) = \int_0^\infty e^{-a_i s} \overline{\boldsymbol{g}}^t(s)\, ds, \quad i = 1, \dots, n, \tag{3.4}$$

are called *internal variables*. If this is the case, the *minimal representation* of the state space, $\Sigma_R = \Sigma/_\sim$, is a *finite* dimensional vector space and we can choose

$$\sigma_R = [\,\theta, \quad \boldsymbol{g}_{a_1}, \quad \boldsymbol{g}_{a_2}, \quad \dots, \quad \boldsymbol{g}_{a_n}\,] \in R^{3n+1}.$$

Moreover, if $P = (\dot{\theta}_P, \boldsymbol{g}_P) \in \Pi$, the evolution function ρ is described through the following system of ordinary differential equations

$$\begin{aligned} &\dot{\theta}(t) = \dot{\theta}_P(t), \\ &\dot{\boldsymbol{g}}_{a_i}(t) = \frac{1}{a_i} \boldsymbol{g}_P(t) - a_i \boldsymbol{g}_{a_i}(t), \quad i = 1, \dots, n, \quad t \geq 0, \end{aligned} \tag{3.5}$$

with the initial condition

$$\sigma_0 = \sigma(0) = (\theta_\star(0), \boldsymbol{g}_{a_1\star}(0), \boldsymbol{g}_{a_2\star}(0), \dots, \boldsymbol{g}_{a_n\star}(0)), \tag{3.6}$$

where

$$\boldsymbol{g}_{a_i\star}(0) = \int_0^\infty e^{-a_i s} \overline{\boldsymbol{g}}_\star^0(s)\, ds, \quad i = 1, \dots, n.$$

Now, our aim is to verify the complete controllability of system (3.5) − (3.6). System (3.5) − (3.6) is linear of dimension $(n+1)$; the control is the function P.

Let $\mathcal{M}(n, m)$ be the space of all real matrices $n \times m$; we recall the following Theorem (see for instance [12]).

Theorem 3.1 *A linear system*

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u},$$
$$\boldsymbol{x}(0) = \boldsymbol{x}_0$$

with $A \in \mathcal{M}(n,n)$, $B \in \mathcal{M}(n,m)$, $\boldsymbol{u} \in R^m$, $\boldsymbol{x}, \boldsymbol{x}_0 \in R^n$, $m < n$, *is completely controllable if and only if*

$$\operatorname{rank}[A|B] = n$$

("Kalman rank condition") where $[A|B]$ *denotes the matrix*

$$[\,B, \quad AB, \quad A^2B, \quad \dots, \quad A^{n-1}B\,] \in \mathcal{M}(n,nm)$$

which consists of consecutively written columns of matrices B, $A\,B$, A^2B, $\dots$, $A^{n-1}B$.

By Theorem 3.1, the controllability of system (3.5) – (3.6) depends on the rank of the square $(n+1)$ matrix

$$[A|B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dfrac{1}{a_1} & -1 & a_1 & -{a_1}^2 & \dots & (-1)^{n-1}\,{a_1}^{n-2} \\ 0 & \dfrac{1}{a_2} & -1 & a_2 & -{a_2}^2 & \dots & (-1)^{n-1}\,{a_2}^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \dfrac{1}{a_n} & -1 & a_n & -{a_n}^2 & \dots & (-1)^{n-1}\,{a_n}^{n-2} \end{bmatrix}.$$

Since

$$\det([A|B]) = \prod_{l=1}^{n} a_l \prod_{1\le j<i\le n} (a_j - a_i),$$

where $a_l \neq 0$ for any $l = 1,\dots,n$ and $a_i \neq a_j$ for any $i \neq j$, the matrix $[A|B]$ is non singular; therefore the state space Σ_R is completely controllable.

Finally, introduced the following differential operators

$$\mathcal{V} = \sum_{h=0}^{n} v_h \frac{d^h}{dt^h}, \qquad \mathcal{T} = \sum_{h=0}^{n-1} l_h \frac{d^h}{dt^h}, \tag{3.7}$$

we prove the equivalence between (3.2) and the *implicit constitutive equation* (see [9])

$$\mathcal{V}\boldsymbol{q} = \mathcal{T}\boldsymbol{g}. \tag{3.8}$$

Theorem 3.2 *Let* $\mathcal{V}$ *and* $\mathcal{T}$, *as in (3.7), be differential operators of order* n *and* $(n-1)$ *respectively, with constant coefficients. For the sake of simplicity, we assume* $v_0 = 1$; *moreover, for physical reasons, we assume* $v_n \neq 0$. *Implicit constitutive equation (3.8) and integral constitutive equation (3.2) are equivalent, namely every solution of (3.8) is also solution of (3.2), and vice versa.*

Proof We put

$$\mathcal{I}(\tau, \overline{\boldsymbol{g}}^t) = \sum_{i=1}^{n} b_i e^{-a_i\tau} \boldsymbol{g}_{a_i}(t), \quad \tau \ge 0,$$

where, for all $i = 1, \ldots, n$, $\boldsymbol{g}_{a_i}$ is given by (3.4). From (3.2), we have

$$\boldsymbol{q}(t) = \mathcal{I}(0, \overline{\boldsymbol{g}}^t) = \sum_{i=1}^{n} b_i \boldsymbol{g}_{a_i}(t) \tag{3.9}$$

and, deriving n times with respect to t, we obtain

$$\frac{d^m}{dt^m}\, \boldsymbol{q}(t) = \frac{d^m}{dt^m}\, \mathcal{I}(0, \overline{\boldsymbol{g}}^t) = \sum_{i=1}^{n} b_i \, \frac{d^m}{dt^m}\, \boldsymbol{g}_{a_i}(t), \quad m = 1, \ldots, n, \tag{3.10}$$

with

$$\frac{d^m}{dt^m}\, \boldsymbol{g}_{a_i}(t) = (-1)^m {a_i}^m \boldsymbol{g}_{a_i}(t) + \sum_{j=0}^{m-1} (-1)^{m-j+1} {a_i}^{m-j-2} \frac{d^j}{dt^j}\, \boldsymbol{g}(t), \quad i = 1, \ldots, n, \tag{3.11}$$

due to $(3.5)_2$. Because of relation (3.11), system (3.10) can be finally rewritten as

$$\frac{d^m}{dt^m}\, \boldsymbol{q}(t) = (-1)^{m+1} \sum_{i=1}^{n} k_i a_i^{m+1} \boldsymbol{g}_{a_i}(t) + \sum_{i=1}^{n} k_i \sum_{j=0}^{m-1} (-1)^{m-j+2} {a_i}^{m-j-1} \frac{d^j}{dt^j}\, \boldsymbol{g}(t), \quad m = 1, \ldots, n. \tag{3.12}$$

The matrix M, given by the coefficients of $\boldsymbol{g}_{a_i}(t)$, $i = 1, \ldots, n$, is equal to

$$M = \begin{bmatrix} k_1\,(-a_1)^2 & k_2\,(-a_2)^2 & \ldots & k_n\,(-a_n)^2 \\ k_1\,(-a_1)^3 & k_2\,(-a_2)^3 & \ldots & k_n\,(-a_n)^3 \\ \ldots & \ldots & \ldots & \ldots \\ k_1\,(-a_1)^{n+1} & k_2\,(-a_2)^{n+1} & \ldots & k_n\,(-a_n)^{n+1} \end{bmatrix} = \left[\mathrm{diag}\,(k_1, k_2, \ldots, k_n)\,\Lambda\right]^{\top},$$

where

$$\Lambda = \begin{bmatrix} (-a_1)^2 & (-a_1)^3 & \ldots & (-a_1)^{n+1} \\ (-a_2)^2 & (-a_2)^3 & \ldots & (-a_2)^{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ (-a_n)^2 & (-a_n)^3 & \ldots & (-a_n)^{n+1} \end{bmatrix}.$$

Since

$$\det(\Lambda) = \prod_{j=1}^{n} {a_j}^2 \prod_{1 \le l < i \le n} (a_l - a_i),$$

we have

$$\det(M) = \prod_{j=1}^{n} k_j {a_j}^2 \prod_{1 \le l < i \le n} (a_l - a_i);$$

then, being $k_j,\, a_j > 0$ for all $j = 1, \ldots, n$ and $a_l < a_i$ for all $l < i$, the matrix M has non-zero determinant. Hence, eliminating the n terms $g_{a_i}(t)$ from equations (3.12) and substituting into (3.9), we obtain equation (3.8). On the other hand, the substitution of (3.9) into (3.8) leads (3.12).

4 Asymptotic Behavior for Rigid Linear Heat Conductors with Memory via Free Energies

This section is devoted to scrutinize the asymptotic behavior in time of rigid linear heat conductors with memory, *when the memory kernel is finite sum of exponentials*, by means of energy type inequality coming from free energy functionals.

Definition 4.1 A function $\psi\colon\, \Sigma \to R$ is called a *free energy* if the following conditions are satisfied:

(i) for any $t \geq 0$, the function ψ is differentiable and satisfies the inequality

$$\dot{\psi}(t) \leq -\mathbf{g}(t) \cdot \boldsymbol{q}(t);$$

(ii) the function ψ is minimal only at zero integrated histories of the temperature gradient, namely for every $(\theta(t), \bar{\mathbf{g}}^t) \in \Sigma$

$$\psi(\theta(t), \bar{\mathbf{g}}^t) \geq \psi(\theta(t), \mathbf{0}^\dagger(t)),$$

where $\mathbf{0}^\dagger(s) = \mathbf{0}$, for any $s \geq 0$, is the zero integrated history of the temperature gradient.

Since the systems involved are linear, it is natural to assume that the free energy is a quadratic function of the minimal representation of the state, which is of finite dimension.

We consider the following family of free energies that can be written as functions of $\widetilde{\sigma_R} = [\,\mathbf{g}_{a_1}, \quad \mathbf{g}_{a_2}, \quad \ldots, \quad \mathbf{g}_{a_n}\,]$, namely

$$\psi(t) = \frac{1}{2} \sum_{i,j=1}^{n} C_{ij} a_i a_j \mathbf{g}_{a_i}(t) \cdot \mathbf{g}_{a_j}(t). \tag{4.1}$$

Now, we are looking for suitable algebraic conditions on the symmetric matrix $C = [C_{ij}] \in \mathcal{M}(n,n)$, such that $\psi(t)$ is a free energy, according to Definition 4.1. The following Theorem holds.

Theorem 4.1 *Let* $C = [C_{ij}]$, $\Gamma = [\Gamma_{ij}] \in \mathcal{M}(n,n)$. *If*

$$\sum_{i=1}^{n} C_{ij} = \sum_{j=1}^{n} C_{ij} = k_j, \quad j = 1, \ldots, n; \tag{4.2}$$

$$\Gamma_{ij} = C_{ij} a_i a_j \, \frac{a_i + a_j}{2}, \quad i,j = 1, \ldots, n; \tag{4.3}$$

C is symmetric and positive semi-definite, Γ is positive semi-definite,

then (4.1) is a free energy in the sense of Definition 4.1.

Proof By virtue of $(3.5)_2$, condition (i) is equivalent to require that

$$\dot{\psi}(t) = \mathbf{g}(t) \cdot \Bigg(\sum_{i,j=1}^{n} C_{ij} a_j \, \mathbf{g}_{a_j}(t) \Bigg) - \sum_{i,j=1}^{n} C_{ij} {a_i}^2 a_j \, \mathbf{g}_{a_i}(t) \cdot \mathbf{g}_{a_j}(t) \leq \mathbf{g}(t) \cdot \Bigg(\sum_{j=1}^{n} a_j k_j \, \mathbf{g}_{a_j}(t) \Bigg);$$

this inequality is satisfied if and only if

$$\begin{aligned} &\sum_{i,j=1}^{n} C_{ij} a_j \, \mathbf{g}_{a_j}(t) = \sum_{j=1}^{n} a_j k_j \mathbf{g}_{a_j}(t) \\ &\sum_{i,j=1}^{n} C_{ij} {a_i}^2 a_j \mathbf{g}_{a_i}(t) \cdot \mathbf{g}_{a_j}(t) \geq 0. \end{aligned} \tag{4.4}$$

From $(4.4)_1$, we find

$$k_j = \sum_{i=1}^{n} C_{ij} = \sum_{j=1}^{n} C_{ij}, \quad j = 1, \ldots, n.$$

Moreover, observing that

$$\Gamma_{ij} = C_{ij} a_i a_j \, \frac{a_i + a_j}{2}$$

is the symmetric part of the matrix $\Gamma^*_{ij} = C_{ij} {a_i}^2 a_j$, it follows that inequality $(4.4)_2$ is satisfied if and only if the symmetric matrix Γ is positive semi-definite.

With regard to condition (ii), it is easily seen that this holds if and only if the matrix C is positive semi-definite.

Remark 4.1 It is worth noting that in the sequel the matrices C and Γ will be assumed positive definite.

Let $\Omega \subset R^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. The energy equation for a linear rigid heat conductor is

$$\rho_0 \, e_t = -\nabla \cdot \boldsymbol{q} + \rho_0 \, r \quad \text{in} \;\; \Omega \times R^+, \tag{4.5}$$

where $\cdot_t = d \cdot / dt$, ρ_0 is the constant mass density, the internal energy e and the heat flux $\boldsymbol{q}$ are given by (2.1) and (2.2) respectively.

We take for simplicity $\rho_0 \alpha_0 = 1$ and we denote the source $\rho_0 r$ by f; substituting equations (2.1) and (2.2) into (4.5) and assuming the memory kernel as finite sum of exponentials

$$K(s) = \sum_{i=1}^{n} k_i e^{-a_i \, s},$$

the corresponding initial boundary value problem becomes

$$\begin{aligned} &\theta_t(\boldsymbol{x}, t) - \sum_{i=1}^{n} k_i \int_0^{\infty} e^{-a_i \, s} \, \Delta\theta(\boldsymbol{x}\,, t - s) \, ds + f(\theta(\boldsymbol{x}, t)) = 0 && \text{in} \;\; \Omega \times R^+, \\ &\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}) && \text{in} \;\; \Omega, \\ &\theta(\boldsymbol{x}, t) = 0 && \text{in} \;\; \partial\Omega \times R^+. \end{aligned} \tag{4.6}$$

We introduce the vector

$$\boldsymbol{\eta}(t) = (\eta_1(t), \ldots, \eta_n(t)) \,,$$

where

$$\eta_i(t) = \int_0^{\infty} e^{-a_i s} \theta(t - s) \, ds, \quad i = 1, \ldots, n. \tag{4.7}$$

As a consequence, by differentiation with respect to t, we get

$$\eta_{i\,t}(t) = \theta(t) - a_i\,\eta_i(t), \quad i = 1, \dots, n.$$

In view of (4.7), the energy equation in (4.6) transforms into the following system

$$\begin{aligned} \theta_t &= \sum_{i=1}^{n} k_i\,\Delta\eta_i - f(\theta) \qquad\qquad \text{in } \ \Omega \times R^+, \\ \eta_{i\,t} &= \theta - a_i\eta_i, \quad i = 1, \dots, n, \quad \text{in } \ \Omega \times R^+. \end{aligned} \tag{4.8}$$

Initial-boundary conditions are then given by

$$\begin{aligned} \theta(x,0) &= \theta_0(x), \qquad\qquad x \in \Omega, \\ \eta_i(x,0) &= \eta_{i\,0}(x), \quad i = 1, \dots, n, \quad x \in \Omega, \\ \theta(x,t) &= 0 \qquad\qquad (x,t) \in \partial\Omega \times R^+, \\ \eta_i(x,t) &= 0, \quad i = 1, \dots, n, \quad (x,t) \in \partial\Omega \times R^+. \end{aligned} \tag{4.9}$$

With usual notation, we introduce the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$, acting on Ω. Hereafter, $\langle \cdot\,, \cdot \rangle$ denotes the L^2-inner product and $\| \cdot \|$ denotes the L^2-norm. If $C = [C_{ij}] \in \mathcal{M}(n,n)$ is positive definite, we put

$$\mathcal{H} = L^2(\Omega) \times \mathcal{W},$$

where

$$\mathcal{W} = \left\{ \boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_n) \in \left[H_0^1(\Omega)\right]^n : \quad \sum_{i,j=1}^{n} \langle \nabla\eta_i,\, C_{ij}\nabla\eta_j \rangle < +\infty \right\}.$$

The corresponding inner product is given by

$$\langle z_1, z_2 \rangle_{\mathcal{H}} = \langle v_1, v_2 \rangle + \sum_{i,j=1}^{n} \langle \nabla w_i, C_{ij}\nabla w_j \rangle,$$

where $z_i = (v_i, w_i) \in \mathcal{H}$, $i = 1, 2$.

Definition 4.2 Let $T > 0$ and $f \in L^1\big([0,T]; L^2(\Omega)\big)$. We say that a function $\boldsymbol{z}(t) = (\theta(t), \boldsymbol{\eta}(t)) \in C\,([0,T]; \mathcal{H})$ is a solution of system (4.8)–(4.9) in the time interval $[0,T]$, with initial data $\boldsymbol{z}_0 = \boldsymbol{z}(0) = (\theta_0, \boldsymbol{\eta}_0) \in \mathcal{H}$, if the following identities are satisfied

$$\langle \theta_t, \tilde\theta \rangle + \sum_{i=1}^{n} k_i \langle \nabla\eta_i, \nabla\tilde\theta \rangle + \langle f(\theta), \tilde\theta \rangle = 0,$$

$$\sum_{i,j=1}^{n} \langle \eta_{it}, C_{ij}\Delta\tilde\eta_j \rangle - \sum_{i,j=1}^{n} \langle \theta, C_{ij}\Delta\tilde\eta_j \rangle + \sum_{i,j=1}^{n} \langle a_i\eta_i, C_{ij}\Delta\tilde\eta_j \rangle = 0$$

for all $\tilde\theta \in H_0^1(\Omega)$, $\tilde{\boldsymbol{\eta}} \in ([H^2(\Omega)]^n \cap \mathcal{W})$ and a.e. $t \in [0,T]$.

We denote by $\mathcal{S}(t)\boldsymbol{z}_0$ the solution of (4.8) – (4.9) with initial data $\boldsymbol{z}_0$. Because the system is autonomous, $\mathcal{S}(t)$ is a strongly continuous semigroup of the continuous operator on $\mathcal{H}$, related to the system (4.8) – (4.9). The total energy associated to (4.8) – (4.9) at time t is

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\Big[\|\theta(t)\|^2 + \sum_{i,j=1}^{n} \langle \nabla\eta_i(t), C_{ij}\nabla\eta_j(t)\rangle\Big] \\ &= \frac{1}{2}\Big[\int_\Omega |\theta(t)|^2\,d\boldsymbol{x} + \int_\Omega \Big|C^{\frac{1}{2}}\nabla\boldsymbol{\eta}(t)\Big|_2^2\,d\boldsymbol{x}\Big].\end{aligned} \tag{4.10}$$

Then, we obtain the following result.

Theorem 4.2 *Let us suppose that $\boldsymbol{z} = (\theta, \boldsymbol{\eta})$ is a solution of system (4.8) – (4.9) in the sense of Definition 4.2. Let $f \in C^1(R)$ satisfying the following hypotheses*

(h1) $\liminf\limits_{|y|\to\infty} \frac{f(y)}{y} \geq 0$*;*

(h2) *there exists a positive constant β such that $|f'(y)| \leq \beta$, $\forall\, y \in R$.*

If the matrices $C = [C_{ij}]$, $\Gamma = [\Gamma_{ij}] \in \mathcal{M}(n,n)$, defined by (4.2), (4.3) respectively, are positive definite, then there exist positive constants A, Λ, ε such that the relation

$$\mathcal{E}(t) \leq A\, e^{-\varepsilon t}\mathcal{E}(0) + \Lambda \tag{4.11}$$

holds for every $t \geq 0$. In particular, if $f \equiv 0$ then $\Lambda = 0$.

To prove Theorem 4.2 we make use of some preparatory lemmas.

Lemma 4.1 *If $f \in C^1(R)$ satisfies hypotheses (h1) and (h2), then*

(1) *for $\gamma > 0$ there exists a positive constant $\mathcal{C}_\gamma$ such that, $\forall\, y \in H_0^1(R)$*

$$\int_\Omega y\, f(y)\, d\boldsymbol{x} \geq -\gamma \int_\Omega |y|^2\, d\boldsymbol{x} - \mathcal{C}_\gamma; \tag{4.12}$$

(2) $\forall\, y \in R$

$$|f(y)| \leq \beta|y| + |f(0)|. \tag{4.13}$$

Proof Inequality (4.12) follows directly from hypothesis (h1) (cf. [7]); (4.13) is an easy consequence of hypothesis (h2).

Lemma 4.2 *Let $f \in C^1(R)$ satisfying hypothesis (h1); let assume $C, \Gamma \in \mathcal{M}(n,n)$ as in Theorem 4.2. If $\boldsymbol{z} = (\theta, \boldsymbol{\eta})$ is a solution of system (4.8) – (4.9) in the sense of Definition 4.2, then the energy norm (4.10) verifies*

$$\frac{d}{dt}\,\mathcal{E}(t) \leq \gamma \int_\Omega |\theta(t)|^2\, d\boldsymbol{x} + \mathcal{C}_\gamma - \alpha_1 \int_\Omega \Big|C^{\frac{1}{2}}\nabla\boldsymbol{\eta}(t)\Big|_2^2\, d\boldsymbol{x}, \tag{4.14}$$

where γ, $\mathcal{C}_\gamma$, α_1 are positive constants.

Proof If $\boldsymbol{z} = (\theta, \boldsymbol{\eta})$ is a solution of system (4.8) – (4.9), then, recalling (4.2), we have

$$\frac{d}{dt}\,\mathcal{E}(t) = -\langle f, \theta(t)\rangle - \sum_{i,j=1}^{n} \langle \nabla\eta_i(t),\, a_i C_{ij}\nabla\eta_j(t)\rangle.$$

Now, by means of inequality (4.12), we find

$$-\langle f, \theta(t)\rangle \leq \int_\Omega |\theta(t)|^2\, d\boldsymbol{x} + \mathcal{C}_\gamma, \qquad \gamma, \mathcal{C}_\gamma > 0 \tag{4.15}$$

and, thanks to our hypotheses on the matrices $C = [C_{ij}]$ and $\Gamma = [\Gamma_{ij}]$, there exist positive constants $\overline{\alpha_1}$, $\overline{\overline{\alpha_1}}$ such that

$$\begin{aligned} -\sum_{i,j=1}^{n} \langle \nabla\eta_i(t),\, a_i C_{ij} \nabla\eta_j(t)\rangle &= -\sum_{i,j=1}^{n} \left\langle \frac{\nabla\eta_i(t)}{a_i},\, \Gamma_{ij}\frac{\nabla\eta_j(t)}{a_j} \right\rangle \\ &\le -\overline{\alpha_1}\,\|\nabla\boldsymbol{\eta}(t)\|^2 \le -\overline{\alpha_1}\;\overline{\overline{\alpha_1}}\sum_{i,j=1}^{n} \langle \nabla\eta_i(t),\, C_{ij}\nabla\eta_j(t)\rangle. \end{aligned} \tag{4.16}$$

From (4.15) and (4.16), putting $\alpha_1 = \overline{\alpha_1}\;\overline{\overline{\alpha_1}}$, estimate (4.14) follows.

Lemma 4.3 *Suppose that $\boldsymbol{z} = (\theta, \boldsymbol{\eta})$ is a solution of system (4.8)–(4.9) in the sense of Definition 4.2 and assume hypotheses of Theorem 4.2 on f, C and Γ. Introduce the following functional*

$$\mathcal{K}(t) = -\left\langle |\theta(t)|, \sum_{i=1}^{n} k_i\, \eta_i(t) \right\rangle, \quad \forall\, t \ge 0;$$

then we have

$$\begin{aligned} \frac{d}{dt}\mathcal{K}(t) &\le \frac{1}{2}(\nu - M_1)\int_{\Omega} |\theta(t)|^2 d\boldsymbol{x} + \mathcal{C}_0 \\ &+ \left[\alpha_2 + \frac{M_1\alpha_3}{2}\left(1 + \frac{\beta^2}{\nu}\right) + \frac{M_2\alpha_4}{2M_1}\right] \int_{\Omega} \left| C^{\frac{1}{2}} \nabla\boldsymbol{\eta}(t)\right|_2^2 dx\,, \end{aligned} \tag{4.17}$$

for some positive constants α_2, α_3, α_4, ν, M_1, M_2, $\mathcal{C}_0$.

Proof The derivative of $\mathcal{K}(t)$ with respect to t entails

$$\frac{d}{dt}\mathcal{K}(t) = \underbrace{-\mathrm{sgn}(\theta)\left\langle \theta_t, \sum_{i=1}^{n} k_i\eta_i \right\rangle}_{=I_1} \underbrace{-\left\langle |\theta|, \sum_{i=1}^{n} k_i\eta_{i\,t} \right\rangle}_{=I_2}. \tag{4.18}$$

Substituting $(4.8)_1$ in the first term at the right-hand side of (4.18) and using Young inequality, we obtain

$$\begin{aligned} I_1 &= -\mathrm{sgn}(\theta)\left\langle \sum_{j=1}^{n} k_j \Delta\eta_j - f,\, \sum_{i=1}^{n} k_i\eta_i \right\rangle \le \left\| \sum_{i=1}^{n} k_i\, \nabla\eta_i \right\|^2 + \left\langle f,\, \sum_{i=1}^{n} k_i\,\eta_i \right\rangle \\ &\le n\int_{\Omega}\left(\sum_{i,h=1}^{n} \nabla\eta_i \delta_{ih} {k_i}^2 \nabla\eta_h \right) d\boldsymbol{x} + \left\langle f,\, \sum_{i=1}^{n} k_i\eta_i \right\rangle \\ &\le n\left| {K_\star}^{\frac{1}{4}} \left({K_\star}^{-\frac{1}{4}} C {K_\star}^{-\frac{1}{4}} \right)^{-\frac{1}{2}} \right|_2^2 \int_{\Omega} \left| C^{\frac{1}{2}}\nabla\boldsymbol{\eta} \right|_2^2 d\boldsymbol{x} + \underbrace{\left\langle f,\, \sum_{i=1}^{n} k_i\eta_i \right\rangle}_{=I_3}, \end{aligned}$$

where $K_\star = [\delta_{ih} {k_i}^2] = \text{diag}(k_i^2)$. From (4.13), applying Young and Poincaré inequalities, we find

$$
\begin{aligned}
I_3 &\le \left\langle \beta|\theta| + |f(0)|, \sum_{i=1}^n k_i \eta_i \right\rangle \\
&\le \int_\Omega \left[|\theta| \left(\beta \sum_{i=1}^n \left| (k_i)^{\frac{1}{2}} \, (k_i^{\frac{1}{2}} \, \eta_i) \right| \right) \right] d\boldsymbol{x} + \int_\Omega \left(|f(0)| \sum_{i=1}^n \left| (k_i)^{\frac{1}{2}} (k_i^{\frac{1}{2}} \eta_i) \right| \right) d\boldsymbol{x} \\
&\le \frac{\nu}{2} \|\theta\|^2 + \frac{1}{2} \left(1 + \frac{\beta^2}{\nu} \right) \left(\sum_{i=1}^n k_i \right) \left(\sum_{i=1}^n \frac{k_i}{\lambda_0^i} \|\nabla \eta_i\|^2 \right) + \frac{1}{2} |f(0)|^2 \operatorname{vol}(\Omega) \\
&\le \frac{\nu}{2} \|\theta\|^2 + \frac{1}{2} \left(1 + \frac{\beta^2}{\nu} \right) \left(\sum_{i=1}^n k_i \right) \left(\sum_{i,h=1}^n \langle \nabla \eta_i, \, \delta_{ih} \frac{k_i}{\lambda_0^i} \nabla \eta_h \rangle \right) + \frac{1}{2} |f(0)|^2 \operatorname{vol}(\Omega) \\
&\le \frac{\nu}{2} \|\theta\|^2 + \frac{1}{2} \left(1 + \frac{\beta^2}{\nu} \right) \left(\sum_{i=1}^n k_i \right) \left| {K_\lambda}^{\frac{1}{4}} \left({K_\lambda}^{-\frac{1}{4}} C {K_\lambda}^{-\frac{1}{4}} \right)^{-\frac{1}{2}} \right|_2^2 \int_\Omega \left| C^{\frac{1}{2}} \nabla \boldsymbol{\eta} \right|_2^2 d\boldsymbol{x} \\
&\quad + \frac{1}{2} |f(0)|^2 \operatorname{vol}(\Omega),
\end{aligned}
$$

where ν and $\lambda_0^i, \; i = 1, \ldots, n,$ are positive constants and $K_\lambda = \left[\delta_{ih} \, \dfrac{k_i}{\lambda_0^i} \right] = \text{diag}\left(\dfrac{k_i}{\lambda_0^i} \right)$.

By means of $(4.8)_2$, the second term in (4.18) can be written as

$$
I_2 = -\left\langle |\theta|, \sum_{i=1}^n k_i \; (\theta - a_i \, \eta_i) \right\rangle = -\left(\sum_{i=1}^n k_i \right) \|\theta\|^2 + \underbrace{\left\langle |\theta|, \sum_{i=1}^n k_i a_i \eta_i \right\rangle}_{= I_4}.
$$

By virtue of Young and Poincaré inequalities, we have

$$
\begin{aligned}
I_4 &\le \int_\Omega |\theta| \sum_{i=1}^n \left| (k_i a_i)^{\frac{1}{2}} (k_i \, a_i)^{\frac{1}{2}} \eta_i \right| d\boldsymbol{x} \\
&\le \frac{\delta}{2} \|\theta\|^2 + \frac{1}{2\delta} \left(\sum_{i=1}^n k_i a_i \right) \left(\sum_{i=1}^n \frac{k_i a_i}{\lambda_0^i} \; \|\nabla \eta_i\|^2 \right) \\
&\le \frac{\delta}{2} \|\theta\|^2 + \frac{1}{2\delta} \left(\sum_{i=1}^n k_i \, a_i \right) \left(\sum_{i,h=1}^n \left\langle \nabla \eta_i, \, \delta_{ih} \frac{k_i a_i}{\lambda_0^i} \nabla \eta_h \right\rangle \right) \\
&\le \frac{\delta}{2} \|\theta\|^2 + \frac{1}{2\delta} \left(\sum_{i=1}^n k_i a_i \right) \left| K_a^{\frac{1}{4}} \left(K_a^{-\frac{1}{4}} C K_a^{-\frac{1}{4}} \right)^{-\frac{1}{2}} \right|_2^2 \int_\Omega \left| C^{\frac{1}{2}} \nabla \boldsymbol{\eta} \right|_2^2 d\boldsymbol{x},
\end{aligned}
$$

where δ is a positive constant and $K_a = \left[\delta_{ih} \, \dfrac{k_i a_i}{\lambda_0^i} \right] = \text{diag}\left(\dfrac{k_i \, a_i}{\lambda_0^i} \right)$. Choosing $\delta = \sum\limits_{i=1}^n k_i$, we find

$$
\begin{aligned}
I_2 &\le -\frac{1}{2} \left(\sum_{i=1}^n k_i \right) \|\theta\|^2 \\
&\quad + \frac{1}{2} \left(\sum_{i=1}^n k_i \right)^{-1} \left(\sum_{i=1}^n k_i a_i \right) \left| K_a^{\frac{1}{4}} \left(K_a^{-\frac{1}{4}} C K_a^{-\frac{1}{4}} \right)^{-\frac{1}{2}} \right|_2^2 \int_\Omega \left| C^{\frac{1}{2}} \nabla \boldsymbol{\eta} \right|_2^2 d\boldsymbol{x}.
\end{aligned}
$$

Finally, collecting the previous inequalities and putting

$$\alpha_2 = n\Big|K_\star^{\frac{1}{4}}\Big(K_\star^{-\frac{1}{4}}CK_\star^{-\frac{1}{4}}\Big)^{-\frac{1}{2}}\Big|_2^2, \qquad \alpha_3 = \Big|K_\lambda^{\frac{1}{4}}\Big(K_\lambda^{-\frac{1}{4}}CK_\lambda^{-\frac{1}{4}}\Big)^{-\frac{1}{2}}\Big|_2^2,$$

$$\alpha_4 = \Big|K_a^{\frac{1}{4}}\big(K_a^{-\frac{1}{4}}CK_a^{-\frac{1}{4}}\big)^{-\frac{1}{2}}\Big|_2^2, \qquad M_1 = \sum_{i=1}^n k_i,$$

$$M_2 = \sum_{i=1}^n k_i a_i, \qquad \mathcal{C}_0 = \frac{1}{2}\,|f(0)|^2\,\mathrm{vol}(\Omega),$$

we obtain (4.17).

At this point, we can prove Theorem 4.2.

Proof For $N > 0$ we introduce the following functional

$$\mathcal{L}(t) = N\mathcal{E}(t) + \mathcal{K}(t), \quad \forall\, t \geq 0;$$

it is easily seen that, if we choose

$$N > \max\{1,\, M_1\alpha_3\},$$

there exist positive constants

$$\gamma_1 = \min\{N-1,\, N - M_1\alpha_3\}, \quad \gamma_2 = \max\{N+1,\, N+M_1\alpha_3\}$$

such that

$$\gamma_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq \gamma_2\mathcal{E}(t), \quad \forall\, t \geq 0. \tag{4.19}$$

Moreover, collecting inequalities (4.14) and (4.17), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq -\bigg(\frac{M_1}{2} - \frac{\nu}{2} - N\gamma\bigg)\|\theta\|^2 + \tilde{\Lambda}(N,\gamma,\Omega)\\ - \bigg[N\alpha_1 - \alpha_2 - \frac{M_1\alpha_3}{2}\bigg(1+\frac{\beta^2}{\nu}\bigg) - \frac{M_2\alpha_4}{2M_1}\bigg]\int_\Omega \Big|C^{\frac{1}{2}}\,\nabla\boldsymbol{\eta}\Big|_2^2\,d\boldsymbol{x},\end{aligned}$$

where $\tilde{\Lambda}(N,\gamma,\Omega) = N\mathcal{C}_\gamma + \mathcal{C}_0$. Taking now

$$\nu = \frac{M_1}{2},$$

choosing N large enough such that

$$N \geq N^* = \frac{1}{\alpha_1}\bigg(\alpha_2 + \frac{M_1\alpha_3}{2} + \alpha_3\beta^2 + \frac{M_2\alpha_4}{2M_1} + \frac{M_1}{8}\bigg)$$

and letting

$$\gamma = \frac{M_1}{8N},$$

we have

$$\frac{d}{dt}\mathcal{L}(t) \le -\frac{M_1}{8}\|\theta\|^2 - \frac{M_1}{8}\int_\Omega \left|C^{\frac{1}{2}}\,\nabla\boldsymbol{\eta}\right|_2^2 d\boldsymbol{x} + \tilde{\Lambda}\left(N, \frac{M_1}{8N}, \Omega\right). \tag{4.20}$$

By means of (4.19), inequality (4.20) yields

$$\frac{d}{dt}\,\mathcal{L}(t) \le -\frac{M_1}{4}\,\mathcal{E}(t) + \tilde{\Lambda} \le -\varepsilon\mathcal{L}(t) + \tilde{\Lambda}\,,$$

where

$$\varepsilon = \frac{M_1}{4\gamma_2}\,.$$

By virtue of the Gronwall Lemma, we obtain

$$\mathcal{L}(t) \le \mathcal{L}(0)\,\mathrm{e}^{-\varepsilon t} + \frac{\tilde{\Lambda}}{\varepsilon}\left(1 - \mathrm{e}^{-\varepsilon t}\right)\,, \quad \forall\, t \ge 0. \tag{4.21}$$

Finally, from (4.19) and (4.21), it follows that

$$\mathcal{E}(t) \le \frac{1}{\gamma_1}\,\mathcal{L}(t) \le \frac{\gamma_2}{\gamma_1}\,\mathcal{E}(0)\,\mathrm{e}^{-\varepsilon t} + \frac{\tilde{\Lambda}}{\varepsilon\gamma_1}$$

holds for every $t \ge 0$, so that taking

$$A = \frac{\gamma_2}{\gamma_1}, \qquad \Lambda = \frac{\tilde{\Lambda}}{\varepsilon\gamma_1}$$

our conclusion follows.

Now, we state the main result of this section.

Theorem 4.3 *Assume $C = [C_{ij}] \in \mathcal{M}(n,n)$ and $f \in C^1(R)$ as in Theorem 4.2. The uniform energy estimate (4.11) implies the existence of a bounded absorbing set $\mathcal{B}^* \subset \mathcal{H}$ for the semigroup $\mathcal{S}(t)$. That is, if $\mathcal{B}^*$ is any ball of $\mathcal{H}$ of radius less than $\sqrt{2\Lambda}$, for any bounded set $\mathcal{B} \subset \mathcal{H}$ there exists $t(\mathcal{B}) \ge 0$ such that*

$$\mathcal{S}(t)\mathcal{B} \subset \mathcal{B}^*, \quad \forall\, t \ge t(\mathcal{B}).$$

Proof The existence of an absorbing set for $\mathcal{S}(t)$ follows directly by (4.11) (see for example [8]).

References

[1] Amendola, G. and Carillo, S. Thermal work and minimum free energy in a heat conductor with memory. *Quarterly J. Mech. Appl. Math.* **57** (2004) 429–446.

[2] Cattaneo, C. Sulla conduzione del calore. *Atti Sem. Mat. Fis. Univ. Modena* **3** (1948) 83–101.

[3] Coleman, B.D. Thermodynamics of materials with memory. *Arch. Rational Mech. Anal.* **17** (1964) 1–46.

[4] Fabrizio, M., Giorgi, C. and Naso, M.G. Viscoelastic solids of exponential type. I. Minimal representations and controllability of the state space. *Meccanica* **39**(6) (2004) 531–546.

[5] Fabrizio, M., Giorgi, C. and Naso, M.G. Viscoelastic solids of exponential type. II. Free energies, stability and attractors. *Meccanica* **39**(6) (2004) 547–561.

[6] Fabrizio, M., Gentili, G. and Reynolds, D.W. On rigid heat conductors with memory. *Int. J. Eng. Sci.* **36** (1998) 765–782.

[7] Ghidaglia, J.M. and Temam, R. Attractors for damped nonlinear hyperbolic equations. *J. Math. Pures Appl.* **66**(3) (1987) 273–319.

[8] Giorgi, C., Muñoz Rivera, J.E. and Pata, V. Global attractors for a semilinear hyperbolic equation in viscoelasticity. *J. Math. Anal. Appl.* **260**(1) (2001) 83–99.

[9] Graffi, D. and Fabrizio, M. On the notion of state for viscoelastic materials of “rate” type. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **83** (1990) 201–208.

[10] Gurtin, M.E. and Pipkin, A.C. A general theory of heat conduction with finite wave speeds. *Arch. Rational Mech. Anal.* **31** (1968) 113–126.

[11] McCarthy, M. Constitutive equations for thermomechanical materials with memory. *Int. J. Eng. Sci.* **8** (1970) 467–474.

[12] Zabczyk, J. *Mathematical control theory: an introduction.* Birkhäuser Boston Inc., Boston, MA, 1992.