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NONLINEAR DYNAMICS & SYSTEMS THEORY

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# Nonlinear Dynamics and Systems Theory

## An International Journal of Research and Surveys



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# Nonlinear Dynamics and Systems Theory
An International Journal of Research and Surveys







# NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys


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# NONLINEAR DYNAMICS AND SYSTEMS THEORY

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**PERSONAGE IN SCIENCE**

# Professor A.A.Martynyuk

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*On March 6, 2006, the Corresponding Member of the National Academy of Sciences of Ukraine, Habilitation Doctor and Ph.D. of physical and mathematical sciences, Professor Anatoly Andreevich Martynyuk turns 65. The Editorial Board of the International Scientific Journal "Nonlinear Dynamics and Systems Theory" congratulates him on the occasion of his 65th birthday and wishes him a great health and new significant achievements in his scientific endeavors. In this regard, the Editorial Board of "Nonlinear Dynamics and Systems Theory" publishes a biographical sketch highlighting Martynyuk's research and scholarly activities.*

## 1 A Brief Survey of Martynyuk's Life

Anatoly A. Martynyuk was born in the family of a railwayman, Andrey Gerasimovich Martynyuk, who lived in Ukraine (Cherkassy region). In 1958, Martynyuk graduated from a high school and the same year he was admitted to the Department of Physics and Mathematics of the Cherkassy State Pedagogical Institute (now B. Khmelnitzky Cherkassy State University). Martynyuk graduated from the Institute with an honor Master of Science degree and for one year he was employed as an instructor of physics and mathematics at a Polessye high school.

In September 1964, Martynyuk was admitted to the post-graduate school of the Institute of Mechanics of Acad. of Sci. of Ukr. SSR (now the S.P.Timoshenko Institute of Mechanics Nat. Acad. of Sci. of Ukraine) chaired by Professor A.N. Golubentzev. Martynyuk's Master's dissertation was focused on the problems of finite stability (on a given time interval). This research was supported both by Professor A.N. Golubentzev and the Department of Differential Equations of the Institute of Mathematics (the Head of the Department the Corresponding Member of Ac. of Sci. of Ukr. SSR, Prof. Yu.D. Sokolov). Martynyuk successfully defended his Master's thesis in the Institute of Mathematics in 1967. In 1969-1973, he worked for his doctorate under Yu.A.Mitropolskii. In 1973 he

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defended his highest degree of Habilitation (with the Academician Yu.A. Mitropolskii serving on his committee). Shortly thereafter, Martynyuk was employed at the Institute of Mathematics, Department of the Presidium Acad. of Sci. of Ukr. SSR, Division of Mathematics, Mechanics and Cybernetics of Acad. of Sci. of Ukr. SSR. In 1978 he founded the Department of Stability of Processes at the Institute of Mechanics of Acad. of Sci. of Ukr. SSR and since then he has been the Head of this Department. In 1988 Martynyuk was elected a Corresponding Member of the Acad. of Sci. of Ukr. SSR, and in 1981 he was presented the prestigious N.M. Kryloff's award of the Acad. of Sci. of Ukr. SSR for his celebrated series of works on nonlinear mechanics.

## 2 Main Directions of the Scientific Investigations

Martynyuk represents the scientific school of Bogolyubov–Mitropolskii, going back to A.M. Liapunov. Thorough his first studies on the theory of the stability of motion, carried out under the influence of works by Ye.A. Barbashin, N.G. Chetaev, I.G. Malkin, N.N. Krasovskii, K.P. Persidskii, V.I. Zubov. The main directions of the scientific research by Martynyuk are:

* construction of approximate solutions of differential equations systems,
* nonclassical motion stability theories (technical, practical, stability in the whole),
* applications of integral inequalities in the stability theory,
* development of the comparison technique in nonlinear dynamics,
* stability analysis of large scale systems,
* topological dynamics (the method of limiting equations),
* creation of the method of the matrix-valued Liapunov functions,
* qualitative analysis of mathematical models in biology.

We shall outline in brief the development of the above directions in the works by Martynyuk.

### 2.1 Construction of approximate solutions of differential equations

In a series of his papers Martynyuk treats systems of ordinary differential equations (autonomous or nonautonomous) and proposes construction of solutions in the form of power series or series in Poincaré variable

$$w = \left(e^{\nu(t-t_0)} - 1\right)\left(e^{\nu(t-t_0)} + 1\right)^{-1}, \qquad \nu = \frac{\pi}{2h}. \tag{2.1}$$

In transformation (2.1) the interior of the strip of width $2h$ in the $\tau$-plane is transformed conformally onto the interior of the unit circle $\{|w| = 1\}$ in the $w$-plane.

If the domain containing solution of the initial system is totaly embedded into the domain of asymptotic stability of the system under investigation, the corresponding series converge for all values of the variable $t$. He constructs new recurrent formulas for calculation of coefficients of these series and studies stability of approximate solutions obtained as a result of approximate solutions obtained as a result of application of a finite number of terms of the series.

Monograph [I] provides generalization of the results [5–7, 9, 16, 18, 21, 22, 25, 30, 34] and presents some applications in dynamics of mechanical systems.

### 2.2 Nonclassical theories of motion stability

Martynyuk's research of nonclassical stability theory is the tantamount of stability analysis of solutions to nonlinear systems, in which the domain interval of the independent (time) variable is a fixed interval. His results are about technical stability, stability on a finite interval and practical stability. Stability problems in the nonclassical sense arise in aviation, rocket building, robot technical systems construction and alike.

In the early sixties, different methods were adapted to the problems of nonclassical stability theory by many investigators. In particular, Martynyuk introduced the "locally large Liapunov function" and proved general theorems on technical stability of the continuous nonlinear systems and systems with delay in general.

Under various assumptions imposed on the system

$$\frac{dx}{dt} = X(t,x), \qquad x(t_0) = x_0, \tag{2.2}$$

where $t \in R_+$, $x \in R^n$ and $X\colon R_+ \times R^n \to R^n$, numerous sufficient conditions for the technical stability of motion were established by means of the Liapunov functions of the form

$$v(t,x) = e^{-Dt} x^{\mathrm{T}} K x, \tag{2.3}$$

where $D$ is the diameter of the domain of admissible motion deviations and $K$ is a $n \times n$-matrix of definite sign.

The development of some ideas and results of [1–4, 8, 17, 20, 27–29, 39, 47, 48] enabled Martynyuk to obtain new tests for practical stability of motion for some classes of systems of equations presented in [60, 68–70, 75, 86, 91]. An efficient application of the direct Liapunov method in the practical stability problems by Martynyuk yielded significant extensions of this method, which are as follows:

(i) an extension of the class of auxiliary functions suitable for the studying practical stability of motion;
(ii) elimination of the property of having a fixed sign of the total derivative of an auxiliary function along with solutions of the system under investigation;
(iii) establishing a relationship between the quantitative values of the auxiliary function in given (finite) domains of the phase space and decrement (increment) of this function, along with solutions of the system under investigation.

The generalized results on practical stability of non-linear systems are found in monographs [V, X]. The criteria of practical stability presented in [X] was obtained with V. Lakshmikantham and S. Leela and include discrete and impulsive systems, systems of integral differential and functional-differential equations, reaction-diffusion equations, controlled systems, and systems with multi-valued right-hand sides.

Major recent developments in this direction are summarized in monographs [XVIII] published in Chinese. For the application of the obtained results in the dynamics of wheeled transport vehicles and rocket dynamics, the reader is referred to monographs [IV] and [XXIII].

### 2.3 Applications of the integral inequalities in the stability theory

It is rather difficult to study the behavior of solutions, with unbounded jumps of systems of the type

$$\frac{dx}{dt} = f(t,x) + g(t,x), \qquad x(t_0) = x_0, \tag{2.4}$$

where $x \in R^n$, $f \in C(R_+ \times R^n, R^n)$, $g(t,x) \in C(R_+ \times R^n, R^n)$. In system (2.4), the vector function $g(t,x)$ can characterize the terms of higher order of smallness compared to the function $f(t,x)$ or to persistent perturbations. Boundedness, asymptotic behavior, oscillations and stability of solutions to system (2.4) are of great interest.

A utilization of integral inequalities is at the heart of a fundamental approach when analyzing the above mentioned properties of solutions to system (2.4) and its particular cases.

Furthermore, an application of integral inequalities for a rough estimation of the qualitative behavior of solutions to linear and nonlinear systems of differential equations represents an essential part in the theory of motion stability.

In the papers [46, 58, 63], the author applied integral inequalities to problems of qualitative analysis of motion in the theory of motion stability. The idea leads to

(i) the utilization of known techniques and the development of new ones in order to reduce system (2.4) to a form suitable for application of integral inequalities;
(ii) developing the method of estimating the nonlinear terms in system (2.4) corresponding to the structure of the employed integral inequalities;
(iii) the investigation of general properties of systems with lumped and distributed parameters, such as boundedness, continuous dependence on the initial values and parameters, stability via Liapunov and Lagrange and stability under persistent perturbation, as well as nonclassical problems of stability theory.

Nonlinear systems of (2.4) type are investigated in [III] under various assumptions imposed on dynamical properties of solutions of nonlinear (linear) approximation to system (2.4).

For further progress of integral inequalities techniques in qualitative analysis of solutions to nonlinear systems of differential equations the reader is referred to monograph [IX], while some applications are found in monograph [XIV].

### 2.4 Comparison technique and averaging method in nonlinear dynamics

Difficulties in analyzing nonlinear systems (2.2) or (2.4) under their high dimensions stipulate a new method of qualitative analysis referred to as the comparison method. As it is known, this method is based on the construction of the comparison equation (system)

$$\frac{du}{dt} = G(t,u), \quad u(t_0) = u_0 \geq 0, \tag{2.5}$$

where $u \in R^m_+$, $G \in C(R_+ \times R_+, R^m)$, $G(t,0) = 0$ for all $t \geq t_0$ whose maximal $u^+(t;t_0,u_0)$ (minimal $u^-(t;t_0,u_0)$) solution is correlated with the solution $x(t;t_0,x_0)$ of system (2.2) as

$$\begin{aligned} Q(t, x(t;t_0,x_0)) &\leq u^+(t;t_0,u_0), \\ Q(t, x(t;t_0,x_0)) &\geq u^-(t;t_0,u_0), \end{aligned} \tag{2.6}$$

where $Q \in C(R_+ \times R^n, R_+^m)$, $Q(t,0) = 0$ for all $t \in R_+$.

In many fundamental works, researchers suggested constructing comparison system (2.5) and comparison functions $Q(t,\cdot)$ , which allow one to analyze stability of the state $x = 0$ of system (2.2) in terms of the solution $u = 0$ to the comparison system.

In Martynyuk's monograph [II], the development of the comparison technique is associated with the analysis of systems of the type

$$\frac{dx_s}{dt} = f_s(t, x_s) + g_s(t, x_1, \dots, x_m), \qquad s = 1, 2, \dots, m, \tag{2.7}$$

where $x_s \in R^{n_s}$, $f_s \in C(R_+ \times R^{n_s}, R^{n_s})$, $g_s \in C(R_+ \times R^{n_1} \times \dots \times R^{n_m}, R^{n_s})$, $f_s(t,0) = 0$, $g(t, 0, \dots, 0) = 0$. Here the comparison technique is based on the integral inequalities

$$\varphi_s(t) \le \psi_s(t) + \int_{t_0}^{t} \sigma_s(\tau, \varphi_1(\tau), \dots, \varphi_m(\tau))\, d\tau, \quad s = 1, 2, \dots, m, \tag{2.8}$$

and the comparison system

$$\begin{gathered} \frac{du_s}{dt} = \sigma_s(t, \psi_1(t) + u_1, \dots, \psi_m(t) + u_m), \\ s = 1, 2, \dots, m, \end{gathered} \tag{2.9}$$

where $\sigma(t, \psi(t) + y)$ is continuous on the open domain $D = \{(t, y) \colon a < t < b,\ y \in R^m\}$ and satisfies the condition of quasimonotonicity.

For inequalities (2.8) and some additional conditions, the estimates

$$\varphi_s(t) \le \psi_s(t) + u_s^+(t, t_0, u_{0s}), \quad s = 1, 2, \dots, m, \tag{2.10}$$

are valid, where $u_s^+(t, t_0, u_{0s})$ is the maximal solution of system (2.9).

Using this type of the comparison technique, the problems on technical stability with respect to separate coordinates and technical stability of multidimensional system were solved in [27]. The applications of this type of the comparison technique to various problems of qualitative analysis of solutions to nonlinear equations are found in the papers [35–37, 50, 55, 57]. In particular, Martynyuk, in his monograph [XIV], studied many problems concerning the qualitative behavior of solutions to equations in the standard form, systems with quick and slow variables, systems with small persistent perturbations, and singularly-perturbed systems. For other results in the direction see [41, 43, 49, 52–54, 61, 62, 66, 74, 78, 81, 95, 93, 99, 100].

### 2.5 Stability analysis of large-scale dynamical systems

Stability of large-scale system of (2.2) type or more general systems modeled by equations in a Banach space, has been discussed in many well-known monographs. An application of vector Liapunov functions or vector norms leads to comparison systems of (3.1) type or other types with the common property of the right-hand side being quasimonotone.

This way, we arrive at a stability problem of a quasimonotone system in a cone. The other important stability problem of large scale systems was an efficient account of the influence of small interconnections between the subsystems in the case when the subsystems are not asymptotically stable.

In the papers [32, 33], for the large scale system

$$\frac{dx_s}{dt} = f_s(t, x_s) + \mu g_s(t, x_1, \ldots, x_m), \qquad s = 1, 2, \ldots, m, \tag{2.11}$$

it was proposed to apply the vector function

$$V(t, x) = (v_1(t, x_1), \ldots, v_m(t, x_m))^{\mathrm{T}},$$

whose components $v_s \in C(R_+ \times R^{n_s}, R_+)$ are constructed for the independent subsystems

$$\frac{dx_s}{dt} = f_s(t, x_s), \qquad x_s(t_0) = x_{s0}, \tag{2.12}$$

of system (2.11). The functions

$$\psi_s(t) = \int_{t_0}^{t} (\nabla v_s(t, x_s))^{\mathrm{T}} g_s(s, \overline{x}_1(s), \ldots, \overline{x}_m(s))\, ds, \qquad s = 1, 2, \ldots, m, \tag{2.13}$$

yield the solutions $\overline{x}_1(s), \ldots, \overline{x}_m(s)$ of subsystems (2.12) and they estimate the influence of interconnections $g_s(t, x_1, \ldots, x_m)$ on the dynamics of whole system (2.11).

Comparison system (2.9), with functions (2.13), lead to estimates (2.10) to be obtained. The latter are a source of various sufficient stability conditions for system of (2.11) type.

In monographs [XVII, XIX], Martynyuk finds estimates (2.8) – (2.12) and develops stability theory of large-scale systems (3.3) under various assumptions on the dynamical properties of subsystems (2.12) and he establishes interconnection functions between them.

In monographs [VI, VII], he developed new aggregation forms for large-scale systems under nonclassical structural perturbations. See also many results in [77, 80, 82, 85, 87, 110–113, 122, 128].

### 2.6 Limiting equations and stability theory

The Poincaré and Liapunov ideas on qualitative solutions to the systems of differential equations with no direct integration, combined with abstract theory of dynamical systems, gave rise to a new direction in the theory of equations, which is based on the notion of limiting equation (system).

Stability or other steady state dynamical properties of system (2.2) are associated with the limiting behavior of solutions as $t \to \infty$ and, therefore, are determined by the limiting characteristics of (2.2) for $t \to \infty$.

It appeared to be fruitful to consider the translations

$$\frac{dx}{dt} = f^{\tau}(x, t) \tag{2.15}$$

of equation (2.2), where $f^\tau(x,t) = f(x,t+\tau)$. These results in a family of equations (2.15) are parametrized by the shift $\tau$. The translation convergence when $t \to +\infty$ under some topology, enables one to study the asymptotic behavior of solutions to initial system (2.2). The equation obtained as a result of this convergence is referred to as the limiting equation.

Monographs [XI, XV] generalize in this direction the theory of motion stability. The authors treat a stability problem of nonautonomous systems modelled by ordinary differential equations, integral equations, equations with infinite delay, systems with small forces, integro-differential systems, abstract compact and uniform dynamical processes, dynamical processes on the space of convergence, asymptotically autonomous evolutionary equations of parabolic and hyperbolic type in Banach spaces, etc. Moreover, the method of limiting equations is applied here to investigate large-scale systems with weakly interacting subsystems. Besides, both stability and instability of large-scale systems are studied. The topics include stability with respect to a subset of variables. See also [115, 119].

## 2.7 The Liapunov's matrix-valued functions method

At the end of the 1970s, Martynyuk began research in the field of matrix-valued Liapunov functions. He proposed an approach to problems of stability based on the two-index system of functions

$$U(t,x) = [u_{ij}(t,x)], \quad i,\, j = 1,2,\dots,m, \tag{2.16}$$

where $u_{ii} \in C(R_+ \times R^n, R_+)$ and $u_{ij} \in C(R_+ \times R^n, R)$ for all $i \neq j$, which is suitable for the construction of Liapunov functions.

Both the scalar function

$$v(t,x,\eta) = \eta^{\mathrm{T}} U(t,x)\eta, \quad \eta \in R^m, \tag{2.17}$$

and the vector function

$$V(t,x,w) = AU(t,x)w, \quad w \in R^m, \tag{2.18}$$

with $A$ being a constant matrix $m \times m$, can be constructed in terms of matrix-valued function (2.16). The function (2.16) together with (2.17) and (2.18) were put by Martynyuk in the basis of the direct Liapunov method and comparison principle with matrix-valued function (see [64, 71–73, 79, 83, 94–97] and the monographs [XIV, XVII, XIX, XX, XXIV]).

The application of function (2.16) in the direct Liapunov method is beneficial in studying stability of large-scale systems (3.3), with no use of the comparison systems of (3.1) or (3.5) type. This enables one to bypass the quasimonotonicity condition when studying stability of large-scale systems, and as a by-product, it preserves the vector function

$$V(t,x) = \operatorname{diag}[u_{11}(t,x),\dots,u_{mm}(t,x)], \tag{2.19}$$

which is the principle diagonal of matrix-valued function (2.16).

The non-diagonal elements $u_{ij}(t,x_i,x_j)$ are constructed for all $(i \neq j) \in [1,m]$ in light of the interconnection functions $g_s(t,x_1,\dots,x_m)$ acting between the subsystems.

In nutshell, the development of the Liapunov matrix functions method rendered by Martynyuk is as follows:

* the discovery of a two-index system of functions as a structure suitable for construction of Liapunov functions;
* the introduction of the formalism of matrix Liapunov functions with the property of having fixed sign of the matrix-valued functions and their derivatives by virtue of the system of motion equations;
* the formulation of the invariance principle in terms of the matrix-valued functions and stability of solutions to the autonomous systems;
* the analytical construction of the matrix and hierarchical matrix-valued Liapunov functions.

As a result of the development of these powerful techniques, Martynyuk and his students established a new efficient stability condition for some classes of systems of equations. Namely,

(a) systems with lumped parameters;
(b) singularly perturbed systems including Lur'e–Postnikov systems;
(c) system with random parameters including singularly perturbed stochastic systems;
(d) impulsive systems;
(e) large-scale discrete systems;
(f) hybrid systems;
(g) large-scale power systems modelled by ODE;
(h) uncertain systems;
(i) systems with delay;
(j) systems in Banach and metric spaces;
(k) systems modelling the population dynamics (generalization of Kolmogorov model);
(l) classes (a), (b), (d) and (e) under nonclassical structural perturbations.

Recently, Martynyuk developed the method of matrix Liapunov functions for the investigation of polystability of motion, stability with respect to two measures, stability analysis of discontinuous systems, and polydynamics of nonlinear system on time scales (see [78, 84, 88-90, 101–106, 108, 118, 123–129].

## 2.8 Analysis of mathematical models in biology

The work of Martynyuk in this direction deals with the analysis of qualitative properties of solutions to the Lotka–Volterra system of equations and its generalizations in the form of the Kolmogorov system of equations

$$\begin{aligned} \frac{dx_i}{dt} &= \beta_i(x_i)F_i(t, x_1, \dots, x_n, \mu), \\ x_i(t_0) &= x_0 \geq 0, \quad i = 1, 2, \dots, n. \end{aligned} \tag{2.20}$$

Here $\beta_i$ are the functions infinitely many times differentiable on $R_+ = [0, \infty)$, $\beta_i(0) = 0$, $\beta_i'(x_i) > 0$ for $x_i > 0$, $\beta_i^j(x_i) \geq 0$, $j = 2, 3, \dots$ and $F_i \in C(R_+ \times R_+^n \times M^k, R)$, where $M^k = [0, 1] \times \dots \times [0, 1]$, $i = 1, 2, \dots, n$. System (2.20) is a multiplicatively and additively perturbed Kolmogorov system of equations that models the population dynamics.

Also, Martynyuk established the boundedness conditions for the population growth with respect to two measures, as well as the stability conditions for the population quantity. See also [110, 116].

## 3 Organizing and Scholarly Activity

Alongside the intensive scientific research, Martynyuk carries out great organizing and scholarly activity. He initiated the publication of "The Lectures on Theoretical Mechanics" of A.M.Liapunov in 1982. He also performs a considerable work as an Editor of the International Series of Scientific Monographs "Stability and Control: Theory, Methods, and Applications" at the Taylor and Francis Publishers (Great Britain). Since 1992, they published 22 volumes in this Series which have gained a world-wide recognition. He is founder of a new International journal "Nonlinear Dynamics and Systems Theory" since 2001, published in English.

Martynyuk serves on editorial boards of six international academic journals: *International Applied Mechanics*, *Elektronnoe Modelirovanie*, and *Nonlinear Oscillations* published in Russian and *Journal of Applied Mathematics and Stochastic Analysis* (USA), *Differential Equations and Dynamical Systems* (India), and *International Journal of Innovative, Computing & Control* (Japan) published in English.

He supervised 21 doctoral and 2 habilitation theses in physical and mathematical sciences. All of his former students are presently employed in different countries of the former Soviet Union.

## 4 Sports and Hobbies

Besides doing an incredible amount of scientific work, Martynyuk takes time to enjoy being with his family, his children and his grandchildren. On weekends he leaves his work for a cycle ride or walking about the forest suburbs of Kiev. Contacts with a wildlife is a source of delight and inspiration for him. Home library of Martynyuk contains about 2000 volumes of scientific literature, fiction and poetry. The books on history, philosophy, natural sciences and art are of his particular interest. He also collects postage-stamps and a series of periodicals "The Great Painters".

After the Chernobyl nuclear accident in 1986 Martynyuk actively opposed to unfounded decision of Political Bureau of Communist Party on construction of 28 nuclear power blocks in Ukraine. His article "A warning to the careless mankind" (see "Vecherniy Kiev", No. 273, November 29, 1989) produced a perceptible effect on the scientific and public society.

Professor Martynyuk and his apprentices proceed with the investigations in chosen areas of applied mathematics and mechanics providing the world science with new interesting results.

The American Biographical Institute recognized Martynyuk as an "Outstanding Man of the 20th Century" and awarded him the "2000 Millennium Medal of Honor".

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# Construction of Lyapunov's Functions for a Class of Nonlinear Systems

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**Abstract:** The conditions of absolute stability for a certain class of nonlinear systems are investigated. It is proved that the systems considered are absolutely stable iff for these systems there exist Lyapunov's functions of the special form. The results obtained are used for the stability analysis of complex systems in critical cases.

**Keywords:** *Nonlinear systems; Lyapunov's functions; absolute stability; large scale systems.*

**Mathematics Subject Classification (2000):** 34D20, 93D20, 93D30.

## 1 Introduction

One of the important problems arising in the investigation of nonlinear systems is the problem of absolute stability [1, 3, 8]. This problem is of both theoretical and applied significance. The main approach for the determination of conditions for the absolute stability is the Lyapunov direct method. By means of this approach, the criteria of absolute stability for many types of systems are obtained. However, it should be noted that until now there are no general methods of construction of Lyapunov's functions for nonlinear systems.

In the present paper a certain class of differential equations systems is investigated. The method of construction of Lyapunov's functions for these systems is suggested. The main goal of the paper is to prove that for the absolute stability of systems considered it is necessary and sufficient that the Lyapunov's functions in the given form exist satisfying the assumptions of the Lyapunov asymptotic stability theorem [3].

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## 2 Statement of the Problem

Consider the system of differential equations

$$\dot{x}_s = \sum_{j=1}^{n} p_{sj} f_j(x_j), \quad s = 1, \ldots, n. \tag{2.1}$$

Here $p_{sj}$ are constant coefficients, functions $f_j(x_j)$ are defined and continuous for $x_j \in (-\infty, +\infty)$ and possess the property $x_j f_j(x_j) > 0$ for $x_j \neq 0$. Hence, system (2.1) has the zero solution. Equations of this kind are widely used in the design of automatic control systems [3, 10].

The problem of absolute stability for system (2.1) was investigated in the works [3, 10, 14]. For the solution of this problem in [3] it was suggested to construct Lyapunov's function in the form

$$V = \sum_{s=1}^{n} \lambda_s \int_0^{x_s} f_s(\tau)\, d\tau, \tag{2.2}$$

where $\lambda_s$ are positive constants. Thus, $V$ is a positive definite function. In [3, 14] the sufficient conditions are obtained under which one may choose numbers $\lambda_s$ for the function

$$\frac{dV}{dt}\bigg|_{(2.1)} = \sum_{s,j=1}^{n} \lambda_s p_{sj} f_s(x_s) f_j(x_j)$$

to be negative definite.

Suppose that coefficients $p_{sj}$ in (2.1) satisfy the conditions

$$p_{ss} < 0, \qquad p_{sj} \geq 0 \text{ for } s \neq j. \tag{2.3}$$

For instance, inequalities (2.3) are valid if (2.1) is obtained as a comparison system for complex system [5, 11].

In this case the criterion of absolute stability for (2.1) was established by S.K. Persidsky [10]. It is proved that system (2.1) is absolutely stable if and only if there exist positive constants $\theta_1, \ldots, \theta_n$ such that

$$\sum_{j=1}^{n} p_{sj}\theta_j < 0, \quad s = 1, \ldots, n. \tag{2.4}$$

It should be noted that the existence of a positive solution for (2.4) is equivalent to the fulfillment of the Sevast'yanov–Kotelyanskij conditions [11]:

$$(-1)^k \det{(p_{sj})}_{s,j=1}^{k} > 0, \quad k = 1, \ldots, n. \tag{2.5}$$

On the other hand, it is known [11] that if inequalities (2.5) are valid, then one may choose numbers $\lambda_s$ for the function $W = \sum\limits_{s,j=1}^{n} \lambda_s p_{sj} y_s y_j$ to be negative definite. Thus, system (2.1) is absolutely stable if and only if for this system there exists Lyapunov's function in the form (2.2), satisfying the assumptions of the Lyapunov asymptotic stability theorem.

The main goal of the present paper is to extend the above results to the system of the form

$$\dot{x}_s = a_s f_s(x_s) + \sum_{j=1}^{k_s} b_{sj} f_1^{\alpha_{s1}^{(j)}}(x_1) \dots f_n^{\alpha_{sn}^{(j)}}(x_n), \quad s = 1, \dots, n. \tag{2.6}$$

Here $a_s$ and $b_{sj}$ are constant coefficients, functions $f_j(x_j)$ possess the same properties as in system (2.1), $\alpha_{si}^{(j)}$ are nonnegative rationals with odd denominators.

## 3 Construction of Lyapunov's Functions

Let the inequalities $\sum\limits_{i=1}^{n} \alpha_{si}^{(j)} > 0,\ j = 1, \dots, k_s,\ s = 1, \dots, n$, be valid. The fulfillment of this assumption provides the existence of the zero solution for system (2.6). Furthermore, we suppose that coefficients $a_s$ and $b_{sj}$ satisfy the conditions

$$a_s < 0, \quad b_{sj} > 0. \tag{3.1}$$

**Definition 3.1** We call (2.6) *absolutely stable* if the zero solution of this system is asymptotically stable for any admissible functions $f_j(x_j)$.

Let us investigate the conditions of absolute stability for (2.6). Along with equations (2.6), consider the system of inequalities

$$a_s\theta_s + \sum_{j=1}^{k_s} b_{sj} \theta_1^{\alpha_{s1}^{(j)}} \dots \theta_n^{\alpha_{sn}^{(j)}} < 0, \quad s = 1, \dots, n. \tag{3.2}$$

**Definition 3.2** We shall say that (2.6) satisfies the *Martynyuk–Obolenskij condition* [9] (MO-condition) if for any $\delta > 0$ there exists solution $\theta_1, \dots, \theta_n$ of system (3.2) such that $0 < \theta_s < \delta,\ s = 1, \dots, n$.

Let us note that in the case, where $f_j(x_j)$ are nondecreasing functions, (2.6) is the Wazewskij's system [5, 11]. In the paper [9] the autonomous Wazewskij's systems were treated. The criterion for the asymptotic stability in the positive cone of the zero solution was obtained. Using this result, we get that the MO-condition is a necessary one for the absolute stability for system (2.6).

To prove sufficiency of this condition for the absolute stability, construct Lyapunov's function in the form

$$\widetilde{V} = \sum_{s=1}^{n} \lambda_s \int_0^{x_s} f_s^{\mu_s}(\tau) d\tau. \tag{3.3}$$

Here $\lambda_s > 0$ are constant coefficients, $\mu_s > 0$ are rationals with odd numerators and denominators.

Function $\widetilde{V}$ is positive definite. By differentiating $\widetilde{V}$ with respect to (2.6), one arrives to

$$\frac{d\widetilde{V}}{dt}\Big|_{(2.6)} = \sum_{s=1}^{n} \lambda_s a_s f_s^{\mu_s+1}(x_s) + \sum_{s=1}^{n} \lambda_s f_s^{\mu_s}(x_s) \sum_{j=1}^{k_s} b_{sj} f_1^{\alpha_{s1}^{(j)}}(x_1) \dots f_n^{\alpha_{sn}^{(j)}}(x_n).$$

Our aim is to determine the conditions under which one may choose coefficients $\lambda_s$ and exponents $\mu_s$ for the function

$$\widetilde{W} = \sum_{s=1}^{n} \lambda_s a_s y_s^{\mu_s+1} + \sum_{s=1}^{n} \lambda_s y_s^{\mu_s} \sum_{j=1}^{k_s} b_{sj} y_1^{\alpha_{s1}^{(j)}} \dots y_n^{\alpha_{sn}^{(j)}} \tag{3.4}$$

to be negative definite.

Let us denote $h_s = 1/(\mu_s + 1)$, $s = 1, \dots, n$. By the use of generally-homogeneous functions properties [12], we get that $\widetilde{W}$ may be negative definite only in the case, where the inequalities

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j)} h_i \geq 0, \quad j = 1, \dots, k_s, \quad s = 1, \dots, n, \tag{3.5}$$

are valid.

*Remark 3.1* Let positive rationals $h_1, \dots, h_n$ with odd numerators and even denominators satisfy conditions (3.5). Suppose that for some values of indices $j$ and $s$ corresponding inequalities in (3.5) are strict. In this case one may construct, instead of (3.4), new function $\widehat{W}$ by setting $b_{sj} = 0$ for all such $j$ and $s$. If there exist positive coefficients $\lambda_1, \dots, \lambda_n$ for which $\widehat{W}$ is negative definite, then for these values of $\lambda_1, \dots, \lambda_n$ function $\widetilde{W}$ possesses the same property [12].

*Remark 3.2* If there exist positive rationals $h_1, \dots, h_n$ for which all the inequalities in (3.5) are strict, i.e.

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j)} h_i > 0, \quad j = 1, \dots, k_s, \quad s = 1, \dots, n, \tag{3.6}$$

then for corresponding values of $\mu_s$ and for any admissible values of $a_s$, $b_{sj}$ and $\lambda_s$ function $\widetilde{W}$ will be negative definite.

## 4 Auxiliary Results

In this section we will investigate the relationship between the fulfillment of the MO-condition and the existence of positive solutions for systems (3.5) and (3.6).

**Lemma 4.1** *If there exists a positive solution for (3.6), then system (2.6) satisfy the MO-condition.*

*Proof* Let for positive constants $h_1, \dots, h_n$ inequalities (3.6) be valid. Then the numbers $\theta_s = \tau^{h_s}$, $s = 1, \dots, n$, satisfy conditions (3.2) for sufficiently small values of $\tau > 0$.

**Lemma 4.2** *Let (2.6) satisfies the MO-condition. Then for any set of indices* $j_1, \dots, j_n$ $(j_s \in \{1, \dots, k_s\},\ s = 1, \dots, n)$ *there exists a positive solution for the system*

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j_s)} h_i \geq 0, \quad s = 1, \dots, n. \tag{4.1}$$

*Proof* For specified values of indices $j_1, \dots, j_n$ consider the inequalities

$$a_s\theta_s + b_{sj_s}\theta_1^{\alpha_{s1}^{(j_s)}} \dots \theta_n^{\alpha_{sn}^{(j_s)}} < 0, \quad s = 1, \dots, n. \tag{4.2}$$

If for (2.6) the MO-condition is fulfilled, then in any neighborhood of the state $(\theta_1, \dots, \theta_n)^* = (0, \dots, 0)^*$ there exists a positive vector $(\tilde\theta_1, \dots, \tilde\theta_n)^*$ satisfying (4.2). Along with (4.1), we investigate the system

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j_s)} h_i = c_s, \quad s = 1, \dots, n, \tag{4.3}$$

where $c_s$ are nonnegative constants. Let us apply the Gaussian elimination procedure [4] to linear system (4.3). This procedure generates equivalent systems of equations with the coefficients changed in the similar way as the orders of $\theta_1, \dots, \theta_n$ under the successive elimination of these variables from (4.2).

Since in any neighborhood of the state $(\theta_1, \dots, \theta_n)^* = (0, \dots, 0)^*$ there exists a positive solution for inequalities (4.2), one may assume, without loss of generality, that application of the Gaussian elimination procedure to the system (4.3) yields the system

$$\sum_{i=s}^{n} \beta_{si} h_i = \tilde c_s, \quad s = 1, \dots, r,$$
$$\sum_{i=r+1}^{n} \beta_{si} h_i = \tilde c_s, \quad s = r+1, \dots, n.$$

Here $1 \leq r < n$; $\beta_{ss} < 0$ for $s = 1, \dots, r$; $\beta_{si} \geq 0$ for $s = 1, \dots, r,\ i = s+1, \dots, n$ and for every $s = 1, \dots, r$ there exists $i_s > s$ such that $\beta_{si_s} > 0$; $\beta_{si} \geq 0$ for $s, i = r+1, \dots, n$; $\tilde c_s \geq 0$ for $s = 1, \dots, n$.

Let $\tilde h_{r+1}, \dots, \tilde h_n$ be arbitrary positive numbers,

$$\tilde h_s = -\frac{1}{\beta_{ss}} \sum_{i=s+1}^{n} \beta_{si} \tilde h_i, \quad s = 1, \dots, r.$$

For these values of $\tilde h_1, \dots, \tilde h_n$ we get $c_s = \tilde c_s = 0$ for $s = 1, \dots, r$ and $c_s = \tilde c_s \geq 0$ for $s = r+1, \dots, n$. Hence, the vector $(\tilde h_1, \dots, \tilde h_n)^*$ is a positive solution for (4.1).

**Lemma 4.3** *If (2.6) satisfies the MO-condition, then there exists a positive solution for system (3.5).*

*Proof* Consider the system

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j)} h_i = c_s^{(j)}, \quad j = 1, \dots, k_s, \quad s = 1, \dots, n, \tag{4.4}$$

where $c_s^{(j)}$ are nonnegative constants. This system may be splitted into $n$ subsystems. Let us apply to (4.4) the modified Gaussian elimination procedure. On the $s$-th step of this procedure we keep in the $s$-th subsystem only equations with negative coefficients of $h_s$. Each of the equations kept is used for the elimination of $h_s$ from the $(s+1)$-th, etc., and $n$-th subsystems. This results in a new set of subsystems with (generally) the number of equations other than that in the initial system.

According to Lemma 4.2, for any set of indices $j_1, \dots, j_n$, system (4.1) possesses a positive solution. Hence, one may assume, without loss of generality, that after the application of the above procedure we obtain the system

$$\sum_{i=s}^{n} \beta_{si}^{(j)} h_i = \tilde{c}_s^{(j)}, \quad j = 1, \dots, q_s, \quad s = 1, \dots, r.$$

Here $1 \leq r < n$, $\tilde{c}_s^{(j)} \geq 0$, $\beta_{ss}^{(j)} < 0$, $\beta_{si}^{(j)} \geq 0$ for $i = s+1, \dots, n$, and for any $j$ and $s$ there exists $i_{sj} > s$ such that $\beta_{si_{sj}}^{(j)} > 0$, $j = 1, \dots, q_s$, $s = 1, \dots, n$.

It can be easily shown that if $\tilde{h}_{r+1}, \dots, \tilde{h}_n$ are arbitrary positive numbers and

$$\tilde{h}_s = -\max_{j=1,\dots,q_s} \frac{1}{\beta_{ss}^{(j)}} \sum_{i=s+1}^{n} \beta_{si}^{(j)} \tilde{h}_i, \quad s = 1, \dots, r,$$

then the vector $(\tilde{h}_1, \dots, \tilde{h}_n)^*$ is a positive solution for (3.5).

*Remark 4.1* Since systems of inequalities (3.5), (3.6) are linear, the investigation of conditions for the existence of positive solutions for them is a much more simple problem than for nonlinear system (3.2).

*Remark 4.2* The proof of Lemma 4.3 contains a constructive algorithm for finding a positive solution for (3.5). Moreover, let us note that using this algorithm one may choose $\tilde{h}_{r+1}, \dots, \tilde{h}_n$ for the numbers $\mu_s = 1/\tilde{h}_s - 1$, $s = 1, \dots, n$, to be positive rationals with odd numerators and denominators.

## 5 Criterion for Absolute Stability

We will find now the necessary and sufficient conditions for system (2.6) to be absolutely stable.

**Theorem 5.1** *System (2.6) is absolutely stable if and only if for this system there exists Lyapunov's function in the form (3.3) satisfying the assumptions of the Lyapunov asymptotic stability theorem.*

*Proof* *Sufficiency* Suppose that there exists Lyapunov's function in the form (3.3) with negative definite derivative with respect to (2.6). Then for arbitrary admissible functions $f_j(x_j)$ the zero solution of the system considered is asymptotically stable. Hence, (2.6) is absolutely stable.

*Necessity* If (2.6) is absolutely stable, then for this system the MO-condition is fulfilled [9]. According to Lemma 4.3, there exist positive rationals $\mu_1, \dots, \mu_n$ with odd numerators and denominators such that for the numbers $\tilde{h}_s = 1/(\mu_s + 1)$, $s = 1, \dots, n$, inequalities (3.5) are valid. We shall take these values of $\mu_1, \dots, \mu_n$ as exponents in Lyapunov's function (3.3). Let us show that one may choose positive constants $\lambda_1, \dots, \lambda_n$ for the function (3.4) to be negative definite.

Consider a positive solution $(\tilde{\theta}_1, \dots, \tilde{\theta}_n)^*$ of (3.2). Let us denote $z_s = y_s/\tilde{\theta}_s$, $\gamma_s = \tilde{\theta}_s^{\mu_s}\lambda_s$, $s = 1, \dots, n$. Then function $\widetilde{W}$ takes the form

$$\widetilde{W} = \sum_{s=1}^{n} \gamma_s \hat{a}_s z_s^{\mu_s+1} + \sum_{s=1}^{n} \gamma_s z_s^{\mu_s} \sum_{j=1}^{k_s} \hat{b}_{sj} z_1^{\alpha_{s1}^{(j)}} \dots z_n^{\alpha_{sn}^{(j)}}.$$

Here $\hat{a}_s = a_s\tilde{\theta}_s$, $\hat{b}_{sj} = b_{sj}\tilde{\theta}_1^{\alpha_{s1}^{(j)}} \dots \tilde{\theta}_n^{\alpha_{sn}^{(j)}}$, and $\hat{a}_s + \sum_{j=1}^{k_s} \hat{b}_{sj} < 0$, $s = 1, \dots, n$.

We will assume, without loss of generality (v. Remark 3.1), that for the numbers $\tilde{h}_1, \dots, \tilde{h}_n$, corresponding to chosen values of $\mu_1, \dots, \mu_n$, all the inequalities in (3.5) turn to equalities.

Let $\mathbf{D} = \{d_{si}\}_{s,i=1}^{n}$, where

$$d_{ss} = \hat{a}_s + \sum_{j=1}^{k_s} \hat{b}_{sj}\alpha_{ss}^{(j)}, \quad d_{si} = \sum_{j=1}^{k_i} \hat{b}_{ij}\alpha_{is}^{(j)} \quad \text{for} \quad s \neq i.$$

Matrix $\mathbf{D}$ is the Metzler matrix [5, 11].

It can be easily shown that the inequality $\mathbf{D}^*\mathbf{h} < \mathbf{0}$ possesses the solution $\tilde{\mathbf{h}} = (\tilde{h}_1, \dots, \tilde{h}_n)^*$. Hence [11], there exists a positive solution $\tilde{\gamma} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_n)^*$ for the inequality $\mathbf{D}\gamma < \mathbf{0}$.

By the use of the Jensen inequality [6], one gets that for such values of coefficients $\tilde{\gamma}_1, \dots, \tilde{\gamma}_n$ the relations

$$\begin{aligned}\widetilde{W} &\leq \sum_{s=1}^{n} \tilde{\gamma}_s \hat{a}_s z_s^{\mu_s+1} + \sum_{s=1}^{n} \tilde{\gamma}_s \sum_{j=1}^{k_s} \hat{b}_{sj} \left( \frac{\mu_s}{\mu_s+1} z_s^{\mu_s+1} + \sum_{i=1}^{n} \frac{\alpha_{si}^{(j)}}{\mu_i+1} z_i^{\mu_i+1} \right) \\ &= \sum_{s=1}^{n} \frac{\tilde{\gamma}_s\mu_s}{\mu_s+1} z_s^{\mu_s+1} \left( \hat{a}_s + \sum_{j=1}^{k_s} \hat{b}_{sj} \right) + \sum_{s=1}^{n} \frac{z_s^{\mu_s+1}}{\mu_s+1} \sum_{i=1}^{n} d_{si}\tilde{\gamma}_i \leq -c \sum_{s=1}^{n} \frac{z_s^{\mu_s+1}}{\mu_s+1}\end{aligned}$$

are valid. Here $c$ is a positive constant. This completes the proof.

*Corollary 5.1* System (2.6) is absolutely stable if and only if it satisfies the MO-condition.

*Remark 5.1* Corollary 5.1 is similar to the criterion for the asymptotic stability obtained in [9] for autonomous Wazewskij's systems. However, in comparison with this criterion, in the present paper it has been proved that only the MO-condition is a sufficient one for the asymptotic stability of the zero solution of (2.6), i.e. the other assumptions from [9] (concerning the uniqueness of solutions, isolation of the equilibrium position at the origin and nondecreasement of the functions $f_j(x_j)$) are redundant.

*Corollary 5.2* Let system (2.6) satisfy the MO-condition. If there exist parameters $\mu_1, \dots, \mu_n$ such that for corresponding values of $h_1, \dots, h_n$ all the inequalities in (3.5) turn to equalities, and $\int_0^{x_s} f_s^{\mu_s}(\tau)\, d\tau \to +\infty$ as $|x_s| \to \infty$, $s = 1, \dots, n$, then the zero solution of (2.6) is globally asymptotically stable.

It should be noted that Remark 3.1 makes possible, in some cases, to simplify the MO-condition verifying. Let positive rationals $h_1,\dots,h_n$ satisfy system (3.5). Then one may assume that in (2.6) $b_{sj}=0$ if for these values of $s$ and $j$ the corresponding inequality in (3.5) is strict. By the use of Remark 3.1, we get that the fulfillment of the MO-condition for such reduced system is equivalent to that one for the initial system (2.6).

*Example 5.1* Let system (2.6) be of the form

$$\begin{aligned}\dot x_1 &= a_1 f_1(x_1) + b_{11} f_2^{2/3}(x_2) f_3^{1/3}(x_3),\\ \dot x_2 &= a_2 f_2(x_2) + b_{21} f_1(x_1) + b_{22} f_3^{3}(x_3),\\ \dot x_3 &= a_3 f_3(x_3) + b_{31} f_1(x_1) + b_{32} f_2^{3}(x_2).\end{aligned} \tag{5.1}$$

Consider inequalities (3.5) corresponding to (5.1). We get

$$\begin{aligned}-h_1 + \frac{2}{3}h_2 + \frac{1}{3}h_3 &\ge 0,\\ -h_2 + h_1 &\ge 0,\\ -h_2 + 3h_3 &\ge 0,\\ -h_3 + h_1 &\ge 0,\\ -h_3 + 3h_2 &\ge 0.\end{aligned} \tag{5.2}$$

By the use of the procedure of successive elimination of variables, it can be easily shown that if positive constants $h_1$, $h_2$, $h_3$ satisfy (5.2), then $h_1=h_2=h_3$. For such values of variables the third and the fifth inequalities in (5.2) are strict, and the others turn to equalities. Hence, for (5.1) the MO-condition is fulfilled if and only if this condition is fulfilled for the reduced system

$$\begin{aligned}\dot x_1 &= a_1 f_1(x_1) + b_{11} f_2^{2/3}(x_2) f_3^{1/3}(x_3),\\ \dot x_2 &= a_2 f_2(x_2) + b_{21} f_1(x_1),\\ \dot x_3 &= a_3 f_3(x_3) + b_{31} f_1(x_1).\end{aligned} \tag{5.3}$$

Verifying the MO-condition for (5.3), we obtain that for (5.1) to be absolutely stable it is necessary and sufficient that the inequality $a_1^3a_2^2a_3 > b_{11}^3b_{21}^2b_{31}$ holds.

*Remark 5.2* In a similar way, the criterion for absolute stability can be obtained for the case when the inequalities $b_{sj}>0$ in (3.1) are replaced by the connecting coefficients $b_{sj}$ and a basis $\omega_1,\dots,\omega_n$: $b_{sj}\omega_s\omega_1^{\alpha_{s1}^{(j)}}\dots\omega_n^{\alpha_{sn}^{(j)}}>0$ for $j=1,\dots,k_s$, $s=1,\dots,n$ [10]. Here every constant $\omega_1,\dots,\omega_n$ takes either the value $+1$ or $-1$.

*Example 5.2* Consider the system

$$\begin{aligned}\dot x_1 &= a_1\, f_1(x_1) + b_1\, f_n^{\alpha_1}(x_n),\\ \dot x_i &= a_i\, f_i(x_i) + b_i\, f_{i-1}^{\alpha_i}(x_{i-1}), \quad i=2,\dots,n-1,\\ \dot x_n &= a_n\, f_n(x_n) + b_n\, f_1^{\nu_1}(x_1)\dots f_{n-1}^{\nu_{n-1}}(x_{n-1}),\end{aligned} \tag{5.4}$$

where $a_j$ and $b_j$ are constant coefficients, $a_j<0$, $b_j\neq 0$, functions $f_j(x_j)$ possess the same properties as in (2.6), $\alpha_i$ and $\nu_i$ are rationals with odd denominators, $\alpha_i>0$, $\nu_i\ge 0$, $\nu_1+\dots+\nu_{n-1}>0$, $j=1,\dots,n$, $i=1,\dots,n-1$.

By the use of Remark 3.2, we obtain that under the condition

$$\alpha_1\nu_1 + \alpha_1\alpha_2\nu_2 + \dots + \alpha_1 \dots \alpha_{n-1}\, \nu_{n-1} > 1$$

system (5.4) is absolutely stable for any admissible values of coefficients $a_j$ and $b_j$.

Next, consider the case, where

$$\alpha_1\nu_1 + \alpha_1\alpha_2\nu_2 + \dots + \alpha_1 \dots \alpha_{n-1}\, \nu_{n-1} = 1. \tag{5.5}$$

It can be easily shown that for the existence of the basis $\omega_1, \dots, \omega_n$ such that

$$b_1\omega_1\omega_n^{\alpha_1} > 0, \quad b_i\omega_i\omega_{i-1}^{\alpha_i} > 0, \quad i = 2, \dots, n-1, \quad b_n\omega_n\omega_1^{\nu_1} \dots \omega_{n-1}^{\nu_{n-1}} > 0$$

it is necessary and sufficient that the inequality

$$b_1^{\xi_1} b_2^{\xi_2} \dots b_{n-1}^{\xi_{n-1}}\, b_n > 0 \tag{5.6}$$

is fulfilled. Here $\xi_i = \nu_i + \alpha_{i+1}\xi_{i+1}$, $i = 1, \dots, n-2$, $\xi_{n-1} = \nu_{n-1}$.

Making the substitution $z_j = \omega_j x_j$, $j = 1, \dots, n$, in (5.4) and applying Corollary 5.1 for the system obtained, we get that under conditions (5.5) and (5.6) system (5.4) is absolutely stable if and only if the inequality

$$\left(-\frac{b_1}{a_1}\right)^{\xi_1} \left(-\frac{b_2}{a_2}\right)^{\xi_2} \dots \left(-\frac{b_{n-1}}{a_{n-1}}\right)^{\xi_{n-1}} \left(-\frac{b_n}{a_n}\right) < 1$$

is valid.

## 6 Stability Analysis for Large Scale Systems in Critical Cases

Let us now show that the results obtained in the present paper may be used to refine some of the known conditions of stability for large scale systems.

Consider the system

$$\dot{x}_s = F_s(x_s) + \sum_{j=1}^{k_s} Q_{sj}(t, x), \quad s = 1, \dots, n, \tag{6.1}$$

where $x_s \in R^{m_s}$, $x = (x_1^*, \dots, x_n^*)^*$; the elements of the vectors $F_s(x_s)$ are continuously differentiable homogeneous functions of the orders $\sigma_s > 1$; the vector functions $Q_{sj}(t, x)$ are continuous for $t \geq 0$, $\|x\| < H$ ($H$ is a positive constant, $\|\cdot\|$ is the Euclidean norm of a vector) and satisfy the inequalities

$$\|Q_{sj}(t, x)\| \leq c_{sj}\, \|x_1\|^{\beta_{s1}^{(j)}} \dots \|x_n\|^{\beta_{sn}^{(j)}}, \quad c_{sj} > 0, \quad \beta_{si}^{(j)} \geq 0.$$

We will assume that (6.1) has the zero solution.

This system describes the dynamics of a complex system composed of $n$ interconnected subsystems [1, 5]. Here $x_s$ are state vectors, the functions $F_s(x_s)$ define the interior connections of subsystems while the functions $Q_{sj}(t, x)$ characterize the interaction between the subsystems.

Suppose that the zero solutions of isolated systems

$$\dot{x}_s = F_s(x_s), \quad s = 1, \ldots, n, \tag{6.2}$$

are asymptotically stable. We will look for the conditions under which the zero solution of (6.1) is also asymptotically stable.

In the papers [2, 7], approaches to studying stability for (6.1) are suggested. For this purpose, methods of the Lyapunov vector [7] or scalar [2] functions are used.

It is known [13] that for isolated systems (6.2) there exist Lyapunov's functions $V_s(x_s)$, which are continuously differentiable positive homogeneous functions of orders $\gamma_s - \sigma_s + 1$, $s = 1, \ldots, n$. Here $\gamma_s$ are arbitrary numbers such that $\gamma_s > \sigma_s$. These functions satisfy the inequalities

$$a_{1s}\|x_s\|^{\gamma_s-\sigma_s+1} \le V_s(x_s) \le a_{2s}\|x_s\|^{\gamma_s-\sigma_s+1},$$
$$\left\|\frac{\partial V_s}{\partial x_s}\right\| \le a_{3s}\|x_s\|^{\gamma_s-\sigma_s}, \qquad \left(\frac{\partial V_s}{\partial x_s}\right)^* F_s \le -a_{4s}\|x_s\|^{\gamma_s}$$

for all $x_s \in R^{m_s}$, where $a_{1s}, a_{2s}, a_{3s}, a_{4s}$ are positive constants. By differentiating $V_s(x_s)$ with respect to (6.1), one can deduce that the estimations

$$\left.\frac{dV_s}{dt}\right|_{(6.1)} \le -a_{4s}\|x_s\|^{\gamma_s} + a_{3s}\|x_s\|^{\gamma_s-\sigma_s} \sum_{j=1}^{k_s} c_{sj}\, \|x_1\|^{\beta_{s1}^{(j)}} \ldots \|x_n\|^{\beta_{sn}^{(j)}}$$

are valid for $t \ge 0$, $\|x\| < H$, $s = 1, \ldots, n$.

According to approach suggested in [7], the Lyapunov vector function is chosen in the form $V = (V_1, \ldots, V_n)^*$. Using this function, we construct the comparison system

$$\dot{u}_s = -\tilde{a}_s u_s^{\frac{\gamma_s}{\gamma_s-\sigma_s+1}} + u_s^{\frac{\gamma_s-\sigma_s}{\gamma_s-\sigma_s+1}} \sum_{j=1}^{k_s} \tilde{b}_{sj} u_1^{\frac{\beta_{s1}^{(j)}}{\gamma_1-\sigma_1+1}} \ldots u_n^{\frac{\beta_{sn}^{(j)}}{\gamma_n-\sigma_n+1}}, \quad s = 1, \ldots, n, \tag{6.3}$$

for (6.1). Here

$$\tilde{a}_s = a_{4s} a_{2s}^{-\frac{\gamma_s}{\gamma_s-\sigma_s+1}}, \qquad \tilde{b}_{sj} = a_{3s} c_{sj} a_{1s}^{-\frac{\gamma_s-\sigma_s}{\gamma_s-\sigma_s+1}} a_{11}^{-\frac{\beta_{s1}^{(j)}}{\gamma_1-\sigma_1+1}} \ldots a_{1n}^{-\frac{\beta_{sn}^{(j)}}{\gamma_n-\sigma_n+1}}.$$

System (6.3) is the Wazewskij one [5]. By analogy with the proof of Theorem 5.1, it can be easily shown that for the zero solution of (6.3) to be asymptotically stable it is sufficient that the corresponding MO-condition is fulfillment. Hence, if in any neighborhood of the state $(\theta_1, \ldots, \theta_n)^* = (0, \ldots, 0)^*$ there exists a positive solution for the system of inequalities

$$-\tilde{a}_s\theta_s + \sum_{j=1}^{k_s} \tilde{b}_{sj} \theta_1^{\beta_{s1}^{(j)}/\sigma_1} \ldots \theta_n^{\beta_{sn}^{(j)}/\sigma_n} < 0, \quad s = 1, \ldots, n, \tag{6.4}$$

then the zero solution of (6.1) is asymptotically stable.

Let us now show that the condition obtained for the asymptotic stability of the zero solution may be weakened by using the results of the previous section. Consider the Lyapunov scalar function

$$\widetilde{V} = \sum_{s=1}^{n} \lambda_s V_s,$$

where $\lambda_s$ are positive coefficients, $V_s$ are positive homogeneous functions of orders $\gamma_s - \sigma_s + 1$ corresponding to isolated subsystems (6.2). For all $t \geq 0$ and $\|x\| < H$ we get

$$\frac{d\widetilde{V}}{dt}\Big|_{(6.1)} \leq -\sum_{s=1}^{n} \lambda_s a_{4s} \|x_s\|^{\gamma_s} + \sum_{s=1}^{n} \lambda_s a_{3s} \|x_s\|^{\gamma_s - \sigma_s} \sum_{j=1}^{k_s} c_{sj} \, \|x_1\|^{\beta_{s1}^{(j)}} \dots \|x_n\|^{\beta_{sn}^{(j)}}.$$

Hence, to prove the asymptotic stability of the zero solution for (6.1) it is sufficient to show that one may choose positive coefficients $\lambda_1, \dots, \lambda_n$ for the function

$$\widetilde{W} = -\sum_{s=1}^{n} \lambda_s a_{4s} y_s^{\mu_s + 1} + \sum_{s=1}^{n} \lambda_s a_{3s} y_s^{\mu_s} \sum_{j=1}^{k_s} c_{sj} \, y_1^{\beta_{s1}^{(j)}/\sigma_1} \dots y_n^{\beta_{sn}^{(j)}/\sigma_n}$$

to be negative definite. Here $\mu_s = \gamma_s/\sigma_s - 1$.

Suppose that parameters $\gamma_1, \dots, \gamma_n$ satisfy the inequalities

$$-\frac{\sigma_s}{\gamma_s} + \sum_{i=1}^{n} \frac{\beta_{si}^{(j)}}{\gamma_i} \geq 0, \qquad j = 1, \dots, k_s, \quad s = 1, \dots, n. \tag{6.5}$$

In this case, by analogy with the proof of Theorem 5.1, we get that the following theorem is valid.

**Theorem 6.1** *If in any neighborhood of the state* $(\theta_1, \dots, \theta_n)^* = (0, \dots, 0)^*$ *there exists a positive solution for the system of inequalities*

$$-a_{4s}\theta_s + a_{3s} \sum_{j=1}^{k_s} c_{sj} \, \theta_1^{\beta_{s1}^{(j)}/\sigma_1} \dots \theta_n^{\beta_{sn}^{(j)}/\sigma_n} < 0, \quad s = 1, \dots, n, \tag{6.6}$$

*then the zero solution of (6.1) is asymptotically stable.*

*Remark 6.1* Coefficients $\tilde{a}_s$, $\tilde{b}_{sj}$, $a_{3s}$, $a_{4s}$ in (6.4) and (6.6) depend, in general, on the chosen values of $\gamma_1, \dots, \gamma_n$.

*Remark 6.2* For given values of $\gamma_1, \dots, \gamma_n$, Theorem 6.1 provides one with more precise conditions of asymptotic stability in comparison with those obtained via the Lyapunov vector function. However, in (6.6), compared with (6.4), it is assumed that $\gamma_1, \dots, \gamma_n$ satisfy additional restrictions (6.5).

*Example 6.1* Let the system

$$\begin{aligned} \dot{x}_1 &= -\rho^2 x_1 - x_1^2 x_2, \\ \dot{x}_2 &= 100 x_1^3 - 100\rho^2 x_2 + a x_3^9, \\ \dot{x}_3 &= -x_3^9 + b\rho^3 \end{aligned} \tag{6.7}$$

be given. Here $\rho = \sqrt{x_1^2 + x_2^2}$, $a$ and $b$ are constants. System (6.7) describes the interaction of two isolated subsystems

$$\begin{aligned} \dot{x}_1 &= -\rho^2 x_1 - x_1^2 x_2, \\ \dot{x}_2 &= 100 x_1^3 - 100\rho^2 x_2, \end{aligned} \qquad \dot{x}_3 = -x_3^9.$$

Consider the functions

$$V_1 = 50\,x_1^2 + \frac{1}{2}\,x_2^2, \quad V_2 = x_3^\gamma,$$

where $\gamma > 1$ is a rational with even numerator and odd denominator. Differentiating these functions with respect to (6.7), one gets

$$\begin{aligned} \dot V_1 &= -100\rho^4 + a x_2 x_3^9, \\ \dot V_2 &= -\gamma x_3^{\gamma+8} + \gamma b \rho^3 x_3^{\gamma-1}. \end{aligned}$$

Hence, the differential inequalities

$$\begin{aligned} \dot V_1 &\le -\frac{1}{25} V_1^2 + |a|(2V_1)^{1/2} V_2^{9/\gamma}, \\ \dot V_2 &\le -\gamma V_2^{1+8/\gamma} + \gamma |b| (2V_1)^{3/2} V_2^{1-1/\gamma} \end{aligned} \tag{6.8}$$

are valid. Verifying the MO-condition for the comparison system corresponding to (6.8), it can be shown that if the inequality

$$|ab| < 1/100 \tag{6.9}$$

holds, then the zero solution of (6.7) is asymptotically stable.

This condition for the asymptotic stability of the zero solution may be weakened by the use of Theorem 6.1. Taking into account the additional restriction (6.5), we get $\gamma = 4$. Hence, system of inequalities (6.6) for (6.7) is of the form

$$\begin{aligned} -100\theta_1 + |a|\theta_2 < 0, \\ -4\theta_2 + 4|b|\theta_1 < 0. \end{aligned} \tag{6.10}$$

According to Theorem 6.1, the zero solution of (6.7) is asymptotically stable if in any neighborhood of the state $(\theta_1, \theta_2)^* = (0,0)^*$ there exists a positive solution for system (6.10). Eliminating variables $\theta_1$, $\theta_2$ from (6.10), we obtain new sufficient condition for asymptotic stability: $|ab| < 100$, which is more precise than (6.9).

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# On The Dependence of Fixed Point Sets of Pseudo-Contractive Multifunctions. Application to Differential Inclusions

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**Abstract:** A weakened notion of multivalued contraction mapping is introduced. Some fixed point results relying on this notion are presented. The associated fixed points sets are shown to enjoy a Lipschitzian behaviour with respect to the graphs of the multifunctions. Applications are given to the dependence of solutions of differential inclusions of the form $\dot{x}(t) \in R(t, x(t))$ on initial values or on the right-hand sides or on parameters.

**Keywords:** *Differential inclusions; fixed points; iterative methods; sensitivity; stability.*

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## 1 Introduction

Studies about the behavior of fixed points are far less abundant than existence results (let us mention [23 – 25, 29]). However such studies are important; for instance they can be used to describe the dependence of solutions to differential inclusions or partial differential equations on some parameters or on boundary data.

Since in general the fixed points are not unique, one is led to use concepts of convergence of sets. Such concepts abound (see [1, 6, 8, 21, 31] for instance). But since we are interested in quantitative estimates and not only in qualitative results, we are led to use a recent variant of the Pompeiu–Hausdorff distance or hemi-metric (see [2, 3, 7, 20, 26, 27, 31]). In these developments, briefly recalled below, the stringent convergence relying on the Pompeiu–Hausdorff hemi-metric is replaced by a convergence relativized

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to bounded sets (the so-called bounded hemi-convergence or bounded convergence or Attouch–Wets convergence). This more realistic approach is justified by a number of facts and results: in finite dimensional spaces, bounded convergence coincides with the classical Painlevé–Kuratowski convergence; convergence in norms of operators is equivalent to bounded convergence of their graphs (see [20, 34]). Moreover the continuity of several usual operations can be obtained for this type of convergence (see [7, 31] for example).

Although the dependence of fixed point sets is a subject which is not limited to the case of Lipschitzian multifunctions, we only consider this case here ; for other approaches see for instance [4]. The reason lies in the fact that in the Lipschitzian case one disposes of an estimate about the distance of a given base point to the fixed point sets ([19, 13, 32]); in [32] a first step towards the study of the dependence of the fixed point sets was made. Here we complete this study in a more symmetric and systematic way (Section 2). Moreover we show how these results can be illustrated by an application to differential inclusions (Section 3). In particular we reveal a connection with a famous result of Filippov (see [5, 14, 15, 35, 36, 37]): while we just give a new method to get the existence theorem, our perturbation results seem to be new.

In the sequel $(X,d)$ is a metric space. Given $x_0 \in X,\ r>0,$ we denote by $B(x_0,r)$ (resp. $U(x_0,r)$) the closed (resp. open) ball with center $x_0$ and radius $r$. Given a base point $x_0 \in X$ and given subsets $C,\ D \subset X$, we set, for $r>0$,

$$e_r(C,D) = e(C \cap U(x_0,r), D)$$

and

$$h_r(C,D) = \max\{e_r(C,D), e_r(D,C)\}$$

with $e(\varnothing, D) = 0$,

$$e(C,D) = \sup_{x\in C} d(x,D) \quad \text{if} \quad D \neq \varnothing, \quad e(C,\varnothing) = +\infty \quad \text{if} \quad C \neq \varnothing,$$
$$d(x,D) = \inf_{z\in D} d(x,z) \quad \text{with the convention} \quad \inf_{\varnothing} = +\infty.$$

In the preceding definition we used open balls $U(x_0,r)$ for technical reasons: many proofs are simpler when using these balls. The reader would easily convince himself that the use of closed balls would not produce any significant change in the results of this paper. Since it is the use of the whole family $(h_r)_{r>0}$ which is important, it is clear that the choice of balls is unessential. We shall also use the classical Pompeiu–Hausdorff metric

$$h(C,D) = \max\{e(C,D),\, e(D,C)\}.$$

A multifunction $F$ from a set $X$ to a set $Y$ is considered as a subset of $X \times Y$. For all $x \in X,\ F(x)$ denotes the (possibly empty) set of $y \in Y$ such that $(x,y) \in F$. The multifunction $F^{-1} \subset Y \times X$ is defined by $F^{-1} = \{(y,x) \colon (x,y) \in F\}$. A fixed point of a multifunction $F\colon X \rightrightarrows X$ is an element $x \in X$ such that $x \in F(x)$. We denote by $\Phi_F$ the set of fixed points of $F$. Given $\theta \in R_+,$ we say that a multifunction $F\colon\ X \rightrightarrows X$ is *pseudo-$\theta$-Lipschitzian* with respect to the subset $U \subset X$ whenever for all $x, x' \in U$

$$e(F(x) \cap U,\, F(x')) \leq \theta d(x,x').$$

It is said to be *pseudo-$\theta$-contractive* with respect to $U$ if it is pseudo-$\theta$-Lipschitzian with respect to $U$ for some $\theta \in [0,1)$. The multifunction $F \subset X \times X$ is said to be $\theta$-Lipschitzian whenever

$$h(F(x), F(x')) \leq \theta d(x, x')$$

for all $x, x' \in X$; it is said to be *$\theta$-contractive* if it is $\theta$-Lipschitzian with $\theta \in (0,1)$. The limit inferior of a sequence $(C_n)$ of closed subsets of a metric space $(X,d)$ is the set of those $x \in X$ such that $\lim_{n\to\infty} d(x, C_n) = 0$. Equivalently it is the set of $x \in X$ for which there exists a sequence $(x_n)$ converging to $x$ such that $x_n \in C_n$ eventually.

In the sequel a product $X \times Y$ of metric spaces will be endowed with the box distance given by

$$d((x,y),(x',y')) = \max\{d(x,x'), d(y,y')\}.$$

*Remark 1.1* Let $F \subset X \times X$ be a multifunction such that for some $x_0 \in X$, $r > 0$ and $\theta \in R_+$ the multifunction $F_r(x) = F(x) \cap U(x_0, r)$ is $\theta$-Lipschitzian on $U(x_0,r)$. Then $F$ is pseudo-$\theta$-Lipschitzian with respect to $U(x_0,r)$ since for any $x$, $x' \in X$

$$e(F(x) \cap U(x_0,r), F(x')) \leq e(F_r(x), F_r(x')).$$

Nevertheless the converse is false as shown by the following simple example. Let $\theta \in [0,4)$, $r = 1$ and let $f\colon R \to R$ be the $\theta$-Lipschitzian function defined by $f(x) = \theta|x| + 1 - \theta/2$. Then $f$ is pseudo-$\theta$-Lipschitzian with respect to $U(0,1)$ since $f$ is $\theta$-Lipschitzian and $\{f(x)\} \cap U(0,1)$ is either empty or equal to $\{f(x)\}$. Now

$$e\Big(\{f(0)\} \cap U(0,1), \Big\{f\Big(\frac{1}{2}\Big)\Big\} \cap U(0,1)\Big) = +\infty$$

since $\{f(0)\} \cap U(0,1) \neq \varnothing$ and $\{f(\frac{1}{2})\} \cap U(0,1) = \varnothing$.

## 2 Fixed Points of Pseudo-Contractive Multifunctions

In this section we consider the behavior of the fixed point set $\Phi_F = \{x \in X\colon x \in F(x)\}$ of a pseudo-contractive multifunction $F\colon X \rightrightarrows X$ as $F$ is perturbed. The existence of fixed points for such multifunctions is well known (see [13, 19 (Lemma 1, p.31), 32 (Proposition 2.5)]). In many cases they are obtained by iterative techniques of one sort or another (see [22, 28]). Such results extend widely the well known result of S.B. Nadler in [28]. Since the estimates of the existence result are crucial for what follows, we give a proof for the convenience of the reader.

**Proposition 2.1** *Let $(X,d)$ be a complete metric space and let $F\colon X \rightrightarrows X$ be a multifunction with closed nonempty values which is assumed to be pseudo-$\theta$-contractive with respect to some ball $U(x_0,r)$ with $r > (1-\theta)^{-1} d(x_0, F(x_0))$. Then for any $\beta > d(x_0, F(x_0))$ such that $\beta(1-\theta)^{-1} \leq r$, there exists a sequence $(x_n)_{n\in N} \subset U(x_0,r)$ such that*

$$x_{n+1} \in F(x_n) \quad \text{and} \quad d(x_{n+1}, x_n) \leq \theta^n \beta \quad \text{for all} \quad n \in N. \tag{1}$$

*Moreover, for any sequence $(x_n)_{n\in N}$ of $U(x_0,r)$ satisfying (1), its limit $\overline{x}$ belongs to $U(x_0,r)$ and is a fixed point of $F$ yielding that $\Phi_F$ is nonempty and*

$$d(x_0, \Phi_F) \leq (1-\theta)^{-1} d(x_0, F(x_0)).$$

*Proof* Let $\beta > d(x_0, F(x_0))$ be such that $\beta(1-\theta)^{-1} \leq r$. Since $d(x_0, F(x_0)) < \beta$, we can find $x_1 \in F(x_0)$ with $d(x_0, x_1) < \beta$. As $\beta < r$, we get $x_1 \in U(x_0, r)$. Assuming that $\theta = 0$, we get $x_1 \in F(x_0) \cap U(x_0, r) \subset F(x_1)$ thus, setting $x_n = x_1$ for all $n \geq 1$, we are done. Assume now that $\theta \neq 0$ and suppose we have constructed a finite sequence $x_1, \ldots, x_n$ in $U(x_0, r)$ with $x_i \in F(x_{i-1})$ and $d(x_i, x_{i-1}) < \theta^{i-1}\beta$ for $i = 1, \ldots, n$. As $x_n \in F(x_{n-1}) \cap U(x_0, r)$ we have

$$d(x_n, F(x_n)) \leq e_r(F(x_{n-1}), F(x_n)) \leq \theta d(x_{n-1}, x_n) < \theta^n \beta,$$

so that we can find $x_{n+1} \in F(x_n)$ with $d(x_n, x_{n+1}) \leq \theta^n \beta$. Then

$$d(x_{n+1}, x_0) \leq \sum_{p=1}^{n+1} d(x_p, x_{p-1}) \leq \sum_{p=1}^{n+1} \theta^{p-1}\beta \leq (1-\theta)^{-1}\beta,$$

hence $x_{n+1} \in U(x_0, r)$. The sequence $(x_n)$ is thus well defined and is a Cauchy sequence in $B(x_0, r)$. Let $\overline{x}$ be its limit. We have

$$d(\overline{x}, x_0) \leq \lim_{n\to\infty} d(x_{n+1}, x_0) \leq (1-\theta)^{-1}\beta,$$

so that $\overline{x} \in B(x_0, r)$ and

$$d(\overline{x}, F(\overline{x})) \leq d(\overline{x}, x_n) + d(x_n, F(\overline{x})) \leq d(\overline{x}, x_n) + \theta d(x_{n-1}, \overline{x})$$

since $x_n \in F(x_{n-1}) \cap U(x_0, r)$. Hence $d(\overline{x}, F(\overline{x})) = 0$ and $\overline{x} \in F(\overline{x})$. Thus $\overline{x} \in \Phi_F$ and $d(x_0, \overline{x}) \leq (1-\theta)^{-1}\beta$. Letting $\beta$ decrease to $d(x_0, F(x_0))$, we get the announced result.

The Nadler's fixed point theorem ([28, Theorem 5]) follows readily from Proposition 2.1. Observe that no boundedness assumption on the values is required.

**Corollary 2.1** ([28, Theorem 5]) *Let $(X, d)$ be a complete metric space and let $F\colon X \rightrightarrows X$ be a multifunction with nonempty graph and closed values which is assumed to be $\theta$-contractive. Then $F$ admits a fixed point.*

*Proof* Let us choose $x_0 \in X$ such that $F(x_0)$ is nonempty and $r \geq 0$ such that $r > (1-\theta)^{-1}d(x_0, F(x_0))$. We can apply Proposition 2.1 which proves the corollary.

Proposition 2.1 is of local character. If one is interested in a global result, one can use the following proposition.

**Proposition 2.2** *Let $(X, d)$ be a complete metric space and let $F\colon X \rightrightarrows X$ be a multifunction with closed nonempty values. Assume that for some $x_0 \in X$ and for all $r > 0$ the multifunction $F$ is $\theta_r$-contractive on $U(x_0, r)$ for some $\theta_r \in [0, 1)$. Then $F$ has a fixed point in $X$ if and only if*

$$\inf_{r>0} \inf_{x \in U(x_0,r)} \frac{(1-\theta_r)d(x, x_0) + d(x, F(x))}{r(1-\theta_r)} < 1.$$

*Proof* Taking $x \in \Phi_F$ and $r > d(x_0, x)$ we see that the condition is necessary. Let us show it is sufficient. By assumption, we can choose $r > 0$ and $x_1 \in U(x_0, r)$ such that

$$(1-\theta_r)d(x_1, x_0) + d(x_1, F(x_1)) < r(1-\theta_r),$$

yielding

$$d(x_1, F(x_1)) < (r - d(x_1, x_0))(1 - \theta_r)$$

and $F$ is pseudo-$\theta_r$-contractive with respect to $U(x_1, r - d(x_1, x_0))$. Thus we can apply Proposition 2.1 with $x_1$ and $r - d(x_1, x_0)$ instead of $x_0$ and $r$ respectively, from which we get $\Phi_F \neq \varnothing$.

It is of interest to study the sensitivity of the fixed points sets $\Phi_F$ when $F$ varies in the power set $2^{(X\times X)}$ (hyperspace of subsets of $X \times X$) endowed with some topology. We turn now to this question. It is natural to choose $(x_0, x_0)$ as a base point in $X \times X$.

**Proposition 2.3** *Let $(X, d)$ be a complete metric space. Let $F\colon X \rightrightarrows X$ be a multifunction with closed nonempty values which is assumed to be pseudo-$\theta$-contractive with respect to $U(x_0, r)$. Then for any $s \in (0, r)$ and for any multifunction $G\colon X \rightrightarrows X$ satisfying*

$$e_s(G, F) < (1 - \theta)(1 + \theta)^{-1}(r - s)$$

*one has*

$$e_s(\Phi_G, \Phi_F) \leq (1 - \theta)^{-1}(1 + \theta)e_s(G, F) < r - s.$$

*Proof* Let $t > e_s(G, F)$ be such that $t < (1 - \theta)(1 + \theta)^{-1}(r - s)$ and let $y \in \Phi_G \cap U(x_0, s)$ (if there is no such $y$, there is nothing to prove). Since $(y, y) \in G \cap U((x_0, x_0), s)$, there exists $(w, z) \in F$ with $d(y, w) < t$ and $d(y, z) < t$. Due to the choice of $t$, we have $t < r - s$ , thus $w, z \in U(x_0, r)$, whence we get

$$d(y, F(y)) \leq d(y, z) + d(z, F(y)) \leq d(y, z) + e_r(F(w), F(y)) \leq t(1 + \theta) < (1 - \theta)(r - s).$$

As $F$ is pseudo-$\theta$-contractive with respect to $U(y, r - s)$, it follows from the preceding estimate and from Proposition 2.1 that

$$d(y, \Phi_F) \leq (1 - \theta)^{-1}d(y, F(y)) \leq (1 - \theta)^{-1}(1 + \theta)t,$$

hence the result, letting $t$ decrease to $e_s(G, F)$.

If instead of an estimate on the excess of the graph of $G$ to the graph of $F$ one assumes a uniform estimate on the images, one gets a more precise result about the fixed points sets.

**Proposition 2.4** *Let $(X, d)$ be a complete metric space. Let $F\colon X \rightrightarrows X$ be a multifunction with closed nonempty values which is pseudo-$\theta$-contractive with respect to $U(x_0, r)$. Then for any $s \in (0, r)$ and for any multifunction $G\colon X \rightrightarrows X$ satisfying*

$$e_s(G(x), F(x)) < (1 - \theta)(r - s) \quad \textit{for each} \quad x \in U(x_0, s)$$

*one has*

$$e_s(\Phi_G, \Phi_F) \leq (1 - \theta)^{-1} \sup_{x \in U(x_0, r)} e_s(G(x), F(x)) \leq (1 - \theta)^{-1}e_s(G, F).$$

*Proof* Let $y \in \Phi_G \cap U(x_0, s)$ and let $t > e_s(G(y), F(y))$ be such that $t < (1-\theta)(r-s)$. Since $y \in G(y) \cap U(x_0, s)$ we can pick $z \in F(y)$ such that $d(y, z) < t$. Since $F$ is pseudo-$\theta$-contractive with respect to $U(y, r - s)$, it follows from Proposition 2.1 that

$$d(y, \Phi_F) \leq (1 - \theta)^{-1}d(y, F(y)) \leq (1 - \theta)^{-1}t,$$

hence the result, taking the infimum over $t > e_s(G, F)$.

By using Proposition 2.3, we obtain the following result on the dependence of the fixed point set $\Phi_F$ when the graph of $F$ is perturbed. Some care is needed in order to obtain a significant result since the conclusion of Proposition 2.3 does not prevent from emptiness of $\Phi_G$. Here we adopt a parametric formulation which is equivalent to the preceding framework (take for $\Lambda$ the set of graphs of multifunctions which are pseudo-$\theta$ contractive provided with the topology associated with $(h_r)_{r\geq 0}$).

**Theorem 2.1** *Let $(X, d)$ be a complete metric space and let $\Lambda$ be a topological space. Let $F \subset \Lambda \times X \times X$ be a multifunction such that for some $x_0 \in X$, $\theta \in [0, 1)$, $r > 0$ and for all $\lambda \in \Lambda$ the multifunction $F_\lambda = F(\lambda, \cdot) \subset X \times X$ is nonempty, closed-valued and pseudo-$\theta$-contractive with respect to $U(x_0, r)$. Assume $r > r_0 = (1-\theta)^{-1} d(x_0, F(\lambda_0, x_0))$ for some $\lambda_0 \in \Lambda$ and*

$$\lim_{\lambda \to \lambda_0} h_r(F(\lambda, \cdot), F(\lambda_0, \cdot)) = 0. \tag{2}$$

*Then for any $s \in (r_0, r)$ there exists a neighborhood $\Lambda_0$ of $\lambda_0$ such that for all $\lambda \in \Lambda_0$ one has $\Phi_{F(\lambda,\cdot)} \cap U(x_0, s) \neq \varnothing$ and*

$$h_s(\Phi_{F(\lambda,\cdot)}, \Phi_{F(\lambda_0,\cdot)}) \leq (1-\theta)^{-1}(1+\theta) h_s(F(\lambda, \cdot), F(\lambda_0, \cdot)). \tag{3}$$

*Proof* Let $t \in (r_0, s)$ be such that $s - t < r - s$. Proposition 2.1 ensures that $\Phi_{F(\lambda_0,\cdot)} \cap U(x_0, t)$ is nonempty. Let $\Lambda_0$ be a neighborhood of $\lambda_0$ such that for $\lambda \in \Lambda_0$ and $\delta_\lambda = h_s(F(\lambda_0, \cdot), F(\lambda, \cdot))$ one has

$$\delta_\lambda < (1-\theta)(1+\theta)^{-1}(s-t) < (1-\theta)(1+\theta)^{-1}(r-s).$$

We obtain from Proposition 2.3 applied to $G = F(\lambda_0, \cdot)$, $F(\lambda, \cdot)$ that

$$e_s(\Phi_{F(\lambda_0,\cdot)}, \Phi_{F(\lambda,\cdot)}) \leq \delta_\lambda (1-\theta)^{-1}(1+\theta) \leq s - t. \tag{4}$$

Since $\Phi_{F(\lambda_0,\cdot)} \cap U(x_0, t)$ is nonempty, we get

$$\Phi_{F(\lambda,\cdot)} \cap U(x_0, s) \neq \varnothing$$

for all $\lambda \in \Lambda_0$. Interchanging the role played by $F(\lambda_0, \cdot)$ and $F(\lambda, \cdot)$ and applying again Proposition 2.3 we obtain that for all $\lambda \in \Lambda_0$

$$e_s(\Phi_{F(\lambda,\cdot)}, \Phi_{F(\lambda_0,\cdot)}) \leq (1-\theta)^{-1}(1+\theta) e_s(F(\lambda, \cdot), F(\lambda_0, \cdot)),$$

which combined with (4) gives estimate (3).

For multivalued contractions in the usual sense we have the following result.

**Corollary 2.2** *Let $(X, d)$ be a complete metric space and let $\Lambda$ be a topological space. Let $F \subset \Lambda \times X \times X$ be a multifunction such that for some $\theta \in [0, 1)$ and for all $\lambda \in \Lambda$, the multifunction $F(\lambda, \cdot) \subset X \times X$ is nonempty closed-valued and $\theta$-contractive. Assume that*

$$\lim_{\lambda \to \lambda_0} h(F(\lambda, \cdot), F(\lambda_0, \cdot)) = 0,$$

*or, more generally that for every $r > 0$ (2) holds. Let $x_0 \in X$. Then for all $t \geq 0$ there exists a neighborhood $\Lambda_0$ of $\lambda_0$ such that for all $\lambda \in \Lambda_0$ we have*

$$h_t(\Phi_{F(\lambda,\cdot)}, \Phi_{F(\lambda_0,\cdot)}) \leq (1-\theta)^{-1}(1+\theta)h(F(\lambda,\cdot), F(\lambda_0,\cdot)).$$

*Proof* Choose $r, s$ with $s \geq t$, $r > s > r_0 := (1-\theta)^{-1}d(x_0, F(\lambda_0, x_0))$ and apply Theorem 2.1, using the fact that $h_t \leq h_s$.

*Remark 2.1* The preceding corollary represents a slight sharpening of the result of Markin in [24]. Indeed this author proves that if $A$ is a closed bounded subset of a Hilbert space $H$ and if $(F_n)$ is a sequence of $\theta$-contractive multifunctions from $A$ to $A$ with nonempty closed convex values such that $\lim_{n\to\infty} h(F_n(x), F(x)) = 0$ uniformly on $A$ then $\lim_{n\to\infty} h(\Phi_F, \Phi_{F_n}) = 0$. Let us set $X = A$ and let us introduce $x_0 \in A$ and $s \geq 0$ such that $A \subset U(x_0, s)$ hence $U_X(x_0, s) = A$. Observe that $\lim_{n\to\infty} h(F_n(x), F(x)) = 0$ uniformly on $A$ implies $\lim_{n\to\infty} h(F, F_n) = 0$ since for all $(x, y) \in F$ one has $d((x, y), F_n) \leq d(y, F_n(x)) \leq h(F(x), F_n(x))$ and the same inequality exchanging $F$ and $F_n$ yielding

$$h(F, F_n) \leq \sup_{x\in A} h(F(x), F_n(x)).$$

From Corollary 2.2 we get

$$\lim_{n\to\infty} h_s(\Phi_F, \Phi_{F_n}) = 0$$

which turns to

$$\lim_{n\to\infty} h(\Phi_F, \Phi_{F_n}) = 0$$

since $\Phi_F \cap U(x_0, s) = \Phi_F$ and $\Phi_{F_n} \cap U(x_0, s) = \Phi_{F_n}$. Moreover we do not need any convexity assumption and we get a quantitative estimate. Corollary 2.2 also improves [23, Theorem 1].

At this stage a natural question arises: is a limit of pseudo-Lipschitzian multifunctions also pseudo-Lipschitzian? The answer is positive and easy for a sequence of $\theta$-Lipschitzian multifunctions which pointwise converges with respect to the Pompeiu–Hausdorff metric. The question is more delicate when graph convergence is used. In this setting, a partial answer is given in the following proposition.

**Proposition 2.5** *Let $(F_n) \subset X \times X$ be a sequence of multifunctions from a metric space $X$ into $X$. Assume that for some $x_0 \in X$, $r > 0$, $\theta \in R_+$ the multifunctions $F_n$ are pseudo-$\theta$-Lipschitzian with respect to $U(x_0, r)$. Let $F \subset X \times X$ be a closed multifunction such that*

$$\lim_{n\to\infty} e_{(2\theta+1)r}(F_n, F) = 0 \quad \textit{and} \quad F \subset \liminf_{n\to\infty} F_n.$$

*Then $F$ is pseudo-$\theta$-Lipschitzian with respect to $U(x_0, r)$ whenever one of the following conditions holds*

(a) *$F$ is closed and for any compact set $K \subset U(x_0, r)$ the set $F(K)$ is relatively compact;*

(b) *$X$ is a reflexive Banach space and $F$ is sequentially $s \times w$-closed, where $w$ and $s$ denote respectively the weak and the strong topology on $X$.*

*Proof* Let $(x,y) \in F \cap U((x_0,x_0),r)$ and let $x' \in U(x_0,r)$. Since $F \subset \liminf_{n\to\infty} F_n$ there exists a sequence $(x_n,y_n) \in F_n$ which converges to $(x,y)$. Given $\alpha > 0$ such that $d(x_0,y) < r-\alpha$, we may suppose $d(x_0,x_n) < r$ and $d(x_0,y_n) < r-\alpha$ for $n$ large enough, so that there exists $z_n' \in F_n(x')$ with $d(z_n',y_n) < \theta d(x',x_n) + \alpha < 2\theta r + \alpha$ for $n$ large enough. Thus $d(x_0,z_n') < (2\theta+1)r$ for such $n$'s and one has

$$(x',z_n') \in F_n \cap U((x_0,x_0),(2\theta+1)r);$$

thus there exists a sequence $(x_n',y_n') \subset F$ such that $(d(x_n',x'))$ and $(d(y_n',z_n'))$ converge to 0.

(a) Let $K := \{x_n'\} \cup \{x'\}$ and let $y' \in F(K)$ be the limit of a convergent subsequence of $(y_n')$. Since $F$ is closed, one has $y' \in F(x')$ and $d(y',y) \le \lim_n d(y_n',y_n) = \lim_n d(z_n',y_n) \le \lim_n \theta d(x',x_n) = \theta d(x',x)$, hence $d(y,F(x')) \le \theta d(x,x')$ and then

$$e(F(x)\cap U(x_0,r),F(x')) \le \theta d(x,x').$$

(b) As the sequence $(y_n')$ is bounded, there exists a subsequence which converges weakly to some $y' \in F(x')$ in view of our closedness assumption. Using the weak lower semicontinuity of the norm we also get $d(y,y') \le \theta d(x,x')$ so that

$$e(F(x)\cap U(x_0,r),F(x')) \le \theta d(x,x').$$

## 3 Applications to Differential Inclusions

In the sequel, we apply the stability result obtained in the previous section to the case where fixed points are solutions of differential inclusions in some functional spaces. Let us present the data of the problem. Let $E$ be Banach space whose closed unit ball is denoted by $B$ and let $T \subset R$ be an interval endowed with the Lebesgue measure, with end points $t_0 \in R$ and $t_1 \in R \cup \{+\infty\}$. Following [38], we say that a multifunction with nonempty values $G\colon T \rightrightarrows E$ is measurable if there exists a sequence $(g_n)_n$ of measurable mappings from $T$ into $E$ such that $g_n(t) \in G(t)$ a.e. on $T$ for all $n \in \mathbb{N}$ and $G(t) \subset \overline{\bigcup_{n\in\mathbb{N}}\{g_n(t)\}}$ a.e. on $T$.

We are interested in the behavior of the set of solutions to the differential inclusion

$$\dot{x}(t) \in R(t,x(t)) \tag{5}$$

where $R\colon T\times E \rightrightarrows E$ is a multifunction. In (5) the solution $x(\cdot)$ is assumed to belong to the space $X = W^{1,1}(T,E)$ of continuous functions $x\colon T \to E$ such that there exists $u \in \mathcal{L}^1(T,E)$ (the space of Bochner integrable functions from $T$ into $E$) such that

$$x(t) = x(t_0) + \int_{t_0}^{t} u(s)\,ds \quad \text{for all} \quad t \in T$$

and $x$ is said to be a solution if $u(t) \in R(t,x(t))$ a.e. $t \in T$. Given $x_0 \in X$ and $\xi \in B(x_0(t_0),\delta)$ with $\delta > 0$, we denote by $S_R(\xi)$ the set of solutions $x$ of (5) such that $x(t_0) = \xi$. We shall make a frequent use of Lemma 3.2 of [38] (see also [11] and Lemma 1.3 in [15] when $E$ is separable).

**Lemma 3.1** *Let $G\colon T \rightrightarrows E$ be a measurable multifunction with values in a Banach space. Let $v_0\colon T \to E$ and $\gamma\colon T \to ]0,+\infty[$ be measurable. Then there exists a measurable mapping $v\colon T \to E$ such that*

$$v(t) \in G(t) \quad \textit{and} \quad \|v(t) - v_0(t)\| \le d(v_0(t), G(t)) + \gamma(t) \quad \textit{almost everywhere on } T.$$

Let us give to $E$ the base point $\xi_0$ and to $X = W^{1,1}(T,E)$ the base point $x_0$ with $x_0(t) = x_0(t_0) + \int_{t_0}^{t} u_0(s)\,ds$ and let $k \in \mathcal{L}^1(T) = \mathcal{L}^1(T,R)$, $k(t) \ge 0$ a.e. In the sequel we shall endow the space $X = W^{1,1}(T,E)$ with the norm

$$\|x\|_X = \|x(t_0)\| + \int_T e^{-m(s)}\|u(s)\|\,ds, \tag{6}$$

and the associated distance $d_X$, where

$$m(s) = \int_{t_0}^{s} k(\tau)\,d\tau \tag{7}$$

and $u \in \mathcal{L}^1(T,X)$ is such that $x(s) = x(t_0) + \int_{t_0}^{s} u(\tau)\,d\tau$ on $T$. This norm is equivalent to the usual norm $x \longmapsto \|x(t_0)\| + \int_T \|u(s)\|\,ds$ since

$$e^{-m(t_1)}\left(\|x(t_0)\| + \int_T \|u(s)\|\,ds\right) \le \|x\|_X \le \|x(t_0)\| + \int_T \|u(s)\|\,ds.$$

Our approach relies on the following lemma which refines a trick in [9] (see also [18]).

**Lemma 3.2** *Given $k$ and $m$ as in (7), let $\theta(t) = 1 - e^{-m(t)}$. For $i = 1, 2$, let $x_i \in W^{1,1}(T,E)$, with $x_i(s) = x_i(t_0) + \int_{t_0}^{s} u_i(\tau)\,d\tau$, $u_i \in \mathcal{L}^1(T,E)$. Then for all $t \in T$*

$$\int_{t_0}^{t} e^{-m(s)}k(s)\|x_2(s) - x_1(s)\|\,ds \le \theta(t)\left(\|x_1(t_0) - x_2(t_0)\| + \int_{t_0}^{t} e^{-m(s)}\|u_2(s) - u_1(s)\|\,ds\right).$$

*Proof* Setting

$$I(t) = \int_{t_0}^{t} e^{-m(s)}k(s)\|x_2(s) - x_1(s)\|\,ds,$$

one has

$$\begin{aligned} I(t) &\le \int_{t_0}^{t} e^{-m(s)}k(s)\Big(\|x_1(t_0) - x_2(t_0)\| + \int_{t_0}^{s} \|u_2(\tau) - u_1(\tau)\|\,d\tau\Big)\,ds \\ &\le \theta(t)\|x_2(t_0) - x_1(t_0)\| + \int_{t_0}^{t}\left(\int_{\tau}^{t} e^{-m(s)}k(s)\,ds\right)\|u_2(\tau) - u_1(\tau)\|\,d\tau. \end{aligned}$$

Observing that

$$\int_{\tau}^{t} e^{-m(s)}k(s)\,ds = e^{-m(\tau)} - e^{-m(t)} \le (1 - e^{-m(t)})e^{-m(\tau)},$$

we get the result of the lemma.

### 3.1 A variant of the Filippov's theorem

Let us assume that the multifunction $R\colon T \times E \to E$ and the data $x_0 \in X,\ k \in \mathcal{L}^1(T)$, $\delta > 0$ satisfy the following assumptions in which $b(t) := re^{m(t)}$ for some $r > 0,\ m(t) := \int_{t_0}^{t} k(s)\,ds$ and $T_b := \cup_{t\in T}\{t\} \times B(x_0(t), b(t))$:

for each $(t,e) \in T_b$ the set $R(t,e)$ is closed, nonempty and $R(\cdot,e)$ is measurable; (8)

for a.e. $t \in T$ the multifunction $R(t,\cdot)$ is $k(t)$-Lipschitzian on $B(x_0(t), b(t))$; (9)

$$\gamma(\cdot) = d\left(u_0(\cdot), R(\cdot, x_0(\cdot))\right) \in \mathcal{L}^1(T) \tag{10}$$

We also suppose

$$e^{m(t_1)}\left(\delta + \int_{T} e^{-m(s)}\gamma(s)\,ds\right) \le r. \tag{11}$$

**Proposition 3.1** *Let $R\colon T\times E \rightrightarrows E$ be a multifunction with closed nonempty values satisfying assumptions (8) – (10) where relation (11) holds. Then for all $\xi \in B(x_0(t_0), \delta)$ the set $S_R(\xi)$ of solutions of*

$$\begin{aligned} \dot{x}(t) &\in R(t, x(t)) \quad \textit{a.e. on } T, \\ x(t_0) &= \xi, \end{aligned} \tag{12}$$

*is nonempty and one has $d(x_0, S_R(\xi)) = \inf\{\|x - x_0\|_X : x \in S_R(\xi)\} \le r$.*

Here the Lipschitz assumption (9) bears on a ball with a variable radius $b(t)$ instead of a ball with a fixed radius $\sup_{t\in T} b(t)$ as in [6, Theorem 10.4.1], [14], [37, Theorem 2.4.3]. Our conclusion involves an estimate of the $W^{1,1}$ norm of $x - x_0$ and, more importantly, we avoid the following assumption

(H) There exists $\sigma \in \mathcal{L}^1(T)$ such that $R(t,\xi) \subset \sigma(t)B$ for all $\xi \in E$ and $t \in T$

made in [18, 37] which excludes unbounded right hand sides. However, we do not get a point-wise estimate of the derivative of $x - x_0$ as in [14], [6, Theorem 10.4.1].

*Proof* Given $\xi \in B(x_0(t_0), \delta)$, let $F\colon X \rightrightarrows X$ be the multifunction defined by

$$y \in F(x) \Longleftrightarrow \begin{cases} y(s) = \xi + \int_{t_0}^{s} v(\tau)\,d\tau \quad \text{for all} \quad s \in T \\ v \in \mathcal{L}^1(T,E) \quad \text{is such that} \quad v(s) \in R(s, x(s)) \quad \text{a.e. on} \quad T. \end{cases}$$

It is clear that $x \in X$ is a solution of (12) if and only if $x$ is a fixed point of $F$. The existence of such a fixed point is ensured by Proposition 2.1 and the following lemma.

**Lemma 3.3** *Given $\xi \in B(x_0(t_0), \delta)$, and $r > 0$ as in relation (11), the multifunction $F\colon U(x_0, r) \rightrightarrows X$ defined above is closed, nonempty-valued, and is $\theta(t_1)$-contractive on $U(x_0, r)$ with $\theta(t) = 1 - e^{-m(t)}$. Moreover one has $d(x_0, F(x_0)) < r(1 - \theta(t_1))$.*

*Proof* Given $x \in U(x_0, r)$, with $x(t) = x(t_0) + \int_{t_0}^{t} u(\tau)\, d\tau$ for $t \in T$, we have

$$\begin{aligned}\|x(t) - x_0(t)\| &\le \|x(t_0) - x_0(t_0)\| + e^{m(t)} \int_{t_0}^{t} e^{-m(\tau)} \|u(\tau) - u_0(\tau)\|\, d\tau \\ &\le e^{m(t)} \|x - x_0\|_X < e^{m(t)} r = b(t),\end{aligned}$$

so that $x(t) \in B(x_0(t), b(t))$ for each $t \in T$. From Theorem 2.2 of [38], the multifunction $s \mapsto R(s, x(s))$ is measurable on $T$. Moreover, using (9) and (10), one sees that $d(u_0(s), R(s, x(s))) \le \bar{\gamma}(s)$ a.e. on $T$ with $\bar{\gamma}(s) = \gamma(s) + k(s)\|x(s) - x_0(s)\|$. As $\bar{\gamma} \in \mathcal{L}^1(T)$, Lemma 3.1 yields the existence of an integrable mapping $u\colon T \to E$ such that $u(s) \in R(s, x(s))$ a.e. on $T$, hence $F(x) \neq \varnothing$. It is easily shown that $F(x)$ is closed.

Now let us prove that $F$ is $\theta(t_1)$-contractive on $U(x_0, r)$ with $\theta(t) = 1 - e^{-m(t)}$. For $i = 1, 2$, let $x_i \in U(x_0, r)$ with $x_i(s) = x_i(t_0) + \int_{t_0}^{s} u_i(\tau)\, d\tau$, $u_i \in \mathcal{L}^1(T, E)$, let $y_1 \in F(x_1)$ with $y_1(s) = \xi + \int_{t_0}^{s} v_1(\tau)\, d\tau$, $v_1(\tau) \in R(\tau, x_1(\tau))$ a.e. on $T$, and let $\varepsilon > 0$. Given $\alpha \in \mathcal{L}^1(T)$ with $\alpha(\tau) > 0$ p.p. and $\int_T \alpha(\tau)\, d\tau < \varepsilon$, we have

$$d(v_1(s), R(s, x_2(s))) \le k(s)\|x_1(s) - x_2(s)\| \quad \text{a.e. on} \quad T.$$

Thus we derive from Lemma 3.1 the existence of a measurable mapping $v_2 : T \to E$ such that $v_2(s) \in R(t, x_2(s))$ a.e. on $T$ and

$$\|v_2(s) - v_1(s)\| \le k(s)\|x_2(s) - x_1(s)\| + \alpha(s) \quad \text{a.e. on} \quad T.$$

Setting $y_2(s) := \xi + \int_{t_0}^{s} v_2(\tau)\, d\tau$, we get $y_2 \in F(x_2)$ and using Lemma 3.2

$$\begin{aligned}\|y_2 - y_1\|_X &= \int_T e^{-m(s)} \|v_2(s) - v_1(s)\|\, ds \\ &\le \int_T e^{-m(s)} \big(k(s)\|x_2(s) - x_1(s)\| + \alpha(s)\big)\, ds \\ &\le \theta(t_1) \left( \|x_1(t_0) - x_2(t_0)\| + \int_T e^{-m(s)} \|u_2(s) - u_1(s)\|\, ds + \varepsilon \right) \\ &\le \theta(t_1)\big(\|x_1 - x_2\|_X + \varepsilon\big).\end{aligned}$$

Taking the infimum over $\varepsilon$, it follows that $d(y_1, F(x_2)) \le \theta(t_1)\|x_2 - x_1\|_X$. Taking the supremum on $y_1 \in F(x_1)$ one obtains that $e(F(x_1), F(x_2)) \le \theta(t)\|x_2 - x_1\|_X$ and then, interchanging $x_1$ and $x_2$

$$h(F(x_1), F(x_2)) \le \theta(t_1)\|x_2 - x_1\|_X. \tag{13}$$

Now let us estimate $d(x_0, F(x_0))$. Let $\varepsilon > 0$ be such that

$$\delta + \int_T e^{-m(s)}\gamma(s)ds + \varepsilon < re^{-m(t_1)},$$

and let $\alpha \in \mathcal{L}^1(T)$ be such that $\int_T \alpha < \varepsilon$ and $\alpha(t) > 0$ for each $t \in T$. Applying again Lemma 3.1, we get a measurable mapping $v\colon\ T \to E$ such that $v(s) \in R(s, x_0(s))$ a.e. on $T$ and

$$\|v(s) - u_0(s)\| \leq \gamma(s) + \alpha(s) \quad \text{a.e. on} \quad T.$$

Setting

$$y(t) := \xi + \int_{t_0}^{t} v(s)ds,$$

one has $y \in F(x_0)$ and

$$\begin{aligned}\|y - x_0\|_X &\leq \delta + \int_T e^{-m(s)}\left(\gamma(s) + \alpha(s)\right) ds \\ &\leq \delta + \int_T e^{-m(s)}\gamma(s)ds + \varepsilon < r(1 - \theta(t_1)).\end{aligned}$$

From this fact the quoted authors give a result on the dependence of the solution of (12) with respect to the initial value. In fact, it is possible to obtain a stronger result and to allow a variation of the right-hand side. Given a multifunction $R$ which satisfy (8) and (9) and given $\xi \in E$, again we denote by $S_R(\xi)$ the set of solutions of (12) and we endow $W^{1,1}(T, E)$ with the norm $\|\cdot\|_X$, providing it with the base point $x_0$.

**Proposition 3.2** *Let $R_1$, $R_2$ be multifunctions which satisfy (8) and (9). Let us set*

$$\rho(t) = \sup\{h(R_1(t, z), R_2(t, z)):\ z \in B(x_0(t), b(t))\}, \tag{14}$$

*let us assume that $\rho \in \mathcal{L}^1(T)$ and let $s \in (0, r)$, $\xi_1, \xi_2 \in E$ be such that*

$$e^{m(t_1)}\left(\|\xi_1 - \xi_2\| + \int_{t_0}^{t_1} e^{-m(t)}\rho(t)\, dt\right) < r - s. \tag{15}$$

*Then*

$$h_s(S_{R_1}(\xi_1), S_{R_2}(\xi_2)) \leq e^{m(t_1)}\left(\|\xi_1 - \xi_2\| + \int_{t_0}^{t_1} \rho(t)\, dt\right).$$

*Proof* It suffices to check the assumptions of Proposition 2.4 in which $F$, $G$ are replaced with the multifunctions $F_1$ and $F_2$ defined as in the proof of Proposition 3.1 with $\xi$ and $R$ replaced with $\xi_1$, $R_1$ and $\xi_2$, $R_2$ respectively. Now, given $u \in \mathcal{L}^1(T, E)$, $x \in B(x_0, s)$ and $y_1 \in F_1(x)$, as in the proof of Lemma 3.3 we have $x(t) \in B(x_0(t), b(t))$

for each $t \in T$. Taking $v \in \mathcal{L}^1(T,E)$ such that $v(t) \in R_1(t,x(t))$ for a.e. $t \in T$ and $y_1(t) = \xi_1 + \int_{t_0}^{t} v(s)ds$ we have $d(v(t), R_2(t,x(t))) \le \rho(t)$ and for any $\varepsilon > 0$ such that

$$e^{m(t_1)}\left(\|\xi_1 - \xi_2\| + \int_{t_0}^{t_1} e^{-m(t)}\rho(t)\,dt\right) + \varepsilon < r - s,$$

we can find $\alpha \in \mathcal{L}^1(T)$, $v_2 \in \mathcal{L}^1(T,E)$ such that $\alpha(t) > 0$ for each $t \in T$,

$$\int_T e^{-m(t)}\alpha(t)\,dt \le e^{-m(t_1)}\varepsilon$$

and

$$v_2(t) \in R_2(t,x(t)), \|v_2(t) - v(t)\| \le \rho(t) + \alpha(t) \quad \text{a.e.} \quad t \in T.$$

Then, for $y_2(t) = \xi_2 + \int_{t_0}^{t} v_2(s)\,ds$ we have

$$\begin{aligned}\|y_1 - y_2\|_X &= \|\xi_1 - \xi_2\| + \int_T e^{-m(s)}\|v_2(s) - v_1(s)\|\,ds \\ &\le \|\xi_1 - \xi_2\| + \int_{t_0}^{t_1} e^{-m(t)}(\rho(t) + \alpha(t))\,dt \le e^{-m(t_1)}(r - s).\end{aligned}$$

Thus $e_s(F_2(x), F_1(x)) < (1-\theta)(r-s)$ for each $x \in U(x_0,s)$, where $\theta = 1 - e^{-m(t_1)}$. Since $S_{R_i}(\xi_i)$ is the set of fixed points of $F_i$ for $i = 1,2$, the result follows from Proposition 2.4 and the fact that the roles of $F_1$ and $F_2$ are symmetric.

*Remark 3.1* This perturbation result can also be deduced from Proposition 3.1 by replacing $x_0$ and $r$ with $x_1 \in S_{R_1}(\xi_1) \cap B(x_0,s)$ and $r - s$ respectively. As in the proof of Lemma 3.3, we have $x_1(t) \in B(x_0(t), se^{m(t)})$ for each $t \in T$ and $B(x_1(t), (r - s)e^{m(t)}) \subset B(x_0(t), b(t))$ for $t \in T$; moreover we have $d(u_1(t), R_2(t,x_1(t))) \le \rho(t)$, where $x_1(t) = \xi_1 + \int_{t_0}^{t} u_1(s)ds$. Then assumptions (8), (9) and (10) are satisfied with $r$ and $x_0$ replaced respectively by $r - s$ and $x_1$. Thus, applying the quoted existence result, we get the conclusion of the proposition.

### 3.2 Stability of global solutions

We can also derive a stability result for the set $S_R(\xi)$ when the right-hand side $R$ and the initial value $\xi$ vary. Let $\Lambda$ be a topological space and let $R\colon \Lambda \times T \times E \rightrightarrows E$ be a family of multifunctions with closed nonempty values parametrized by $\lambda \in \Lambda$. Let us introduce the following assumptions

- ($a_\Lambda$) $R(\lambda, \cdot, x)$ is measurable for all $\lambda \in \Lambda$, $x \in E$;
- ($b_\Lambda$) $R(\lambda, t, \cdot)$ is $k(t)$-Lipschitz for all $\lambda \in \Lambda$ a.e. with $k \in \mathcal{L}^1(T)$;
- ($c_\Lambda$) there exists $\xi_0 \in E$ and $\lambda_0 \in \Lambda$ such that

$$d(0, R(\lambda_0, t, \xi_0)) \in \mathcal{L}^1(T).$$

**Theorem 3.1** *Let $\Lambda$ be a topological space and let $R\colon \Lambda\times T\times E \rightrightarrows E$ be a family of multifunctions with closed nonempty values parametrized by $\Lambda$. Assume that assumptions $(a_\Lambda)$, $(b_\Lambda)$ and $(c_\Lambda)$ are satisfied. For all $s>0$, $\lambda\in\Lambda$, let $\varepsilon_s(\cdot,\cdot)$ be a function defined on $T\times\Lambda$ such that for all $(t,\lambda)\in T\times\Lambda$*

$$\sup_{z\in B(\xi_0,s)} h(R(\lambda,t,z),R(\lambda_0,t,z))<\varepsilon_s(t,\lambda).$$

*Assume that for all $s>0$ and for all $\lambda\in\Lambda$*

$$\varepsilon_s(\cdot,\lambda)\in\mathcal{L}^1(T) \quad \textit{and} \quad \varepsilon_s(\cdot,\lambda) \quad \textit{converges to} \quad 0 \quad \textit{in} \quad \mathcal{L}^1(T) \quad \textit{as} \quad \lambda\to\lambda_0. \tag{16}$$

*Then there exist a constant $c>0$ such that for all $r$, $s$ with*

$$e^{m(t_1)}\int_{t_0}^{t_1} d(0,R(\lambda_0,t,\xi_0))\,dt<s<r$$

*and for all $\xi\in E$ one has*

$$S_{R(\lambda,\cdot,\cdot)}(\xi)\cap U(0,s)\neq\varnothing$$

*and there exist neighborhoods $\Lambda_0$ of $\lambda_0$ and $\Xi_0$ of $\xi_0$ such that for all $\lambda\in\Lambda_0$ and $\xi\in\Xi_0$ one has*

$$h_s\big(S_{R(\lambda_0,\cdot,\cdot)}(\xi_0),S_{R(\lambda,\cdot,\cdot)}(\xi)\big)\le c\left(\|\xi-\xi_0\|+\|\varepsilon_\sigma(\cdot,\lambda)\|_{L^1(T)}\right),$$

*with $\sigma=re^{m(t_1)}$.*

*Proof* Let $X=W^{1,1}(T,E)$ endowed with the norm (6). For all $(\lambda,\xi)\in\Lambda\times E$, one has

$$d(0,R(\lambda,t,\xi))\le\rho(t)$$

with $\rho(t)=d(0,R(\lambda_0,t,\xi_0))+\varepsilon_s(t,\lambda)+k(t)\|\xi-\xi_0\|\in\mathcal{L}^1(T)$. Thus we can define a multifunction $F\colon \Lambda\times E\times X\rightrightarrows X$ with nonempty closed values by $y\in F(\lambda,\xi,x)$ if and only if there exists $v\in\mathcal{L}^1(T,E)$ with $v(t)\in R(\lambda,t,x(t))$ a.e. and

$$y(t)=\xi+\int_{t_0}^{t} v(s)\,ds \quad \text{for all} \quad t\in T.$$

Relying on Lemma 3.3, we obtain that the multifunction $F(\lambda,\xi,\cdot)$ is $\theta$-Lipschitz with $\theta=1-e^{-m(t_1)}$. Moreover one easily checks that

$$d(0,F(\lambda_0,\xi_0,0)\le\int_{t_0}^{t_1} d(0,R(\lambda_0,t,\xi_0))\,dt.$$

Let us set

$$r_0=(1-\theta)d(0,F(\lambda_0,\xi_0,0)\le e^{-m(t_1)}\int_{t_0}^{t_1} d(0,R(\lambda_0,t,\xi_0))\,dt.$$

Let $(\lambda, \xi) \in \Lambda \times E$ and let $(y, x) \in F(\lambda_0, \xi_0, \cdot) \cap U((0,0), r)$. It follows that $x(t)$ remains in $B(\xi_0, \sigma)$ with $\sigma = e^{m(t_1)} r$. For almost all $t \in T$ we have $v(t) \in R(\lambda_0, t, x(t))$ then

$$d(v(t), R(\lambda, t, x(t))) < \varepsilon_\sigma(t, \lambda)$$

thus, from Lemma 3.1 there exists a measurable function $w\colon T \to E$ such that for all $t \in T$

$$w(t) \in R(\lambda, t, x(t)) \quad \text{and} \quad \|w(t) - v(t)\| \le \varepsilon_\sigma(t, \lambda) \quad \text{a.e.}$$

Observe that $w \in \mathcal{L}^1(T, E)$ and that $z \in F(\lambda, \xi, x)$ where $z(t) = \xi + \int_{t_0}^{t} w(s)\, ds$, yielding

$$d((y, x), F(\lambda, \xi, \cdot)) \le \|y - z\| \le \|\xi - \xi_0\| + \int_{t_0}^{t_1} e^{-m(t)} \varepsilon_\sigma(t, \lambda)\, dt.$$

Choosing $(0,0)$ as base point in $X \times X$ and interchanging $(\lambda, \xi)$ and $(\lambda_0, \xi_0)$ we get

$$h_r(F(\lambda_0, \xi_0, \cdot), F(\lambda, \xi, \cdot)) \le \|\xi - \xi_0\| + \int_{t_0}^{t_1} e^{-m(t)} \varepsilon_\sigma(t, \lambda)\, dt,$$

hence

$$\lim_{(\lambda,\xi)\to(\lambda_0,\xi_0)} h_r(F(\lambda_0, \xi_0, \cdot), F(\lambda, \xi, \cdot)) = 0$$

and the result follows, applying Theorem 2.1 and observing that

$$S_{R(\lambda,\cdot,\cdot)}(\xi) = \Phi_{F(\lambda,\xi,\cdot)}.$$

In the particular case when there is no explicit dependence on the parameter $\lambda$ we get a slight improvement of the result of [25] and [23].

**Corollary 3.1** *Let $R\colon T \times E \rightrightarrows E$ be a multifunction with closed nonempty values. Assume that assumptions (a), (b) and (c) are satisfied. Then there exists a constant $c \ge 0$ such that, for all $s > e^{m(t_1)} \int_{t_0}^{t_1} d(0, R(\lambda_0, t, \xi_0))\, dt$ there exist a neighborhood $\Xi_0$ of $\xi_0$ such that for all $\xi \in \Xi_0$ one has $S_R(\xi) \cap U(x_0, s) \ne \varnothing$ and*

$$h_s(S_R(\xi_0), S_R(\xi)) \le c\|\xi - \xi_0\|.$$

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# An Analysis of Clattering Impacts of a Falling Rod

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**Abstract:** This paper deals with both analytical and quantitative analysis of multiple impacts of a two-dimensional rod. The successions of clattering sequence of a rod dropping to the floor are modeled and analyzed to find out the impact responses as it collides with the ground. The model is described by a system of ordinary differential equations, with a classical contact problem. We conduct a comparison study of the cases where the effect of the gravity is neglected, versus the cases where the gravity is considered. This mathematical analysis can further provide useful information for durability study of the impact on mobile electronic device.

**Keywords:** *Two-dimensional rod; clattering impacts; analytical and quantitative analysis.*

**Mathematics Subject Classification (2000):** 70E18, 70F40, 70B10, 70G10.

## 1 Introduction

In a pioneering study of Goyal, *et al.* [1, 2], it was found that when a two-dimensional rod was dropped at a small angle to the ground, the second impact might be as large as twice of the initial impact under some assumptions. For its consequence in applications, their surprising result stirred some interest on this otherwise classical problem.

In the related literature, mathematical issues of one impact or first impact have been considered in a number of papers, see for example, [3 − 5] for rigid body collisions. Even in single-impact cases, the topic remains a focus of much discussion [6 − 8] as many theoretical contact dynamics issues involving frictions started to get resolved recently. Recent attention has been directed to detect and calculate the micro-collisions that occur

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in a short time interval, when the bodies are allowed to be flexible [9, 10]. These micro-collisions are consequence of the elastic oscillations during one impact, and occur in a relatively short period of time. During the sequence of micro-collisions, the location and posture of the bodies change very little.

The study of multiple-impacts, however, is only an emerging area. Goyal, *et al.* (1998) [1, 2] used transition matrix method to calculate the clattering sequence and its impacts. In a surprising way, they showed that when a two dimensional rod with uniform density is dropped to the ground at a very small angle, the second impact can be as large as twice of the first impact. Of course, this result is derived based on a number of assumption and simplifications such as full restitution and ignoring the effect of gravity, etc.

In this paper, we provide a study of the entire multiple-impact sequence of a two-dimensional rod with/without consideration of gravity, and using a general restitution coefficient. Our methodology allows us to consider a prototype problem for cell phone multi-impact dropping by several initial postures. We prove a number of assumptions required in Goyal's study are in fact valid, and interesting application is found in studying of clattering phenomenon of falling rigid bodies referred in [1, 2]. This model is a first step towards model study for the design and optimization of electronic components for mobile electronic product, future modeling considerations will involve flexible or multiple-body impacts.

We outline our article as follows. In Section 2, we state the basic rigid body dynamics equation. Section 3 includes impacts of analysis in absence of gravity. We give a comparison study to see the effect of gravity in Section 4. Discussion and conclusion are in Section 5.

## 2 Collision Equations for a Falling Rod

The model presented in this section is based on the linear impulse-momentum principle, the angular impulse-momentum principle for the rigid body, and some impact parameters that relates the pre- and post-impact variables, such as the coefficient of restitution, which is defined as the ratio of the post-impact relative normal velocity to the pre-impact relative normal velocity at the impact location. The limitation of the model is such that only sliding friction can occur. We assume that there is no sticking during the impact process. When sticking does occur, the situation becomes very complex. We defer discussion to Section 5.

We consider two rigid bodies having masses $m_1$ and $m_2$ respectively. We denote the initial velocities, before collision, in lower cases, and after collision, with capital letters. Collision equations are the following:

$$m_i\,(\overrightarrow{V}_i - \overrightarrow{v}_i) = \overrightarrow{P}_i, \quad i = 1, 2, \tag{1}$$

$$\overrightarrow{H}_i - \overrightarrow{h}_i = \overrightarrow{d}_i \times \overrightarrow{P}_i, \quad i = 1, 2, \tag{2}$$

where for body $i = 1, 2$, we denoted: $m_i$ is the mass, $\overrightarrow{v}_i$ and $\overrightarrow{V}_i$ are the pre- and post-impact velocity, $\overrightarrow{P}_i$ is the impulse, $\overrightarrow{h}_i$ and $\overrightarrow{H}_i$ are the pre- and post- impact angular momentum, $\overrightarrow{d}_i$ is the position vector from the mass center to the collision contact point.

We can write:

$$\overrightarrow{P}_i = P_n\,(\overrightarrow{n} + \mu\,\overrightarrow{t}), \tag{3}$$

where $\mu$ is the sliding friction coefficient, $\overrightarrow{n}$ and $\overrightarrow{t}$ are the normal and tangential unit vectors of the contact surface.

The post-impact relative velocity $\overrightarrow{V}_r$ and pre-impact relative velocity $\overrightarrow{v}_r$ at the collision contact point are related by:

$$\overrightarrow{V}_r \cdot \overrightarrow{n} = -e\,\overrightarrow{v}_r \cdot \overrightarrow{n}, \tag{4}$$

where $e$ is the coefficient of restitution.

Related to the center of mass, velocity and angular velocity, $\overrightarrow{V}_r$ and $\overrightarrow{v}_r$ can be written as:

$$\overrightarrow{V}_r = \overrightarrow{V}_1 + \overrightarrow{\Omega}_1 \times \overrightarrow{d}_1 - (\overrightarrow{V}_2 + \overrightarrow{\Omega}_2 \times \overrightarrow{d}_2), \tag{5}$$

$$\overrightarrow{v}_r = \overrightarrow{v}_1 + \overrightarrow{\omega}_1 \times \overrightarrow{d}_1 - (\overrightarrow{v}_2 + \overrightarrow{\omega}_2 \times \overrightarrow{d}_2), \tag{6}$$

where $\overrightarrow{\omega}_i$ and $\overrightarrow{\Omega}_i$ are the vectors of the pre- and post- impact angular velocities, respectively. For two-dimensional case, $\overrightarrow{\omega}_i = \omega_i \overrightarrow{k}$ and $\overrightarrow{\Omega}_i = \Omega_i \overrightarrow{k}$, where $\overrightarrow{k}$ is the unit vector normal to the two-dimensional work plane.

The equations (1)–(6) form a closed system. Solving the equations above, we derive (see [4] for example):

$$\begin{aligned}
V_{1n} &= v_{1n} + \frac{\overline{m}(1+e)q}{m_1} v_{rn}, & V_{1t} &= v_{1t} + \frac{\mu\overline{m}(1+e)q}{m_1} v_{rn},\\
V_{2n} &= v_{2n} - \frac{\overline{m}(1+e)q}{m_2} v_{rn}, & V_{2t} &= v_{2t} - \frac{\mu\overline{m}(1+e)q}{m_2} v_{rn},\\
\Omega_1 &= \omega_1 + \frac{\overline{m}(1+e)q(d_{1t}-\mu d_{1n})}{I_1} v_{rn}, & \Omega_2 &= \omega_2 - \frac{\overline{m}(1+e)q(d_{2t}-\mu d_{2n})}{I_2} v_{rn},
\end{aligned} \tag{7}$$

In the above solution, we denoted:

$$\overline{m} = \frac{m_1 m_2}{m_1 + m_2}, \qquad v_{rn} = (v_{2n} - d_{2t}\omega_2) - (v_{1n} + d_{1t}\omega_1),$$

$$q = \left[1 + \frac{\overline{m}d_{1t}^2}{I_1} + \frac{\overline{m}d_{2t}^2}{I_2} - \mu\left(\frac{\overline{m}d_{1t}d_{1n}}{I_1} + \frac{\overline{m}d_{2t}d_{2n}}{I_2}\right)\right]^{-1},$$

$$e = -\frac{V_{2n} - V_{1n}}{v_{2n} - v_{1n}}, \qquad \mu = \frac{P_t}{P_n}.$$

The formula for $e$ is called the Newton's Law of Restitution. The value $\mu$ is the relative ratio of impulses (tangential over normal), and it reflects the friction coefficient, as long as no sticking is happening during the impact. The terms $I_1$ and $I_2$ represent the mass moment of inertia with respect to center of mass, for the two rigid bodies. The subscripts "n" and "t" in the equations (7) stand for the normal and tangential components of the velocity vector and the position vectors respectively. The Figure 2.1 shows the position vectors from the mass center to the collision contact point, $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$, together with their normal and tangential components.

If a planar barrier collision occurs, for simplicity, let the moving body be the body 1 and the barrier be the body 2. All velocities related to body 2 are set to zero. The above approach is now applied to the multiple impacts of a falling rod, see Figure 2.2. In this

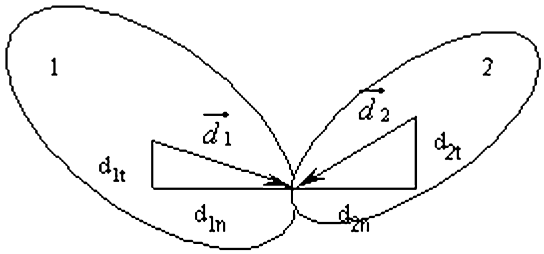


**Figure 2.1.** Rigid collision between two bodies.

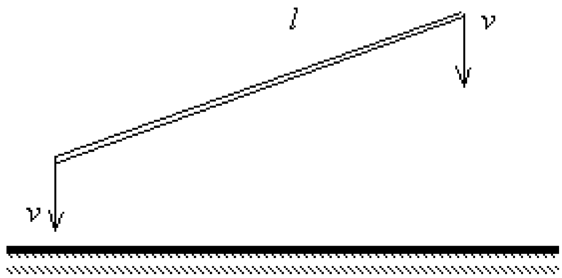


**Figure 2.2.** A rod colliding with the ground.

study, we consider a rod with uniform density. The mass of the rod $m_1 = 1$, the length of rod $l = 1$, the moment of inertia of the rod $I_1 = 1/12$, the friction coefficient $\mu = 0$, and the restitution coefficient $e \in [0, 1]$. The mass of the ground $m_2 = \infty$.

Hence, for our case, the equations (7) will reduce to

$$V_n = v_n + (1+e)qv_{rn}, \quad \Omega = \omega + 12(1+e)qd_t v_{rn}$$

with

$$q = \frac{1}{1+12d_t^2}, \quad v_{rn} = -(v_n + d_t\omega), \quad d_t - \mu d_n = d_t = -\frac{\cos\alpha}{2}.$$

We dropped the index {1, 2} in the previous text because we will refer just to the normal and angular velocity of the rod relative to the ground. The tangential velocity remains zero at all the time. Further, we will be interested in the angle at the moment of the impact, and a qualitative estimation of the impact. We will be having the initial velocity $v$ at the moment right before the first impact, as a unit.

## 3 The First Three Impacts, Disregarding the Effect of Gravity

We assume the impact sequence occurs without gravity. The clattering sequence terminates when the rod will no longer collide with the ground. The impact contact angles at the first three impacts are denoted as $\alpha$, $\beta$ and $\gamma$, as shown in Figure 3.1.

Following from the equations (7), for the first bounce, the quantities can be calculated as

$$V_n^I = \frac{e - 3\cos^2\alpha}{1+3\cos^2\alpha}v, \quad \Omega^I = -\frac{6(1+e)\cos\alpha}{1+3\cos^2\alpha}v, \tag{8}$$

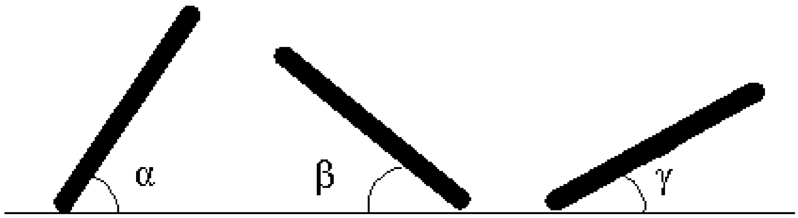


**Figure 3.1**. The succession of the first three impacts of the falling rod. The acute angle between the rod and the ground will be considered at all the times.

where $\alpha$ is the initial drop angle. Then the first impact is

$$P_n^I = V_n^I + v = \frac{1+e}{1+3\cos^2\alpha}\, v_{rn}^I, \tag{9}$$

where $v_{rn}^I = v$.

Let us consider $h^I$ the vertical height of the rod's center of mass at first impact, and $h^{II}$ the vertical height at the center of mass at second impact, without considering the gravity. These heights are related to the contact angles as

$$h^I = \frac{\sin\alpha}{2}\,, \qquad h^{II} = \frac{\sin\beta}{2}\,.$$

For the second impact, we have the equation

$$h^I + V_n^I T^I = h^{II},$$

where $T^I$ is the duration of airborne. It can be analytically written as

$$T^I = \frac{-(\alpha+\beta)}{\Omega^I}\,.$$

We can determine the angle $\beta$ numerically, for a given initial angle $\alpha$, using the height relation, so that

$$\sin\alpha + \frac{e - 3\cos^2\alpha}{3(1+e)\cos\alpha}\,(\alpha+\beta) = \sin\beta. \tag{10}$$

The new velocities for the second bounce are

$$V_n^{II} = V_n^I + \frac{1+e}{1+3\cos^2\beta}\, v_{rn}^{II}, \qquad \Omega^{II} = \Omega^I + \frac{6(1+e)\cos\beta}{1+3\cos^2\beta}\, v_{rn}^{II},$$

where

$$v_{rn}^{II} = -V_n^I - \frac{\cos\beta}{2}\,\Omega^I.$$

This gives the relation between the velocities of first two impacts

$$\begin{aligned} V_n^{II} &= \frac{-e+3\cos^2\beta}{1+3\cos^2\beta}\, V_n^I + \frac{-\frac{1+e}{2}\cos\beta}{1+3\cos^2\beta}\,\Omega^I, \\ \Omega^{II} &= \frac{-6(1+e)\cos\beta}{1+3\cos^2\beta}\, V_n^I + \frac{1-3e\cos^2\beta}{1+3\cos^2\beta}\,\Omega^I. \end{aligned} \tag{11}$$

Hence, by substituting equations (8) into equations (11), we derive

$$V_n^{II} = \frac{-(e-3\cos^2\alpha)(e-3\cos^2\beta)+3(1+e)^2\cos\alpha\cos\beta}{(1+3\cos^2\alpha)(1+3\cos^2\beta)}\,v,$$
$$\Omega^{II} = \frac{-6(1+e)(\cos\alpha+e\cos\beta)(1-3\cos\alpha\cos\beta)}{(1+3\cos^2\alpha)(1+3\cos^2,\beta)}\,v,$$
$$v_{rn}^{II} = v+(1+e)\,\frac{-1+3\cos\alpha\cos\beta}{(1+3\cos^2\alpha)}\,v.$$

The second angle, $\beta$, is numerically determined by solving equation (10) using Mathematica [11], and the impulse for second impact is

$$P_n^{II} = V_n^{II} - V_n^{I} = \frac{1+e}{1+3\cos^2\beta}\,v_{rn}^{II}.$$

The third impact can be calculated in a similar way. The height at the center of mass at the third impact will be $h^{III} = \frac{sin\gamma}{2}$, where $\gamma$ is the third impact angle between the rod and the floor.

At the third impact

$$h^{II} + V_n^{II}T^{II} = h^{III}, \tag{12}$$

where $T^{II} = \dfrac{\beta+\gamma}{\Omega^{II}}$ is the elapsed time between the second and the third impacts. Therefore, we obtain that

$$V_n^{II}T^{II} = \frac{-(e-3\cos^2\alpha)(e-3\cos^2\beta)+3(1+e)^2\cos\alpha\cos\beta}{6(1+e)(\cos\alpha+e\cos\beta)(-1+3\cos\alpha\cos\beta)}\,(\beta+\gamma). \tag{13}$$

Using the relations in equations (12) and (13), we obtain the following equation that relates $\alpha$, $\beta$ and $\gamma$ for a general value of the restitution coefficient $e$

$$\sin\beta + \frac{-(e-3\cos^2\alpha)(e-3\cos^2\beta)+3(1+e)^2\cos\alpha\cos\beta}{3(1+e)(\cos\alpha+e\cos\beta)(-1+3\cos\alpha\cos\beta)}\,(\beta+\gamma) = \sin\gamma. \tag{14}$$

Once the angle $\beta$ is obtained by equation (10) for any given $\alpha$, the angle $\gamma$ can be computed numerically by equation (14).

Now we find the center of mass' velocity and angular velocity, $V_n$ and $\Omega$, for the third bounce:

$$V_n^{III} = V_n^{II} + \frac{1+e}{1+3\cos^2\gamma}\,v_{rn}^{III}, \qquad \Omega^{III} = \Omega^{II} + \frac{6(1+e)\cos\gamma}{1+3\cos^2\gamma}\,v_{rn}^{III},$$

where

$$v_{rn}^{III} = -V_n^{II} + \frac{cos\gamma}{2}\,\Omega^{II}.$$

Hence,

$$\begin{aligned} V_n^{III} &= \frac{-e+3\cos^2\gamma}{1+3\cos^2\gamma}\,V_n^{II} + \frac{\frac{1+e}{2}\cos\gamma}{1+3\cos^2\gamma}\,\Omega^{II}, \\ \Omega^{III} &= \frac{-6(1+e)\cos\gamma}{1+3\cos^2\gamma}\,V_n^{II} + \frac{1+3(e+2)\cos^2\gamma}{1+3\cos^2\gamma}\,\Omega^{II}. \end{aligned} \tag{15}$$

To derive an explicit expression of $V_n^{III}$ and $\Omega^{III}$, we substitute the expression of $V_n^{II}$ and $\Omega^{II}$ to get

$$
\begin{aligned}
V_n^{III} = {} & \frac{1}{(1+3\cos^2\alpha)(1+3\cos^2\beta)(1+3\cos^2\gamma)}\,[(e-3\cos^2\alpha)(e-3\cos^2\beta)(e-3\cos^2\gamma) \\
& -3(1+e)^2(\cos\alpha\cos\beta(e-3\cos^2\gamma)+\cos\beta\cos\gamma(e-3\cos^2\alpha) \\
& +\cos\gamma\cos\alpha(e-3\cos^2\beta))]\,v, \\
\Omega^{III} = {} & \frac{6(1+e)}{(1+3\cos^2\alpha)(1+3\cos^2\beta)(1+3\cos^2\gamma)}\,[-3(1+e)^2\cos\alpha\cos\beta\cos\gamma \\
& +\cos\gamma(e-3\cos^2\alpha)(e-3\cos^2\beta)+\cos\alpha(1-3e\cos^2\beta)(1+3(1+e)\cos^2\gamma) \\
& +\cos\beta(1+3(2+e)\cos^2\gamma)(e-3\cos^2\alpha)]v.
\end{aligned}
$$

Also, the contact velocity at the third impact is

$$
\begin{aligned}
v_{rn}^{III} = {} & v + \frac{(e^2-1)-3(e+1)(\cos^2\alpha+\cos^2\beta)-3(e+1)^2\cos\alpha\cos\beta}{(1+3\cos^2\alpha)(1+3\cos^2\beta)}\,v \\
& +3(e+1)\frac{\cos\gamma(\cos\alpha+e\cos\beta)(-1+3\cos\alpha\cos\beta)}{(1+3\cos^2\alpha)(1+3\cos^2\beta)}\,v,
\end{aligned}
$$

and the impulse at the third impact is

$$
P_n^{II} = V_n^{III} - V_n^{II} = \frac{1+e}{1+3\cos^2\gamma}\,v_{rn}^{III}.
$$

We give numerical examples of the formulae for the impact sequence.

For complete restitution case with $e = 1$, given a small angle $\alpha$, the angle $\beta$ should be less than or equal to $\alpha$, as long as $1-3\cos^2\alpha < 0$. We have the equality $\alpha = \beta$ at 54.74°. Numerically, solution $\beta$ exists until the rod drops on an angle of $\alpha = 58.49°$. Also, up to this value, the impulse keeps a positive value. There is no solution for $\beta$ afterwards. From physical point of view, the rod impact sequence ends with just one impact for $\alpha > 58.49°$.

The impulse for the third impact decreases from 0.5 to 0, and it reaches the zero value for $\alpha = 24.79°$. Afterwards, the third impact ceases to exist. The results for full restitution are expressed graphically in Figure 3.2 and Figure 3.3.

In engineering applications it was found the restitution $e = 0.5$ is of significance. We show the impact results for half restitution $(e = 0.5)$ in a comparison study below.

The results when the restitution coefficient is 0.5 are similar to the full restitution case, although the rebounds at both ends are slower due to energy loss. We can obtain solution for $\beta$ until the rod drops on an angle of $\alpha = 67.21°$. There is no solution for $\beta$ afterwards.

The impulse for the third impact reaches the zero value for $\alpha = 35.00°$. The results for half restitution are expressed graphically in Figure 3.4 and Figure 3.5.

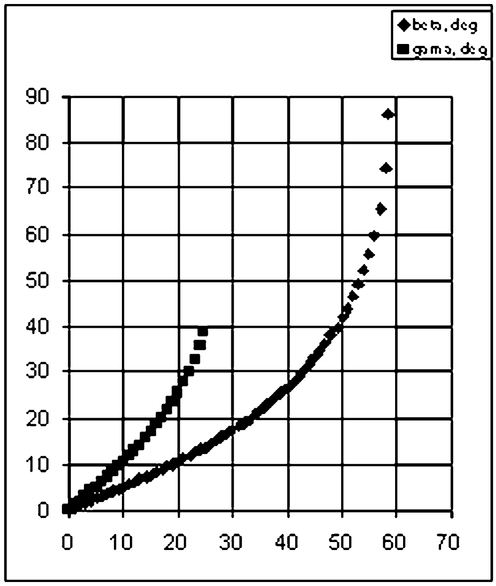


**Figure 3.2**. The dropping angles at the second and the third impact are shown as functions of the angle $\alpha$, when e=1. When $\alpha$ is small, $\beta$ is roughly half of angle $\alpha$, and $\gamma$ is nearly the same as angle $\alpha$.

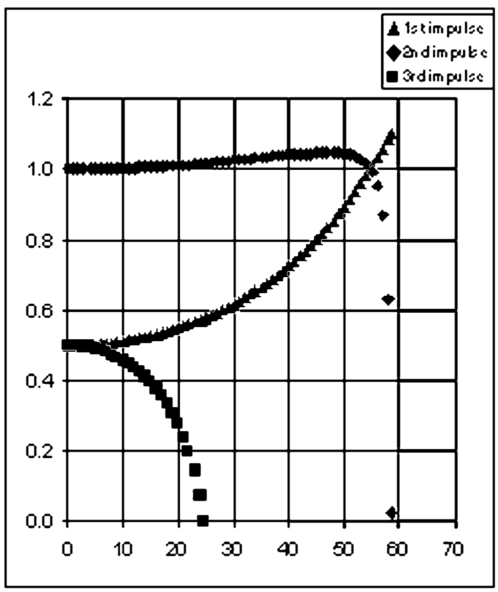


**Figure 3.3**. The impulses at the first, second and third impacts are shown as functions of the initial angle $\alpha$, when e=1. When $\alpha$ is small, the second impact is nearly twice of first one, and the third impact is about the same as the first one. The first two impulses become equal at $\alpha = 54.74°$.

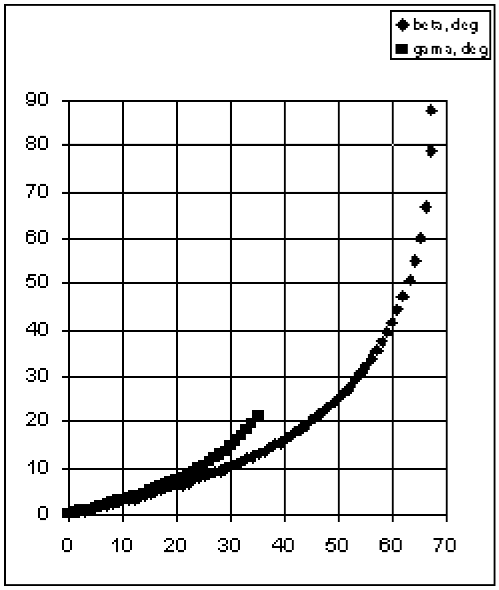


**Figure 3.4**. The dropping angle at the second and the third impact as function of initial angle, when e=0.5. They are smaller than those for full restitution. The angles where second and third impact terminate are relative higher values, when e=0.5.

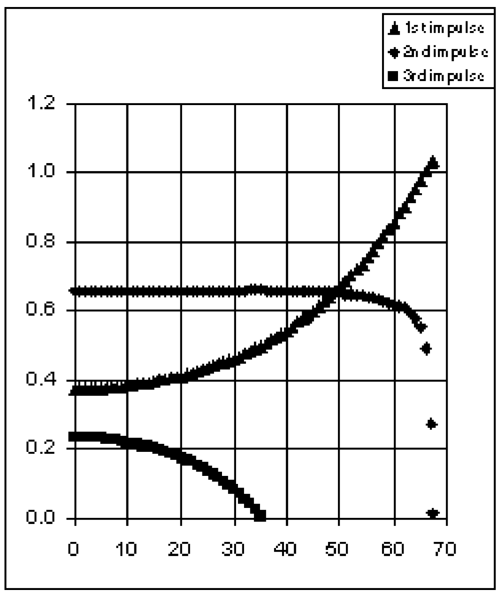


**Figure 3.5**. The impulses at the first, second and third impact are presented as function of initial angle $\alpha$, when e=0.5. The impact with half restitution involves energy loss during the impact process. Still, the second impact shows much larger impulse when the angle $\alpha$ is relatively small.

## 4 The First Three Impacts, with the Gravity

In previous studies [1, 2], it is generally assumed there is no gravity. The validity of such an assumption needs to be checked. In this section, we compare quantitatively the effect of gravity for the impacts sequence. Now, with gravitational force, the impact sequence does not end in finite number, as the rod will fall back again and again. We will still define the clattering sequence as the same number of impact as the case without gravity.

In order to determine the new angles $\beta$ and $\gamma$, we will use the following equations

$$\begin{aligned} h^I + V_n^I T^I - \frac{1}{2}\, g {T^I}^2 &= h^{II}, \\ h^{II} + V_n^{II} T^{II} - \frac{1}{2}\, g {T^{II}}^2 &= h^{III}, \end{aligned} \tag{16}$$

respectively.

From (16), we use

$$V_n^I = \frac{e - 3\cos^2\alpha}{1 + 3\cos^2\alpha}\, v,$$

and

$$T^I = \frac{-(\alpha+\beta)}{\Omega^I} = \frac{1 + 3\cos^2\alpha}{6(1+e)\cos\alpha}\,(\alpha+\beta)\,\frac{1}{v}.$$

Hence the new angle relation for first and second impact is expressed as

$$\sin\alpha + 2\left(\frac{e - 3\cos^2\alpha}{6(1+e)\cos\alpha}\,(\alpha+\beta)\right) - \frac{1}{2}\frac{g}{v^2}\left(\frac{1+3\cos^2\alpha}{6(1+e)\cos\alpha}\,(\alpha+\beta)\right)^2 = \sin\beta. \tag{17}$$

To derive the relation from second angle to third angle, we use

$$V_n^{II} = \frac{-(e - 3\cos^2\alpha)(e - 3\cos^2\beta) + 3(1+e)^2\cos\alpha\cos\beta}{(1+3\cos^2\alpha)(1+3\cos^2\beta)}\, v$$

and

$$T^{II} = \frac{\beta+\gamma}{\Omega^{II}} = \frac{-(1+3\cos^2\alpha)(1+3\cos^2\beta)}{6(1+e)(\cos\alpha + e\cos\beta)(1 - 3\cos\alpha\cos\beta)}\,(\beta+\gamma)\,\frac{1}{v}\,.$$

Hence

$$\begin{aligned} &\sin\beta + 2\left(\frac{(e-3\cos^2\alpha)(e-3\cos^2\beta) - 3(1+e)^2\cos\alpha\cos\beta}{(1+3\cos^2\alpha)(1+3\cos^2\beta)}\,(\beta+\gamma)\right) \\ &\quad - \frac{1}{2}\frac{g}{v^2}\left(\frac{-(1+3\cos^2\alpha)(1+3\cos^2\beta)}{6(1+e)(\cos\alpha+e\cos\beta)(1-3\cos\alpha\cos\beta)}\,(\beta+\gamma)\right)^2 = \sin\gamma. \end{aligned} \tag{18}$$

Using the equations (17) and (18), we can find the angles $\beta$ and $\gamma$, respectively, given velocity $v$.

For example, as we are motivated by the cell phone dropping problem, that phone typically starts a free fall from the pocket. Supposing it drops from a height of one meter, we can find $v$ and go on to find the impact angles

$$\frac{1}{2} g\, t^2 = 1 \;\Rightarrow\; t = \sqrt{\frac{2}{g}},$$

$$v = g\, t \;\Rightarrow\; v = \sqrt{2\, g} \;\Rightarrow\; \frac{g}{v^2} = \frac{1}{2}.$$

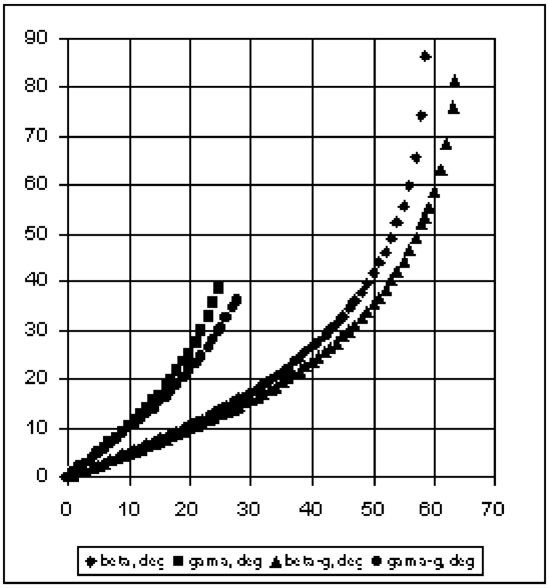


**Figure 4.1**. The second and the third angles, for total restitution and in both cases, with and without gravity, as a function of initial angle $\alpha$.

So, by plugging in 1/2 for $g/v^2$ value in the above equations, we find the impact angles and impulses as shown in Figures 4.1 – 4.4 below.

As we observe in Figure 4.1 that the second and third angles change very little for small initial angles by the effect of gravity. Both angles $\beta$ and $\gamma$ are smaller in the case with gravity, and also the second and third clattering moment exists for slightly wider ranges of intervals of $\alpha$, than in the case when gravity is not considered. The difference between the values for $\beta$ and also the difference of the values for $\gamma$, in the cases without and with gravity, is less than one degree for roughly half of the interval of existence of $\beta$ and $\gamma$ respectively, which is 12 and 25 degrees respectively.

The results for the impulse are similar, in the sense that for the same landmarks (say at 12 degree and 25 degree), the difference between the values of impulse in the two cases is less than 0.003 for the second impact, and less than 0.005 for the third impact, while the ranges of the impulses for both cases are at (1.000, 1.018) for the second impact when $0° \leq \beta \leq 12°$, and are at (0.435, 0.500) for the third impact when $0° \leq \gamma \leq 25°$, as we see in Figure 4.2.

For both figures, the discrepancy is present when the clattering sequence takes longer time to finish.

When the restitution coefficient equals 0.5, we also compare the results.

As we observe in Figure 4.3, that is similar to the cases with total restitution, the angles $\beta$ and $\gamma$ change very little for small angles of $\alpha$ by the gravity effect. Both impact angles $\beta$ and $\gamma$ are smaller in the case with gravity though, and also the second and third clattering moment exists for a wider interval for $\alpha$ than in the case without gravity. The difference between the values for $\beta$ and also the difference for the values for $\gamma$, in the cases without and with gravity, is less than one degree for roughly half of the interval of existence of $\beta$ and $\gamma$ respectively, which is 17 and 23 degrees respectively, comparing to 12 and 25 in the case with total restitution.

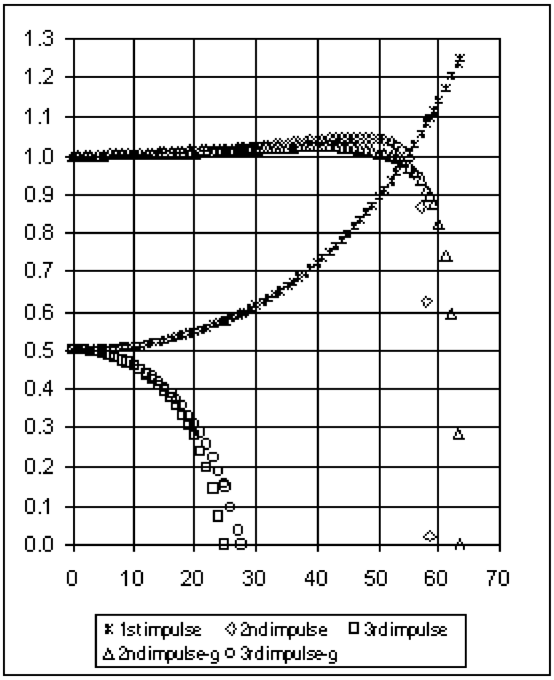


**Figure 4.2.** The impulses of the first, second and third impact, for total restitution and in both cases, with and without gravity, are shown as a function of initial angle $\alpha$.

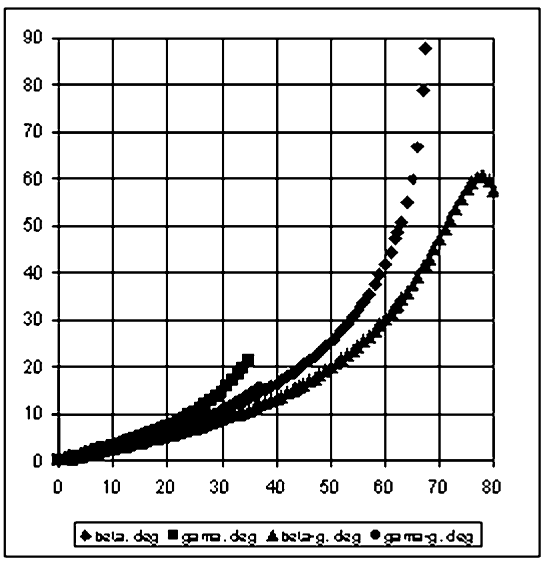


**Figure 4.3.** The second and the third angle for e=0.5, in both cases, with and without gravity, are shown as a function of initial angle $\alpha$.

The results for impulses are also similar, in the sense that for the same landmarks (at 17 and 23 degree respectively), the difference between the values in the without gravity/with gravity cases is less than 0.001 for the second impact, and less than 0.01 for the third impact. The ranges of the impulses are both at (0.6563, 0.6593) for the second impact when $0° \leq \beta \leq 17°$, and at (0.1930, 0.2344) for the third impact when $0° \leq \beta \leq 23°$, as we observe in Figure 4.4.

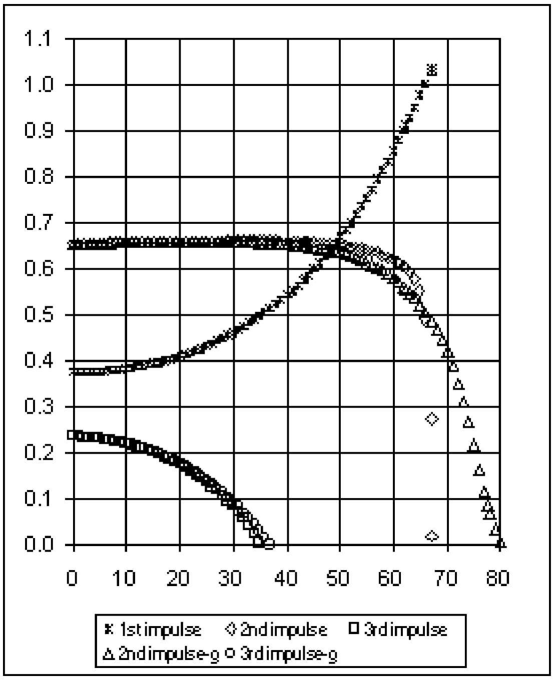


**Figure 4.4**. The impulse of the first, second and third impact, for e=0.5, in both cases, with and without gravity, is shown as a function of initial angle $\alpha$.

## 5 Discussions

The overall aim of this article is to study analytically the issues surrounding clattering. Our discussions are limited to a rod with a uniformly distributed mass. Our study confirms the results of Goyal, *et al.* [1, 2] that if a rod falls to ground in a small angle, then its clattering impact series has a much larger second impact than the initial one. Furthermore, our analytic study finds that same phenomenon is happening to angles as large as 54 degree. In realistic situations, the range might be small when energy dissipation and softness of the ground are included in consideration as we indicated in the case study of $e = 0.5$.

In both situations of $e = 0.5$ and $e = 1.0$ without gravity, there is no forth impact. With gravity, the forth impact will occur, but it does not belong to the same clattering sequence of the first three impacts. So we restrict our discussion to first three impacts.

Through the comparison study at Section 4, we find that gravity plays only a minor role in our clattering problems. Though friction is not considered in this study, we understand that the fiction is a much complex issues. Some initial study indicated that with a certain friction on the ground, when drop angle is small, sticking might occur during the impact process. If the initial rotation is also included, then there is possibility of revered sliding as well as sticking, as discussed in [8]. These topics as well as the clattering of multiple-body and flexible body remain subject of further study.

## Acknowledgement

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# A Study of Nonlocal History-Valued Retarded Differential Equations Using Analytic Semigroups

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**Abstract:** In this paper we study a semi-linear retarded differential equation with a nonlocal history condition considered in an arbitrary Banach space. Using the theory of analytic semigroups we establish the existence, uniqueness and regularity of solutions. We also give an example to illustrate the applications of the abstract results.

**Keywords:** *Retarded differential equation; analytic semigroup; mild solution; regularity of a mild solution.*

**Mathematics Subject Classification (2000):** 34K30, 34G20, 47H06.

## 1 Introduction

Let $X$ be a Banach space with its norm denoted by $\|\cdot\|$ and for $t \in [0,T]$, $0 < T < \infty$, let $\mathcal{C}_t = C([-\tau, t]; X)$, $0 < \tau < \infty$, be the Banach space of all continuous functions from $[-\tau, t]$ into $X$ endowed with the supremum norm

$$\|\psi\|_t = \sup_{-\tau \le \theta \le t} \|\psi(\theta)\|.$$

Let $A$ be a linear operator defined from $D(A) \subset X$ into $X$ be such that $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\colon t \ge 0\}$ of bounded linear operators in $X$. It follows that the fractional power $A^\alpha$ of $A$ is defined for $0 \le \alpha \le 1$ and $D(A^\alpha)$ is a Banach space endowed with the graph norm of $A^\alpha$. Let $X_\alpha$ be the Banach space $D(A^\alpha)$ endowed with the norm

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha),$$

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equivalent to the graph norm of $A^\alpha$. For $u \in \mathcal{C}_T$ and $t \in [0,T]$, let $u_t \in \mathcal{C}_0$ be given by $u_t(\theta) = u(t+\theta)$. Consider the following semilinear retarded differential equation with a nonlocal history condition:

$$\begin{aligned} u'(t) + Au(t) &= f(t, u(t), u_t), \quad t \in (0,T], \\ H(u_0) &= \phi \quad \text{on} \quad [-\tau, 0], \end{aligned} \tag{1.1}$$

where the nonlinear map $f$ is defined from $[0,T] \times X_\alpha \times \mathcal{C}_0^\alpha$ into $X$, $\mathcal{C}_0^\alpha$ being the space of all continuous functions from $[-\tau, 0]$ into $D(A^\alpha)$ endowed with the norm

$$\|\psi\|_{0,\alpha} = \sup_{-\tau \le \theta \le 0} \|A^\alpha \psi(\theta)\|, \quad \psi \in \mathcal{C}_0^\alpha,$$

and the map $H$ is defined from $\mathcal{C}_0$ into $\mathcal{C}_0$, $\phi \in \mathcal{C}_0$.

The theory of functional differential equations with the history conditions of the type considered in (1.1) may be applied to the epidemic population dynamic models. For such related works we refer to Alaoui [8] and references cited therein.

For the earlier works on existence, uniqueness and stability of various types of solutions of differential and functional differential equations with nonlocal conditions we refer to Byszewski and Akca [6], Byszewski and Lakshmikantham [2], Byszewski [3], Balachandran and Chandrasekaran [4], Lin and Liu [5] and references cited in these papers.

Bahuguna [11] has considered the existence of mild, strong and classical solutions of (1.1) for the particular case $F(t,u,\psi) \equiv F(t,u)$ under different conditions on the operator $A$ and local Lipschitz-like condition from $[0,T] \times X$ into $X$. Here we consider the case when $-A$ is the infinitesimal generator of an analytic semigroup and on $f$ we consider a Lipschitz-like condition from $[0,T] \times D(A^\alpha) \times C_0^\alpha$ into $X$ for some $0 < \alpha < 1$.

We first establish the local existence and uniqueness of a mild solution of (1.1) for every $\chi \in \mathcal{C}_T$ satisfying $H(\chi_0) = \phi$. Finally, we establish a global existence result in the sense that classical solution $u$ of (1.1) exists on $[-\tau, T]$ for any arbitrary finite positive $T$.

## 2 Preliminaries

We continue to use the notations of the earlier section. We note that if $-A$ is the infinitesimal generator of an analytic semigroup then $-(A+cI)$ is invertible and generates a bounded analytic semigroup for $c > 0$ large enough. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence for convenience, we suppose that

$$\|S(t)\| \le M \quad \text{for} \quad t \ge 0$$

and

$$0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0 \le \alpha \le 1$, $A^\alpha$ can be defined as a closed linear invertible operator with its domain $D(A^\alpha)$ being dense in $X$. We have $X_\beta \hookrightarrow X_\alpha$ for $0 < \alpha < \beta$ and the embedding is continuous.

It can be seen easily that $C_t^\alpha = C([-\tau, t]; X_\alpha)$, for all $t \in [0, T]$, is a Banach space endowed with the supremum norm,

$$\|\psi\|_{t,\alpha} = \sup_{-\tau \le \eta \le t} \|\psi(\eta)\|_\alpha, \quad \psi \in \mathcal{C}_t^\alpha.$$

For $0 \le \alpha \le 1$, we define $A^{-\alpha}\colon C_0 \to C_0$ by $(A^{-\alpha}\psi)(t) = A^{-\alpha}(\psi(t))$ for any $\psi \in C_0$.

We assume the following conditions on the functions $f$, $H$ and $\phi$.

**A1.** There exist $\chi \in \mathcal{C}_T^\alpha$ such that $H(\chi_0) = \phi$ on $[-\tau, 0]$ and the function $\chi$ is locally Hölder continuous on $[-\tau, 0]$.

**A1.** The nonlinear map $f\colon [0, T] \times X_\alpha \times \mathcal{C}_0^\alpha \to X$ satisfies a local Lipschitz-like condition,

$$\|f(t, x, \psi) - f(s, y, \tilde{\psi})\| \le L_f(r)[|t - s|^\theta + \|x - y\|_\alpha + \|\psi - \tilde{\psi}\|_{0,\alpha}],$$

for all $t, s \in [0, T]$, a fixed $\theta$, $0 \le \theta \le 1$ and $\psi, \tilde{\psi} \in B_r(\mathcal{C}_0^\alpha, A^\alpha \chi)$, $x, y \in B_r(X_\alpha, \chi(s))$ where $L_f\colon R_+ \to R_+$ is a nondecreasing function and for $z_0$ in a Banach space $(Z, \|\cdot\|_Z)$ and $r, r_1 > 0$,

$$B_r(Z, z_0) = \{z \in Z\colon \|z - z_0\|_Z \le r\}.$$

## 3 Existence of Mild Solutions

Let $\widetilde{T}$ be any number such that $0 < \widetilde{T} \le T$. A function $u \in \mathcal{C}_{\widetilde{T}}$ is called a mild solution of (1.1) on $[-\tau, \widetilde{T}]$ if it satisfies the integral equation given by

$$u(t) = \begin{cases} \chi(t), & t \in [-\tau, 0], \\ S(t)\chi(0) + \int_0^t S(t - s) f(s, u(s), u_s)\, ds, & t \in [0, \widetilde{T}], \end{cases}$$

where $\chi \in \mathcal{C}_T$ is such that $H(\chi_0) = \phi$ on $[-\tau, 0]$. As pointed out earlier, we may suppose without loss of generality that the analytic semigroup generated by $-A$ is bounded and that $-A$ is invertible. Furthermore, we assume that $0 < T < \infty$. With these simplifications we have the following theorem.

**Theorem 3.1** *Suppose that the operator $-A$ generates the analytic semigroup $S(t)$ with $\|S(t)\| \le M$, $t \ge 0$ and that $0 \in \rho(-A)$. If the conditions A1, A2 and $\chi(t) \in D(A^\beta)$ for all $t \in [-\tau, 0]$, $\alpha < \beta$ are satisfies then (1.1) has a local mild solution on $[-\tau, t_0]$ for some $0 < t_0 \le T$. The solution is unique if and only if $\chi$ is unique on $[-\tau, 0]$ satisfying $H(\chi_0) = \phi$ on $[-\tau, 0]$.*

*Proof* We establish the existence of a mild solution $u$ on $[-\tau, t_0]$ for some $0 < t_0 \le T$. For any $0 < \widetilde{T} \le T$, we define a mapping $F$ from $\mathcal{C}_{\widetilde{T}}$ into $\mathcal{C}_{\widetilde{T}}$ given by,

$$(F\psi)(t) = \begin{cases} A^\alpha \chi(t), & t \in [-\tau, 0], \\ S(t)A^\alpha\chi(0) + \int_0^t A^\alpha S(t - s) f(s,\, A^{-\alpha}\psi(s),\, A^{-\alpha}\psi_s)\, ds, & t \in [0, \widetilde{T}]. \end{cases} \tag{3.1}$$

Clearly $F$ is well defined. For $R>0$, let

$$S=\{\psi\in B_R(\mathcal{C}_{t_0},A^\alpha\chi):\ \psi(t)=A^\alpha\chi(t),\ t\in[-\tau,0]\}.$$

Choose $0<t_0\le T$ such that for $0<t\le t_0$, we have,

$$\|(S(t)-I)A^\alpha\chi(0)\|\le\frac{R}{3},$$
$$\|A^\alpha\chi(t)-A^\alpha\chi(0)\|\le\frac{R}{3}$$

and

$$t_0<\left[\frac{R}{3}{C_\alpha}^{-1}(1-\alpha)\{L_f(R)(T+2R+2\|\chi\|_{T,\alpha})+\beta\}^{-1}\right]^{\frac{1}{1-\alpha}}, \tag{3.2}$$

where $C_\alpha$ is a positive constant depending on $\alpha$ satisfying $\|A^\alpha S(t)\|\le C_\alpha t^{-\alpha}$, for $t>0$ and $\beta=\|f(0,0,0)\|$. Clearly, $F\colon S\to\mathcal{C}_{t_0}$. We first show that $F\colon S\to S$. For any $\psi\in\mathcal{C}_{t_0}$, $F\psi=A^\alpha\chi$ on $[-\tau,0]$. Thus, to show that $F$ maps from $S$ into $S$ we only need to show that,

$$\|F\psi-A^\alpha\chi\|_{t_0}\le R.$$

For this, we have $(F\psi)(t)-A^\alpha\chi(t)=0$ if $t\in[-\tau,0]$ and for $t\in[0,t_0]$,

$$\begin{aligned}(F\psi)(t)-A^\alpha\chi(t)&=[(S(t)-I)A^\alpha\chi(0)]+[A^\alpha\chi(0)-A^\alpha\chi(t)]\\&\quad+\int_0^t A^\alpha S(t-s)[f(s,A^{-\alpha}\psi(s),A^{-\alpha}\psi_s)-f(0,0,0)]\,ds,\\&\quad+\int_0^t A^\alpha S(t-s)f(0,0,0)\,ds.\end{aligned}$$

Hence we have,

$$\begin{aligned}\|(F\psi)(t)-A^\alpha\chi(t)\|&\le\|(S(t)-I)A^\alpha\chi(0)\|+\|A^\alpha\chi(0)-A^\alpha\chi(t)\|\\&\quad+\int_0^t\|A^\alpha S(t-s)\|\|f(s,A^{-\alpha}\psi(s),A^{-\alpha}\psi_s)-f(0,0,0)\|\,ds,\\&\quad+\int_0^t\|A^\alpha S(t-s)\|\|f(0,0,0)\|\,ds,\quad t\in[0,t_0] \qquad (3.3)\\&\le\frac{2R}{3}+C_\alpha[L_f(R)(T+2R+2\|\chi\|_{T,\alpha})+\beta]\int_0^t(t-s)^{-\alpha}\,ds\\&\le\frac{2R}{3}+\frac{C_\alpha}{(1-\alpha)}[L_f(R)(T+2R+2\|\chi\|_{T,\alpha})+\beta]t_0^{1-\alpha}.\end{aligned}$$

From (3.2) and (3.3), we have $\|(F\psi)-A^\alpha\chi\|_{t_0}\le R$ for $0<t<t_0$. Hence the map $F\colon S\to S$. Now, we show that $F$ is a contraction mapping on $S$. For this, we have $\|(F\psi)(t)-(F\tilde{\psi})(t)\|=0$ for all $t\in[-\tau,0]$ and

$$(F\psi)(t)-(F\tilde{\psi})(t)=\int_0^t A^\alpha S(t-s)[f(s,A^{-\alpha}\psi(s),A^{-\alpha}\psi_s)-f(s,A^{-\alpha}\tilde{\psi}(s),A^{-\alpha}\tilde{\psi}_s)]\,ds,$$

for $t \in [0, t_0]$ and $\psi, \tilde{\psi} \in S$. Hence,

$$\|(F\psi)(t) - (F\tilde{\psi})(t)\| \le \int_0^t \|A^\alpha S(t-s)\| \, \|f(s, A^{-\alpha}\psi(s), A^{-\alpha}\psi_s) - f(s, A^{-\alpha}\tilde{\psi}(s), A^{-\alpha}\tilde{\psi}_s)\| ds$$

for all $\psi, \tilde{\psi} \in S$ and $t \in [-\tau, t_0]$. From A2 we have,

$$\|(F\psi)(t) - (F\tilde{\psi})(t)\| \le \frac{1}{R}\frac{C_\alpha}{(1-\alpha)}[L_f(R)(T+2R)+\beta]t_0^{1-\alpha}\|\psi - \tilde{\psi}\|_{t_0} \le \frac{1}{3}\|\psi - \tilde{\psi}\|_{t_0},$$

for all $\psi, \tilde{\psi} \in S$. Thus $F$ is a strict contraction map on $S$ and therefore by the Banach contraction principle there exists a unique fixed point $\psi$ of $F$ in $S$ i.e. there is a unique $\psi \in S$ such that $(F\psi)(t) = \psi(t)$, for all $t \in [-\tau, t_0]$, that is,

$$\psi(t) = \begin{cases} A^\alpha \chi(t), & t \in [-\tau, 0], \\ S(t)A^\alpha\chi(0) + \int_0^t A^\alpha S(t-s) f(s, A^{-\alpha}\psi(s), A^{-\alpha}\psi_s)\, ds, & t \in [0, t_0]. \end{cases}$$

Thus, if we take $u = A^{-\alpha}\psi$, then for $t \in [-\tau, t_0]$, we have

$$u(t) = \begin{cases} \chi(t), & t \in [-\tau, 0], \\ S(t)\chi(0) + \int_0^t S(t-s) f(s, u(s), u_s)\, ds, & t \in [0, t_0], \end{cases} \tag{3.4}$$

and $H(u_0) = \phi$ on $[-\tau, 0]$. Hence $u$ given by the above equation is a mild solution of equation (1.1). Now we show that a mild solution $u \in \mathcal{C}_{t_0}$ of (1.1) on $[-\tau, t_0]$ with $u = \chi$ on $[-\tau, 0]$ is unique. Let $u_1$ and $u_2$ be two such functions. Let $u = u_1 - u_2$. Then $u = 0$ on $[-\tau, 0]$ and for $t \in [0, t_0]$, we have

$$\|u(t)\|_\alpha \le 2L_f(\widetilde{R}) \int_0^t (t-s)^{-\alpha} \sup_{-\tau \le \theta \le 0} \|u(s+\theta)\|_\alpha ds, \tag{3.5}$$

where $\widetilde{R} = \max\{\|u_1\|_{t_0}, \|u_2\|_{t_0}\}$. Let $\bar{\theta} \in [-t, 0]$ and $t \in [0, t_0]$ and let us assume that $t_0 \le \tau$, hence we have $0 \le t \le \tau$. For $t \le -\bar{\theta}$, we have $u(t+\bar{\theta}) = 0$. For $t \ge -\bar{\theta}$, we have

$$\|u(t+\bar{\theta})\|_\alpha \le 2L_f(\widetilde{R}) \int_0^{t+\bar{\theta}} (t+\bar{\theta}-s)^{-\alpha} \sup_{-\tau \le \theta \le 0} \|u(s+\theta)\|_\alpha ds.$$

Now we put $\bar{\theta} - s = -\eta$, in the above inequality to obtain

$$\|u(t+\bar{\theta})\|_\alpha \le 2L_f(\widetilde{R}) \int_{-\bar{\theta}}^{t} (t-\eta)^{-\alpha} \sup_{-\tau \le \theta \le 0} \|u(\eta+\bar{\theta}+\theta)\|_\alpha \, d\eta.$$

Let $\theta = \gamma - \bar{\theta}$ in the above inequality to get

$$\|u(t+\bar{\theta})\|_\alpha \le 2L_f(\widetilde{R}) \int_{-\bar{\theta}}^{t} (t-\eta)^{-\alpha} \sup_{-\tau+\bar{\theta} \le \gamma \le 0} \|u(\eta+\gamma)\|_\alpha d\eta.$$

Since $u(\eta+\gamma)=0$ on $[-\tau+\bar{\theta},-\tau]$, the above inequality may be written as

$$\begin{aligned}\|u_t(\bar{\theta})\|_\alpha &\leq 2L_f(\widetilde{R})\int_{-\bar{\theta}}^{t}(t-\eta)^{-\alpha}\sup_{-\tau\leq\gamma\leq 0}\|u_\eta(\gamma)\|_\alpha d\eta\\ &\leq 2L_f(\widetilde{R})\int_{0}^{t}(t-\eta)^{-\alpha}\sup_{-\tau\leq\gamma\leq 0}\|u_\eta(\gamma)\|_\alpha d\eta.\end{aligned} \tag{3.6}$$

Taking supremum on $\bar{\theta}$ over $[-\tau,0]$, we get

$$\|u_t\|_{0,\alpha}\leq 2L_f(\widetilde{R})\int_{0}^{t}(t-\eta)^{-\alpha}\|u_\eta\|_{0,\alpha}d\eta. \tag{3.7}$$

Now by applying the Gronwall's inequality to the above inequality we get the required result.

## 4 Regularity of Mild Solutions

In this section we establish the regularity of the mild solutions to (1.1).

**Theorem 4.1** *Suppose that $-A$ generates the analytic semigroup $S(t)$ such that $\|S(t)\|\leq M$ for $t\geq 0$, and $0\in\rho(-A)$. Further suppose that the conditions A1 and A2 hold and $\chi(t)\in D(A)$ for all $t\in[-\tau,0]$. Then (1.1) has a local classical solution and it is unique if and only if $\chi$ is unique on $[-\tau,0]$ satisfying $H(\chi_0)=\phi$ on $[-\tau,0]$.*

*Proof* From Theorem 3.1, it follows that there exist $t_0$, $0<t_0\leq T$ and a function $u$ such that $u$ is a unique mild solution to equation (1.1) on $[-\tau,t_0]$ given by

$$u(t)=\begin{cases}\chi(t), & t\in[-\tau,0],\\ S(t)\chi(0)+\int_0^t S(t-s)f(s,u(s),u_s)\,ds, & t\in[0,t_0].\end{cases}$$

Let $\psi(t)=A^\alpha u(t)$. Then

$$\psi(t)=\begin{cases}A^\alpha\chi(t), & t\in[-\tau,0],\\ S(t)A^\alpha\chi(0)+\int_0^t A^\alpha S(t-s)f(s,A^{-\alpha}\psi(s),A^{-\alpha}\psi_s)\,ds, & t\in[0,t_0].\end{cases}$$

As $u$ is unique hence $\psi(t)$ is also unique. Since $\psi(t)$ is continuous on $[-\tau,t_0]$ and the map $f$ satisfy assumption A2, it follows that $f$ is continuous, and therefore bounded on $[0,t_0]$. Let $N_1=L_f(R)(T+2R+2\|\chi\|_{T,\alpha})+\beta$. Now we want to show that $f$ is locally Hölder continuous on $(0,t_0]$. From Theorem 2.6.13 in Pazy [1], it follows that for every $0<\beta<1-\alpha$, $t>s>0$ and every $0<h<1$, we have

$$\begin{aligned}\|(S(h)-I)A^\alpha S(t-s)\| &\leq C_\beta h^\beta\|A^{\alpha+\beta}S(t-s)\|\\ &\leq Ch^\beta(t-s)^{-(\alpha+\beta)}.\end{aligned} \tag{4.1}$$

Next, for $0 < t < t+h \le t_0$, we have

$$
\begin{aligned}
\|\psi(t+h)-\psi(t)\| \le \|(S(h)-I)S(t)A^{\alpha}\chi(0)\| \\
+ \int_0^t \|(S(h)-I)A^{\alpha}S(t-s)\|\,\|f(s,A^{-\alpha}\psi(s)A^{-\alpha}\psi_s)\|ds \\
+ \int_t^{t+h} \|A^{\alpha}S(t+h-s)\|\|f(s,A^{-\alpha}\psi(s),A^{-\alpha}\psi_s)\|ds.
\end{aligned}
\tag{4.2}
$$

Now,

$$
\|(S(h)-I)S(t)A^{\alpha}\chi(0)\| \le Ct^{-(\alpha+\beta)}h^{\beta} \le M_1h^{\beta}, \tag{4.3}
$$

where $M_1$ depends on $t$ and blows up as $t$ decreases to zero. Furthermore

$$
\int_0^t \|(S(h)-I)A^{\alpha}S(t-s)\|\,\|f(s,A^{-\alpha}\psi(s),A^{-\alpha}\psi_s)\|ds \le Ch^{\beta}N_1\int_0^t (t-s)^{-(\alpha+\beta)}ds \le M_2h^{\beta}, \tag{4.4}
$$

where $M_2$ is independent of $t$. Also,

$$
\int_t^{t+h} \|A^{\alpha}S(t+h-s)\|\,\|f(s,A^{-\alpha}\psi(s),A^{-\alpha}\psi_s)\|ds \le C_{\alpha}N_1 \int_t^{t+h} (t+h-s)^{-\alpha}ds \le M_3h^{\beta}, \tag{4.5}
$$

where $M_3$ is independent of $t$.

Hence inequalities (4.1)–(4.5) imply that there exists a constant $C_1$ such that

$$
\|\psi(t)-\psi(s)\| \le C_1|t-s|^{\beta}, \tag{4.6}
$$

for all $0 < t,\ s < t_0 < T$, thus $\psi$ is locally Hölder continuous on $(0,t_0]$. Now, assumptions A1 and A2 together with (4.6) imply that there exist constants $C_2 \ge 0$ and $0<\gamma<1$ such that for all $0<t,\ s<t_0<T$, we have

$$
\|f(t,A^{-\alpha}\psi(t),A^{-\alpha}\psi_t) - f(s,A^{-\alpha}\psi(s),A^{-\alpha}\psi_s)\| \le C_2|t-s|^{\gamma}. \tag{4.7}
$$

Hence $f$ is locally Hölder continuous on $(0,t_0]$.

Let $h(t) = f(t,A^{-\alpha}\psi(t),A^{-\alpha}\psi_t)$. Consider the following initial value problem

$$
\begin{aligned}
\frac{dw(t)}{dt} + Aw(t) = h(t), \quad t\in(0,t_0], \\
w(0) = \chi(0).
\end{aligned}
\tag{4.8}
$$

By Corollary 4.3.3 in Pazy [1], (4.8) has a unique solution $w\in C^1((0,t_0];X)$ given by

$$
w(t) = S(t)\chi(0) + \int_0^t S(t-s)h(s)ds, \quad t\in[0,t_0]. \tag{4.9}
$$

Let $w(t) = \chi(t)$ on $[-\tau, 0]$. Clearly for each $t \in [0, t_0]$ each term of the right-hand side of (4.9) belongs to $D(A)$ and hence belongs to $D(A^\alpha)$. Applying $A^\alpha$ to both sides of (4.9)and using the fact that $u \in \mathcal{C}_{t_0}$ with $u(t) = \chi(t)$ on $[-\tau, 0]$ and satisfying

$$u(t) = S(t)\chi(0) + \int_0^t S(t-s)f(s, u(s), u_s)ds, \quad t \in (0, t_0],$$

is unique, we have that $A^\alpha w(t) = \psi(t) = A^\alpha u(t)$ for all $t \in [-\tau, t_0]$. Thus we have

$$\psi(t) = \begin{cases} A^\alpha \chi(t), & t \in [-\tau, 0], \\ S(t)A^\alpha \chi(0) + \int_0^t A^\alpha S(t-s) f(s, A^{-\alpha}\psi(s), A^{-\alpha}\psi_s)\, ds, & t \in [0, t_0]. \end{cases}$$

Thus if we put $u(t) = A^{-\alpha}\psi(t)$ in the above equation then we get

$$u(t) = \begin{cases} \chi(t), & t \in [-\tau, 0], \\ S(t)\chi(0) + \int_0^t S(t-s) f(s, u(s), u_s)\, ds, & t \in [0, t_0], \end{cases}$$

it follows that $u$ is the unique classical solution to (1.1) on $[-\tau, t_0]$.

## 5 Global Existence

**Theorem 5.1** *Suppose that $0 \in \rho(-A)$ and the operator $-A$ generates the analytic semigroup $S(t)$ with $\|S(t)\| \leq M$ for $t \geq 0$, the conditions A1, A2 are satisfied and $\chi(t) \in D(A^\alpha)$ for all $t \in [-\tau, 0]$. If there is a continuous nondecreasing real valued function $k(t)$ such that*

$$\|f(t, x, y)\| \leq k(t)(1 + \|x\|_\alpha + \|y\|_{0,\alpha}) \quad \text{for} \quad t \geq 0, \quad x \in X_\alpha, \quad y \in \mathcal{C}_0^\alpha, \tag{5.1}$$

*then the initial value problem (1.1) has a unique solution $u$ which exists for all $t \in [-\tau, T]$.*

*Proof* By Theorem 3.1 we can continue the solution of (1.1) as long as $\|u(t)\|_\alpha$ stays bounded. It is therefore sufficient to show that if $u$ exist on $[-\tau, T[$ then $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$. Since if $t \in [-\tau, 0]$ then we have

$$\|u(t)\|_\alpha \leq \|\chi\|_{0,\alpha}.$$

For $t \in [0, T[$, we have

$$\|u(t)\|_\alpha \leq C_1 + C_2 \int_0^t (t-s)^{-\alpha} \sup_{-\tau \leq \theta \leq 0} \|u(s+\theta)\|_\alpha ds, \tag{5.2}$$

where

$$C_1 = M\|A^\alpha \chi(0)\| + \frac{2k(T)C_\alpha T^{1-\alpha}}{(1-\alpha)} \quad \text{and} \quad C_2 = 2k(T)C_\alpha.$$

*First Case* $(0 \le t \le \tau)$ We replace $t$ by $t+\bar\theta$, where $-t \le \bar\theta \le 0$ in the above inequality (5.2), so we get

$$\|u(t+\bar\theta)\|_\alpha \le C_1 + C_2 \int_0^{t+\bar\theta} (t+\bar\theta-s)^{-\alpha} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_\alpha \, ds. \tag{5.3}$$

In the above inequality we put $\eta = s-\bar\theta$ and $\nu = \theta+\bar\theta$, thus inequality (5.3) after some simplification becomes

$$\|u(t+\bar\theta)\|_\alpha \le C_1 + C_2 \int_{-\bar\theta}^{t} (t-\eta)^{-\alpha} \sup_{-\tau+\bar\theta\le\nu\le 0} \|u(\eta+\nu)\|_\alpha \, d\eta.$$

Since $u(\eta+\nu) = \chi(\eta+\nu)$ for $-\tau+\bar\theta \le \nu \le -\tau$, we have

$$\begin{aligned}\|u(t+\bar\theta)\|_\alpha &\le C_1 + C_2 \int_{-\bar\theta}^{t} (t-\eta)^{-\alpha} \sup_{-\tau+\bar\theta\le\nu\le -\tau} \|u(\eta+\nu)\|_\alpha d\eta \\ &\quad + C_2 \int_{-\bar\theta}^{t} (t-\eta)^{-\alpha} \sup_{-\tau\le\nu\le 0} \|u(\eta+\nu)\|_\alpha d\eta \\ &\le C_3 + C_2 \int_{-\bar\theta}^{t} (t-\eta)^{-\alpha} \sup_{-\tau\le\nu\le 0} \|u(\eta+\nu)\|_\alpha d\eta,\end{aligned} \tag{5.4}$$

where

$$C_3 = C_1 + \int_0^t (t-\eta)^{-\alpha} \sup_{-\tau+\bar\theta\le\nu\le-\tau} \|\chi(\eta+\nu)\|_\alpha d\eta.$$

Now, the inequality (5.4) leads to

$$\sup_{-t\le\bar\theta\le 0} \|u(t+\bar\theta)\|_\alpha \le C_3 + C_2 \int_0^t (t-\eta)^{-\alpha} \sup_{-\tau\le\nu\le 0} \|u(\eta+\nu)\|_\alpha d\eta. \tag{5.5}$$

From the above inequality we get

$$\begin{aligned}\|u_t\|_{0,\alpha} &= \sup_{-\tau\le\bar\theta\le 0} \|u(t+\bar\theta)\|_\alpha \\ &\le \sup_{-\tau\le\bar\theta\le -t} \|u(t+\bar\theta)\|_\alpha + \sup_{-t\le\bar\theta\le 0} \|u(t+\bar\theta)\|_\alpha \\ &\le \sup_{-\tau\le\bar\theta\le -t} \|\chi(t+\bar\theta)\|_\alpha + C_3 + C_2 \int_0^t (t-\eta)^{-\alpha} \|u_\eta\|_{0,\alpha} d\eta.\end{aligned} \tag{5.6}$$

Hence applying the Gronwall's inequality to the above inequality (5.6), we get

$$\|u_t\|_{0,\alpha} \leq M_1 \quad \text{for all} \quad t \in [-\tau, \tau]. \tag{5.7}$$

*Second Case* $(\tau \leq t \leq 2\tau)$ In this case we have

$$\begin{aligned} \|u(t)\|_\alpha &\leq C_1 + C_2 \int_0^\tau (t-s)^{-\alpha} \sup_{-\tau\leq\theta\leq 0} \|u(s+\theta)\|_\alpha \, ds \\ &\quad + C_2 \int_\tau^t (t-s)^{-\alpha} \sup_{-\tau\leq\theta\leq 0} \|u(s+\theta)\|_\alpha \, ds \\ &\leq C_4 + C_2 \int_\tau^t (t-s)^{-\alpha} \sup_{-\tau\leq\theta\leq 0} \|u(s+\theta)\|_\alpha ds, \end{aligned} \tag{5.8}$$

where

$$C_4 = C_1 + C_2 \int_0^\tau (t-s)^{-\alpha} \sup_{-\tau\leq\theta\leq 0} \|u(s+\theta)\|_\alpha ds.$$

We replace $t$ by $t+\bar{\theta}$ in (5.8) where $\bar{\theta} \in [\tau - t, 0]$, we get

$$\|u(t+\bar{\theta})\|_\alpha \leq C_4 + C_2 \int_\tau^{t+\bar{\theta}} (t+\bar{\theta}-s)^{-\alpha} \sup_{-\tau\leq\theta\leq 0} \|u(s+\theta)\|_\alpha \, ds. \tag{5.9}$$

In (5.9) we put $\eta = s - \bar{\theta}$ and $\nu = \theta + \bar{\theta}$ to obtain

$$\begin{aligned} \|u(t+\bar{\theta})\|_\alpha &\leq C_4 + C_2 \int_{\tau-\bar{\theta}}^{t} (t-\eta)^{-\alpha} \sup_{-t\leq\nu\leq -\tau} \|u(\eta+\nu)\|_\alpha \, d\eta \\ &\quad + C_2 \int_{\tau-\bar{\theta}}^{t} (t-\eta)^{-\alpha} \sup_{-\tau\leq\nu\leq 0} \|u(\eta+\nu)\|_\alpha \, d\eta. \end{aligned} \tag{5.10}$$

Since $u(\eta+\nu) = \chi(\eta+\nu)$ for $\tau - \bar{\theta} \leq \eta \leq t$ and $-t \leq \nu \leq -\tau$, inequality (5.10) implies that

$$\|u(t+\bar{\theta})\|_\alpha \leq C_5 + C_2 \int_0^t (t-\eta)^{-\alpha} \|u_\eta\|_{0,\alpha} \, d\eta, \tag{5.11}$$

where $C_5 = C_4 + C_2 \|\chi\|_{0,\alpha} (T^{1-\alpha}/(1-\alpha))$. Now taking supremum on $\bar{\theta}$ over $[\tau - t, 0]$ we get

$$\sup_{\tau-t\leq\bar{\theta}\leq 0} \|u(t+\bar{\theta})\|_\alpha \leq C_5 + C_2 \int_0^t (t-\eta)^{-\alpha} \|u_\eta\|_{0,\alpha} \, d\eta. \tag{5.12}$$

Now,

$$\begin{aligned}\|u\|_{0,\alpha} &= \sup_{-\tau\le\bar\theta\le 0}\|u(t+\bar\theta)\|_\alpha \\ &\le \sup_{-\tau\le\bar\theta\le\tau-t}\|u(t+\bar\theta)\|_\alpha + \sup_{\tau-t\le\bar\theta\le 0}\|u(t+\bar\theta)\|_\alpha \\ &\le M_1 + C_5 + C_2\int_0^t (t-\eta)^{-\alpha}\|u_\eta\|_{0,\alpha}d\eta.\end{aligned} \tag{5.13}$$

Hence applying the Gronwall's inequality to the above inequality (5.13), we get

$$\|u(t)\|_\alpha \le M_2 \quad for\ all \quad t\in[-\tau,2\tau]. \tag{5.14}$$

Hence by repeating the above process we get the required result. This completes the proof of the theorem.

## 6 Applications

Let $X = L^2((0,1);R)$, and $\tau > 0$. Consider the partial differential equations

$$\begin{aligned}&\partial_t w(t,x) - \partial_x^2 w(t,x) = f_1(t,x),\\ &+\int_0^1 h_1(w(t,x),\partial_x w(t,x))\,dx \int_{-\tau}^0 k(-\theta)g(w(t+\theta,x),\partial_x w(t+\theta,x))\,d\theta,\\ &x\in(0,1),\quad t>0,\\ &\int_{-\tau}^0 k(-\theta)g(w(\theta,x),\frac{\partial w}{\partial x}(\theta,x))\,d\theta = \phi(x),\quad x\in(0,1),\\ &w(t,0)=w(t,1)=0,\quad t\in[0,T],\quad 0<T<\infty,\end{aligned} \tag{6.1}$$

where $g$ and $h_1$ are real valued smooth functions and $k$ is a square integrable function.

We define an operator $A$ as follows:

$$Au = -u'' \quad \text{with} \quad u\in D(A) = \{u\in H_0^1(0,1)\cap H^2(0,1):\ u''\in X\}. \tag{6.2}$$

Here clearly the operator $A$ is self-adjoint, with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$. Now we take $\alpha = 1/2$, $D(A^{1/2})$ is the Banach space endowed with the norm

$$\|x\|_{1/2} = \|A^{1/2}x\|,\quad x\in D(A^{1/2}),$$

and we denote this space by $X_{1/2}$. Also, for $t\in[0,T]$, we denote

$$C_t^{1/2} = C([-\tau,t];\ D(A^{1/2})),$$

endowed with the sup norm

$$\|\psi\|_{t,1/2} = \sup_{-\tau\le\eta\le t} \|\psi(\eta)\|_{1/2}, \quad \psi \in \mathcal{C}_t^{1/2}.$$

We observe some properties of the operators $A$ and $A^{1/2}$ defined by (6.2) (cf. [9] for more details). For $u \in D(A)$ and $\lambda \in R$, with $Au = -u'' = \lambda u$, we have $\langle Au, u\rangle = \langle \lambda u, u\rangle$; that is,

$$\langle -u'', u\rangle = |u'|^2_{L^2} = \lambda |u|^2_{L^2}$$

so $\lambda > 0$. A solution $u$ of $Au = \lambda u$ is of the form

$$u(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

and the conditions $u(0) = u(1) = 0$ imply that $C = 0$ and $\lambda = \lambda_n = n^2\pi^2$, $n \in N$. Thus, for each $n \in N$, the corresponding solution is given by

$$u_n(x) = D\sin(\sqrt{\lambda_n}x).$$

We have $\langle u_n, u_m\rangle = 0$ for $n \neq m$ and $\langle u_n, u_n\rangle = 1$ and hence $D = \sqrt{2}$. For $u \in D(A)$, there exists a sequence of real numbers $\{\alpha_n\}$ such that

$$u(x) = \sum_{n\in N} \alpha_n u_n(x), \quad \sum_{n\in N} (\alpha_n)^2 < +\infty \quad \text{and} \quad \sum_{n\in N} (\lambda_n)^2(\alpha_n)^2 < +\infty.$$

We have

$$A^{1/2}u(x) = \sum_{n\in N} \sqrt{\lambda_n}\,\alpha_n\, u_n(x)$$

with $u \in D(A^{1/2})$; that is, $\sum_{n\in N} \lambda_n(\alpha_n)^2 < +\infty$.

The equation (6.1) can be reformulated as the following abstract equation in $X = L^2((0,1); R)$:

$$\begin{aligned} \frac{du(t)}{dt} + Au(t) &= f(t, u(t), u_t) \quad t > 0, \\ H(u_0) &= \phi, \end{aligned} \tag{6.3}$$

where $u(t) = w(t,\cdot)$ that is $u(t)(x) = w(t,x)$, $u_t(\theta)(x) = w(t+\theta, x)$, $t \in [0,T]$, $\theta \in [-\tau, 0]$, $x \in (0,1)$, the operator $A$ is as define in equation (6.2), the function $f\colon [0,T] \times X_{1/2} \times C_0^{1/2} \to X$ is given by

$$f(t,\psi,\xi)(x) = f_1(t,x) + \int_0^1 h(\psi(x),\psi'(x))\,dx \int_{-\tau}^0 k(-\theta) g(\xi(\theta)(x), \partial_x(\xi(\theta)(x)))\,d\theta, \tag{6.4}$$

and the function $H\colon \mathcal{C}_0 \to \mathcal{C}_0$ is given by

$$H(\psi)(x) = \int_{-\tau}^0 k(-\theta) g\Big(\psi(\theta,x), \frac{\partial\psi}{\partial x}(\theta,x)\Big) d\theta.$$

Also $f_1$, defined from $[0,T]\times(0,1)$ into $R$, is such that $f_1(0,\cdot)\in L^2(0,1)$ and satisfies the following property

$$|f_1(t,x)-f_1(s,x)|\le k_1(x)|t-s|^\theta, \quad \text{for all} \quad t,s\in R \quad \text{a.e.} \quad x\in(0,1),$$

where $k_1\in L^2(0,1)$. It may be verified that the assumptions of Theorem 3.1 are satisfied which ensures the existence of solutions of (6.3) as well as that of (6.1).

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# Designing by Control Law without Model for Dynamic IS-LM Model

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**Abstract:** IS-LM model is used for comparative static analysis and many dynamic factors are not considered, so dynamic analysis is introduced to IS-LM model to analyze economy more deeply. Control without model proposed by Mr. Han Zhi-gang has many advantages, such as strong adaptability, strong tracking ability, strong anti-disturbance ability, time lag controlling and so on, so it is fit for macro-economy dynamic analysis. The property of maximal energy saving of control law without model makes it possible to save more fund when government uses finance policy and currency policy.

**Keywords:** *Dynamic IS-LM; control without model.*

**Mathematics Subject Classification (2000):** 91B28, 91B62, 91B64.

## 1 Introduction

As it is well known the IS-LM model is the core of modern macro-economy [1]. The IS-LM model can be used to analyze every kind of problems of the public finance policy and currency policy and the match of these policies so that the national macro-economy can attain the aim of high economy growth rate and low inflation rate. But this model has some weakness, it is a kind of static balanced analysis, and does not consider many dynamic factors (for example, time lag) within economy, so it is difficult to do more in-depth analysis of economy. The macro-economic system is a complicated one, and is nonlinear with time lag. It is difficult to establish an available mathematical model, and along with the economic reformation going deep and system innovating, the model's structure changes constantly too.

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Therefore, the control law for regulating macro-economy must not only adapt to the model parameters change, and also adapt to the variety of the model structure. Control without model is an effective tool for solving this kind of problem. The mathematical model of dynamic system generally need to be established before designing the control law. The classical method requests this kind of mathematical model be established in advance, at least its construction must be certain in advance, and the model parameters are as accurate as possible. When designing control law without model, it breaks the restrict that control law must be based on the accurate mathematical model. The process of establishing model goes along with feedback control [2]. The initial mathematical model can be not accurate, but it must guarantee the control law designed having proper astringency.

The control law without model works along with establishing model. After getting new data, the model is established again and control is established again. Going on like this, and making the model gradually accurate the performance of system under the control law improves. Real-time establishing model and feedback controlling become integral. The means make the control law have structure self-adaptability with real-time identification i.e. real-time feedback control used by control without model combines identification with control law designing. At the same time we can prove that the control law without model has the property of maximal "energy saving" so that government can use minimum fund to do the same thing when government makes use of finance policy and currency policy.

## 2 Some Control Without Model Theory

In reference [3], the following lemma has been proved.

**Lemma 1** *For any dynamic system with one step delay, if input-output data* $\{u(k-2), y(k-1)\}$, $\{u(k-1), y(k)\}$ *are given and* $u(k-2) \neq u(k-1)$, *then there exists a vector* $\varphi(k)$ *such that*

$$y(k) - y(k-1) = \varphi(k-1)^{\tau}[u(k-1) - u(k-2)] \tag{1}$$

*where* $(^{\tau})$ *is a symbol of transposition,* $y(k)$ *is one-dimension output of system and* $u(k)$ *is input vector,* $\varphi(r)$ *is called pseudo-gradient.*

By the following way, $\varphi(k)$ can be estimated.

Let

$$z(k) = y(k) - y(k-1),$$
$$\phi(k) = u(k-1) - u(k-2).$$

Using the above notation, we now can rewrite (1) as

$$z(k) = \phi(k)^{\tau}\varphi(k-1). \tag{2}$$

Real-time observed values $y(k)$ and $u(k-1)$, $z(k)$ and $\phi(k)$ are obtained. Therefore $\varphi(k-1)$ is estimated by the value $\hat{\varphi}(k-1)$ as follows

$$\begin{aligned}
\hat{\varphi}(k-1) &= \hat{\varphi}_1(k-1) + \frac{\delta}{\eta_k + \|\phi(k)\|^2}\,\phi(k)\{z(k) - \phi(k)^\tau \hat{\varphi}_1(k-1)\},\\
\hat{\varphi}_1(k-1) &= \hat{\varphi}_1(k-2) + M(k-1)\{z(k) - \phi(k)^\tau \hat{\varphi}_1(k-2)\},\\
M(k-1) &= \frac{p(k-2)\phi(k)}{\lambda + \phi(k)^\tau p(k-2)\phi(k)}\,,\\
p(k-1) &= \frac{1}{\lambda}[I - M(k-1)\phi(k)^\tau]\,p(k-2)
\end{aligned} \tag{3}$$

where $\eta_k$ is a suitable small positive value and $\delta$ is a proper constant.

Then we find forecasting value of $\hat{\varphi}(k-1)$ signed as $\hat{\varphi}^*(k)$. A simple method is

$$\hat{\varphi}^*(k) = \hat{\varphi}(k-1).$$

When we design control law, also sign $\hat{\varphi}^*(k)$ as $\hat{\varphi}(k)$. So using the basic form of control law without model

$$u(k) = u(k-1) + \frac{\lambda_k}{a + \|\hat{\varphi}(k)\|^2}\,\hat{\varphi}(k)\{y_0(k+1) - y(k)\}, \tag{4}$$

where $\lambda_k$ is called control parameter, $y_0(k+1)$ is expectation output at $k+1$ time, and $a$ is suitable small positive constant which makes denominator not equal to zero, we can obtain control vector $u(k)$. It acts on the system, so we can obtain new output $y(k+1)$ and a new group of data $\{y(k+1), u(k)\}$.

The next theorem shows that control variation has the property of minimum.

**Theorem 1** *If $y_0(k+1)$, $y(k)$, $u(k-1)$, $\varphi(k)$ are known and $\|\varphi(k)\|^2 \neq 0$, then vector of control is defined by*

$$u(k) = u(k-1) + \frac{1}{\|\varphi(k)\|^2}\,\varphi(k)\{y_0(k+1) - y(k)\}$$

*and satisfies the conditions*

$$\begin{aligned}
y_0(k+1) - y(k) &= \varphi(k)^\tau [u(k) - u(k-1)],\\
\|u(k) - u(k-1)\|^2 &= \min_u \|u - u(k-1)\|^2.
\end{aligned}$$

*Proof* The Lagrangian multiplier can be used here.
Let

$$f(u,\lambda) = \|u - u(k-1)\|^2 + \lambda\{y_0(k+1) - y(k) - \varphi(k)^\tau [u - u(k-1)]\}$$

be Lagrangian function. For this case, it can be shown that

$$\begin{aligned}
\frac{\partial f}{\partial u} &= 2(u - u(k-1)) - \lambda\varphi(k),\\
\frac{\partial f}{\partial \lambda} &= y_0(k+1) - y(k) - \varphi(k)^\tau [u - u(k-1)].
\end{aligned}$$

Let

$$\frac{\partial f}{\partial u} = 0 \quad \text{and} \quad \frac{\partial f}{\partial \lambda} = 0.$$

We can compute

$$\begin{gathered} u - u(k-1) = \frac{\lambda}{2}\,\varphi(k), \\ y_0(k+1) - y(k) = \varphi(k)^\tau [u - u(k-1)] = \frac{\lambda}{2}\,\|\varphi(k)\|^2 . \end{gathered} \tag{5}$$

Hence, when $\|\varphi(k)\|^2 \neq 0$ we obtain

$$\lambda = \frac{2}{\|\varphi(k)\|^2}\,\{y_0(k+1) - y(k)\}.$$

Thus, from (5) it can be obtained that

$$u(k) = u(k-1) + \frac{1}{\|\varphi(k)\|^2}\varphi(k)\{y_0(k+1) - y(k)\}$$

which satisfies the conclusion of Theorem 1, since there is only minimum point for the function $\|u - u(k-1)\|^2$.

Above is the case of multi-input and single-output. Reference [4] extended it to MIMO system. Suppose that the dimension of system output variable $y(k)$ is $n$, the dimension of input (control) variable $u(k)$ is $m$, and that $n \le m$. Suppose the time lag of system is 1, so the model can be written as

$$y(k+1) - y(k) = \varphi(k)[\hat{u}(k) - \hat{u}(k-1)],$$

where

$$\varphi(k) = \begin{bmatrix} \varphi_1(k)^\tau \\ \varphi_2(k)^\tau \\ \cdots\cdots \\ \varphi_n(k)^\tau \end{bmatrix} = \begin{bmatrix} \varphi_{11}(k) & \varphi_{12}(k) & \ldots & \varphi_{1m}(k) \\ \varphi_{21}(k) & \varphi_{22}(k) & \ldots & \varphi_{2m}(k) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{n1}(k) & \varphi_{n2}(k) & \ldots & \varphi_{nm}(k) \end{bmatrix},$$

i.e. $\varphi(k)$ is called pseudo-gradient matrix. Set

$$r(k) = \operatorname{rank}\{\varphi(k)\}.$$

Apparently $r(k) \le n$. Suppose $D_t(k)$ is $r(k)$ full-rank submatrix of $\varphi(k)$, $t = 1, 2, \ldots, N$, $N$ is the number of $r(k)$ full-rank submatrix of $\varphi(k)$. Let $\|D_t(k)\|$ denote a kind of norm of $D_t(k)$. There must be one $r(k)$ full-rank submatrix

$$D(k) = \begin{bmatrix} \varphi_{i_1 j_1}(k) & \varphi_{i_1 j_2}(k) & \ldots & \varphi_{i_1 j_r}(k) \\ \varphi_{i_2 j_1}(k) & \varphi_{i_2 j_2}(k) & \ldots & \varphi_{i_2 j_r}(k) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{i_r j_1}(k) & \varphi_{i_r j_2}(k) & \ldots & \varphi_{i_r j_r}(k) \end{bmatrix}$$

that makes

$$\|D(k)\| = \max_{1 \le t \le N} \|D_t(k)\| .$$

We call $D(k)$ dominant $r(k)$ full-rank submatrix of $\varphi(k)$. Its corresponding $u_{j_1}(k)$, $u_{j_2}(k), \dots, u_{j_r}(k)$ are called dominant control variables. Its corresponding output variables are $y_{i_1}(k), y_{i_2}(k), \dots, y_{i_r}(k)$, set

$$\begin{aligned} y^*(k+1) &= (y_{i_1}(k+1), y_{i_2}(k+1), \dots, y_{i_r}(k+1))^\tau, \\ u^*(k) &= (u_{j_1}(k), u_{j_2}(k), \dots, u_{j_r}(k))^\tau. \end{aligned}$$

Eliminating $y^*(k+1)$ from $y(k+1)$, the rest can be written as vector $y^-(k+1)$. Similarly eliminating $u^*(k)$ from $u(k)$, the rest can be written as vector $u^-(k)$. Ordering $y(k+1)$ and $u(k)$ properly, there exists

$$y(k+1) = (y^*(k+1)^\tau, y^-(k+1)^\tau)^\tau, \quad u(k) = (u^*(k)^\tau, u^-(k)^\tau)^\tau.$$

So we can acquire MIMO control law without model

$$\begin{aligned} &\hat{u}^*(k) = \hat{u}^*(k-1) + \frac{\lambda_k}{a + |\widehat{D}(k)|}\hat{D}^*(k)\{y^*(k+1) - y^*(k)\}, \\ &\hat{u}^-(k) = \hat{u}^-(k-1), \end{aligned} \tag{6}$$

where $\widehat{D}^*(k)$ denotes adjoint of $\widehat{D}(k)$, $|\hat{D}(k)|$ denotes determinant of $\hat{D}(k)$, and $y^*(k+1)$ denotes expectation value of the component determined by $y(k+1)$ independently. We have the matrix

$$\lambda_k = \begin{bmatrix} \lambda_1(k) & & & 0 \\ & \lambda_2(k) & & \\ & & \ddots & \\ 0 & & & \lambda_r(k) \end{bmatrix},$$

where $\lambda_1(k), \lambda_2(k), \dots, \lambda_r(k)$ are proper parameters, $\lambda_k$ is called control parameter matrix.

## 3 Control without Model Application in Macro-Economy

In the model IS-LM, finance policy variable (M) and currency policy variable (G) can be taken as control(input) variables and Gross Domestic Product (GDP) and nominal interest rate (i) can be taken as output variables. Nominal interest rate is equal to actual interest rate plus inflation rate. What shows economy running well is high economic growth rate and low inflation rate, so the control aim of macro-economy system can be

$$aim\colon \quad \begin{cases} i(t) = i^*, \\ Y(t) = Y^*(1+\alpha)^t, \end{cases}$$

where $i^*$ and $Y^*$ are given constants, $\alpha$ is given economy growth rate. According to actual situation of China, annual interest rate is 2.25% and expected inflation rate is under 3%, so set $i^* = 5\%$ and economic growth rate is $\alpha = 8\%$. So the model may be written as

$$\begin{bmatrix} Y(k+1) - Y(k) \\ i(k+1) - i(k) \end{bmatrix} = \varphi(k) \begin{bmatrix} G(k) - G(k-1) \\ M(k) - M(k-1) \end{bmatrix}.$$

According to the data of National Bureau of Statistics of China, see Table 3.1, by formula (3), $\hat{\varphi}(k)$ can be obtained. Suppose target value is

$$\begin{bmatrix} Y_0(k+1) \\ i_0(k+1) \end{bmatrix} = \begin{bmatrix} 117.25\,(1+0.08) \\ 5\% \end{bmatrix}.$$

By formula (6), $\widehat{G}(k)$ and $\widehat{M}(k)$ that meet the target can be obtained. In formula (6), $\lambda_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ denotes neutral finance policy and balanced currency policy.

**Table 3.1.** 1990 – 2003 statistics datum.

| year | Gross Domestic Product GDP billion yuan | Finance Payout G billion yuan | Money Supply M2 billion yuan | Consumer Price Index CPI | Annual Interest % |
|---|---|---|---|---|---|
| 1990 | 1854.79 | 308.359 | 1529.34 | 103.1 | 8.64 |
| 1991 | 2161.78 | 338.662 | 1934.99 | 103.4 | 7.92 |
| 1992 | 2663.81 | 374.22 | 2540.22 | 106.4 | 7.56 |
| 1993 | 3463.44 | 464.23 | 3487.98 | 114.7 | 9.26 |
| 1994 | 4675.94 | 579.262 | 4692.35 | 124.1 | 10.98 |
| 1995 | 5847.81 | 682.372 | 6075.05 | 117.1 | 10.98 |
| 1996 | 6788.46 | 793.755 | 7609.49 | 108.3 | 9.21 |
| 1997 | 7446.26 | 923.356 | 9099.53 | 102.8 | 7.17 |
| 1998 | 7834.52 | 1079.818 | 10449.85 | 99.2 | 5.03 |
| 1999 | 8206.75 | 1318.767 | 11989.79 | 98.6 | 2.89 |
| 2000 | 8946.81 | 1588.65 | 13461.04 | 100.4 | 2.25 |
| 2001 | 9731.48 | 1890.258 | 15830.19 | 100.7 | 2.25 |
| 2002 | 10517.23 | 2205.315 | 18500.70 | 99.2 | 2.03 |
| 2003 | 11725.19 | 2464.995 | 22122.28 | 101.2 | 1.98 |

Note: Annual Interest is arithmetic mean.

Going on with

$$\begin{bmatrix} \widehat{Y}(k+1) \\ \hat{i}(k+1) \end{bmatrix} = \begin{bmatrix} Y(k) \\ i(k) \end{bmatrix} + \hat{\varphi}(k) \begin{bmatrix} \widehat{G}(k) - G(k-1) \\ \widehat{M}(k) - M(k-1) \end{bmatrix},$$

estimated values of next year can be obtained. Repeating formulas (3),(6) graph 1 can be obtained. From the graph we can draw the following conclusions:

1. From (c) and (d), we can see that system tracking ability is very good, estimated values superimpose with target values.
2. Estimated finance payout amplitude is 9.4% on the average, money supply amplitude is 14.8% on the average. They are less than the average value 18.0% and 16.2% of past 5 years.

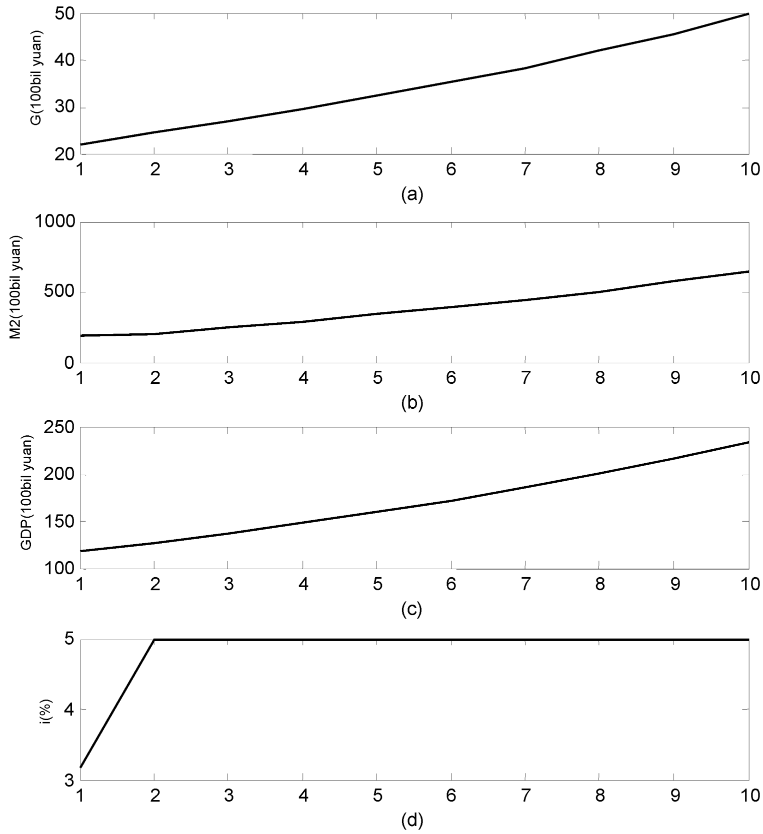


**Figure 3.1**. System simulation curve.

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# Generic Well-Posedness of Linear Optimal Control Problems without Convexity Assumptions

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**Abstract:** The Tonelli existence theorem in the calculus of variations and its subsequent modifications were established for integrands $f$ which satisfy convexity and growth conditions. In our previous work a generic well-posedness result (with respect to variations of the integrand of the integral functional) without the convexity condition was established for a class of optimal control problems satisfying the Cesari growth condition. In this paper we extend this generic well-posedness result to two classes of linear optimal control problems.

**Keywords:** *Complete metric space; generic property; integrand; linear optimal control problem.*

**Mathematics Subject Classification (2000):** 49J99, 90C31.

## 1 Introduction

The Tonelli existence theorem in the calculus of variations [11] and its subsequent generalizations and extensions (e.g. [2, 3, 6, 9, 10]) were established for integrands $f$ which satisfy convexity and growth conditions. Moreover, certain convexity assumptions are also necessary for properties of lower semicontinuity of integral functionals which are crucial in most of the existence proofs, although there are some interesting theorems without convexity (see [2, Ch. 16] and [1, 7, 8]).

In [13] it was shown that the convexity condition is not needed generically, and not only for the existence but also for well-posedness of the problem (with respect to some natural topology in the space of integrands). More precisely, in [13] we considered a class of optimal control problems (with the same system of differential equations, the same functional constraints and the same boundary conditions) which is identified with the corresponding complete metric space of cost functions (integrands), say $\mathcal{M}$. We did

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not impose any convexity assumptions. These integrands are only assumed to satisfy the Cesari growth condition. The main result in [13] establishes the existence of an everywhere dense $G_\delta$-set $\mathcal{F} \subset \mathcal{M}$ such that for each integrand in $\mathcal{F}$ the corresponding optimal control problem has a unique solution.

The next steps in this area of research were done in [5, 12, 14]. In [5] we introduced a general variational principle having its prototype in the variational principle of Deville, Godefroy and Zizler [4]. A generic existence result in the calculus of variations without convexity assumptions was then obtained as a realization of this variational principle. It was also shown in [5] that some other generic well-posedness results in optimization theory known in the literature and their modifications are obtained as a realization of this variational principle. Note that the generic existence result in [5] was established for variational problems but not for optimal control problems and that the topologies in the spaces of integrands in [13] and [5] are different.

In [12] we suggested a modification of the variational principle in [5] and applied it to classes of optimal control problems with various topologies in the corresponding spaces of integrands. As a realization of this principle we established a generic existence result for a class of optimal control problems in which the constraint maps are also subject to variations as well as the cost functions [12]. In [14] we applied the variational principle obtained in [12] and established generic well-posedness results for two classes of variational problems in which the values at the end points are also subject to variations as well as the cost functions. In the present paper we establish generic well-posedness results for two classes of linear optimal control problems in which the right-hand side of the governing linear differential equations is also subject to variations.

## 2 Main Results

In this paper we use the following notations and definitions. Let $k \geq 1$ be an integer. We denote by mes$(E)$ the Lebesgue measure of a measurable set $E \subset R^k$, by $|\cdot|$ the Euclidean norm in $R^k$ and by $\langle \cdot, \cdot \rangle$ the scalar product in $R^k$. We use the convention that $\infty - \infty = 0$. For any $f \in C^q(R^k)$ we set

$$\|f\|_{C^q} = \|f\|_{C^q(R^k)} = \sup_{z \in R^k} \{|\partial^{|\alpha|} f(z)/\partial x_1^{\alpha_1} \dots \partial x_k^{\alpha_k}|\colon$$
$$\alpha_i \geq 0 \text{ is an integer}, \ i = 1, \dots, k, \ \ |\alpha| \leq q\},$$

where $|\alpha| = \sum_{i=1}^k \alpha_i$.

For each function $f \colon Y \to [-\infty, \infty]$, where $Y$ is nonempty, we set $\inf(f) = \inf\{f(y)\colon y \in Y\}$.

In this paper we usually consider topological spaces with two topologies where one is weaker than the other. (Note that they can coincide.) We refer to them as the weak and the strong topology, respectively. If $(X, d)$ is a metric space with a metric $d$ and $Y \subset X$, then usually $Y$ is also endowed with the metric $d$ (unless another metric is introduced in $Y$). Assume that $X_1$ and $X_2$ are topological spaces and that each of them is endowed with a weak and a strong topology. Then for the product $X_1 \times X_2$ we also introduce a pair of topologies: a weak topology which is the product of the weak topologies of $X_1$ and $X_2$ and a strong topology which is the product of the strong topologies of $X_1$ and $X_2$. If $Y \subset X_1$, then we consider the topological subspace $Y$ with the relative weak and strong topologies (unless other topologies are introduced). If $(X_i, d_i)$, $i = 1, 2$, are metric spaces with the metric $d_1$ and $d_2$ respectively, then the space $X_1 \times X_2$ is endowed with the metric $d$ defined by

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2), \quad (x_i, y_i) \in X \times Y, \quad i = 1, 2.$$

Let $0 \le T_1 < T_2 < \infty$ and let $m, n$ be natural numbers. Denote by $X$ the set of all pairs of functions $(x, u)$, where $x\colon [T_1, T_2] \to R^n$ is an absolutely continuous (a.c.) function and $u\colon [T_1, T_2] \to R^m$ is a measurable function.

To be more precise, we have to define elements of $X$ as classes of pairs equivalent in the sense that $(x_1, u_1)$ and $(x_2, u_2)$ are equivalent if and only if $x_2(t) = x_1(t)$ for all $t \in [T_1, T_2]$ and $u_2(t) = u_1(t)$ for almost every $t \in (T_1, T_2)$.

For the set $X$ we consider the metric $\rho$ defined by

$$\rho((x_1, u_1), (x_2, u_2)) = \inf_{\epsilon > 0}\{\text{mes}\{t \in [T_1, T_2] \colon |x_1(t) - x_2(t)| + |u_1(t) - u_2(t)| \ge \epsilon\} \le \epsilon\},$$
$$(x_1, u_1),\ (x_2, u_2) \in X. \tag{2.1}$$

For each $z \in R^n$, each matrix $A$ of dimension of $n \times n$ and each matrix $B$ of dimension $n \times m$ denote by $X(z, A, B)$ the set of all $(x, u) \in X$ such that

$$x(T_1) = z, \tag{2.2}$$

$$x'(t) = Ax(t) + Bu(t), \quad t \in (T_1, T_2) \ \text{(a.e.)}. \tag{2.3}$$

Denote by $\mathcal{M}$ the set of all functions $f\colon (T_1, T_2) \times R^n \times R^m \to R^1$ with the following properties:

(i) $f$ is measurable with respect to the $\sigma$-algebra generated by products of Lebesgue measurable subsets of $(T_1, T_2)$ and Borel subsets of $R^n \times R^m$;

(ii) $f(t, \cdot, \cdot)$ is lower semicontinuous for almost every $t \in (T_1, T_2)$;

(iii) for each $\epsilon > 0$ there exists an integrable scalar function $\psi_\epsilon(t) \ge 0$, $t \in (T_1, T_2)$, such that

$$|u| + |x| \le \psi_\epsilon(t) + \epsilon f(t, x, u) \ \text{ for all } \ (t, x, u) \in (T_1, T_2) \times R^n \times R^m;$$

(iv) for each $\epsilon, M > 0$ there exists $\delta > 0$ such that for almost every $t \in (T_1, T_2)$ the inequality $|f(t, x_1, u_1) - f(t, x_2, u_2)| \le \epsilon$ holds for each $x_1, x_2 \in R^n$ and each $u_1, u_2 \in R^m$ satisfying

$$|x_i|, |u_i| \le M, \quad i = 1, 2 \ \text{ and } \ |x_1 - x_2|, |u_1 - u_2| \le \delta;$$

(v) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that for almost every $t \in (T_1, T_2)$ the inequality

$$|f(t, x_1, u) - f(t, x_2, u)| \le \epsilon \max\{|f(t, x_1, u)|, |f(t, x_2, u)|\} + \epsilon$$

is valid for each $x_1, x_2 \in R^n$ and each $u \in R^m$ satisfying

$$|x_1|, |x_2| \le M, \quad |u| \ge \Gamma, \quad |x_1 - x_2| \le \delta;$$

(vi) there is a constant $c_f > 0$ such that $|f(t, 0, 0)| \le c_f$ for almost every $t \in (T_1, T_2)$.

The growth condition used in (iii) was proposed by Cesari [2] and its equivalents and modifications are rather common in the literature. It follows from property (i) that for any $f \in \mathcal{M}$ and any $(x,u) \in X$ the function $f(t,x(t),u(t))$, $t \in (T_1,T_2)$, is measurable. Properties (iv) and (vi) imply that for each $M>0$ there is $c_M>0$ such that for almost every $t \in (T_1,T_2)$ the inequality $|f(t,x,u)| \le c_M$ holds for each $x \in R^n$ and each $u \in R^m$ satisfying $|x|, |u| \le M$.

It is an elementary exercise to show that a function $f = f(t,x,u) \in C^1((T_1,T_2) \times R^n \times R^m)$ belongs to $\mathcal{M}$ if (iii) and (vi) are true and the following conditions hold:

(a) for each $M>0$

$$\sup\{|\partial f/\partial x(t,x,u)| + |\partial f/\partial u(t,x,u)| \colon t \in (T_1,T_2),\\ x \in R^n,\ u \in R^m \text{ and } |x|,|u| \le M\} < \infty;$$

(b) there exist an increasing function $\psi\colon [0,\infty) \to [0,\infty)$ and a bounded (on bounded subsets of $[0,\infty)$) function $\psi_0\colon [0,\infty) \to [0,\infty)$ such that for each $(t,x,u) \in (T_1,T_2) \times R^n \times R^m$,

$$|\partial f/\partial x(t,x,u)| \le \psi_0(|x|)\psi(|u|)$$

and

$$\psi(|u|) \le f(t,x,u).$$

Denote by $\mathcal{M}^l$ (respectively $\mathcal{M}^c$) the set of all lower semicontinuous (respectively continuous) functions $f \in \mathcal{M}$. Now we equip the set $\mathcal{M}$ with the strong and weak topologies. For the space $\mathcal{M}$ we consider the uniformity determined by the following base:

$$E_{\mathcal{M}}(\epsilon) = \{(f,g) \in \mathcal{M} \times \mathcal{M} \colon |f(t,x,u) - g(t,x,u)| \le \epsilon,\\ (t,x,u) \in (T_1,T_2) \times R^n \times R^m\}, \tag{2.4}$$

where $\epsilon > 0$. It is easy to see that the uniform space $\mathcal{M}$ with this uniformity is metrizable (by a metric $d_{\mathcal{M}}$) and complete. This uniformity generates in $\mathcal{M}$ the strong topology. Clearly $\mathcal{M}^l$ and $\mathcal{M}^c$ are closed subsets of $\mathcal{M}$ with this topology.

For each $\epsilon > 0$ we set

$$E_{\mathcal{M}w}(\epsilon) = \Big\{(f,g) \in \mathcal{M} \times \mathcal{M} \colon \quad \text{there exists a nonnegative } \ \phi \in L^1(T_1,T_2)\\ \text{such that } \int_{T_1}^{T_2} \phi(t)\,dt \le 1, \quad \text{and for almost every } \ t \in (T_1,T_2),\\ |f(t,x,u) - g(t,x,u)| < \epsilon + \epsilon \max\{|f(t,x,u)|, |g(t,x,u)|\} + \epsilon\phi(t)\\ \text{for each } x \in R^n \text{ and each } u \in R^m\Big\}. \tag{2.5}$$

From [12, Lemma 1.1] (see also Lemma 4.1 below) it follows that for the set $\mathcal{M}$, there exists a uniformity which is determined by the base $\mathcal{E}_{\mathcal{M}w}(\epsilon)$, $\epsilon > 0$. This uniformity induces in $\mathcal{M}$ the weak topology.

For each $f \in \mathcal{M}$ define $I^{(f)}\colon X \to R^1 \cup \{\infty\}$ by

$$I^{(f)}(x,u) = \int_{T_1}^{T_2} f(t,x(t),u(t))\,dt, \quad (x,u) \in X. \tag{2.6}$$

Now we define subspaces of $\mathcal{M}$ which consist of integrands differentiable with respect to the control variable $u$.

Let $k \geq 1$ be an integer. Denote by $\mathcal{M}_k$ the set of all $f \in \mathcal{M}$ such that for each $(t,x) \in (T_1,T_2) \times R^n$ the function $f(t,x,\cdot) \in C^k(R^m)$. We consider the topological subspace $\mathcal{M}_k \subset \mathcal{M}$ with the relative weak topology. The strong topology on $\mathcal{M}_k$ is induced by the uniformity which is determined by the following base:

$$\begin{aligned} E_{\mathcal{M}k}(\epsilon) = \{(f,g) \in \mathcal{M}_k \times \mathcal{M}_k \colon |f(t,x,u) - g(t,x,u)| \leq \epsilon \\ \text{for all } \ (t,x,u) \in (T_1,T_2) \times R^n \times R^m \ \text{ and} \\ \|f(t,x,\cdot) - g(t,x,\cdot)\|_{C^k(R^m)} \leq \epsilon \quad \text{for all} \quad (t,x) \in (T_1,T_2) \times R^n\}, \end{aligned} \tag{2.7}$$

where $\epsilon > 0$. It is easy to see that the space $\mathcal{M}_k$ with this uniformity is metrizable (by a metric $d_{\mathcal{M},k}$) and complete. Define

$$\mathcal{M}_k^l = \mathcal{M}_k \cap \mathcal{M}^l, \quad \mathcal{M}_k^c = \mathcal{M}_k \cap \mathcal{M}^c.$$

Clearly $\mathcal{M}_k^l$ and $\mathcal{M}_k^c$ are closed sets in $\mathcal{M}_k$ with the strong topology.

Finally we define subspaces of $\mathcal{M}$ which consist of integrands differentiable with respect to the state variable $x$ and the control variable $u$. Denote by $\mathcal{M}_k^*$ the set of all $f\colon (T_1,T_2)\times R^n\times R^m \to R^1$ in $\mathcal{M}$ such that for each $t \in (T_1,T_2)$ the function $f(t,\cdot,\cdot) \in C^k(R^n\times R^m)$. We consider the topological subspace $\mathcal{M}_k^* \subset \mathcal{M}$ with the relative weak topology. The strong topology in $\mathcal{M}_k^*$ is induced by the uniformity which is determined by the following base:

$$\begin{aligned} E^*_{\mathcal{M}k}(\epsilon) = \{(f,g) \in \mathcal{M}_k^* \times \mathcal{M}_k^* \colon |f(t,x,u) - g(t,x,u)| \leq \epsilon \\ \text{for all } \ (t,x,u) \in (T_1,T_2) \times R^n \times R^m \ \text{ and} \\ \|f(t,\cdot,\cdot) - g(t,\cdot,\cdot)\|_{C^k(R^{n+m})} \leq \epsilon \quad \text{for all} \quad t \in (T_1,T_2)\}, \end{aligned} \tag{2.8}$$

where $\epsilon > 0$. It is easy to see that the space $\mathcal{M}_k^*$ with this uniformity is metrizable (by a metric $d^*_{\mathcal{M},k}$) and complete. Define

$$\mathcal{M}_k^{*l} = \mathcal{M}_k^* \cap \mathcal{M}^l, \quad \mathcal{M}_k^{*c} = \mathcal{M}_k^* \cap \mathcal{M}^c.$$

Clearly $\mathcal{M}_k^{*l}$ and $\mathcal{M}_k^{*c}$ are closed sets in $\mathcal{M}_k^*$ with the strong topology.

Let $\mathcal{A}_1$ be one of the following spaces:

$$\mathcal{M}, \ \mathcal{M}^l, \ \mathcal{M}^c, \ \mathcal{M}_k, \ \mathcal{M}_k^l, \ \mathcal{M}_k^c, \ \mathcal{M}_k^*, \ \mathcal{M}_k^{*l}, \ \mathcal{M}_k^{*c}.$$

Denote by $\mathcal{A}_{21}$ the set of all matrices $A$ of dimension of $n \times n$. For each $A = (a_{ij})_{i,j=1}^n$ set

$$\|A\| = \max\{|a_{ij}| \colon i,j = 1,\dots,n\}.$$

The space $\mathcal{A}_{21}$ is equipped with the metric $d_{21}$ defined by

$$d_{21}(A,B) = \|A - B\|$$

where $A, B \in \mathcal{A}_{21}$.

Denote by $\mathcal{A}_{22}$ the set of all matrices $A$ of dimension of $n \times m$. For each

$$A = (a_{ij} : \ \ i = 1,\dots,n, \ \ j = 1,\dots,m)$$

set

$$\|A\| = \max\{|a_{ij}|:\ i = 1, \dots, n,\quad j = 1, \dots, m\}.$$

The space $\mathcal{A}_{22}$ is equipped with the metric $d_{22}$ defined by

$$d_{22}(A, B) = \|A - B\|$$

for each $A, B \in \mathcal{A}_{22}$.

Let $\mathcal{A}_{23} = R^n$ be equipped with the metric

$$d_{23}(x, y) = |x - y|,\quad x, y \in R^n.$$

Let $z \in R^n$, $\mathcal{A}_2 = \mathcal{A}_{21} \times \mathcal{A}_{22}$ and let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$.

For each $a_2 = (A, B) \in \mathcal{A}_2$ set

$$S_{a_2} = X(z, A, B).$$

For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ we define $J_a\colon X \to R^1 \cup \{\infty\}$ by

$$J_a(x, u) = I^{(a_1)}(x, u),\quad (x, u) \in S_{a_2},\quad J_a(x, u) = \infty,\quad (x, u) \in X \backslash S_{a_2}. \tag{2.9}$$

It follows from Propositions 4.1 and 4.2 of [12] that $J_a$ is lower semicontinuous for all $a \in \mathcal{A}_1 \times A_2$. It is not difficult to see that for each $a \in \mathcal{A}$, $\inf(J_a)$ is finite. We will establish the following result.

**Theorem 2.1** *There exists a set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$, $\inf(J_a)$ is finite and attained at a unique point $(x_a, u_a) \in X$ and the following assertion holds:*

*For each $\epsilon > 0$ there exist a neighborhood $\mathcal{V}$ of $a$ in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(J_b)$ is finite and if $(z, v) \in X$ satisfies $J_b(z, v) \le \inf(J_b) + \delta$, then $\rho((x_a, u_a), (z, v)) \le \epsilon$ and $|J_b(z, v) - J_a(x_a, u_a)| \le \epsilon$.*

Now we will state our second main result.

Let $\mathcal{A}_2 = \mathcal{A}_{21} \times \mathcal{A}_{22} \times \mathcal{A}_{23}$ and let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. For each $a_2 = (A, B, z) \in \mathcal{A}_2$ we set

$$S_{a_2} = X(z, A, B).$$

For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ we define $\widehat{J}_a\colon X \to R^1 \cup \{\infty\}$ by

$$\widehat{J}_a(x, u) = I^{(a_1)}(x, u),\quad (x, u) \in S_{a_2},\quad \widehat{J}_a(x, u) = \infty,\quad (x, u) \in X \setminus S_{a_2}.$$

It follows from Propositions 4.1 and 4.2 of [12] that $\widehat{J}_a$ is lower semicontinuous for all $a \in \mathcal{A}_1 \times A_2$. It is not difficult to see that for each $a \in \mathcal{A}$, $\inf(\widehat{J}_a)$ is finite. We will establish the following result.

**Theorem 2.2** *There exists a set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$, $\inf(\widehat{J}_a)$ is finite and attained at a unique point $(x_a, u_a) \in X$ and the following assertion holds:*

*For each $\epsilon > 0$ there exist a neighborhood $\mathcal{V}$ of $a$ in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(\widehat{J}_b)$ is finite and if $(z, v) \in X$ satisfies $\widehat{J}_b(z, v) \le \inf(\widehat{J}_b) + \delta$, then $\rho((x_a, u_a), (z, v)) \le \epsilon$ and $|\widehat{J}_b(z, v) - \widehat{J}_a(x_a, u_a)| \le \epsilon$.*

## 3 Variational Principles

We consider a metric space $(X, \rho)$ which is called the domain space and a complete metric space $(\mathcal{A}, d)$ which is called the data space. We always consider the set $X$ with the topology generated by the metric $\rho$. For the space $\mathcal{A}$ we consider the topology generated by the metric $d$. This topology will be called the strong topology and denoted by $\tau_s$. In addition to the strong topology we also consider a weaker topology on $\mathcal{A}$ which is not necessarily Hausdorff. This topology will be called the weak topology and denoted by $\tau_w$. We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function $f_a$ on $X$ is associated with values in $\overline{R} = [-\infty, \infty]$. In our study we use the following basic hypotheses about the functions.

**(H1)** For any $a \in \mathcal{A}$, any $\epsilon > 0$ and any $\gamma > 0$ there exist a nonempty open set $\mathcal{W}$ in $\mathcal{A}$ with the weak topology, $x \in X$, $\alpha \in R^1$ and $\eta > 0$ such that

$$\mathcal{W} \cap \{b \in \mathcal{A} \colon d(a, b) < \epsilon\} \neq \varnothing$$

and for any $b \in \mathcal{W}$

(i) $\inf(f_b)$ is finite;

(ii) if $z \in X$ is such that $f_b(z) \le \inf(f_b) + \eta$, then $\rho(z, x) \le \gamma$ and $|f_b(z) - \alpha| \le \gamma$.

**(H2)** If $a \in \mathcal{A}$, $\inf(f_a)$ is finite, $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence and the sequence $\{f_a(x_n)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges in $X$.

Let $a \in \mathcal{A}$. We say that the minimization problem for $f_a$ on $(X, \rho)$ is strongly well-posed with respect to $(\mathcal{A}, \tau_w)$ if $\inf(f_a)$ is finite and attained at a unique point $x_a \in X$ and the following assertion holds:

For each $\epsilon > 0$ there exist a neighborhood $\mathcal{V}$ of $a$ in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(f_b)$ is finite and if $z \in X$ satisfies $f_b(z) \le \inf(f_b) + \delta$, then $\rho(x_a, z) \le \epsilon$ and $|f_b(z) - f_a(x_a)| \le \epsilon$.

(In a slightly different setting a similar property was introduced in [15].)

The following result was established in [12, Theorem 2.1].

**Theorem 3.1** *Assume that (H1) and (H2) hold. Then there exists a set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$ the minimization problem for $f_a$ on $(X, \rho)$ is strongly well posed with respect to $(\mathcal{A}, \tau_w)$.*

Now we assume that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ where $(\mathcal{A}_i, d_i)$, $i = 1, 2$, are complete metric spaces and

$$d((a_1, a_2), (b_1, b_2)) = d_1(a_1, b_1) + d_2(a_2, b_2), \quad (a_1, a_2), (b_1, b_2) \in \mathcal{A}.$$

For the space $\mathcal{A}_2$ we consider the topology induced by the metric $d_2$ (the strong and weak topologies coincide) and for the space $\mathcal{A}_1$ we consider the strong topology which is induced by the metric $d_1$ and a weak topology which is weaker than the strong topology. The strong topology of $\mathcal{A}$ is the product of the strong topology of $\mathcal{A}_1$ and the topology of $\mathcal{A}_2$ and the weak topology of $\mathcal{A}$ is the product of the weak topology of $\mathcal{A}_1$ and the topology of $\mathcal{A}_2$.

Assume that with every $a \in \mathcal{A}_1$ a function $\phi_a : X \to R^1 \cup \{\infty\}$ is associated and with every $a \in \mathcal{A}_2$ a nonempty set $S_a \subset X$ is associated. For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ define $f_a : X \to R^1 \cup \{\infty\}$ by

$$f_a(x) = \phi_{a_1}(x) \quad \text{for all} \quad x \in S_{a_2}, \quad f_a(x) = \infty \quad \text{for all} \quad x \in X \setminus S_{a_2}. \tag{3.1}$$

Fix $\theta \in \mathcal{A}_2$. We use the following hypotheses.

**(A1)** For each $a \in \mathcal{A}$, $\inf(f_a)$ is finite and $f_a$ is lower semicontinuous.

**(A2)** For each $a_1 \in \mathcal{A}_1$, each $\epsilon > 0$ and each $D > 0$ there exists a neighborhood $\mathcal{V}$ of $a_1$ in $\mathcal{A}_1$ with the weak topology such that for each $b \in \mathcal{V}$ and each $x \in X$ satisfying $\min\{\phi_{a_1}(x), \phi_b(x)\} \le D$ the inequality $|\phi_{a_1}(x) - \phi_b(x)| \le \epsilon$ holds.

**(A3)** For each $(a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, each $\gamma \in (0, 1)$ and each $r \in (0, 1)$ there exist $\bar{a}_1 \in \mathcal{A}_1$, $\bar{x} \in S_{a_2}$, $\delta > 0$ such that $d_1(\bar{a}_1, a_1) < r$ and for each $x \in S_{a_2}$ satisfying $\phi_{\bar{a}_1}(x) \le \inf(f_{(\bar{a}_1, a_2)}) + \delta$ the inequality $\rho(x, \bar{x}) \le \gamma$ is valid.

**(A4)** For each $a_1 \in \mathcal{A}_1$, each $M, D > 0$ and each $\epsilon \in (0, 1)$ there exists a number $\delta > 0$ such that for each $a_2 \in \mathcal{A}_2$ satisfying $d_2(a_2, \theta) \le M$, each $x \in S_{a_2}$ satisfying $\phi_{a_1}(x) \le D$ and each $\xi \in \mathcal{A}_2$ satisfying $d_2(a_2, \xi) \le \delta$ there exists $y \in S_\xi$ such that $\rho(x, y) \le \epsilon$ and $|\phi_{a_1}(x) - \phi_{a_1}(y)| \le \epsilon$.

The following result was proved in [14, Proposition 1.1].

**Proposition 3.1** Assume that (A1)–(A4) hold. Then (H1) holds.

## 4 Proofs of Theorems 2.1 and 2.2

The following result was proved in [12, Lemma 1.1].

**Lemma 4.1** *Let* $a, b \in R^1$, $\epsilon \in (0, 1)$, $\Delta \ge 0$ *and let*

$$|a - b| < (1 + \Delta)\epsilon + \epsilon \max\{|a|, |b|\}.$$

*Then*

$$|a - b| < (1 + \Delta)(\epsilon + \epsilon^2(1 - \epsilon)^{-1}) + \epsilon(1 - \epsilon)^{-1} \min\{|a|, |b|\}.$$

Analogously to Proposition 4.4 of [12] we can prove the following result.

**Proposition 4.1** Let $f \in \mathcal{M}$, $\epsilon \in (0, 1)$ and $D > 0$. Then there exists a neighborhood $\mathcal{V}$ of $f$ in $\mathcal{M}$ with the weak topology such that for each $g \in \mathcal{V}$ and each $(x, u) \in X$ satisfying $\min\{I^f(x, u),\ I^g(x, u)\} \le D$ the inequality $|I^f(x, u) - I^g(x, u)| \le \epsilon$ is valid.

We preface the proofs of our main results by the following lemma.

**Lemma 4.2** *Let* $f \in \mathcal{M}$, $M, D > 0$ *and let* $\epsilon \in (0,1)$. *Then there exists a number* $\delta > 0$ *such that for each* $z \in R^n$, $A \in \mathcal{A}_{21}$, $B \in \mathcal{A}_{22}$ *satisfying*

$$|z| \le M \quad and \quad \|A\|, \|B\| \le M, \tag{4.1}$$

*each*

$$(x,u) \in X(z,A,B) \tag{4.2}$$

*which satisfies*

$$I^{(f)}(x,u) \le D \tag{4.3}$$

*and each* $\xi \in R^n$, $P \in \mathcal{A}_{21}$ *and* $Q \in \mathcal{A}_{22}$ *satisfying*

$$|z-\xi|,\ \|A-P\|,\ \|B-Q\| \le \delta \tag{4.4}$$

*there exists* $(y,v) \in X(\xi,P,Q)$ *such that*

$$v(t) = u(t), \quad t \in (T_1,T_2) \quad a.e., \tag{4.5}$$

$$|x(t)-y(t)| \le \epsilon, \quad t \in [T_1,T_2], \tag{4.6}$$

$$|I^{(f)}(x,u) - I^f(y,v)| \le \epsilon. \tag{4.7}$$

*Proof* By property (iii) (see the definition of $\mathcal{M}$) there is an integrable scalar function $\psi_1(t) \ge 0$, $t \in (T_1,T_2)$, such that

$$|x| + |u| \le \psi_1(t) + f(t,x,u) \quad \text{for all} \quad (t,x,u) \in (T_1,T_2) \times R^n \times R^m. \tag{4.8}$$

Choose a positive number $d_0$ such that

$$d_0 > \sup\{\|e^{\tau C}\| :\ \tau \in [0, T_2-T_1],\ C \in \mathcal{A}_{21} \ \text{and} \ \|C\| \le M+1\}. \tag{4.9}$$

Set

$$\|\psi_1\| = \int_{T_1}^{T_2} \psi_1(t)\,dt. \tag{4.10}$$

Inequality (4.8) implies that for each $(t,x,u) \in (T_1,T_2) \times R^n \times R^m$

$$|f(t,x,u)| \le f(t,x,u) + 2\psi_1(t). \tag{4.11}$$

Choose a number

$$M_0 > 2 + M(\|\psi_1\| + D + 1). \tag{4.12}$$

We show that the following property holds:

*(P)* If $z \in R^n$, $A \in \mathcal{A}_{21}$ and $B \in \mathcal{A}_{22}$ satisfy (4.1) and $(x,u) \in X(z,A,B)$ satisfies (4.3), then

$$|x(t)| \le M_0 - 2 \quad \text{for all} \quad t \in [T_1,T_2]. \tag{4.13}$$

Assume that $z \in R^n$, $A \in \mathcal{A}_{21}$ and $B \in \mathcal{A}_{22}$ satisfy (4.1) and that $(x, u) \in X(z, A, B)$ satisfies (4.3). Then it follows from the definition of $X(z, A, B)$, (2.2), (2.3), (4.1), (4.3), (4.8), (4.10) and (4.12) that for each $t \in [T_1, T_2]$

$$\begin{aligned}
|x(t)| &\le |x(T_1)| + \left| \int_{T_1}^{t} [Ax(s) + Bu(s)]\, ds \right| \\
&\le |x(T_1)| + \|A\| \int_{T_1}^{t} |x(s)|\, ds + \|B\| \int_{T_1}^{t} |u(s)| ds \le M + M \int_{T_1}^{t} (|x(s)| + |u(s)|)\, ds \\
&\le M \left( 1 + \int_{T_1}^{T_2} (|x(s)| + |u(s)|)\, ds \right) \\
&\le M \left( 1 + \int_{T_1}^{T_2} f(s, x(s), u(s))\, ds + \int_{T_1}^{T_2} \psi_1(s)\, ds \right) \\
&\le M(1 + D + \|\psi_1\|) \le M_0 - 2.
\end{aligned}$$

Thus property (P) holds.

Choose a positive number

$$\epsilon_0 < \epsilon \,(T_2 - T_1 + D + 2\|\psi_1\| + 1)^{-1}/4 \tag{4.14}$$

and a positive number $\epsilon_1 < 1$ for which

$$\epsilon_1 + \epsilon_1 (1 - \epsilon_1)^{-1} < \epsilon_0 / 8. \tag{4.15}$$

In view of property (v) (see the definition of $\mathcal{M}$) there exist $\Gamma_0$, $\delta_0 > 0$ such that for almost every $t \in (T_1, T_2)$

$$|f(t, x_1, u) - f(t, x_2, u)| \le \epsilon_1 \max\{|f(t, x_1, u)|, |f(t, x_2, u)|\} + \epsilon_1 \tag{4.16}$$

for each $u \in R^m$ and each $x_1, x_2 \in R^n$ which satisfy

$$|x_i| \le M_0, \quad i = 1, 2, \quad |u| \ge \Gamma_0, \quad |x_1 - x_2| \le 4\delta_0. \tag{4.17}$$

By property (iv) (see the definition of $\mathcal{M}$) there exists a positive number

$$\delta_1 < \min\{\delta_0, \epsilon_1, 1\} \tag{4.18}$$

such that for almost every $t \in (T_1, T_2)$ the inequality

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \le \epsilon_0 \tag{4.19}$$

holds for each $x_1, x_2 \in R^n$ and each $u_1, u_2 \in R^m$ such that

$$|x_i|, |u_i| \le M_0 + \Gamma_0 + 1, \quad i = 1, 2, \quad |x_1 - x_2|, |u_1 - u_2| \le \delta_1. \tag{4.20}$$

Let $\delta_2 > 0$ satisfy

$$(\delta_2 d_0 + M\delta_2)\left(1 + D + \int_{T_1}^{T_2} \psi_1(t)\,dt\right) < \delta_1/4.$$

Choose $\delta > 0$ such that

$$\delta < \min\{1, \delta_1, \delta_2\} \tag{4.22}$$

and that for each $A, P \in \mathcal{A}_{21}$ satisfying

$$\|A\| \le M, \quad \|A - P\| \le \delta$$

and each $\tau \in [0,\, T_2 - T_1]$ the inequality

$$\|e^{\tau P} - e^{\tau A}\| \le \delta_2 \tag{4.23}$$

holds.

Assume that $z \in R^n$, $A \in \mathcal{A}_{21}$ and $B \in \mathcal{A}_{22}$ satisfy (4.1), $(x, u) \in X$ satisfy (4.2), (4.3) and $\xi \in R^n$, $P \in \mathcal{A}_{21}$ and $Q \in \mathcal{A}_{22}$ satisfy (4.4). It follows from (4.2), (2.2) and (2.3) that

$$x(T_1) = z, \tag{4.24}$$

$$x'(t) = Ax(t) + Bu(t), \quad t \in (T_1, T_2) \quad \text{a.e.} \tag{4.25}$$

Relations (4.24) and (4.25) imply that

$$x(t) = e^{(t-T_1)A} z + \int_{T_1}^{t} e^{(t-s)A} Bu(s)\,ds, \quad t \in [T_1, T_2]. \tag{4.26}$$

In view of (4.3) and (4.8) $\int_{T_1}^{T_2} |u(t)|\,dt < \infty$. Define

$$y(t) = e^{(t-T_1)P} \xi + \int_{T_1}^{t} e^{(t-s)P} Qu(s)\,ds, \quad t \in [T_1, T_2]. \tag{4.27}$$

It is not difficult to see that

$$(y, u) \in X(\xi, P, Q). \tag{4.28}$$

It follows from (4.27), (4.26), (4.1), (4.4), (4.22), (4.9) and the choice of $\delta$ (see (4.23)) that for each $t \in [T_1, T_2]$

$$|y(t) - x(t)| = \left| e^{(t-T_1)A} z + \int_{T_1}^{t} e^{(t-s)A} Bu(s)\,ds - e^{(t-T_1)P}\xi - \int_{T_1}^{t} e^{(t-s)P} Qu(s)\,ds \right|$$

$$\le |e^{(t-T_1)P}\xi - e^{(t-T_1)P} z| + |e^{(t-T_1)P} z - e^{(t-T_1)A} z|$$

$$+\left|\int_{T_1}^{t} e^{(t-s)P}Qu(s)ds - \int_{T_1}^{T} e^{(t-s)P}Bu(s)ds\right| + \left|\int_{T_1}^{t} e^{(t-s)P}Bu(s)ds - \int_{T_1}^{t} e^{(t-s)A}Bu(s)ds\right|$$

$$\leq |\xi - z| \sup\{\|e^{\tau C}x\| \colon \tau \in [0, T_2 - T_1],\ C \in \mathcal{A}_{21},\ \|C\| \leq M+1\} + |z|\delta_2$$

$$+ \int_{T_1}^{t} \|e^{(t-s)P}\|\|B-Q\||u(s)|ds + \left(\int_{T_1}^{t} \|B\||u(s)|ds\right) \sup\{\|e^{\tau P} - e^{\tau A}\| \colon\ \tau \in [0, T_2 - T_1]\}$$

$$\leq \delta d_0 + M\delta_2 + d_0\delta \int_{T_1}^{t} |u(s)|ds + \delta_2 M \int_{T_1}^{t} |u(s)|ds$$

$$\leq \delta d_0 + M\delta_2 + \left(\int_{T_1}^{T_2} |u(t)|dt\right)(d_0\delta + \delta_2 M). \tag{4.29}$$

Relations (4.8) and (4.3) imply that

$$\int_{T_1}^{T_2} |u(t)|\,dt \leq \int_{T_1}^{T_2} f(t, x(t), u(t))\,dt + \int_{T_1}^{T_2} \psi_1(t)\,dt \leq D + \int_{T_1}^{T_2} \psi_1(t)\,dt. \tag{4.30}$$

In view of (4.29), (4.30), (4.22) and (4.21) for each $t \in [T_1, T_2]$

$$|y(t) - x(t)| \leq (\delta d_0 + M\delta_2)\left(1 + D + \int_{T_1}^{T_2} \psi_1(t)\,dt\right) < \delta_1/4. \tag{4.31}$$

By property (P), (4.1), (4.2) and (4.3)

$$|x(t)| \leq M_0 - 2, \quad t \in [T_1, T_2]. \tag{4.32}$$

Set

$$\Omega = \{t \in (T_1, T_2) \colon |u(t)| \geq \Gamma_0\}. \tag{4.33}$$

We will estimate

$$\int_{T_1}^{T_2} |f(t, x(t), u(t)) - f(t, y(t), u(t))|\,dt.$$

Clearly

$$\begin{aligned}\int_{T_1}^{T_2} |f(t, x(t), u(t)) - f(t, y(t), u(t))|\,dt &\leq \int_{\Omega} |f(t, x(t), u(t)) - f(t, y(t), u(t))|\,dt \\ &+ \int_{[T_1,T_2]\setminus\Omega} |f(t, x(t), u(t)) - f(t, y(t), u(t))|\,dt.\end{aligned} \tag{4.34}$$

It follows from (4.33), (4.32), (4.31) and the choice of $\Gamma_0, \delta_0$ (see (4.16)–(4.18)) that for almost every $t \in \Omega$

$$|f(t,x(t),u(t)) - f(t,y(t),u(t))| \le \epsilon_1 + \epsilon_1 \max\{|f(t,x(t),u(t))|, |f(t,y(t),u(t))|\}. \quad (4.35)$$

In view of (4.35), (4.15) and Lemma 4.1 for almost every $t \in \Omega$

$$\begin{aligned} |f(t,x(t),u(t)) - f(t,y(t),u(t))| &\le \epsilon_1 + \epsilon_1^2(1-\epsilon_1)^{-1} + \epsilon_1(1-\epsilon_1)^{-1}|f(t,x(t),u(t))| \\ &< \epsilon_0/8 + (\epsilon_0/8)|f(t,x(t),u(t))|. \end{aligned}$$

Combined with (4.8), (4.3), (4.10) and (4.14) this inequality implies that

$$\begin{aligned} \int_\Omega |f(t,x(t),u(t)) - f(t,y(t),u(t))|\,dt &\le \int_{T_1}^{T_2} [\epsilon_0/8 + (\epsilon_0/8)|f(t,x(t),u(t))|\,dt \\ &\le (\epsilon_0/8)(T_2 - T_1) + (\epsilon_0/8)\int_{T_1}^{T_2} (f(t,x(t),u(t)) + 2\psi_1(t))\,dt \\ &\le (\epsilon_0/8)(T_2 - T_1) + (\epsilon_0/8)(D + 2\|\psi_1)\|) < \epsilon/8. \end{aligned} \quad (4.36)$$

It follows from the choice of $\delta_1$ (see (4.18)–(4.20)), (4.33), (4.32) and (4.31) that for almost every $t \in (T_1, T_2)\backslash\Omega$

$$|f(t,x(t),u(t)) - f(t,y(t),u(t))| \le \epsilon_0.$$

Together with (4.14) this implies that

$$\int_{(T_1,T_2)\backslash\Omega} |f(t,x(t),u(t)) - f(t,y(t),u(t))| \le \epsilon_0(T_2 - T_1) < \epsilon/4.$$

Combined with (4.36) and (4.31) this inequality implies that

$$|I^f(x,u) - I^f(y,u)| \le \epsilon/2.$$

This completes the proof of Lemma 4.2.

*Proofs of Theorems 2.1 and 2.2* By Theorem 3.1 and Proposition 3.1 we need only to show that the hypotheses (A1) – (A4) and (H2) hold. We have already noted in Section 2 that (A1) is valid. (H2) follows from Proposition 4.2 of [12]. Proposition 4.1 implies (A2). (A3) follows from Lemma 5.1 of [12]. Lemma 4.2 implies (A4). This completes the proofs of Theorems 2.1 and 2.2.

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# Decentralized $\mathcal{H}_2$ Controller Design for Descriptor Systems: An LMI Approach

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**Abstract:** This paper considers a decentralized $\mathcal{H}_2$ control problem for multi-channel linear time-invariant (LTI) descriptor systems. Our interest is to design a *low order* dynamic output feedback controller. The control problem is reduced to a feasibility problem of a bilinear matrix inequality (BMI) with respect to variables of a coefficient matrix defining the controller, a Lyapunov matrix and a matrix related to the descriptor matrix. Under a matching condition between the descriptor matrix and the measurement output matrix (or the control input matrix), we propose to set the Lyapunov matrix in the BMI as block diagonal appropriately so that the BMI is reduced to LMIs.

**Keywords:** *Multi-channel descriptor system; $\mathcal{H}_2$ control; decentralized control; bilinear matrix inequality (BMI); linear matrix inequality (LMI).*

**Mathematics Subject Classification (2000):** 93B40, 93B50, 93C05, 93C15, 93D25.

## 1 Introduction

It is well known that descriptor systems (also known as singular systems or implicit systems) have high abilities in representing dynamical systems. They can preserve physical parameters in the coefficient matrices, and describe the dynamic part, static part, and even improper part of the system in the same form. In this sense, descriptor systems are much superior to systems represented by state-space models.

There have been reported many works on descriptor systems, e.g., [2, 13, 10]. Among these works, Ref. [10] applied the LMI approach (e.g., [2]) to stabilization and $\mathcal{H}_\infty$ control

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problems for descriptor systems. Since the LMI-type conditions proposed there contain equality constraints, which are not desirable in real applications, Ref. [17] derived strict LMI conditions for stability, robust stabilization and $\mathcal{H}_\infty$ control of linear descriptor systems. Since the strict LMIs are definite ones without equality constraints, they are highly tractable and reliable when we use recent popular softwares for solving LMIs. Later, Ref. [8] extended the consideration to $\mathcal{H}_2$ control problem for descriptor systems and derived a strict LMI condition which is necessary and sufficient for $\mathcal{H}_2$ control.

Concerning decentralized control of descriptor systems, Ref. [9] considered a decentralized stabilization problem for large-scale interconnected descriptor systems, which are special cases of multi-channel descriptor systems. In that context, the design problem was reduced to feasibility of a BMI, and to solve the BMI, a homotopy-based method was proposed, where the interconnections between subsystems are increased gradually from zeros to the given magnitudes. Ref. [20] extended the results in [17] to decentralized $\mathcal{H}_\infty$ control for descriptor systems and proposed strict LMI conditions for designing low order decentralized controller. However, to the best of our knowledge, there is very few existing result considering decentralized $\mathcal{H}_2$ controller design for multi-channel descriptor systems.

Motivated by the above observations, we consider low order decentralized $\mathcal{H}_2$ controller design for multi-channel descriptor systems in this paper. More precisely, for the multi-channel descriptor systems under consideration, in addition to the requirement that the controller should be decentralized (composed of local controllers), we require that the sum of the orders of local controllers should be smaller than the order of the system to be controlled. As pointed out in many references [4, 7], the problem of computing a low order controller is quite difficult. In [18], the homotopy-based algorithm was also extended to low order decentralized $\mathcal{H}_\infty$ controller design for multi-channel LTI systems, by augmenting the matrix variable defining the decentralized controller of desired low order to a matrix variable defining a full order decentralized controller. Although the homotopy-based method in [18] can also be applied for the present problem by some modifications, the convergence of the algorithm depends on how to choose the initial full order centralized controller, and the random search of such a centralized controller introduced in [18] needs huge computational efforts in general.

In this paper, we first apply the existing results in [8] for $\mathcal{H}_2$ control of linear descriptor systems, to express the existence condition of decentralized $\mathcal{H}_2$ controllers with desired orders as a BMI with respect to variables of a coefficient matrix defining the controller, a Lyapunov matrix and a matrix related to the descriptor matrix. As also pointed out in [18], although it is not difficult to obtain such a BMI, there has been no guaranteed method for solving general BMIs, especially of large size [6, 10]. Here, under a matching condition between the descriptor matrix and the measurement output matrix (or the control input matrix), we apply and modify the method developed in [12, 19, 13] so that the BMI on hand is reduced to an LMI [2] which is sufficient to the BMI but much more tractable. More precisely, we propose to set the Lyapunov matrix variable in the BMI as block diagonal appropriately corresponding to the controller's desired order. Because the structure of the block diagonal matrix variables can be set freely, we can consider the controller's order arbitrarily.

The remainder of this paper is organized as follows. In Section 2 we formulate our control problem and rewrite compactly the closed-loop decentralized control system composed of the original descriptor system and the local controllers, by defining some notations. In Section 3, under a matching condition between the descriptor matrix and the measurement output matrix, we derive the first LMI condition for existence of desired controllers by setting the Lyapunov matrix variable in the BMI as block diagonal appropriately. In Section 4, under a matching condition between the descriptor matrix and the control input matrix, we derive the second LMI condition.

## 2 Problem Formulation

We consider the $N$-channel LTI descriptor system described by

$$\begin{aligned} E\dot{x} &= Ax + B_1 w + \sum_{i=1}^{N} B_{2i} u_i, \\ z &= C_1 x, \\ y_i &= C_{2i} x, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

where $x \in R^n$ is the descriptor variable, $w \in R^h$ is the disturbance input, $z \in R^p$ is the controlled output, $u_i \in R^{m_i}$ and $y_i \in R^{q_i}$ are the control input and the measurement output of channel $i$ $(i = 1, 2, \ldots, N)$. The matrices $E$, $A$, $B_1$, $B_{2i}$, $C_1$, $C_{2i}$ are constant and of appropriate size, $N > 1$ is the number of subsystems. The matrix $E$ may be singular and we denote its rank by $r = \operatorname{rank} E \leq n$. Without loss of generality, we assume that for every $i$, $B_{2i}$ is of full column rank, and $C_{2i}$ is of full row rank. Furthermore, to ensure fitness of the $\mathcal{H}_2$ control problem, we assume that the system (1) satisfies the following condition [16, 8]

$$\ker E \subset \ker C_1. \tag{2}$$

For the system (1), we consider a decentralized output feedback controller

$$\begin{aligned} \dot{x}_{ci} &= A_{ci} x_{ci} + B_{ci} y_i, \\ u_i &= C_{ci} x_{ci} + D_{ci} y_i \end{aligned} \tag{3}$$

where $x_{ci} \in R^{n_{ci}}$ is the state of the $i$-th local controller, $n_{ci}$ is a *specified dimension*, and $A_{ci}$, $B_{ci}$, $C_{ci}$, $D_{ci}$, $i = 1, 2, \ldots, N$, are constant matrices to be determined. Since we are interested in designing a low order decentralized controller, we require that $n_c = \sum_{i=1}^{N} n_{ci} < \bar{n} \leq n$, where $\bar{n}$ is the order of the system described by the transfer function $C_1(sE - A)^{-1} B_1$.

The closed-loop system obtained by applying the controller (3) to the system (1) is

$$\begin{aligned} E\dot{x} &= \left( A + \sum_{i=1}^{N} B_{2i} D_{ci} C_{2i} \right) x + \sum_{i=1}^{N} B_{2i} C_{ci} x_{ci} + B_1 w, \\ \dot{x}_{ci} &= B_{ci} C_{2i} x + A_{ci} x_{ci}, \\ z &= C_1 x. \end{aligned} \tag{4}$$

By $T_{zw}(s)$, we denote the transfer function from $w$ to $z$ in the above closed-loop system. Then, the control problem of this paper is stated as follows:

**Decentralized $\mathcal{H}_2$ control problem.** *For a specified scalar* $\gamma > 0$*, design a low order decentralized controller* (3) *for the system* (1) *so that the resultant closed-loop system* (4) *is stable and* $\|T_{zw}(s)\|_2 < \gamma$. If such a decentralized controller exists, we say the descriptor system (1) is stabilizable with $\mathcal{H}_2$ norm $\gamma$ via a decentralized controller (3).

We collect the controller state $x_{ci}$ and the coefficient matrices $A_{ci}$, $B_{ci}$, $C_{ci}$, $D_{ci}$ as

$$\begin{aligned} x_c &= [x_{c1}^{\mathrm T} \quad x_{c2}^{\mathrm T} \quad \dots \quad x_{cN}^{\mathrm T}]^{\mathrm T}, \\ A_{cD} &= \mathrm{diag}\,\{A_{c1},\, A_{c2},\, \dots,\, A_{cN}\}, \\ B_{cD} &= \mathrm{diag}\,\{B_{c1},\, B_{c2},\, \dots,\, B_{cN}\}, \\ C_{cD} &= \mathrm{diag}\,\{C_{c1},\, C_{c2},\, \dots,\, C_{cN}\}, \\ D_{cD} &= \mathrm{diag}\,\{D_{c1},\, D_{c2},\, \dots,\, D_{cN}\}, \end{aligned}$$

and define the matrices

$$\begin{aligned} B_2 &= [\, B_{21} \quad B_{22} \quad \dots \quad B_{2N}\,], \\ C_2 &= [\, C_{21}^{\mathrm T} \quad C_{22}^{\mathrm T} \quad \dots \quad C_{2N}^{\mathrm T}\,]^{\mathrm T} \end{aligned}$$

to describe the closed-loop system (4) as

$$\begin{aligned} E\dot x &= (A + B_2 D_{cD} C_2)x + B_2 C_{cD} x_c + B_1 w, \\ \dot x_c &= B_{cD} C_2 x + A_{cD} x_c, \\ z &= C_1 x. \end{aligned} \tag{5}$$

Since it is reasonable to consider the case where all the input/output channels are independent, we assume that $B_2$ is of full column rank and $C_2$ is of full row rank.

We further write the matrices $A_{cD}$, $B_{cD}$, $C_{cD}$ and $D_{cD}$ in a single matrix

$$G_D = \begin{bmatrix} A_{cD} & B_{cD} \\ C_{cD} & D_{cD} \end{bmatrix} \tag{6}$$

and introduce the notations

$$\begin{aligned} [\,\widetilde E \quad \tilde A\,] &= \left[\begin{array}{cc|cc} E & 0 & A & 0_{n\times n_c} \\ 0 & I_{n_c} & 0_{n_c\times n} & 0_{n_c\times n_c} \end{array}\right], \\ [\,\widetilde B_1 \quad \widetilde B_2\,] &= \left[\begin{array}{c|cc} B_1 & 0_{n\times n_c} & B_2 \\ 0_{n_c\times h} & I_{n_c} & 0_{n_c\times m} \end{array}\right], \\ \begin{bmatrix} \widetilde C_1 \\ \widetilde C_2 \end{bmatrix} &= \left[\begin{array}{cc} C_1 & 0_{p\times n_c} \\ \hline 0_{n_c\times n} & I_{n_c} \\ C_2 & 0_{q\times n_c} \end{array}\right], \end{aligned}$$

where $m = \sum\limits_{i=1}^{N} m_i,\ q = \sum\limits_{i=1}^{N} q_i$. Then, the system (5) is written in a compact form as

$$\begin{aligned} \widetilde E\dot{\tilde x} &= (\tilde A + \widetilde B_2 G_D \widetilde C_2)\tilde x + \widetilde B_1 w, \\ z &= \widetilde C_1 \tilde x, \end{aligned} \tag{7}$$

where $\tilde x = [x^{\mathrm T} x_c^{\mathrm T}]^{\mathrm T} \in R^{n+n_c}$. In this description, only the controller coefficient matrix $G_D$ is unknown, while all the other matrices are given by the system (1) and specified orders of local controllers.

## 3 Controller Design I

We first recall an existing result for $\mathcal{H}_2$ control of linear descriptor systems.

**Lemma 1** [8] *Consider the linear descriptor system described by*

$$\begin{aligned} E\dot{x} &= Ax + Bw, \\ z &= Cx, \end{aligned} \tag{8}$$

*where $x \in R^n$ is the descriptor variable, $w \in R^h$ is the disturbance input, $z \in R^p$ is the controlled output, and $E$, $A$, $B$, $C$ are constant matrices of appropriate size. The matrix $E$ may be singular and* rank $E = r \le n$. *Let matrices $V, U \in R^{n\times(n-r)}$ be of full column rank and composed of bases of* Null $E$ *and* Null $E^{\mathrm{T}}$*, respectively. Assume that the fitness condition (2) is true between $E$ and $C$. Then, for a given positive scalar $\gamma$, the system (8) is stable and $\|C(sE-A)^{-1}B\|_2 < \gamma$ if and only if there exist $P > 0$ and $S$ satisfying the LMIs*

$$A(PE^{\mathrm{T}} + VSU^{\mathrm{T}}) + (PE^{\mathrm{T}} + VSU^{\mathrm{T}})^{\mathrm{T}} A^{\mathrm{T}} + BB^{\mathrm{T}} < 0,$$
$$\mathrm{trace}\,[CPC^{\mathrm{T}}] < \gamma^2.$$

Translating Lemma 1 in terms of the closed-loop system (7), we see that the decentralized $\mathcal{H}_2$ control problem is reduced to solving the matrix inequalities

$$(\tilde{A} + \widetilde{B}_2 G_D \widetilde{C}_2)(\widetilde{P}\widetilde{E}^{\mathrm{T}} + \widetilde{V}\widetilde{S}\widetilde{U}^{\mathrm{T}}) + (\widetilde{P}\widetilde{E}^{\mathrm{T}} + \widetilde{V}\widetilde{S}\widetilde{U}^{\mathrm{T}})^{\mathrm{T}}(\tilde{A} + \widetilde{B}_2 G_D \widetilde{C}_2)^{\mathrm{T}} + \widetilde{B}_1\widetilde{B}_1^{\mathrm{T}} < 0, \tag{9}$$

$$\mathrm{trace}\,[\widetilde{C}_1\widetilde{P}\widetilde{C}_1^{\mathrm{T}}] < \gamma^2 \tag{10}$$

with respect to $G_D$, $\widetilde{P} > 0$ and $\widetilde{S}$, where

$$\widetilde{V} = \begin{bmatrix} V \\ 0_{n_c\times(n-r)} \end{bmatrix}, \quad \widetilde{U} = \begin{bmatrix} U \\ 0_{n_c\times(n-r)} \end{bmatrix}.$$

It is observed from the above that the existence condition (9) for a desired decentralized $\mathcal{H}_2$ controller is a BMI with respect to $(\widetilde{P}, \widetilde{S})$ and $G_D$, and at present there is no globally effective method to solve general BMI problems. Although global optimization approaches using branch and bound methods for general BMIs have been proposed [6, 10], the necessary computational efforts would be prohibitive when their methods are applied to solve our BMI for systems of high dimensions in unlimited regions of the matrix variables in (9). Another algorithm has been proposed in [18] for solving the BMI (9) by using the idea of the homotopy method, where the controller's coefficient matrices are deformed from full matrices defined by a centralized controller, to block diagonal matrices of specified dimensions which describe a decentralized controller. Since the convergence of the algorithm in [18] depends on the choice of the initial centralized controller, a random search has been proposed for such centralized controller. However, for large scale problems, the computation efforts for such random search is still very large. For this reason, we propose to set the Lyapunov matrix variable in (9) as block diagonal appropriately so that the BMI (9) is reduced to an LMI, which is easy to solve by using the existing softwares (for example, the LMI Control Toolbox of MATLAB [5]).

Throughout this section, we assume:

**Assumption 1** There exists a matrix $C_{2e}$ such that $C_2 = C_{2e}E$.

This assumption requires a *matching condition* between the descriptor matrix $E$ and the measurement output matrix $C_2$, which implies that the null space of $E$ is included in that of $C_2$. We note that the measurement output in control systems is the quantity that we can adjust in real implementation, and thus Assumption 1 is not an unrealistic condition.

**Theorem 1** *The system (1) under Assumption 1 is stabilizable with $\mathcal{H}_2$ norm $\gamma$ via a decentralized controller (3) if there exist a matrix $\widetilde{S} \in R^{(n-r)\times(n-r)}$, a positive definite matrix $\widehat{P}$ structured as*

$$\widehat{P} = \begin{bmatrix} \widehat{P}_1 & 0 \\ 0 & \widehat{P}_2 \end{bmatrix}, \qquad \widehat{P}_1 = \begin{bmatrix} \widehat{P}_A & \widehat{P}_B \\ \widehat{P}_B^{\mathrm{T}} & \widehat{P}_D \end{bmatrix},$$
$$\widehat{P}_A = \operatorname{diag}\{\widehat{P}_{A1}, \widehat{P}_{A2}, \ \ldots, \ \widehat{P}_{AN}\},$$
$$\widehat{P}_B = \operatorname{diag}\{\widehat{P}_{B1}, \widehat{P}_{B2}, \ \ldots, \ \widehat{P}_{BN}\},$$
$$\widehat{P}_D = \operatorname{diag}\{\widehat{P}_{D1}, \widehat{P}_{D2}, \ \ldots, \ \widehat{P}_{DN}\}$$

*with $\widehat{P}_{Ai} \in R^{n_{ci}\times n_{ci}}$, $\widehat{P}_{Bi} \in R^{n_{ci}\times q_i}$, $\widehat{P}_{Di} \in R^{q_i\times q_i}$, and a matrix $W$ structured as*

$$W = \begin{bmatrix} W_A & W_B \\ W_C & W_D \end{bmatrix}, \tag{11}$$
$$W_A = \operatorname{diag}\{W_{A1}, W_{A2}, \ \ldots, \ W_{AN}\},$$
$$W_B = \operatorname{diag}\{W_{B1}, W_{B2}, \ \ldots, \ W_{BN}\},$$
$$W_C = \operatorname{diag}\{W_{C1}, W_{C2}, \ \ldots, \ W_{CN}\},$$
$$W_D = \operatorname{diag}\{W_{D1}, W_{D2}, \ \ldots, \ W_{DN}\}$$

*with $W_{Ai} \in R^{n_{ci}\times n_{ci}}$, $W_{Bi} \in R^{n_{ci}\times q_i}$, $W_{Ci} \in R^{m_i\times n_{ci}}$, $W_{Di} \in R^{m_i\times q_i}$, such that the LMIs*

$$\Phi_1 + \Phi_1^{\mathrm{T}} + \widehat{B}_1\widehat{B}_1^{\mathrm{T}} < 0, \tag{12}$$
$$\Phi_1 = \hat{A}(\widehat{P}\widehat{E}^{\mathrm{T}} + \hat{V}\widetilde{S}\hat{U}^{\mathrm{T}}) + \widehat{B}_2\,[\,W \quad 0\,]\,\widehat{E}^{\mathrm{T}},$$
$$\operatorname{trace}[\hat{C}_1\widehat{P}\hat{C}_1^{\mathrm{T}}] < \gamma^2 \tag{13}$$

*hold. Here, $\widehat{E} = T^{-1}\widetilde{E}T$, $\hat{A} = T^{-1}\tilde{A}T$, $\widehat{B}_1 = T^{-1}\widetilde{B}_1$, $\widehat{B}_2 = T^{-1}\widetilde{B}_2$, $\hat{C}_1 = \widetilde{C}_1T$, $\hat{V} = T^{-1}\widetilde{V}$, $\hat{U} = T^{-1}\widetilde{U}$, and $T \in R^{(n+n_c)\times(n+n_c)}$ is a nonsingular matrix satisfying*

$$\widetilde{C}_2T = [\,I_{n_c+q} \quad 0\,]\,. \tag{14}$$

*When the LMIs (12) – (13) are feasible, one desired controller is computed as*

$$G_D = W\widehat{P}_1^{-1}. \tag{15}$$

*Proof* We first note that since we have assumed in the previous section that $C_2$ is of full row rank, $\widetilde{C}_2$ is also of full row rank, and thus there always exists a nonsingular matrix $T$ such that (14) is satisfied. Although such a matrix is not unique, we can see later that the choice of $T$ does not affect the feasibility of the LMIs (12)−(13).

Pre-multiplying the first LMI (12) by $T$ and post-multiplying it by $T^{\mathrm{T}}$, and then substituting all the notations we defined together with $\widetilde{P} = T\widehat{P}T^{\mathrm{T}}$, we obtain

$$\begin{aligned} \tilde{\Phi}_1 + \tilde{\Phi}_1^{\mathrm{T}} + \widetilde{B}_1\widetilde{B}_1^{\mathrm{T}} < 0 \\ \tilde{\Phi}_1 = \tilde{A}(\widetilde{P}\widetilde{E}^{\mathrm{T}} + \widetilde{V}\widetilde{S}\widetilde{U}^{\mathrm{T}}) + \widetilde{B}_2\,[\,W \quad 0\,]\,T^{\mathrm{T}}\widetilde{E}^{\mathrm{T}}. \end{aligned} \tag{16}$$

It is easy to confirm from (14) and (15) that

$$[\,W \quad 0\,] = G_D\widetilde{C}_2T\widehat{P},$$

and that

$$\widetilde{C}_2\widetilde{V} = \begin{bmatrix} 0_{n_c\times n} & I_{n_c} \\ C_2 & 0_{q\times n_c} \end{bmatrix} \begin{bmatrix} V \\ 0_{n_c\times(n-r)} \end{bmatrix} = \begin{bmatrix} 0 \\ C_2V \end{bmatrix} = \begin{bmatrix} 0 \\ C_{2e}EV \end{bmatrix} = 0.$$

Thus, we obtain from (16) that

$$\begin{aligned} \tilde{A}(\widetilde{P}\widetilde{E}^{\mathrm{T}} + \widetilde{V}\widetilde{S}\widetilde{U}^{\mathrm{T}}) + (\widetilde{P}\widetilde{E}^{\mathrm{T}} + \widetilde{V}\widetilde{S}\widetilde{U}^{\mathrm{T}})^{\mathrm{T}}\tilde{A}^{\mathrm{T}} + \widetilde{B}_2G_D\widetilde{C}_2\widetilde{V}\widetilde{S}\widetilde{U}^{\mathrm{T}} + (\widetilde{B}_2G_D\widetilde{C}_2\widetilde{V}\widetilde{S}\widetilde{U}^{\mathrm{T}})^{\mathrm{T}} \\ + \widetilde{B}_2G_D\widetilde{C}_2\widetilde{P}\widetilde{E}^{\mathrm{T}} + (\widetilde{B}_2G_D\widetilde{C}_2\widetilde{P}\widetilde{E}^{\mathrm{T}})^{\mathrm{T}} + \widetilde{B}_1\widetilde{B}_1^{\mathrm{T}} < 0 \end{aligned}$$

which is exactly the matrix inequality (9). Since the second LMI (13) is the same as (10), we declare that the closed-loop system (7) with (15) is stable with $\mathcal{H}_2$ norm $\gamma$.

What we have to do next is to prove that the controller coefficient matrix $G_D$ given by (15) has the decentralized structure defined in (6). Since we required $\widehat{P} > 0$ in the theorem, we get $\widehat{P}_A > 0$ and $\widehat{P}_D > 0$. Then, it is not difficult to obtain that

$$\widehat{P}_1^{-1} = \begin{bmatrix} \overline{P}_A & \overline{P}_B \\ \overline{P}_B^{\mathrm{T}} & \overline{P}_D \end{bmatrix}$$

where

$$\begin{aligned} \overline{P}_A &= \widehat{P}_A^{-1} + \widehat{P}_A^{-1}\widehat{P}_B(\widehat{P}_D - \widehat{P}_B^{\mathrm{T}}\widehat{P}_A^{\mathrm{T}}\widehat{P}_B)^{-1}\widehat{P}_B^{\mathrm{T}}\widehat{P}_A^{-1} \\ \overline{P}_B &= -\widehat{P}_A^{-1}\widehat{P}_B(\widehat{P}_D - \widehat{P}_B^{\mathrm{T}}\widehat{P}_A^{-1}\widehat{P}_B)^{-1} \\ \overline{P}_D &= (\widehat{P}_D - \widehat{P}_B^{\mathrm{T}}\widehat{P}_A^{-1}\widehat{P}_B)^{-1}. \end{aligned}$$

Since $\widehat{P}_A$, $\widehat{P}_B$, $\widehat{P}_D$ are block diagonal, $\overline{P}_A$, $\overline{P}_B$ and $\overline{P}_D$ are block diagonal too. Then, we obtain from (15) that

$$G_D = W\widehat{P}_1^{-1} = \begin{bmatrix} W_A\overline{P}_A + W_B\overline{P}_B^{\mathrm{T}} & W_A\overline{P}_B + W_B\overline{P}_D \\ W_C\overline{P}_A + W_D\overline{P}_B^{\mathrm{T}} & W_C\overline{P}_B + W_D\overline{P}_D \end{bmatrix}. \tag{17}$$

Since $W_A$, $W_B$, $W_C$, $W_D$ are block diagonal, we see that all the four elements in (17) are block diagonal and thus the above $G_D$ has the decentralized structure specified in (6).

*Remark 1* It is understood from the above proof that the block diagonal structures of $W$ and $\widehat{P}_1$ are designed so that a decentralized controller is obtained, and the block diagonal structure of $\widehat{P}$ is assumed so that the coupling between $G_D$ and $\widetilde{P}$ can be removed by using some equivalent transformation. Although the structures of the variables are complicated at a first glimpse, the matrix inequalities (12)−(13) are linear with respect to $\widetilde{S}$, $\widehat{P}$, $W$, and thus are very easy to solve by using the existing software LMI Control Toolbox [5].

## 4 Controller Design II

In this section, we assume:

**Assumption 2** There exists a matrix $B_{2e}$ such that $B_2 = EB_{2e}$.

This assumption requires a *matching condition* between the descriptor matrix $E$ and the control input matrix $B_2$, which implies that the space spanned by $B_2$ is included in that by $E$. We note that the control input in control systems is the quantity that we can adjust in real implementation, and thus Assumption 2 is not an unrealistic condition.

To proceed, we first derive another form of Lemma 1 for the benefit of the discussion in this section. To do this, we consider the same system (8) as in Lemma 1. Noticing that $\|C(sE-A)^{-1}B\|_2 < \gamma$ is equivalent to $\|B^{\mathrm{T}}(sE^{\mathrm{T}}-A^{\mathrm{T}})^{-1}C^{\mathrm{T}}\|_2 < \gamma$ together with the fact

$$(E^{\mathrm{T}})^{\mathrm{T}}V = 0, \quad (E^{\mathrm{T}})U = 0,$$

we apply Lemma 1 to the dual system of (8), described by $(E^{\mathrm{T}}, A^{\mathrm{T}}, C^{\mathrm{T}}, B^{\mathrm{T}})$, to obtain the following result. It is noted that the result has also appeared in [8].

**Lemma 2** *For a given positive scalar $\gamma$, the system (8) is stable and $\|C(sE-A)^{-1}B\|_2 < \gamma$ if and only if there exist $Q > 0$ and $R$ satisfying the LMIs*

$$A^{\mathrm{T}}(QE + URV^{\mathrm{T}}) + (QE + URV^{\mathrm{T}})^{\mathrm{T}}A + C^{\mathrm{T}}C < 0$$
$$\mathrm{trace}\,[B^{\mathrm{T}}QB] < \gamma^2.$$

Translating Lemma 2 in terms of the closed-loop system (7), we see that the decentralized $\mathcal{H}_2$ control problem is reduced to solving the matrix inequalities

$$\big(\tilde{A} + \widetilde{B}_2 G_D \widetilde{C}_2\big)^{\mathrm{T}}(\widetilde{Q}\widetilde{E} + \widetilde{U}\widetilde{R}\widetilde{V}^{\mathrm{T}}) + (\widetilde{Q}\widetilde{E} + \widetilde{U}\widetilde{R}\widetilde{V}^{\mathrm{T}})^{\mathrm{T}}\big(\tilde{A} + \widetilde{B}_2 G_D \widetilde{C}_2\big) + \widetilde{C}_1^{\mathrm{T}}\widetilde{C}_1 < 0, \tag{18}$$
$$\mathrm{trace}\,[\widetilde{B}_1^{\mathrm{T}}\widetilde{Q}\widetilde{B}_1] < \gamma^2 \tag{19}$$

with respect to $G_D$, $\widetilde{Q} > 0$ and $\widetilde{R}$. Same as in the previous section, the matrix inequality (18) is a BMI with respect to $(\widetilde{Q}, \widetilde{R})$ and $G_D$, there is no globally effective method for solving it. Here, under Assumption 2, we propose to set the Lyapunov matrix variable $\widetilde{Q}$ as block diagonal appropriately so that the BMI (18) is reduced to an LMI.

**Theorem 2** *The system (1) under Assumption 2 is stabilizable with $\mathcal{H}_2$ norm $\gamma$ via a decentralized controller (3) if there exist a matrix $\widetilde{R} \in R^{(n-r)\times(n-r)}$, a positive definite matrix $\widehat{Q}$ structured as*

$$\widehat{Q} = \begin{bmatrix} \widehat{Q}_1 & 0 \\ 0 & \widehat{Q}_2 \end{bmatrix}, \quad \widehat{Q}_1 = \begin{bmatrix} \widehat{Q}_A & \widehat{Q}_B \\ \widehat{Q}_B^{\mathrm{T}} & \widehat{Q}_D \end{bmatrix},$$
$$\widehat{Q}_A = \mathrm{diag}\,\{\widehat{Q}_{A1}, \widehat{Q}_{A2}, \ldots, \widehat{Q}_{AN}\},$$
$$\widehat{Q}_B = \mathrm{diag}\,\{\widehat{Q}_{B1}, \widehat{Q}_{B2}, \ldots, \widehat{Q}_{BN}\},$$
$$\widehat{Q}_D = \mathrm{diag}\,\{\widehat{Q}_{D1}, \widehat{Q}_{D2}, \ldots, \widehat{Q}_{DN}\},$$

*with $\widehat{Q}_{Ai} \in R^{n_{ci}\times n_{ci}}$, $\widehat{Q}_{Bi} \in R^{n_{ci}\times m_i}$, $\widehat{Q}_{Di} \in R^{m_i\times m_i}$, and a matrix $W$ structured as (11) such that the LMIs*

$$\Upsilon_1 + \Upsilon_1^{\mathrm{T}} + \breve{C}_1^{\mathrm{T}}\breve{C}_1 < 0, \tag{20}$$
$$\Upsilon_1 = (\breve{E}^{\mathrm{T}}\widehat{Q} + \breve{V}\widetilde{R}^{\mathrm{T}}\breve{U}^{\mathrm{T}})\breve{A} + \breve{E}^{\mathrm{T}} \begin{bmatrix} W \\ 0 \end{bmatrix} \breve{C}_2$$
$$\mathrm{trace}\,[\breve{B}_1^{\mathrm{T}}\widehat{Q}\breve{B}_1] < \gamma^2 \tag{21}$$

*hold. Here, $\breve{E} = X\widetilde{E}X^{-1}$, $\breve{A} = X\tilde{A}X^{-1}$, $\breve{B}_1 = X\widetilde{B}_1$, $\breve{C}_1 = \widetilde{C}_1X^{-1}$, $\breve{C}_2 = \widetilde{C}_2X^{-1}$, $\breve{V} = \left(X^{-1}\right)^{\mathrm{T}}\widetilde{V}$, $\breve{U} = \left(X^{-1}\right)^{\mathrm{T}}\widetilde{U}$, and $X \in R^{(n+n_c)\times(n+n_c)}$ is a nonsingular matrix satisfying*

$$X\widetilde{B}_2 = \begin{bmatrix} I_{n_c+m} \\ 0 \end{bmatrix}. \tag{22}$$

*When the LMIs (20) – (21) are feasible, one desired controller is computed as*

$$G_D = \widehat{Q}_1^{-1}W. \tag{23}$$

*Proof* We first note that since we have assumed that $B_2$ is of full column rank, $\widetilde{B}_2$ is also of full column rank, and thus there always exists a nonsingular matrix $X$ such that (22) is satisfied. Also, we can see later that the choice of $X$ does not affect the feasibility of the LMIs (20) – (21).

Pre-multiplying the first LMI (20) by $X^{\mathrm{T}}$ and post-multiplying it by $X$, and then substituting all the notations we defined with $\widetilde{Q} = X^{\mathrm{T}}\widehat{Q}X$, we obtain

$$\tilde{\Upsilon}_1 + \tilde{\Upsilon}_1^{\mathrm{T}} + \widetilde{C}_1^{\mathrm{T}}\widetilde{C}_1 < 0, \tag{24}$$
$$\tilde{\Upsilon}_1 = (\widetilde{E}^{\mathrm{T}}\widetilde{Q} + \widetilde{V}\widetilde{R}^{\mathrm{T}}\widetilde{U}^{\mathrm{T}})\tilde{A} + \widetilde{E}^{\mathrm{T}}X^{\mathrm{T}} \begin{bmatrix} W \\ 0 \end{bmatrix} \widetilde{C}_2.$$

According to (22) and (23), we compute

$$\widehat{Q}X\widetilde{B}_2G_D = \begin{bmatrix} \widehat{Q}_1 & 0 \\ 0 & \widehat{Q}_2 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \widehat{Q}_1^{-1}W = \begin{bmatrix} W \\ 0 \end{bmatrix}.$$

Together with the fact

$$\widetilde{B}_2^{\mathrm{T}}\widetilde{U} = \begin{bmatrix} 0 & I_{\hat{n}} \\ B_2^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} U \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ B_{2e}^{\mathrm{T}} E^{\mathrm{T}} U \end{bmatrix} = 0, \tag{25}$$

we obtain (18) easily from (24)–(25).

Since the second LMI (21) is the same as (19), and the decentralized structure of $G_D = \widehat{Q}_1^{-1}W$ can be proved by using the same technique as used in Theorem 1, we conclude that the system (1) is stabilized with $\mathcal{H}_2$ norm $\gamma$ via the decentralized controller (3) given by (23).

*Remark 2* Although Theorems 1 and 2 come up with dual forms, they are not equivalent and are supposed to deal with different cases of Assumption 1 or Assumption 2, respectively. Furthermore, the LMI conditions provided by the theorems are sufficient ones. Therefore, even in the case where both Assumption 1 and Assumption 2 hold and thus both theorems can be applied, the LMI conditions of one theorem would be satisfied while the other would not.

*Remark 3* When it is necessary, we can try to obtain a tight $\mathcal{H}_2$ norm $\gamma$ by considering the generalized eigenvalue problem (EVP) [2]: "minimize $\gamma^2$, s.t. (12) – (13) or (20) – (21), respectively".

## 5 Conclusion

This paper has considered a decentralized $\mathcal{H}_2$ control problem for multi-channel linear time-invariant (LTI) descriptor systems. We first reduce the control problem to a feasibility problem of a bilinear matrix inequality (BMI) with respect to variables of a coefficient matrix defining the controller, a Lyapunov matrix and a matrix related to the descriptor matrix. Then, under a matching condition between the descriptor matrix and the measurement output matrix (or the control input matrix), we has proposed to set the Lyapunov matrix in the BMI as block diagonal appropriately so that the BMI is reduced to LMIs. Since the structure of the block diagonal matrix variables can be set freely, we can consider the controller's order arbitrarily. We suggest that the present approach should be applicable for any controller design problem with controller structure constraints.

Noting that there are several references [1, 14] dealing with $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ control of descriptor systems also using the matrix inequality approach, our future research interest includes the extension of the results in the present paper to the case of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ decentralized control for time-delay descriptor systems. Stochastic or probabilistic control [1] is another interesting issue for descriptor systems.

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