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On The Dependence of Fixed Point Sets of Pseudo-Contractive Multifunctions. Application to Differential Inclusions

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Abstract: A weakened notion of multivalued contraction mapping is introduced. Some fixed point results relying on this notion are presented. The associated fixed points sets are shown to enjoy a Lipschitzian behaviour with respect to the graphs of the multifunctions. Applications are given to the dependence of solutions of differential inclusions of the form $\dot{x}(t) \in R(t, x(t))$ on initial values or on the right-hand sides or on parameters.

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1 Introduction

Studies about the behavior of fixed points are far less abundant than existence results (let us mention [23 − 25, 29]). However such studies are important; for instance they can be used to describe the dependence of solutions to differential inclusions or partial differential equations on some parameters or on boundary data.

Since in general the fixed points are not unique, one is led to use concepts of convergence of sets. Such concepts abound (see [1, 6, 8, 21, 31] for instance). But since we are interested in quantitative estimates and not only in qualitative results, we are led to use a recent variant of the Pompeiu–Hausdorff distance or hemi-metric (see [2, 3, 7, 20, 26, 27, 31]). In these developments, briefly recalled below, the stringent convergence relying on the Pompeiu–Hausdorff hemi-metric is replaced by a convergence relativized

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to bounded sets (the so-called bounded hemi-convergence or bounded convergence or Attouch–Wets convergence). This more realistic approach is justified by a number of facts and results: in finite dimensional spaces, bounded convergence coincides with the classical Painlevé–Kuratowski convergence; convergence in norms of operators is equivalent to bounded convergence of their graphs (see [20, 34]). Moreover the continuity of several usual operations can be obtained for this type of convergence (see [7, 31] for example).

Although the dependence of fixed point sets is a subject which is not limited to the case of Lipschitzian multifunctions, we only consider this case here ; for other approaches see for instance [4]. The reason lies in the fact that in the Lipschitzian case one disposes of an estimate about the distance of a given base point to the fixed point sets ([19, 13, 32]); in [32] a first step towards the study of the dependence of the fixed point sets was made. Here we complete this study in a more symmetric and systematic way (Section 2). Moreover we show how these results can be illustrated by an application to differential inclusions (Section 3). In particular we reveal a connection with a famous result of Filippov (see [5, 14, 15, 35, 36, 37]): while we just give a new method to get the existence theorem, our perturbation results seem to be new.

In the sequel (X, d) is a metric space. Given $x_0 \in X$, $r > 0$, we denote by $B(x_0, r)$ (resp. $U(x_0, r)$) the closed (resp. open) ball with center x_0 and radius r. Given a base point $x_0 \in X$ and given subsets C, $D \subset X$, we set, for $r > 0$,

$$e_r(C, D) = e(C \cap U(x_0, r), D)$$

and

$$h_r(C, D) = \max\{e_r(C, D), e_r(D, C)\}$$

with $e(\varnothing, D) = 0$,

$$e(C, D) = \sup_{x \in C} d(x, D) \quad \text{if} \quad D \neq \varnothing, \quad e(C, \varnothing) = +\infty \quad \text{if} \quad C \neq \varnothing,$$
$$d(x, D) = \inf_{z \in D} d(x, z) \quad \text{with the convention} \quad \inf_{\varnothing} = +\infty.$$

In the preceding definition we used open balls $U(x_0, r)$ for technical reasons: many proofs are simpler when using these balls. The reader would easily convince himself that the use of closed balls would not produce any significant change in the results of this paper. Since it is the use of the whole family $(h_r)_{r>0}$ which is important, it is clear that the choice of balls is unessential. We shall also use the classical Pompeiu–Hausdorff metric

$$h(C, D) = \max\{e(C, D),\, e(D, C)\}.$$

A multifunction F from a set X to a set Y is considered as a subset of $X \times Y$. For all $x \in X$, $F(x)$ denotes the (possibly empty) set of $y \in Y$ such that $(x, y) \in F$. The multifunction $F^{-1} \subset Y \times X$ is defined by $F^{-1} = \{(y, x)\colon (x, y) \in F\}$. A fixed point of a multifunction $F\colon X \rightrightarrows X$ is an element $x \in X$ such that $x \in F(x)$. We denote by Φ_F the set of fixed points of F. Given $\theta \in R_+$, we say that a multifunction $F\colon X \rightrightarrows X$ is *pseudo-θ-Lipschitzian* with respect to the subset $U \subset X$ whenever for all $x, x' \in U$

$$e(F(x) \cap U,\, F(x')) \leq \theta d(x, x').$$

It is said to be *pseudo-θ-contractive* with respect to U if it is pseudo-θ-Lipschitzian with respect to U for some $\theta \in [0,1)$. The multifunction $F \subset X \times X$ is said to be θ-Lipschitzian whenever

$$h(F(x), F(x')) \le \theta d(x, x')$$

for all $x, x' \in X$; it is said to be *θ-contractive* if it is θ-Lipschitzian with $\theta \in (0,1)$. The limit inferior of a sequence (C_n) of closed subsets of a metric space (X,d) is the set of those $x \in X$ such that $\lim_{n\to\infty} d(x, C_n) = 0$. Equivalently it is the set of $x \in X$ for which there exists a sequence (x_n) converging to x such that $x_n \in C_n$ eventually.

In the sequel a product $X \times Y$ of metric spaces will be endowed with the box distance given by

$$d((x,y),(x',y')) = \max\{d(x,x'), d(y,y')\}.$$

Remark 1.1 Let $F \subset X \times X$ be a multifunction such that for some $x_0 \in X$, $r > 0$ and $\theta \in R_+$ the multifunction $F_r(x) = F(x) \cap U(x_0, r)$ is θ-Lipschitzian on $U(x_0, r)$. Then F is pseudo-θ-Lipschitzian with respect to $U(x_0, r)$ since for any x, $x' \in X$

$$e(F(x) \cap U(x_0, r), F(x')) \le e(F_r(x), F_r(x')).$$

Nevertheless the converse is false as shown by the following simple example. Let $\theta \in [0,4)$, $r = 1$ and let $f\colon R \to R$ be the θ-Lipschitzian function defined by $f(x) = \theta|x| + 1 - \theta/2$. Then f is pseudo-θ-Lipschitzian with respect to $U(0,1)$ since f is θ-Lipschitzian and $\{f(x)\} \cap U(0,1)$ is either empty or equal to $\{f(x)\}$. Now

$$e\Big(\{f(0)\} \cap U(0,1), \Big\{f\Big(\frac{1}{2}\Big)\Big\} \cap U(0,1)\Big) = +\infty$$

since $\{f(0)\} \cap U(0,1) \ne \varnothing$ and $\{f(\frac{1}{2})\} \cap U(0,1) = \varnothing$.

2 Fixed Points of Pseudo-Contractive Multifunctions

In this section we consider the behavior of the fixed point set $\Phi_F = \{x \in X\colon\ x \in F(x)\}$ of a pseudo-contractive multifunction $F\colon X \rightrightarrows X$ as F is perturbed. The existence of fixed points for such multifunctions is well known (see [13, 19 (Lemma 1, p.31), 32 (Proposition 2.5)]). In many cases they are obtained by iterative techniques of one sort or another (see [22, 28]). Such results extend widely the well known result of S.B. Nadler in [28]. Since the estimates of the existence result are crucial for what follows, we give a proof for the convenience of the reader.

Proposition 2.1 *Let (X,d) be a complete metric space and let $F\colon X \rightrightarrows X$ be a multifunction with closed nonempty values which is assumed to be pseudo-θ-contractive with respect to some ball $U(x_0, r)$ with $r > (1-\theta)^{-1} d(x_0, F(x_0))$. Then for any $\beta > d(x_0, F(x_0))$ such that $\beta(1-\theta)^{-1} \le r$, there exists a sequence $(x_n)_{n\in N} \subset U(x_0, r)$ such that*

$$x_{n+1} \in F(x_n) \quad \textit{and} \quad d(x_{n+1}, x_n) \le \theta^n \beta \quad \textit{for all} \quad n \in N. \tag{1}$$

Moreover, for any sequence $(x_n)_{n\in N}$ of $U(x_0, r)$ satisfying (1), its limit $\overline{x}$ belongs to $U(x_0, r)$ and is a fixed point of F yielding that Φ_F is nonempty and

$$d(x_0, \Phi_F) \le (1-\theta)^{-1} d(x_0, F(x_0)).$$

Proof Let $\beta > d(x_0, F(x_0))$ be such that $\beta(1-\theta)^{-1} \leq r$. Since $d(x_0, F(x_0)) < \beta$, we can find $x_1 \in F(x_0)$ with $d(x_0, x_1) < \beta$. As $\beta < r$, we get $x_1 \in U(x_0, r)$. Assuming that $\theta = 0$, we get $x_1 \in F(x_0) \cap U(x_0, r) \subset F(x_1)$ thus, setting $x_n = x_1$ for all $n \geq 1$, we are done. Assume now that $\theta \neq 0$ and suppose we have constructed a finite sequence $x_1, \dots, x_n$ in $U(x_0, r)$ with $x_i \in F(x_{i-1})$ and $d(x_i, x_{i-1}) < \theta^{i-1}\beta$ for $i = 1, \dots, n$. As $x_n \in F(x_{n-1}) \cap U(x_0, r)$ we have

$$d(x_n, F(x_n)) \leq e_r(F(x_{n-1}), F(x_n)) \leq \theta d(x_{n-1}, x_n) < \theta^n \beta,$$

so that we can find $x_{n+1} \in F(x_n)$ with $d(x_n, x_{n+1}) \leq \theta^n \beta$. Then

$$d(x_{n+1}, x_0) \leq \sum_{p=1}^{n+1} d(x_p, x_{p-1}) \leq \sum_{p=1}^{n+1} \theta^{p-1}\beta \leq (1-\theta)^{-1}\beta,$$

hence $x_{n+1} \in U(x_0, r)$. The sequence (x_n) is thus well defined and is a Cauchy sequence in $B(x_0, r)$. Let $\overline{x}$ be its limit. We have

$$d(\overline{x}, x_0) \leq \lim_{n\to\infty} d(x_{n+1}, x_0) \leq (1-\theta)^{-1}\beta,$$

so that $\overline{x} \in B(x_0, r)$ and

$$d(\overline{x}, F(\overline{x})) \leq d(\overline{x}, x_n) + d(x_n, F(\overline{x})) \leq d(\overline{x}, x_n) + \theta d(x_{n-1}, \overline{x})$$

since $x_n \in F(x_{n-1}) \cap U(x_0, r)$. Hence $d(\overline{x}, F(\overline{x})) = 0$ and $\overline{x} \in F(\overline{x})$. Thus $\overline{x} \in \Phi_F$ and $d(x_0, \overline{x}) \leq (1-\theta)^{-1}\beta$. Letting β decrease to $d(x_0, F(x_0))$, we get the announced result.

The Nadler's fixed point theorem ([28, Theorem 5]) follows readily from Proposition 2.1. Observe that no boundedness assumption on the values is required.

Corollary 2.1 ([28, Theorem 5]) *Let (X, d) be a complete metric space and let $F\colon X \rightrightarrows X$ be a multifunction with nonempty graph and closed values which is assumed to be θ-contractive. Then F admits a fixed point.*

Proof Let us choose $x_0 \in X$ such that $F(x_0)$ is nonempty and $r \geq 0$ such that $r > (1-\theta)^{-1} d(x_0, F(x_0))$. We can apply Proposition 2.1 which proves the corollary.

Proposition 2.1 is of local character. If one is interested in a global result, one can use the following proposition.

Proposition 2.2 *Let (X, d) be a complete metric space and let $F\colon X \rightrightarrows X$ be a multifunction with closed nonempty values. Assume that for some $x_0 \in X$ and for all $r > 0$ the multifunction F is θ_r-contractive on $U(x_0, r)$ for some $\theta_r \in [0, 1)$. Then F has a fixed point in X if and only if*

$$\inf_{r>0} \inf_{x \in U(x_0, r)} \frac{(1-\theta_r) d(x, x_0) + d(x, F(x))}{r(1-\theta_r)} < 1.$$

Proof Taking $x \in \Phi_F$ and $r > d(x_0, x)$ we see that the condition is necessary. Let us show it is sufficient. By assumption, we can choose $r > 0$ and $x_1 \in U(x_0, r)$ such that

$$(1-\theta_r) d(x_1, x_0) + d(x_1, F(x_1)) < r(1-\theta_r),$$

yielding

$$d(x_1, F(x_1)) < (r - d(x_1, x_0))(1 - \theta_r)$$

and F is pseudo-θ_r-contractive with respect to $U(x_1, r - d(x_1, x_0))$. Thus we can apply Proposition 2.1 with x_1 and $r - d(x_1, x_0)$ instead of x_0 and r respectively, from which we get $\Phi_F \neq \varnothing$.

It is of interest to study the sensitivity of the fixed points sets Φ_F when F varies in the power set $2^{(X\times X)}$ (hyperspace of subsets of $X \times X$) endowed with some topology. We turn now to this question. It is natural to choose (x_0, x_0) as a base point in $X \times X$.

Proposition 2.3 *Let (X, d) be a complete metric space. Let $F\colon X \rightrightarrows X$ be a multifunction with closed nonempty values which is assumed to be pseudo-θ-contractive with respect to $U(x_0, r)$. Then for any $s \in (0, r)$ and for any multifunction $G\colon X \rightrightarrows X$ satisfying*

$$e_s(G, F) < (1 - \theta)(1 + \theta)^{-1}(r - s)$$

one has

$$e_s(\Phi_G, \Phi_F) \leq (1 - \theta)^{-1}(1 + \theta)e_s(G, F) < r - s.$$

Proof Let $t > e_s(G, F)$ be such that $t < (1 - \theta)(1 + \theta)^{-1}(r - s)$ and let $y \in \Phi_G \cap U(x_0, s)$ (if there is no such y, there is nothing to prove). Since $(y, y) \in G \cap U((x_0, x_0), s)$, there exists $(w, z) \in F$ with $d(y, w) < t$ and $d(y, z) < t$. Due to the choice of t, we have $t < r - s$, thus $w, z \in U(x_0, r)$, whence we get

$$d(y, F(y)) \leq d(y, z) + d(z, F(y)) \leq d(y, z) + e_r(F(w), F(y)) \leq t(1 + \theta) < (1 - \theta)(r - s).$$

As F is pseudo-θ-contractive with respect to $U(y, r - s)$, it follows from the preceding estimate and from Proposition 2.1 that

$$d(y, \Phi_F) \leq (1 - \theta)^{-1}d(y, F(y)) \leq (1 - \theta)^{-1}(1 + \theta)t,$$

hence the result, letting t decrease to $e_s(G, F)$.

If instead of an estimate on the excess of the graph of G to the graph of F one assumes a uniform estimate on the images, one gets a more precise result about the fixed points sets.

Proposition 2.4 *Let (X, d) be a complete metric space. Let $F\colon X \rightrightarrows X$ be a multifunction with closed nonempty values which is pseudo-θ-contractive with respect to $U(x_0, r)$. Then for any $s \in (0, r)$ and for any multifunction $G\colon X \rightrightarrows X$ satisfying*

$$e_s(G(x), F(x)) < (1 - \theta)(r - s) \quad \textit{for each} \quad x \in U(x_0, s)$$

one has

$$e_s(\Phi_G, \Phi_F) \leq (1 - \theta)^{-1} \sup_{x \in U(x_0, r)} e_s(G(x), F(x)) \leq (1 - \theta)^{-1}e_s(G, F).$$

Proof Let $y \in \Phi_G \cap U(x_0, s)$ and let $t > e_s(G(y), F(y))$ be such that $t < (1-\theta)(r-s)$. Since $y \in G(y) \cap U(x_0, s)$ we can pick $z \in F(y)$ such that $d(y, z) < t$. Since F is pseudo-θ-contractive with respect to $U(y, r - s)$, it follows from Proposition 2.1 that

$$d(y, \Phi_F) \leq (1 - \theta)^{-1}d(y, F(y)) \leq (1 - \theta)^{-1}t,$$

hence the result, taking the infimum over $t > e_s(G, F)$.

By using Proposition 2.3, we obtain the following result on the dependence of the fixed point set Φ_F when the graph of F is perturbed. Some care is needed in order to obtain a significant result since the conclusion of Proposition 2.3 does not prevent from emptiness of Φ_G. Here we adopt a parametric formulation which is equivalent to the preceding framework (take for Λ the set of graphs of multifunctions which are pseudo-θ contractive provided with the topology associated with $(h_r)_{r\geq 0}$).

Theorem 2.1 *Let (X, d) be a complete metric space and let Λ be a topological space. Let $F \subset \Lambda \times X \times X$ be a multifunction such that for some $x_0 \in X$, $\theta \in [0, 1)$, $r > 0$ and for all $\lambda \in \Lambda$ the multifunction $F_\lambda = F(\lambda, \cdot) \subset X \times X$ is nonempty, closed-valued and pseudo-θ-contractive with respect to $U(x_0, r)$. Assume $r > r_0 = (1-\theta)^{-1} d(x_0, F(\lambda_0, x_0))$ for some $\lambda_0 \in \Lambda$ and*

$$\lim_{\lambda \to \lambda_0} h_r(F(\lambda, \cdot), F(\lambda_0, \cdot)) = 0. \tag{2}$$

Then for any $s \in (r_0, r)$ there exists a neighborhood Λ_0 of λ_0 such that for all $\lambda \in \Lambda_0$ one has $\Phi_{F(\lambda,\cdot)} \cap U(x_0, s) \neq \varnothing$ and

$$h_s(\Phi_{F(\lambda,\cdot)}, \Phi_{F(\lambda_0,\cdot)}) \leq (1-\theta)^{-1}(1+\theta) h_s(F(\lambda, \cdot), F(\lambda_0, \cdot)). \tag{3}$$

Proof Let $t \in (r_0, s)$ be such that $s - t < r - s$. Proposition 2.1 ensures that $\Phi_{F(\lambda_0,\cdot)} \cap U(x_0, t)$ is nonempty. Let Λ_0 be a neighborhood of λ_0 such that for $\lambda \in \Lambda_0$ and $\delta_\lambda = h_s(F(\lambda_0, \cdot), F(\lambda, \cdot))$ one has

$$\delta_\lambda < (1-\theta)(1+\theta)^{-1}(s-t) < (1-\theta)(1+\theta)^{-1}(r-s).$$

We obtain from Proposition 2.3 applied to $G = F(\lambda_0, \cdot)$, $F(\lambda, \cdot)$ that

$$e_s(\Phi_{F(\lambda_0,\cdot)}, \Phi_{F(\lambda,\cdot)}) \leq \delta_\lambda (1-\theta)^{-1}(1+\theta) \leq s - t. \tag{4}$$

Since $\Phi_{F(\lambda_0,\cdot)} \cap U(x_0, t)$ is nonempty, we get

$$\Phi_{F(\lambda,\cdot)} \cap U(x_0, s) \neq \varnothing$$

for all $\lambda \in \Lambda_0$. Interchanging the role played by $F(\lambda_0, \cdot)$ and $F(\lambda, \cdot)$ and applying again Proposition 2.3 we obtain that for all $\lambda \in \Lambda_0$

$$e_s(\Phi_{F(\lambda,\cdot)}, \Phi_{F(\lambda_0,\cdot)}) \leq (1-\theta)^{-1}(1+\theta) e_s(F(\lambda, \cdot), F(\lambda_0, \cdot)),$$

which combined with (4) gives estimate (3).

For multivalued contractions in the usual sense we have the following result.

Corollary 2.2 *Let (X, d) be a complete metric space and let Λ be a topological space. Let $F \subset \Lambda \times X \times X$ be a multifunction such that for some $\theta \in [0, 1)$ and for all $\lambda \in \Lambda$, the multifunction $F(\lambda, \cdot) \subset X \times X$ is nonempty closed-valued and θ-contractive. Assume that*

$$\lim_{\lambda \to \lambda_0} h(F(\lambda, \cdot), F(\lambda_0, \cdot)) = 0,$$

or, more generally that for every $r > 0$ (2) holds. Let $x_0 \in X$. Then for all $t \geq 0$ there exists a neighborhood Λ_0 of λ_0 such that for all $\lambda \in \Lambda_0$ we have

$$h_t(\Phi_{F(\lambda,\cdot)}, \Phi_{F(\lambda_0,\cdot)}) \leq (1-\theta)^{-1}(1+\theta)h(F(\lambda,\cdot), F(\lambda_0,\cdot)).$$

Proof Choose r, s with $s \geq t$, $r > s > r_0 := (1-\theta)^{-1}d(x_0, F(\lambda_0, x_0))$ and apply Theorem 2.1, using the fact that $h_t \leq h_s$.

Remark 2.1 The preceding corollary represents a slight sharpening of the result of Markin in [24]. Indeed this author proves that if A is a closed bounded subset of a Hilbert space H and if (F_n) is a sequence of θ-contractive multifunctions from A to A with nonempty closed convex values such that $\lim\limits_{n\to\infty} h(F_n(x), F(x)) = 0$ uniformly on A then $\lim\limits_{n\to\infty} h(\Phi_F, \Phi_{F_n}) = 0$. Let us set $X = A$ and let us introduce $x_0 \in A$ and $s \geq 0$ such that $A \subset U(x_0, s)$ hence $U_X(x_0, s) = A$. Observe that $\lim\limits_{n\to\infty} h(F_n(x), F(x)) = 0$ uniformly on A implies $\lim\limits_{n\to\infty} h(F, F_n) = 0$ since for all $(x, y) \in F$ one has $d((x, y), F_n) \leq d(y, F_n(x)) \leq h(F(x), F_n(x))$ and the same inequality exchanging F and F_n yielding

$$h(F, F_n) \leq \sup_{x\in A} h(F(x), F_n(x)).$$

From Corollary 2.2 we get

$$\lim_{n\to\infty} h_s(\Phi_F, \Phi_{F_n}) = 0$$

which turns to

$$\lim_{n\to\infty} h(\Phi_F, \Phi_{F_n}) = 0$$

since $\Phi_F \cap U(x_0, s) = \Phi_F$ and $\Phi_{F_n} \cap U(x_0, s) = \Phi_{F_n}$. Moreover we do not need any convexity assumption and we get a quantitative estimate. Corollary 2.2 also improves [23, Theorem 1].

At this stage a natural question arises: is a limit of pseudo-Lipschitzian multifunctions also pseudo-Lipschitzian? The answer is positive and easy for a sequence of θ-Lipschitzian multifunctions which pointwise converges with respect to the Pompeiu–Hausdorff metric. The question is more delicate when graph convergence is used. In this setting, a partial answer is given in the following proposition.

Proposition 2.5 *Let $(F_n) \subset X \times X$ be a sequence of multifunctions from a metric space X into X. Assume that for some $x_0 \in X$, $r > 0$, $\theta \in R_+$ the multifunctions F_n are pseudo-θ-Lipschitzian with respect to $U(x_0, r)$. Let $F \subset X \times X$ be a closed multifunction such that*

$$\lim_{n\to\infty} e_{(2\theta+1)r}(F_n, F) = 0 \quad \textit{and} \quad F \subset \liminf_{n\to\infty} F_n.$$

Then F is pseudo-θ-Lipschitzian with respect to $U(x_0, r)$ whenever one of the following conditions holds

(a) *F is closed and for any compact set $K \subset U(x_0, r)$ the set $F(K)$ is relatively compact;*

(b) *X is a reflexive Banach space and F is sequentially $s \times w$-closed, where w and s denote respectively the weak and the strong topology on X.*

Proof Let $(x,y) \in F \cap U((x_0,x_0),r)$ and let $x' \in U(x_0,r)$. Since $F \subset \liminf_{n\to\infty} F_n$ there exists a sequence $(x_n,y_n) \in F_n$ which converges to (x,y). Given $\alpha > 0$ such that $d(x_0,y) < r-\alpha$, we may suppose $d(x_0,x_n) < r$ and $d(x_0,y_n) < r-\alpha$ for n large enough, so that there exists $z'_n \in F_n(x')$ with $d(z'_n,y_n) < \theta d(x',x_n) + \alpha < 2\theta r + \alpha$ for n large enough. Thus $d(x_0,z'_n) < (2\theta+1)r$ for such n's and one has

$$(x',z'_n) \in F_n \cap U((x_0,x_0),(2\theta+1)r);$$

thus there exists a sequence $(x'_n,y'_n) \subset F$ such that $(d(x'_n,x'))$ and $(d(y'_n,z'_n))$ converge to 0.

(a) Let $K := \{x'_n\} \cup \{x'\}$ and let $y' \in F(K)$ be the limit of a convergent subsequence of (y'_n). Since F is closed, one has $y' \in F(x')$ and $d(y',y) \le \lim_n d(y'_n,y_n) = \lim_n d(z'_n,y_n) \le \lim_n \theta d(x',x_n) = \theta d(x',x)$, hence $d(y,F(x')) \le \theta d(x,x')$ and then

$$e(F(x) \cap U(x_0,r), F(x')) \le \theta d(x,x').$$

(b) As the sequence (y'_n) is bounded, there exists a subsequence which converges weakly to some $y' \in F(x')$ in view of our closedness assumption. Using the weak lower semicontinuity of the norm we also get $d(y,y') \le \theta d(x,x')$ so that

$$e(F(x) \cap U(x_0,r), F(x')) \le \theta d(x,x').$$

3 Applications to Differential Inclusions

In the sequel, we apply the stability result obtained in the previous section to the case where fixed points are solutions of differential inclusions in some functional spaces. Let us present the data of the problem. Let E be Banach space whose closed unit ball is denoted by B and let $T \subset R$ be an interval endowed with the Lebesgue measure, with end points $t_0 \in R$ and $t_1 \in R \cup \{+\infty\}$. Following [38], we say that a multifunction with nonempty values $G\colon T \rightrightarrows E$ is measurable if there exists a sequence $(g_n)_n$ of measurable mappings from T into E such that $g_n(t) \in G(t)$ a.e. on T for all $n \in \mathbb{N}$ and $G(t) \subset \overline{\bigcup_{n\in\mathbb{N}}\{g_n(t)\}}$ a.e. on T.

We are interested in the behavior of the set of solutions to the differential inclusion

$$\dot{x}(t) \in R(t,x(t)) \tag{5}$$

where $R\colon T \times E \rightrightarrows E$ is a multifunction. In (5) the solution $x(\cdot)$ is assumed to belong to the space $X = W^{1,1}(T,E)$ of continuous functions $x\colon T \to E$ such that there exists $u \in \mathcal{L}^1(T,E)$ (the space of Bochner integrable functions from T into E) such that

$$x(t) = x(t_0) + \int_{t_0}^{t} u(s)\,ds \quad \text{for all} \quad t \in T$$

and x is said to be a solution if $u(t) \in R(t,x(t))$ a.e. $t \in T$. Given $x_0 \in X$ and $\xi \in B(x_0(t_0),\delta)$ with $\delta > 0$, we denote by $S_R(\xi)$ the set of solutions x of (5) such that $x(t_0) = \xi$. We shall make a frequent use of Lemma 3.2 of [38] (see also [11] and Lemma 1.3 in [15] when E is separable).

Lemma 3.1 *Let $G\colon T \rightrightarrows E$ be a measurable multifunction with values in a Banach space. Let $v_0\colon T \to E$ and $\gamma\colon T \to]0, +\infty[$ be measurable. Then there exists a measurable mapping $v\colon T \to E$ such that*

$$v(t) \in G(t) \quad \textit{and} \quad \|v(t) - v_0(t)\| \le d(v_0(t), G(t)) + \gamma(t) \quad \textit{almost everywhere on } T.$$

Let us give to E the base point ξ_0 and to $X = W^{1,1}(T,E)$ the base point x_0 with $x_0(t) = x_0(t_0) + \int_{t_0}^{t} u_0(s)\,ds$ and let $k \in \mathcal{L}^1(T) = \mathcal{L}^1(T,R)$, $k(t) \ge 0$ a.e. In the sequel we shall endow the space $X = W^{1,1}(T,E)$ with the norm

$$\|x\|_X = \|x(t_0)\| + \int_T e^{-m(s)}\|u(s)\|\,ds, \tag{6}$$

and the associated distance d_X, where

$$m(s) = \int_{t_0}^{s} k(\tau)\,d\tau \tag{7}$$

and $u \in \mathcal{L}^1(T,X)$ is such that $x(s) = x(t_0) + \int_{t_0}^{s} u(\tau)\,d\tau$ on T. This norm is equivalent to the usual norm $x \longmapsto \|x(t_0)\| + \int_T \|u(s)\|\,ds$ since

$$e^{-m(t_1)}\left(\|x(t_0)\| + \int_T \|u(s)\|\,ds\right) \le \|x\|_X \le \|x(t_0)\| + \int_T \|u(s)\|\,ds.$$

Our approach relies on the following lemma which refines a trick in [9] (see also [18]).

Lemma 3.2 *Given k and m as in (7), let $\theta(t) = 1 - e^{-m(t)}$. For $i = 1, 2$, let $x_i \in W^{1,1}(T,E)$, with $x_i(s) = x_i(t_0) + \int_{t_0}^{s} u_i(\tau)\,d\tau$, $u_i \in \mathcal{L}^1(T,E)$. Then for all $t \in T$*

$$\int_{t_0}^{t} e^{-m(s)}k(s)\|x_2(s) - x_1(s)\|\,ds \le \theta(t)\left(\|x_1(t_0) - x_2(t_0)\| + \int_{t_0}^{t} e^{-m(s)}\|u_2(s) - u_1(s)\|\,ds\right).$$

Proof Setting

$$I(t) = \int_{t_0}^{t} e^{-m(s)}k(s)\|x_2(s) - x_1(s)\|\,ds,$$

one has

$$\begin{aligned} I(t) &\le \int_{t_0}^{t} e^{-m(s)}k(s)\Big(\|x_1(t_0) - x_2(t_0)\| + \int_{t_0}^{s} \|u_2(\tau) - u_1(\tau)\|\,d\tau\Big)\,ds \\ &\le \theta(t)\|x_2(t_0) - x_1(t_0)\| + \int_{t_0}^{t}\left(\int_{\tau}^{t} e^{-m(s)}k(s)\,ds\right)\|u_2(\tau) - u_1(\tau)\|\,d\tau. \end{aligned}$$

Observing that

$$\int_{\tau}^{t} e^{-m(s)}k(s)\,ds = e^{-m(\tau)} - e^{-m(t)} \le (1-e^{-m(t)})e^{-m(\tau)},$$

we get the result of the lemma.

3.1 A variant of the Filippov's theorem

Let us assume that the multifunction $R\colon\, T\times E \to E$ and the data $x_0 \in X,\ k \in \mathcal{L}^1(T)$, $\delta > 0$ satisfy the following assumptions in which $b(t) := re^{m(t)}$ for some $r > 0,\ m(t) := \int_{t_0}^{t} k(s)\,ds$ and $T_b := \cup_{t\in T}\{t\} \times B(x_0(t), b(t))$:

for each $(t,e) \in T_b$ the set $R(t,e)$ is closed, nonempty and $R(\cdot,e)$ is measurable; (8)

for a.e. $t \in T$ the multifunction $R(t,\cdot)$ is $k(t)$-Lipschitzian on $B(x_0(t), b(t))$; (9)

$$\gamma(\cdot) = d\left(u_0(\cdot), R(\cdot, x_0(\cdot))\right) \in \mathcal{L}^1(T) \tag{10}$$

We also suppose

$$e^{m(t_1)}\left(\delta + \int_T e^{-m(s)}\gamma(s)\,ds\right) \le r. \tag{11}$$

Proposition 3.1 *Let $R\colon\, T\times E \rightrightarrows E$ be a multifunction with closed nonempty values satisfying assumptions (8) – (10) where relation (11) holds. Then for all $\xi \in B(x_0(t_0), \delta)$ the set $S_R(\xi)$ of solutions of*

$$\begin{aligned} &\dot{x}(t) \in R(t, x(t)) \quad \text{a.e. on } \ T,\\ &x(t_0) = \xi, \end{aligned} \tag{12}$$

is nonempty and one has $d(x_0, S_R(\xi)) = \inf\{\|x - x_0\|_X :\ x \in S_R(\xi)\} \le r$.

Here the Lipschitz assumption (9) bears on a ball with a variable radius $b(t)$ instead of a ball with a fixed radius $\sup_{t\in T} b(t)$ as in [6, Theorem 10.4.1], [14], [37, Theorem 2.4.3]. Our conclusion involves an estimate of the $W^{1,1}$ norm of $x - x_0$ and, more importantly, we avoid the following assumption

(H) There exists $\sigma \in \mathcal{L}^1(T)$ such that $R(t,\xi) \subset \sigma(t)B$ for all $\xi \in E$ and $t \in T$

made in [18, 37] which excludes unbounded right hand sides. However, we do not get a point-wise estimate of the derivative of $x - x_0$ as in [14], [6, Theorem 10.4.1].

Proof Given $\xi \in B(x_0(t_0), \delta)$, let $F\colon\, X \rightrightarrows X$ be the multifunction defined by

$$y \in F(x) \iff \begin{cases} y(s) = \xi + \int_{t_0}^{s} v(\tau)\,d\tau \quad \text{for all} \quad s \in T \\ v \in \mathcal{L}^1(T,E) \quad \text{is such that} \quad v(s) \in R(s, x(s)) \quad \text{a.e. on} \quad T. \end{cases}$$

It is clear that $x \in X$ is a solution of (12) if and only if x is a fixed point of F. The existence of such a fixed point is ensured by Proposition 2.1 and the following lemma.

Lemma 3.3 *Given $\xi \in B(x_0(t_0), \delta)$, and $r > 0$ as in relation (11), the multifunction $F\colon U(x_0, r) \rightrightarrows X$ defined above is closed, nonempty-valued, and is $\theta(t_1)$-contractive on $U(x_0, r)$ with $\theta(t) = 1 - e^{-m(t)}$. Moreover one has $d(x_0, F(x_0)) < r(1 - \theta(t_1))$.*

Proof Given $x \in U(x_0, r)$, with $x(t) = x(t_0) + \int\limits_{t_0}^{t} u(\tau)\,d\tau$ for $t \in T$, we have

$$\begin{aligned} \|x(t) - x_0(t)\| &\le \|x(t_0) - x_0(t_0)\| + e^{m(t)} \int\limits_{t_0}^{t} e^{-m(\tau)}\, \|u(\tau) - u_0(\tau)\|\, d\tau \\ &\le e^{m(t)}\, \|x - x_0\|_X < e^{m(t)} r = b(t), \end{aligned}$$

so that $x(t) \in B(x_0(t), b(t))$ for each $t \in T$. From Theorem 2.2 of [38], the multifunction $s \mapsto R(s, x(s))$ is measurable on T. Moreover, using (9) and (10), one sees that $d(u_0(s), R(s, x(s))) \le \bar\gamma(s)$ a.e. on T with $\bar\gamma(s) = \gamma(s) + k(s)\|x(s) - x_0(s)\|$. As $\bar\gamma \in \mathcal{L}^1(T)$, Lemma 3.1 yields the existence of an integrable mapping $u\colon T \to E$ such that $u(s) \in R(s, x(s))$ a.e. on T, hence $F(x) \neq \varnothing$. It is easily shown that $F(x)$ is closed.

Now let us prove that F is $\theta(t_1)$-contractive on $U(x_0, r)$ with $\theta(t) = 1 - e^{-m(t)}$. For $i = 1, 2$, let $x_i \in U(x_0, r)$ with $x_i(s) = x_i(t_0) + \int\limits_{t_0}^{s} u_i(\tau)\,d\tau$, $u_i \in \mathcal{L}^1(T, E)$, let $y_1 \in F(x_1)$ with $y_1(s) = \xi + \int\limits_{t_0}^{s} v_1(\tau)\,d\tau$, $v_1(\tau) \in R(\tau, x_1(\tau))$ a.e. on T, and let $\varepsilon > 0$. Given $\alpha \in \mathcal{L}^1(T)$ with $\alpha(\tau) > 0$ p.p. and $\int\limits_T \alpha(\tau)\,d\tau < \varepsilon$, we have

$$d(v_1(s), R(s, x_2(s))) \le k(s)\|x_1(s) - x_2(s)\| \quad \text{a.e. on} \quad T.$$

Thus we derive from Lemma 3.1 the existence of a measurable mapping $v_2 : T \to E$ such that $v_2(s) \in R(t, x_2(s))$ a.e. on T and

$$\|v_2(s) - v_1(s)\| \le k(s)\|x_2(s) - x_1(s)\| + \alpha(s) \quad \text{a.e. on} \quad T.$$

Setting $y_2(s) := \xi + \int\limits_{t_0}^{s} v_2(\tau)\,d\tau$, we get $y_2 \in F(x_2)$ and using Lemma 3.2

$$\begin{aligned} \|y_2 - y_1\|_X &= \int\limits_T e^{-m(s)} \|v_2(s) - v_1(s)\|\, ds \\ &\le \int\limits_T e^{-m(s)} \big(k(s)\|x_2(s) - x_1(s)\| + \alpha(s)\big)\, ds \\ &\le \theta(t_1) \left(\|x_1(t_0) - x_2(t_0)\| + \int\limits_T e^{-m(s)} \|u_2(s) - u_1(s)\|\, ds + \varepsilon \right) \\ &\le \theta(t_1)\big(\|x_1 - x_2\|_X + \varepsilon\big). \end{aligned}$$

Taking the infimum over ε, it follows that $d(y_1, F(x_2)) \le \theta(t_1)\|x_2 - x_1\|_X$. Taking the supremum on $y_1 \in F(x_1)$ one obtains that $e(F(x_1), F(x_2)) \le \theta(t)\|x_2 - x_1\|_X$ and then, interchanging x_1 and x_2

$$h(F(x_1), F(x_2)) \le \theta(t_1)\|x_2 - x_1\|_X. \tag{13}$$

Now let us estimate $d(x_0, F(x_0))$. Let $\varepsilon > 0$ be such that

$$\delta + \int_T e^{-m(s)}\gamma(s)ds + \varepsilon < re^{-m(t_1)},$$

and let $\alpha \in \mathcal{L}^1(T)$ be such that $\int_T \alpha < \varepsilon$ and $\alpha(t) > 0$ for each $t \in T$. Applying again Lemma 3.1, we get a measurable mapping $v\colon T \to E$ such that $v(s) \in R(s, x_0(s))$ a.e. on T and

$$\|v(s) - u_0(s)\| \leq \gamma(s) + \alpha(s) \quad \text{a.e. on} \quad T.$$

Setting

$$y(t) := \xi + \int_{t_0}^{t} v(s)ds,$$

one has $y \in F(x_0)$ and

$$\begin{aligned}\|y - x_0\|_X &\leq \delta + \int_T e^{-m(s)}\left(\gamma(s) + \alpha(s)\right)ds \\ &\leq \delta + \int_T e^{-m(s)}\gamma(s)ds + \varepsilon < r(1 - \theta(t_1)).\end{aligned}$$

From this fact the quoted authors give a result on the dependence of the solution of (12) with respect to the initial value. In fact, it is possible to obtain a stronger result and to allow a variation of the right-hand side. Given a multifunction R which satisfy (8) and (9) and given $\xi \in E$, again we denote by $S_R(\xi)$ the set of solutions of (12) and we endow $W^{1,1}(T, E)$ with the norm $\|\cdot\|_X$, providing it with the base point x_0.

Proposition 3.2 *Let R_1, R_2 be multifunctions which satisfy (8) and (9). Let us set*

$$\rho(t) = \sup\{h(R_1(t, z), R_2(t, z))\colon\ z \in B(x_0(t), b(t))\}, \tag{14}$$

let us assume that $\rho \in \mathcal{L}^1(T)$ and let $s \in (0, r)$, $\xi_1, \xi_2 \in E$ be such that

$$e^{m(t_1)}\left(\|\xi_1 - \xi_2\| + \int_{t_0}^{t_1} e^{-m(t)}\rho(t)\,dt\right) < r - s. \tag{15}$$

Then

$$h_s(S_{R_1}(\xi_1), S_{R_2}(\xi_2)) \leq e^{m(t_1)}\left(\|\xi_1 - \xi_2\| + \int_{t_0}^{t_1} \rho(t)\,dt\right).$$

Proof It suffices to check the assumptions of Proposition 2.4 in which F, G are replaced with the multifunctions F_1 and F_2 defined as in the proof of Proposition 3.1 with ξ and R replaced with ξ_1, R_1 and ξ_2, R_2 respectively. Now, given $u \in \mathcal{L}^1(T, E)$, $x \in B(x_0, s)$ and $y_1 \in F_1(x)$, as in the proof of Lemma 3.3 we have $x(t) \in B(x_0(t), b(t))$

for each $t \in T$. Taking $v \in \mathcal{L}^1(T,E)$ such that $v(t) \in R_1(t,x(t))$ for a.e. $t \in T$ and $y_1(t) = \xi_1 + \int_{t_0}^{t} v(s)ds$ we have $d(v(t), R_2(t,x(t))) \le \rho(t)$ and for any $\varepsilon > 0$ such that

$$e^{m(t_1)}\left(\|\xi_1 - \xi_2\| + \int_{t_0}^{t_1} e^{-m(t)}\rho(t)\,dt\right) + \varepsilon < r - s,$$

we can find $\alpha \in \mathcal{L}^1(T)$, $v_2 \in \mathcal{L}^1(T,E)$ such that $\alpha(t) > 0$ for each $t \in T$,

$$\int_T e^{-m(t)}\alpha(t)\,dt \le e^{-m(t_1)}\varepsilon$$

and

$$v_2(t) \in R_2(t,x(t)), \|v_2(t) - v(t)\| \le \rho(t) + \alpha(t) \quad \text{a.e.} \quad t \in T.$$

Then, for $y_2(t) = \xi_2 + \int_{t_0}^{t} v_2(s)\,ds$ we have

$$\begin{aligned}\|y_1 - y_2\|_X &= \|\xi_1 - \xi_2\| + \int_T e^{-m(s)}\|v_2(s) - v_1(s)\|\,ds \\ &\le \|\xi_1 - \xi_2\| + \int_{t_0}^{t_1} e^{-m(t)}(\rho(t) + \alpha(t))\,dt \le e^{-m(t_1)}(r-s).\end{aligned}$$

Thus $e_s(F_2(x), F_1(x)) < (1-\theta)(r-s)$ for each $x \in U(x_0,s)$, where $\theta = 1 - e^{-m(t_1)}$. Since $S_{R_i}(\xi_i)$ is the set of fixed points of F_i for $i = 1,2$, the result follows from Proposition 2.4 and the fact that the roles of F_1 and F_2 are symmetric.

Remark 3.1 This perturbation result can also be deduced from Proposition 3.1 by replacing x_0 and r with $x_1 \in S_{R_1}(\xi_1) \cap B(x_0,s)$ and $r-s$ respectively. As in the proof of Lemma 3.3, we have $x_1(t) \in B(x_0(t), se^{m(t)})$ for each $t \in T$ and $B(x_1(t), (r-s)e^{m(t)}) \subset B(x_0(t), b(t))$ for $t \in T$; moreover we have $d(u_1(t), R_2(t,x_1(t))) \le \rho(t)$, where $x_1(t) = \xi_1 + \int_{t_0}^{t} u_1(s)ds$. Then assumptions (8), (9) and (10) are satisfied with r and x_0 replaced respectively by $r-s$ and x_1. Thus, applying the quoted existence result, we get the conclusion of the proposition.

3.2 Stability of global solutions

We can also derive a stability result for the set $S_R(\xi)$ when the right-hand side R and the initial value ξ vary. Let Λ be a topological space and let $R\colon \Lambda \times T \times E \rightrightarrows E$ be a family of multifunctions with closed nonempty values parametrized by $\lambda \in \Lambda$. Let us introduce the following assumptions

- (a_Λ) $R(\lambda,\cdot,x)$ is measurable for all $\lambda \in \Lambda$, $x \in E$;
- (b_Λ) $R(\lambda,t,\cdot)$ is $k(t)$-Lipschitz for all $\lambda \in \Lambda$ a.e. with $k \in \mathcal{L}^1(T)$;
- (c_Λ) there exists $\xi_0 \in E$ and $\lambda_0 \in \Lambda$ such that

$$d(0, R(\lambda_0,t,\xi_0)) \in \mathcal{L}^1(T).$$

Theorem 3.1 *Let Λ be a topological space and let $R\colon\ \Lambda\times T\times E \rightrightarrows E$ be a family of multifunctions with closed nonempty values parametrized by Λ. Assume that assumptions (a_Λ), (b_Λ) and (c_Λ) are satisfied. For all $s>0$, $\lambda\in\Lambda$, let $\varepsilon_s(\cdot,\cdot)$ be a function defined on $T\times\Lambda$ such that for all $(t,\lambda)\in T\times\Lambda$*

$$\sup_{z\in B(\xi_0,s)} h(R(\lambda,t,z),R(\lambda_0,t,z))<\varepsilon_s(t,\lambda).$$

Assume that for all $s>0$ and for all $\lambda\in\Lambda$

$$\varepsilon_s(\cdot,\lambda)\in\mathcal{L}^1(T) \quad \text{and} \quad \varepsilon_s(\cdot,\lambda) \quad \text{converges to} \quad 0 \quad \text{in} \quad \mathcal{L}^1(T) \quad \text{as} \quad \lambda\to\lambda_0. \tag{16}$$

Then there exist a constant $c>0$ such that for all r, s with

$$e^{m(t_1)}\int_{t_0}^{t_1} d(0,R(\lambda_0,t,\xi_0))\,dt<s<r$$

and for all $\xi\in E$ one has

$$S_{R(\lambda,\cdot,\cdot)}(\xi)\cap U(0,s)\neq\varnothing$$

and there exist neighborhoods Λ_0 of λ_0 and Ξ_0 of ξ_0 such that for all $\lambda\in\Lambda_0$ and $\xi\in\Xi_0$ one has

$$h_s\big(S_{R(\lambda_0,\cdot,\cdot)}(\xi_0),S_{R(\lambda,\cdot,\cdot)}(\xi)\big)\le c\left(\|\xi-\xi_0\|+\|\varepsilon_\sigma(\cdot,\lambda)\|_{L^1(T)}\right),$$

with $\sigma=re^{m(t_1)}$.

Proof Let $X=W^{1,1}(T,E)$ endowed with the norm (6). For all $(\lambda,\xi)\in\Lambda\times E$, one has

$$d(0,R(\lambda,t,\xi))\le\rho(t)$$

with $\rho(t)=d(0,R(\lambda_0,t,\xi_0))+\varepsilon_s(t,\lambda)+k(t)\|\xi-\xi_0\|\in\mathcal{L}^1(T)$. Thus we can define a multifunction $F\colon\ \Lambda\times E\times X\rightrightarrows X$ with nonempty closed values by $y\in F(\lambda,\xi,x)$ if and only if there exists $v\in\mathcal{L}^1(T,E)$ with $v(t)\in R(\lambda,t,x(t))$ a.e. and

$$y(t)=\xi+\int_{t_0}^{t} v(s)\,ds \quad \text{for all} \quad t\in T.$$

Relying on Lemma 3.3, we obtain that the multifunction $F(\lambda,\xi,\cdot)$ is θ-Lipschitz with $\theta=1-e^{-m(t_1)}$. Moreover one easily checks that

$$d(0,F(\lambda_0,\xi_0,0)\le\int_{t_0}^{t_1} d(0,R(\lambda_0,t,\xi_0))\,dt.$$

Let us set

$$r_0=(1-\theta)d(0,F(\lambda_0,\xi_0,0)\le e^{-m(t_1)}\int_{t_0}^{t_1} d(0,R(\lambda_0,t,\xi_0))\,dt.$$

Let $(\lambda, \xi) \in \Lambda \times E$ and let $(y, x) \in F(\lambda_0, \xi_0, \cdot) \cap U((0,0), r)$. It follows that $x(t)$ remains in $B(\xi_0, \sigma)$ with $\sigma = e^{m(t_1)} r$. For almost all $t \in T$ we have $v(t) \in R(\lambda_0, t, x(t))$ then

$$d(v(t), R(\lambda, t, x(t))) < \varepsilon_\sigma(t, \lambda)$$

thus, from Lemma 3.1 there exists a measurable function $w\colon T \to E$ such that for all $t \in T$

$$w(t) \in R(\lambda, t, x(t)) \quad \text{and} \quad \|w(t) - v(t)\| \le \varepsilon_\sigma(t, \lambda) \quad \text{a.e.}$$

Observe that $w \in \mathcal{L}^1(T, E)$ and that $z \in F(\lambda, \xi, x)$ where $z(t) = \xi + \int_{t_0}^{t} w(s)\, ds$, yielding

$$d((y, x), F(\lambda, \xi, \cdot)) \le \|y - z\| \le \|\xi - \xi_0\| + \int_{t_0}^{t_1} e^{-m(t)} \varepsilon_\sigma(t, \lambda)\, dt.$$

Choosing $(0,0)$ as base point in $X \times X$ and interchanging (λ, ξ) and (λ_0, ξ_0) we get

$$h_r(F(\lambda_0, \xi_0, \cdot), F(\lambda, \xi, \cdot)) \le \|\xi - \xi_0\| + \int_{t_0}^{t_1} e^{-m(t)} \varepsilon_\sigma(t, \lambda)\, dt,$$

hence

$$\lim_{(\lambda,\xi) \to (\lambda_0, \xi_0)} h_r(F(\lambda_0, \xi_0, \cdot), F(\lambda, \xi, \cdot)) = 0$$

and the result follows, applying Theorem 2.1 and observing that

$$S_{R(\lambda,\cdot,\cdot)}(\xi) = \Phi_{F(\lambda,\xi,\cdot)}.$$

In the particular case when there is no explicit dependence on the parameter λ we get a slight improvement of the result of [25] and [23].

Corollary 3.1 *Let $R\colon T \times E \rightrightarrows E$ be a multifunction with closed nonempty values. Assume that assumptions (a), (b) and (c) are satisfied. Then there exists a constant $c \ge 0$ such that, for all $s > e^{m(t_1)} \int_{t_0}^{t_1} d(0, R(\lambda_0, t, \xi_0))\, dt$ there exist a neighborhood Ξ_0 of ξ_0 such that for all $\xi \in \Xi_0$ one has $S_R(\xi) \cap U(x_0, s) \ne \varnothing$ and*

$$h_s(S_R(\xi_0), S_R(\xi)) \le c \|\xi - \xi_0\|.$$

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