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Construction of Lyapunov's Functions for a Class of Nonlinear Systems

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Abstract: The conditions of absolute stability for a certain class of nonlinear systems are investigated. It is proved that the systems considered are absolutely stable iff for these systems there exist Lyapunov's functions of the special form. The results obtained are used for the stability analysis of complex systems in critical cases.

Keywords: *Nonlinear systems; Lyapunov's functions; absolute stability; large scale systems.*

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1 Introduction

One of the important problems arising in the investigation of nonlinear systems is the problem of absolute stability [1, 3, 8]. This problem is of both theoretical and applied significance. The main approach for the determination of conditions for the absolute stability is the Lyapunov direct method. By means of this approach, the criteria of absolute stability for many types of systems are obtained. However, it should be noted that until now there are no general methods of construction of Lyapunov's functions for nonlinear systems.

In the present paper a certain class of differential equations systems is investigated. The method of construction of Lyapunov's functions for these systems is suggested. The main goal of the paper is to prove that for the absolute stability of systems considered it is necessary and sufficient that the Lyapunov's functions in the given form exist satisfying the assumptions of the Lyapunov asymptotic stability theorem [3].

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2 Statement of the Problem

Consider the system of differential equations

$$\dot{x}_s = \sum_{j=1}^{n} p_{sj} f_j(x_j), \quad s = 1, \ldots, n. \tag{2.1}$$

Here p_{sj} are constant coefficients, functions $f_j(x_j)$ are defined and continuous for $x_j \in (-\infty, +\infty)$ and possess the property $x_j f_j(x_j) > 0$ for $x_j \neq 0$. Hence, system (2.1) has the zero solution. Equations of this kind are widely used in the design of automatic control systems [3, 10].

The problem of absolute stability for system (2.1) was investigated in the works [3, 10, 14]. For the solution of this problem in [3] it was suggested to construct Lyapunov's function in the form

$$V = \sum_{s=1}^{n} \lambda_s \int_0^{x_s} f_s(\tau)\, d\tau, \tag{2.2}$$

where λ_s are positive constants. Thus, V is a positive definite function. In [3, 14] the sufficient conditions are obtained under which one may choose numbers λ_s for the function

$$\frac{dV}{dt}\bigg|_{(2.1)} = \sum_{s,j=1}^{n} \lambda_s p_{sj} f_s(x_s) f_j(x_j)$$

to be negative definite.

Suppose that coefficients p_{sj} in (2.1) satisfy the conditions

$$p_{ss} < 0, \qquad p_{sj} \geq 0 \ \text{ for } \ s \neq j. \tag{2.3}$$

For instance, inequalities (2.3) are valid if (2.1) is obtained as a comparison system for complex system [5, 11].

In this case the criterion of absolute stability for (2.1) was established by S.K. Persidsky [10]. It is proved that system (2.1) is absolutely stable if and only if there exist positive constants $\theta_1, \ldots, \theta_n$ such that

$$\sum_{j=1}^{n} p_{sj} \theta_j < 0, \quad s = 1, \ldots, n. \tag{2.4}$$

It should be noted that the existence of a positive solution for (2.4) is equivalent to the fulfillment of the Sevast'yanov–Kotelyanskij conditions [11]:

$$(-1)^k \det{(p_{sj})}_{s,j=1}^{k} > 0, \quad k = 1, \ldots, n. \tag{2.5}$$

On the other hand, it is known [11] that if inequalities (2.5) are valid, then one may choose numbers λ_s for the function $W = \sum\limits_{s,j=1}^{n} \lambda_s p_{sj} y_s y_j$ to be negative definite. Thus, system (2.1) is absolutely stable if and only if for this system there exists Lyapunov's function in the form (2.2), satisfying the assumptions of the Lyapunov asymptotic stability theorem.

The main goal of the present paper is to extend the above results to the system of the form

$$\dot{x}_s = a_s f_s(x_s) + \sum_{j=1}^{k_s} b_{sj} f_1^{\alpha_{s1}^{(j)}}(x_1) \dots f_n^{\alpha_{sn}^{(j)}}(x_n), \quad s = 1, \dots, n. \tag{2.6}$$

Here a_s and b_{sj} are constant coefficients, functions $f_j(x_j)$ possess the same properties as in system (2.1), $\alpha_{si}^{(j)}$ are nonnegative rationals with odd denominators.

3 Construction of Lyapunov's Functions

Let the inequalities $\sum\limits_{i=1}^{n} \alpha_{si}^{(j)} > 0$, $j = 1, \dots, k_s$, $s = 1, \dots, n$, be valid. The fulfillment of this assumption provides the existence of the zero solution for system (2.6). Furthermore, we suppose that coefficients a_s and b_{sj} satisfy the conditions

$$a_s < 0, \quad b_{sj} > 0. \tag{3.1}$$

Definition 3.1 We call (2.6) *absolutely stable* if the zero solution of this system is asymptotically stable for any admissible functions $f_j(x_j)$.

Let us investigate the conditions of absolute stability for (2.6). Along with equations (2.6), consider the system of inequalities

$$a_s \theta_s + \sum_{j=1}^{k_s} b_{sj} \theta_1^{\alpha_{s1}^{(j)}} \dots \theta_n^{\alpha_{sn}^{(j)}} < 0, \quad s = 1, \dots, n. \tag{3.2}$$

Definition 3.2 We shall say that (2.6) satisfies the *Martynyuk–Obolenskij condition* [9] (MO-condition) if for any $\delta > 0$ there exists solution $\theta_1, \dots, \theta_n$ of system (3.2) such that $0 < \theta_s < \delta$, $s = 1, \dots, n$.

Let us note that in the case, where $f_j(x_j)$ are nondecreasing functions, (2.6) is the Wazewskij's system [5, 11]. In the paper [9] the autonomous Wazewskij's systems were treated. The criterion for the asymptotic stability in the positive cone of the zero solution was obtained. Using this result, we get that the MO-condition is a necessary one for the absolute stability for system (2.6).

To prove sufficiency of this condition for the absolute stability, construct Lyapunov's function in the form

$$\widetilde{V} = \sum_{s=1}^{n} \lambda_s \int_0^{x_s} f_s^{\mu_s}(\tau) d\tau. \tag{3.3}$$

Here $\lambda_s > 0$ are constant coefficients, $\mu_s > 0$ are rationals with odd numerators and denominators.

Function $\widetilde{V}$ is positive definite. By differentiating $\widetilde{V}$ with respect to (2.6), one arrives to

$$\frac{d\widetilde{V}}{dt}\Big|_{(2.6)} = \sum_{s=1}^{n} \lambda_s a_s f_s^{\mu_s+1}(x_s) + \sum_{s=1}^{n} \lambda_s f_s^{\mu_s}(x_s) \sum_{j=1}^{k_s} b_{sj} f_1^{\alpha_{s1}^{(j)}}(x_1) \dots f_n^{\alpha_{sn}^{(j)}}(x_n).$$

Our aim is to determine the conditions under which one may choose coefficients λ_s and exponents μ_s for the function

$$\widetilde{W} = \sum_{s=1}^{n} \lambda_s a_s y_s^{\mu_s+1} + \sum_{s=1}^{n} \lambda_s y_s^{\mu_s} \sum_{j=1}^{k_s} b_{sj} y_1^{\alpha_{s1}^{(j)}} \dots y_n^{\alpha_{sn}^{(j)}} \qquad (3.4)$$

to be negative definite.

Let us denote $h_s = 1/(\mu_s + 1)$, $s = 1, \dots, n$. By the use of generally-homogeneous functions properties [12], we get that $\widetilde{W}$ may be negative definite only in the case, where the inequalities

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j)} h_i \geq 0, \quad j = 1, \dots, k_s, \quad s = 1, \dots, n, \qquad (3.5)$$

are valid.

Remark 3.1 Let positive rationals $h_1, \dots, h_n$ with odd numerators and even denominators satisfy conditions (3.5). Suppose that for some values of indices j and s corresponding inequalities in (3.5) are strict. In this case one may construct, instead of (3.4), new function $\widehat{W}$ by setting $b_{sj} = 0$ for all such j and s. If there exist positive coefficients $\lambda_1, \dots, \lambda_n$ for which $\widehat{W}$ is negative definite, then for these values of $\lambda_1, \dots, \lambda_n$ function $\widetilde{W}$ possesses the same property [12].

Remark 3.2 If there exist positive rationals $h_1, \dots, h_n$ for which all the inequalities in (3.5) are strict, i.e.

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j)} h_i > 0, \quad j = 1, \dots, k_s, \quad s = 1, \dots, n, \qquad (3.6)$$

then for corresponding values of μ_s and for any admissible values of a_s, b_{sj} and λ_s function $\widetilde{W}$ will be negative definite.

4 Auxiliary Results

In this section we will investigate the relationship between the fulfillment of the MO-condition and the existence of positive solutions for systems (3.5) and (3.6).

Lemma 4.1 *If there exists a positive solution for (3.6), then system (2.6) satisfy the MO-condition.*

Proof Let for positive constants $h_1, \dots, h_n$ inequalities (3.6) be valid. Then the numbers $\theta_s = \tau^{h_s}$, $s = 1, \dots, n$, satisfy conditions (3.2) for sufficiently small values of $\tau > 0$.

Lemma 4.2 *Let (2.6) satisfies the MO-condition. Then for any set of indices* $j_1, \dots, j_n$ $(j_s \in \{1, \dots, k_s\},\ s = 1, \dots, n)$ *there exists a positive solution for the system*

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j_s)} h_i \ge 0, \quad s = 1, \dots, n. \tag{4.1}$$

Proof For specified values of indices $j_1, \dots, j_n$ consider the inequalities

$$a_s\theta_s + b_{sj_s}\theta_1^{\alpha_{s1}^{(j_s)}} \dots \theta_n^{\alpha_{sn}^{(j_s)}} < 0, \quad s = 1, \dots, n. \tag{4.2}$$

If for (2.6) the MO-condition is fulfilled, then in any neighborhood of the state $(\theta_1, \dots, \theta_n)^* = (0, \dots, 0)^*$ there exists a positive vector $(\tilde{\theta}_1, \dots, \tilde{\theta}_n)^*$ satisfying (4.2). Along with (4.1), we investigate the system

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j_s)} h_i = c_s, \quad s = 1, \dots, n, \tag{4.3}$$

where c_s are nonnegative constants. Let us apply the Gaussian elimination procedure [4] to linear system (4.3). This procedure generates equivalent systems of equations with the coefficients changed in the similar way as the orders of $\theta_1, \dots, \theta_n$ under the successive elimination of these variables from (4.2).

Since in any neighborhood of the state $(\theta_1, \dots, \theta_n)^* = (0, \dots, 0)^*$ there exists a positive solution for inequalities (4.2), one may assume, without loss of generality, that application of the Gaussian elimination procedure to the system (4.3) yields the system

$$\sum_{i=s}^{n} \beta_{si} h_i = \tilde{c}_s, \quad s = 1, \dots, r,$$
$$\sum_{i=r+1}^{n} \beta_{si} h_i = \tilde{c}_s, \quad s = r+1, \dots, n.$$

Here $1 \le r < n$; $\beta_{ss} < 0$ for $s = 1, \dots, r$; $\beta_{si} \ge 0$ for $s = 1, \dots, r$, $i = s+1, \dots, n$ and for every $s = 1, \dots, r$ there exists $i_s > s$ such that $\beta_{si_s} > 0$; $\beta_{si} \ge 0$ for $s, i = r+1, \dots, n$; $\tilde{c}_s \ge 0$ for $s = 1, \dots, n$.

Let $\tilde{h}_{r+1}, \dots, \tilde{h}_n$ be arbitrary positive numbers,

$$\tilde{h}_s = -\frac{1}{\beta_{ss}} \sum_{i=s+1}^{n} \beta_{si}\tilde{h}_i, \quad s = 1, \dots, r.$$

For these values of $\tilde{h}_1, \dots, \tilde{h}_n$ we get $c_s = \tilde{c}_s = 0$ for $s = 1, \dots, r$ and $c_s = \tilde{c}_s \ge 0$ for $s = r+1, \dots, n$. Hence, the vector $(\tilde{h}_1, \dots, \tilde{h}_n)^*$ is a positive solution for (4.1).

Lemma 4.3 *If (2.6) satisfies the MO-condition, then there exists a positive solution for system (3.5).*

Proof Consider the system

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j)} h_i = c_s^{(j)}, \quad j = 1, \dots, k_s, \quad s = 1, \dots, n, \tag{4.4}$$

where $c_s^{(j)}$ are nonnegative constants. This system may be splitted into n subsystems. Let us apply to (4.4) the modified Gaussian elimination procedure. On the s-th step of this procedure we keep in the s-th subsystem only equations with negative coefficients of h_s. Each of the equations kept is used for the elimination of h_s from the $(s+1)$-th, etc., and n-th subsystems. This results in a new set of subsystems with (generally) the number of equations other than that in the initial system.

According to Lemma 4.2, for any set of indices $j_1, \ldots, j_n$, system (4.1) possesses a positive solution. Hence, one may assume, without loss of generality, that after the application of the above procedure we obtain the system

$$\sum_{i=s}^{n} \beta_{si}^{(j)} h_i = \tilde{c}_s^{(j)}, \quad j = 1, \ldots, q_s, \quad s = 1, \ldots, r.$$

Here $1 \leq r < n$, $\tilde{c}_s^{(j)} \geq 0$, $\beta_{ss}^{(j)} < 0$, $\beta_{si}^{(j)} \geq 0$ for $i = s+1, \ldots, n$, and for any j and s there exists $i_{sj} > s$ such that $\beta_{si_{sj}}^{(j)} > 0$, $j = 1, \ldots, q_s$, $s = 1, \ldots, n$.

It can be easily shown that if $\tilde{h}_{r+1}, \ldots, \tilde{h}_n$ are arbitrary positive numbers and

$$\tilde{h}_s = - \max_{j=1,\ldots,q_s} \frac{1}{\beta_{ss}^{(j)}} \sum_{i=s+1}^{n} \beta_{si}^{(j)} \tilde{h}_i, \quad s = 1, \ldots, r,$$

then the vector $(\tilde{h}_1, \ldots, \tilde{h}_n)^*$ is a positive solution for (3.5).

Remark 4.1 Since systems of inequalities (3.5), (3.6) are linear, the investigation of conditions for the existence of positive solutions for them is a much more simple problem than for nonlinear system (3.2).

Remark 4.2 The proof of Lemma 4.3 contains a constructive algorithm for finding a positive solution for (3.5). Moreover, let us note that using this algorithm one may choose $\tilde{h}_{r+1}, \ldots, \tilde{h}_n$ for the numbers $\mu_s = 1/\tilde{h}_s - 1$, $s = 1, \ldots, n$, to be positive rationals with odd numerators and denominators.

5 Criterion for Absolute Stability

We will find now the necessary and sufficient conditions for system (2.6) to be absolutely stable.

Theorem 5.1 *System (2.6) is absolutely stable if and only if for this system there exists Lyapunov's function in the form (3.3) satisfying the assumptions of the Lyapunov asymptotic stability theorem.*

Proof *Sufficiency* Suppose that there exists Lyapunov's function in the form (3.3) with negative definite derivative with respect to (2.6). Then for arbitrary admissible functions $f_j(x_j)$ the zero solution of the system considered is asymptotically stable. Hence, (2.6) is absolutely stable.

Necessity If (2.6) is absolutely stable, then for this system the MO-condition is fulfilled [9]. According to Lemma 4.3, there exist positive rationals $\mu_1, \ldots, \mu_n$ with odd numerators and denominators such that for the numbers $\tilde{h}_s = 1/(\mu_s + 1)$, $s = 1, \ldots, n$, inequalities (3.5) are valid. We shall take these values of $\mu_1,\dots,\mu_n$ as exponents in Lyapunov's function (3.3). Let us show that one may choose positive constants $\lambda_1,\dots,\lambda_n$ for the function (3.4) to be negative definite.

Consider a positive solution $(\tilde\theta_1,\dots,\tilde\theta_n)^*$ of (3.2). Let us denote $z_s=y_s/\tilde\theta_s$, $\gamma_s=\tilde\theta_s^{\mu_s}\lambda_s$, $s=1,\dots,n$. Then function $\widetilde W$ takes the form

$$\widetilde W=\sum_{s=1}^n\gamma_s\hat a_s z_s^{\mu_s+1}+\sum_{s=1}^n\gamma_s z_s^{\mu_s}\sum_{j=1}^{k_s}\hat b_{sj}z_1^{\alpha_{s1}^{(j)}}\dots z_n^{\alpha_{sn}^{(j)}}.$$

Here $\hat a_s=a_s\tilde\theta_s$, $\hat b_{sj}=b_{sj}\tilde\theta_1^{\alpha_{s1}^{(j)}}\dots\tilde\theta_n^{\alpha_{sn}^{(j)}}$, and $\hat a_s+\sum\limits_{j=1}^{k_s}\hat b_{sj}<0$, $s=1,\dots,n$.

We will assume, without loss of generality (v. Remark 3.1), that for the numbers $\tilde h_1,\dots,\tilde h_n$, corresponding to chosen values of $\mu_1,\dots,\mu_n$, all the inequalities in (3.5) turn to equalities.

Let $\mathbf D=\{d_{si}\}_{s,i=1}^n$, where

$$d_{ss}=\hat a_s+\sum_{j=1}^{k_s}\hat b_{sj}\alpha_{ss}^{(j)},\quad d_{si}=\sum_{j=1}^{k_i}\hat b_{ij}\alpha_{is}^{(j)}\quad\text{for}\quad s\neq i.$$

Matrix $\mathbf D$ is the Metzler matrix [5, 11].

It can be easily shown that the inequality $\mathbf D^*\mathbf h<\mathbf 0$ possesses the solution $\tilde{\mathbf h}=(\tilde h_1,\dots,\tilde h_n)^*$. Hence [11], there exists a positive solution $\tilde\gamma=(\tilde\gamma_1,\dots,\tilde\gamma_n)^*$ for the inequality $\mathbf D\gamma<\mathbf 0$.

By the use of the Jensen inequality [6], one gets that for such values of coefficients $\tilde\gamma_1,\dots,\tilde\gamma_n$ the relations

$$\begin{aligned}\widetilde W&\le\sum_{s=1}^n\tilde\gamma_s\hat a_s z_s^{\mu_s+1}+\sum_{s=1}^n\tilde\gamma_s\sum_{j=1}^{k_s}\hat b_{sj}\left(\frac{\mu_s}{\mu_s+1}z_s^{\mu_s+1}+\sum_{i=1}^n\frac{\alpha_{si}^{(j)}}{\mu_i+1}z_i^{\mu_i+1}\right)\\&=\sum_{s=1}^n\frac{\tilde\gamma_s\mu_s}{\mu_s+1}z_s^{\mu_s+1}\left(\hat a_s+\sum_{j=1}^{k_s}\hat b_{sj}\right)+\sum_{s=1}^n\frac{z_s^{\mu_s+1}}{\mu_s+1}\sum_{i=1}^n d_{si}\tilde\gamma_i\le -c\sum_{s=1}^n\frac{z_s^{\mu_s+1}}{\mu_s+1}\end{aligned}$$

are valid. Here c is a positive constant. This completes the proof.

Corollary 5.1 System (2.6) is absolutely stable if and only if it satisfies the MO-condition.

Remark 5.1 Corollary 5.1 is similar to the criterion for the asymptotic stability obtained in [9] for autonomous Wazewskij's systems. However, in comparison with this criterion, in the present paper it has been proved that only the MO-condition is a sufficient one for the asymptotic stability of the zero solution of (2.6), i.e. the other assumptions from [9] (concerning the uniqueness of solutions, isolation of the equilibrium position at the origin and nondecreasement of the functions $f_j(x_j)$) are redundant.

Corollary 5.2 Let system (2.6) satisfy the MO-condition. If there exist parameters $\mu_1,\dots,\mu_n$ such that for corresponding values of $h_1,\dots,h_n$ all the inequalities in (3.5) turn to equalities, and $\int_0^{x_s}f_s^{\mu_s}(\tau)\,d\tau\to+\infty$ as $|x_s|\to\infty$, $s=1,\dots,n$, then the zero solution of (2.6) is globally asymptotically stable.

It should be noted that Remark 3.1 makes possible, in some cases, to simplify the MO-condition verifying. Let positive rationals $h_1, \dots, h_n$ satisfy system (3.5). Then one may assume that in (2.6) $b_{sj} = 0$ if for these values of s and j the corresponding inequality in (3.5) is strict. By the use of Remark 3.1, we get that the fulfillment of the MO-condition for such reduced system is equivalent to that one for the initial system (2.6).

Example 5.1 Let system (2.6) be of the form

$$\begin{aligned} \dot{x}_1 &= a_1 f_1(x_1) + b_{11} f_2^{2/3}(x_2) f_3^{1/3}(x_3), \\ \dot{x}_2 &= a_2 f_2(x_2) + b_{21} f_1(x_1) + b_{22} f_3^3(x_3), \\ \dot{x}_3 &= a_3 f_3(x_3) + b_{31} f_1(x_1) + b_{32} f_2^3(x_2). \end{aligned} \tag{5.1}$$

Consider inequalities (3.5) corresponding to (5.1). We get

$$\begin{aligned} -h_1 + \frac{2}{3} h_2 + \frac{1}{3} h_3 &\geq 0, \\ -h_2 + h_1 &\geq 0, \\ -h_2 + 3h_3 &\geq 0, \\ -h_3 + h_1 &\geq 0, \\ -h_3 + 3h_2 &\geq 0. \end{aligned} \tag{5.2}$$

By the use of the procedure of successive elimination of variables, it can be easily shown that if positive constants h_1, h_2, h_3 satisfy (5.2), then $h_1 = h_2 = h_3$. For such values of variables the third and the fifth inequalities in (5.2) are strict, and the others turn to equalities. Hence, for (5.1) the MO-condition is fulfilled if and only if this condition is fulfilled for the reduced system

$$\begin{aligned} \dot{x}_1 &= a_1 f_1(x_1) + b_{11} f_2^{2/3}(x_2) f_3^{1/3}(x_3), \\ \dot{x}_2 &= a_2 f_2(x_2) + b_{21} f_1(x_1), \\ \dot{x}_3 &= a_3 f_3(x_3) + b_{31} f_1(x_1). \end{aligned} \tag{5.3}$$

Verifying the MO-condition for (5.3), we obtain that for (5.1) to be absolutely stable it is necessary and sufficient that the inequality $a_1^3 a_2^2 a_3 > b_{11}^3 b_{21}^2 b_{31}$ holds.

Remark 5.2 In a similar way, the criterion for absolute stability can be obtained for the case when the inequalities $b_{sj} > 0$ in (3.1) are replaced by the connecting coefficients b_{sj} and a basis $\omega_1, \dots, \omega_n$: $b_{sj} \omega_s \omega_1^{\alpha_{s1}^{(j)}} \dots \omega_n^{\alpha_{sn}^{(j)}} > 0$ for $j = 1, \dots, k_s$, $s = 1, \dots, n$ [10]. Here every constant $\omega_1, \dots, \omega_n$ takes either the value $+1$ or -1.

Example 5.2 Consider the system

$$\begin{aligned} \dot{x}_1 &= a_1\, f_1(x_1) + b_1\, f_n^{\alpha_1}(x_n), \\ \dot{x}_i &= a_i\, f_i(x_i) + b_i\, f_{i-1}^{\alpha_i}(x_{i-1}), \quad i = 2, \dots, n-1, \\ \dot{x}_n &= a_n\, f_n(x_n) + b_n\, f_1^{\nu_1}(x_1) \dots f_{n-1}^{\nu_{n-1}}(x_{n-1}), \end{aligned} \tag{5.4}$$

where a_j and b_j are constant coefficients, $a_j < 0$, $b_j \neq 0$, functions $f_j(x_j)$ possess the same properties as in (2.6), α_i and ν_i are rationals with odd denominators, $\alpha_i > 0$, $\nu_i \geq 0$, $\nu_1 + \dots + \nu_{n-1} > 0$, $j = 1, \dots, n$, $i = 1, \dots, n-1$.

By the use of Remark 3.2, we obtain that under the condition

$$\alpha_1\nu_1 + \alpha_1\alpha_2\nu_2 + \dots + \alpha_1 \dots \alpha_{n-1}\,\nu_{n-1} > 1$$

system (5.4) is absolutely stable for any admissible values of coefficients a_j and b_j.

Next, consider the case, where

$$\alpha_1\nu_1 + \alpha_1\alpha_2\nu_2 + \dots + \alpha_1 \dots \alpha_{n-1}\,\nu_{n-1} = 1. \tag{5.5}$$

It can be easily shown that for the existence of the basis $\omega_1, \dots, \omega_n$ such that

$$b_1\omega_1\omega_n^{\alpha_1} > 0, \quad b_i\omega_i\omega_{i-1}^{\alpha_i} > 0, \quad i = 2, \dots, n-1, \quad b_n\omega_n\omega_1^{\nu_1} \dots \omega_{n-1}^{\nu_{n-1}} > 0$$

it is necessary and sufficient that the inequality

$$b_1^{\xi_1} b_2^{\xi_2} \dots b_{n-1}^{\xi_{n-1}}\, b_n > 0 \tag{5.6}$$

is fulfilled. Here $\xi_i = \nu_i + \alpha_{i+1}\xi_{i+1}$, $i = 1, \dots, n-2$, $\xi_{n-1} = \nu_{n-1}$.

Making the substitution $z_j = \omega_j x_j$, $j = 1, \dots, n$, in (5.4) and applying Corollary 5.1 for the system obtained, we get that under conditions (5.5) and (5.6) system (5.4) is absolutely stable if and only if the inequality

$$\Big(-\frac{b_1}{a_1}\Big)^{\xi_1}\Big(-\frac{b_2}{a_2}\Big)^{\xi_2}\dots\Big(-\frac{b_{n-1}}{a_{n-1}}\Big)^{\xi_{n-1}}\Big(-\frac{b_n}{a_n}\Big) < 1$$

is valid.

6 Stability Analysis for Large Scale Systems in Critical Cases

Let us now show that the results obtained in the present paper may be used to refine some of the known conditions of stability for large scale systems.

Consider the system

$$\dot{x}_s = F_s(x_s) + \sum_{j=1}^{k_s} Q_{sj}(t, x), \quad s = 1, \dots, n, \tag{6.1}$$

where $x_s \in R^{m_s}$, $x = (x_1^*, \dots, x_n^*)^*$; the elements of the vectors $F_s(x_s)$ are continuously differentiable homogeneous functions of the orders $\sigma_s > 1$; the vector functions $Q_{sj}(t, x)$ are continuous for $t \geq 0$, $\|x\| < H$ (H is a positive constant, $\|\cdot\|$ is the Euclidean norm of a vector) and satisfy the inequalities

$$\|Q_{sj}(t, x)\| \leq c_{sj}\, \|x_1\|^{\beta_{s1}^{(j)}} \dots \|x_n\|^{\beta_{sn}^{(j)}}, \quad c_{sj} > 0, \quad \beta_{si}^{(j)} \geq 0.$$

We will assume that (6.1) has the zero solution.

This system describes the dynamics of a complex system composed of n interconnected subsystems [1, 5]. Here x_s are state vectors, the functions $F_s(x_s)$ define the interior connections of subsystems while the functions $Q_{sj}(t, x)$ characterize the interaction between the subsystems.

Suppose that the zero solutions of isolated systems

$$\dot{x}_s = F_s(x_s), \quad s = 1, \dots, n, \tag{6.2}$$

are asymptotically stable. We will look for the conditions under which the zero solution of (6.1) is also asymptotically stable.

In the papers [2, 7], approaches to studying stability for (6.1) are suggested. For this purpose, methods of the Lyapunov vector [7] or scalar [2] functions are used.

It is known [13] that for isolated systems (6.2) there exist Lyapunov's functions $V_s(x_s)$, which are continuously differentiable positive homogeneous functions of orders $\gamma_s - \sigma_s + 1$, $s = 1, \dots, n$. Here γ_s are arbitrary numbers such that $\gamma_s > \sigma_s$. These functions satisfy the inequalities

$$a_{1s}\|x_s\|^{\gamma_s-\sigma_s+1} \le V_s(x_s) \le a_{2s}\|x_s\|^{\gamma_s-\sigma_s+1},$$
$$\left\|\frac{\partial V_s}{\partial x_s}\right\| \le a_{3s}\|x_s\|^{\gamma_s-\sigma_s}, \qquad \left(\frac{\partial V_s}{\partial x_s}\right)^* F_s \le -a_{4s}\|x_s\|^{\gamma_s}$$

for all $x_s \in R^{m_s}$, where $a_{1s}, a_{2s}, a_{3s}, a_{4s}$ are positive constants. By differentiating $V_s(x_s)$ with respect to (6.1), one can deduce that the estimations

$$\left.\frac{dV_s}{dt}\right|_{(6.1)} \le -a_{4s}\|x_s\|^{\gamma_s} + a_{3s}\|x_s\|^{\gamma_s-\sigma_s}\sum_{j=1}^{k_s} c_{sj}\,\|x_1\|^{\beta_{s1}^{(j)}}\dots\|x_n\|^{\beta_{sn}^{(j)}}$$

are valid for $t \ge 0,\ \|x\| < H,\ s = 1, \dots, n$.

According to approach suggested in [7], the Lyapunov vector function is chosen in the form $V = (V_1, \dots, V_n)^*$. Using this function, we construct the comparison system

$$\dot{u}_s = -\tilde{a}_s u_s^{\frac{\gamma_s}{\gamma_s-\sigma_s+1}} + u_s^{\frac{\gamma_s-\sigma_s}{\gamma_s-\sigma_s+1}}\sum_{j=1}^{k_s}\tilde{b}_{sj}u_1^{\frac{\beta_{s1}^{(j)}}{\gamma_1-\sigma_1+1}}\dots u_n^{\frac{\beta_{sn}^{(j)}}{\gamma_n-\sigma_n+1}}, \quad s = 1, \dots, n, \tag{6.3}$$

for (6.1). Here

$$\tilde{a}_s = a_{4s}a_{2s}^{-\frac{\gamma_s}{\gamma_s-\sigma_s+1}}, \qquad \tilde{b}_{sj} = a_{3s}c_{sj}a_{1s}^{-\frac{\gamma_s-\sigma_s}{\gamma_s-\sigma_s+1}}a_{11}^{-\frac{\beta_{s1}^{(j)}}{\gamma_1-\sigma_1+1}}\dots a_{1n}^{-\frac{\beta_{sn}^{(j)}}{\gamma_n-\sigma_n+1}}.$$

System (6.3) is the Wazewskij one [5]. By analogy with the proof of Theorem 5.1, it can be easily shown that for the zero solution of (6.3) to be asymptotically stable it is sufficient that the corresponding MO-condition is fulfillment. Hence, if in any neighborhood of the state $(\theta_1, \dots, \theta_n)^* = (0, \dots, 0)^*$ there exists a positive solution for the system of inequalities

$$-\tilde{a}_s\theta_s + \sum_{j=1}^{k_s}\tilde{b}_{sj}\theta_1^{\beta_{s1}^{(j)}/\sigma_1}\dots\theta_n^{\beta_{sn}^{(j)}/\sigma_n} < 0, \quad s = 1, \dots, n, \tag{6.4}$$

then the zero solution of (6.1) is asymptotically stable.

Let us now show that the condition obtained for the asymptotic stability of the zero solution may be weakened by using the results of the previous section. Consider the Lyapunov scalar function

$$\widetilde{V} = \sum_{s=1}^{n}\lambda_s V_s,$$

where λ_s are positive coefficients, V_s are positive homogeneous functions of orders $\gamma_s - \sigma_s + 1$ corresponding to isolated subsystems (6.2). For all $t \geq 0$ and $\|x\| < H$ we get

$$\frac{d\widetilde{V}}{dt}\Big|_{(6.1)} \leq -\sum_{s=1}^{n} \lambda_s a_{4s} \|x_s\|^{\gamma_s} + \sum_{s=1}^{n} \lambda_s a_{3s} \|x_s\|^{\gamma_s - \sigma_s} \sum_{j=1}^{k_s} c_{sj} \, \|x_1\|^{\beta_{s1}^{(j)}} \dots \|x_n\|^{\beta_{sn}^{(j)}}.$$

Hence, to prove the asymptotic stability of the zero solution for (6.1) it is sufficient to show that one may choose positive coefficients $\lambda_1, \dots, \lambda_n$ for the function

$$\widetilde{W} = -\sum_{s=1}^{n} \lambda_s a_{4s} y_s^{\mu_s + 1} + \sum_{s=1}^{n} \lambda_s a_{3s} y_s^{\mu_s} \sum_{j=1}^{k_s} c_{sj} \, y_1^{\beta_{s1}^{(j)}/\sigma_1} \dots y_n^{\beta_{sn}^{(j)}/\sigma_n}$$

to be negative definite. Here $\mu_s = \gamma_s/\sigma_s - 1$.

Suppose that parameters $\gamma_1, \dots, \gamma_n$ satisfy the inequalities

$$-\frac{\sigma_s}{\gamma_s} + \sum_{i=1}^{n} \frac{\beta_{si}^{(j)}}{\gamma_i} \geq 0, \qquad j = 1, \dots, k_s, \quad s = 1, \dots, n. \tag{6.5}$$

In this case, by analogy with the proof of Theorem 5.1, we get that the following theorem is valid.

Theorem 6.1 *If in any neighborhood of the state $(\theta_1, \dots, \theta_n)^* = (0, \dots, 0)^*$ there exists a positive solution for the system of inequalities*

$$-a_{4s}\theta_s + a_{3s} \sum_{j=1}^{k_s} c_{sj} \, \theta_1^{\beta_{s1}^{(j)}/\sigma_1} \dots \theta_n^{\beta_{sn}^{(j)}/\sigma_n} < 0, \quad s = 1, \dots, n, \tag{6.6}$$

then the zero solution of (6.1) is asymptotically stable.

Remark 6.1 Coefficients $\tilde{a}_s$, $\tilde{b}_{sj}$, a_{3s}, a_{4s} in (6.4) and (6.6) depend, in general, on the chosen values of $\gamma_1, \dots, \gamma_n$.

Remark 6.2 For given values of $\gamma_1, \dots, \gamma_n$, Theorem 6.1 provides one with more precise conditions of asymptotic stability in comparison with those obtained via the Lyapunov vector function. However, in (6.6), compared with (6.4), it is assumed that $\gamma_1, \dots, \gamma_n$ satisfy additional restrictions (6.5).

Example 6.1 Let the system

$$\begin{aligned} \dot{x}_1 &= -\rho^2 x_1 - x_1^2 x_2, \\ \dot{x}_2 &= 100 x_1^3 - 100 \rho^2 x_2 + a x_3^9, \\ \dot{x}_3 &= -x_3^9 + b\rho^3 \end{aligned} \tag{6.7}$$

be given. Here $\rho = \sqrt{x_1^2 + x_2^2}$, a and b are constants. System (6.7) describes the interaction of two isolated subsystems

$$\begin{aligned} \dot{x}_1 &= -\rho^2 x_1 - x_1^2 x_2, \\ \dot{x}_2 &= 100 x_1^3 - 100 \rho^2 x_2, \end{aligned} \qquad \dot{x}_3 = -x_3^9.$$

Consider the functions

$$V_1 = 50\, x_1^2 + \frac{1}{2}\, x_2^2, \quad V_2 = x_3^\gamma,$$

where $\gamma > 1$ is a rational with even numerator and odd denominator. Differentiating these functions with respect to (6.7), one gets

$$\begin{aligned} \dot V_1 &= -100\rho^4 + a x_2 x_3^9, \\ \dot V_2 &= -\gamma x_3^{\gamma+8} + \gamma b \rho^3 x_3^{\gamma-1}. \end{aligned}$$

Hence, the differential inequalities

$$\begin{aligned} \dot V_1 &\le -\frac{1}{25} V_1^2 + |a|(2V_1)^{1/2} V_2^{9/\gamma}, \\ \dot V_2 &\le -\gamma V_2^{1+8/\gamma} + \gamma |b| (2V_1)^{3/2} V_2^{1-1/\gamma} \end{aligned} \tag{6.8}$$

are valid. Verifying the MO-condition for the comparison system corresponding to (6.8), it can be shown that if the inequality

$$|ab| < 1/100 \tag{6.9}$$

holds, then the zero solution of (6.7) is asymptotically stable.

This condition for the asymptotic stability of the zero solution may be weakened by the use of Theorem 6.1. Taking into account the additional restriction (6.5), we get $\gamma = 4$. Hence, system of inequalities (6.6) for (6.7) is of the form

$$\begin{aligned} -100\theta_1 + |a|\theta_2 < 0, \\ -4\theta_2 + 4|b|\theta_1 < 0. \end{aligned} \tag{6.10}$$

According to Theorem 6.1, the zero solution of (6.7) is asymptotically stable if in any neighborhood of the state $(\theta_1, \theta_2)^* = (0,0)^*$ there exists a positive solution for system (6.10). Eliminating variables θ_1, θ_2 from (6.10), we obtain new sufficient condition for asymptotic stability: $|ab| < 100$, which is more precise than (6.9).

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