Nonlinear Dynamics and Systems Theory, **5**(4) (2005) 395–405

Periodic Solution of a Convex Subquadratic Hamiltonian System

N. Kallel[1]* and M. Timoumi[2]

[1] *Institut préparatoire aux études d'ingégnieur de Sfax, Département de Mathématiques, BP 805 Sfax 3018, Tunisie*

[2] *Faculté de sciences de Monastir, Département de Mathématiques, CP 5019, Tunisie*

Received: March 1, 2005; Revised: September 10, 2005

Abstract: In this paper we study the periodic solutions of an autonomous Hamiltonian system

$$(\mathcal{H}) \qquad \dot{x} = JH'(x)$$

where H is convex and superquadratic.

We prove by using the Ambrosetti–Rabinowitz theorem and perturbation techniques that for all $T > 0$ the system $(\mathcal{H})$ has a nontrivial T-periodic solution.

Keywords: *Hamiltonian system; periodic solutions; Palais–Smale condition.*

Mathematics Subject Classification (2000): 34C25, 37J45, 70H05.

1 Introduction

In this paper we consider the Hamiltonian system:

$$(\mathcal{H}) \qquad \dot{x} = JH'(x)$$

where $H\colon R^{2N} \to R$ is a continuously differentiable function and

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

is the standard symplectic matrix.

In 1979, under the following assumptions:

(1) H is strictly convex,
(2) $\forall x \in R^{2N},\ H(x) \geq H(0) = 0$,
(3) $\exists \gamma > 2\colon \forall x \in R^{2N},\ H'(x)x \geq \gamma H(x)$,
(4) $\exists k > 0\colon \forall x \in R^{2N},\ H(x) \leq k|x|^\gamma$,

*Corresponding author: najeh.kallel@ipeis.rnu.tn

Ekeland in [1] proved that the system $(\mathcal{H})$ has for any $T > 0$ at least one nonconstant T-periodic solution.

In the present paper, we try to find the same result under some more general hypotheses. Precisely, we assume that H satisfies the following hypotheses:

(H_1) H is convex;
(H_2) $\forall\, x \in R^{2N},\ x \neq 0,\ H(x) > H(0) = 0$;
(H_3) there exist $\alpha > 2$ and $\beta > 2$ such that:

$$\forall\,(p,q) \in R^{2N}, \quad H(p,q) \leq \frac{1}{\alpha}\frac{\partial H}{\partial p}(p,q)p + \frac{1}{\beta}\frac{\partial H}{\partial q}(p,q)q;$$

(H_4) There exists $l > 0$ such that $\forall\,(p,q) \in R^{2N}$

$$|H'_p(p,q)| \leq l\left(1 + |p|^{\alpha-1} + |q|^{\beta\frac{(\alpha-1)}{\alpha}}\right),$$
$$|H'_q(p,q)| \leq l\left(1 + |q|^{\beta-1} + |p|^{\alpha\frac{(\beta-1)}{\beta}}\right);$$

(H_5) there exist $m > 0,\ n > 0$ such that $\forall(p,q) \in R^{2N}$

$$|H'_p(p,q)| \geq m|p|^{\alpha-1} - n.$$
$$|H'_q(p,q)| \geq m|q|^{\beta-1} - n.$$

Example 1.1 This is an example of Hamiltonian H which verifies the hypotheses $(H_1)-(H_5)$. Let $G, K\colon R^N \to R$ be two functions of class C^1, convex such that:

$$\forall\, x \in R^N, \quad x \neq 0, \quad G(x) > G(0) = 0, \quad K(x) > K(0) = 0,$$
$$\forall\, x \in R^N, \quad \frac{1}{\alpha}G'(x)x \geq G(x), \quad \frac{1}{\beta}K'(x)x \geq K(x),$$
$$\exists\, a, b > 0\colon\ \forall\, x \in R^N, \quad G(x) \leq a|x|^\alpha, \quad K(x) \leq b|x|^\beta.$$

Then the Hamiltonian $H(p,q) = G(p) + K(q)$, verifies the hypotheses $(H_1)-(H_5)$.

Our main result is the following.

Theorem 1.1 *Under the hypotheses* $(H_1)-(H_5)$*, the system* $(\mathcal{H})$ *possesses for any* $T > 0$ *a non constant* T*-periodic solution. Moreover, the energy* h *verifies the condition:*

$$h \leq \frac{\alpha+\beta}{\alpha\beta-\alpha-\beta}\left[\left(\frac{1}{2}-\frac{1}{\alpha}\right)\left[\frac{4\pi}{a^2T}\right]^{\frac{\alpha}{\alpha-2}} + \left(\frac{1}{2}-\frac{1}{\beta}\right)\left[\frac{4\pi}{b^2T}\right]^{\frac{\beta}{\beta-2}}\right] + \frac{(\alpha+\beta)a^\alpha}{\alpha(\alpha\beta-\alpha-\beta)} = \bar{h}$$

with $\dfrac{a^\alpha}{\alpha} = \dfrac{b^\beta}{\beta} = \min\{H(p,q),\ |p|^\alpha + |q|^\beta = 1\}.$

2 Preliminaries

Definition 2.1 Let E be a Banach space and $f\colon E \to R$ be a function of class C^1. The function f satisfies the Palais–Smale condition (PS) if every sequence (x_n) such that $(f(x_n))$ is bounded and $f'(x_n) \to 0$ as n goes to infinity, possesses a convergent subsequence.

Theorem 2.1 (Ambrosetti–Rabinowitz Theorem) [7] *Let E be a Banach space and $f\colon E \to R$ be a function of class C^1. Assume that:*

(i) *there exists $\alpha > 0$ such that:*

$$m(\alpha) = \inf\{f(x)\colon \|x\| = \alpha\} > f(0),$$

(ii) *there exists $z \in E$ such that $\|z\| \geq \alpha$ and $f(z) \leq m(\alpha)$,*

(iii) *f satisfies the Palais–Smale condition (PS).*

Then there exists $\bar{x} \in E$ such that $f'(\bar{x}) = 0$ and $f(\bar{x}) \geq m(\alpha)$. Moreover

$$f(\bar{x}) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E)\colon \gamma(0) = 0$ et $\gamma(1) = z\}$.

We have the version of the theorem of Krasnoselskii [5].

Theorem 2.2 *Let Ω be a measurable bounded set of R^n and $f\colon \Omega \times R^N \times R^N \to \overline{R}$ be a function verifying the following condition.*

For almost every $t \in \Omega$, $f(t,\cdot,\cdot)$ is convex, of class C^1, and that for all $(x,y) \in R^N \times R^N$, $f(\cdot,x,y)$ is measurable.

Let $\alpha, \beta > 1$ be two reals, we assume that there exist $\bar{\xi} \in L^\alpha(0,T;R^N)$, $\bar{\mu} \in L^\beta(0,T;R^N)$, $\bar{u} \in L^{\alpha^*}(0,T;R^N)$, $\bar{v} \in L^{\beta^*}(0,T;R^N)$ where $\alpha^{-1} + \alpha{*}^{-1} = 1$, $\beta^{-1} + \beta{*}^{-1} = 1$, such that

$$\int_\Omega |f(t,\bar{\xi}(t),\bar{\mu}(t))|\,dt < \infty, \qquad \int_\Omega |f^*(t,\bar{u}(t),\bar{v}(t))\,dt < \infty,$$

and there exists a constant $a > 0$ such that for all $t \in \Omega$ and $(p,q) \in R^{2N}$,

$$\left|\frac{\partial f}{\partial p}(t,p,q)\right| \leq a\max\{1,\, |p|^{\alpha-1},\, |q|^{\beta\frac{(\alpha-1)}{\alpha}}\},$$
$$\left|\frac{\partial f}{\partial q}(t,p,q)\right| \leq a\max\{1,\, |p|^{\alpha\frac{(\beta-1)}{\beta}},\, |q|^{\beta-1}\},$$

so the functional

$$F\colon L^\alpha \times L^\beta \to \overline{R},$$
$$(p,q) \mapsto \int_\Omega f(t,p(t),q(t))\,dt$$

is of class C^1 *and*

$$[F'(p,q)](t) = \frac{\partial f}{\partial (p,q)}(t, p(t), q(t)).$$

3 Proof of Theorem 1.1

We will proceed by successive lemmas.

The hypothesis (H_3) is equivalent to the following

$$(H_6) \qquad \forall\, \lambda \geq 1, \quad \forall\, (p,q) \in R^{2N}, \quad H(\lambda^{1/\alpha}p,\, \lambda^{1/\beta}q) \geq \lambda H(p,q).$$

Let $\epsilon_0 \in]0, m[$ be a fixed real. For all $0 < \epsilon \leq \epsilon_0$, we consider the Hamiltonian

$$H_\epsilon(p,q) = H(p,q) + \epsilon(|p|^\alpha + |q|^\beta).$$

It's clear that H_ϵ is strictly convex and verifies $(H_2)-(H_5)$.

Set

$$\frac{a_\epsilon^\alpha}{\alpha} = \frac{b_\epsilon^\beta}{\beta} = \frac{a^\alpha}{\alpha} + \epsilon, \quad l_\epsilon = l + \epsilon, \quad m_\epsilon = m - \epsilon.$$

Lemma 3.1 *Let* α^* *and* β^* *be such that* $\frac{1}{\alpha} + \frac{1}{\alpha^*} = \frac{1}{\beta} + \frac{1}{\beta^*} = 1$, *so*

(7) H_ϵ^* *is of class* C^1;

(8) $\forall (r,s) \in R^{2N}$, $\frac{1}{\alpha^*}(H_\epsilon^*)'_r(r,s)r + \frac{1}{\beta^*}(H_\epsilon^*)'_s(r,s)s \leq {H_\epsilon}^*(r,s)$;

(9) $H_\epsilon(p,q) \geq \frac{{a_\epsilon}^\alpha}{\alpha}|p|^\alpha + \frac{{b_\epsilon}^\beta}{\beta}|q|^\beta$ *for all* $|p|^\alpha + |q|^\beta \geq 1$, $H_\epsilon(p,q) \geq \frac{a_\epsilon^\alpha}{\alpha}(|p|^\alpha + |q|^\beta - 1)$ *for all* $(p,q) \in R^{2N}$;

(10) $\forall\, (r,s) \in R^{2N}$, $H_\epsilon^*(r,s) \leq \frac{a_\epsilon^\alpha}{\alpha} + \frac{1}{\alpha^* a_\epsilon^{\alpha^*}}|r|^{\alpha^*} + \frac{1}{\beta^* b_\epsilon^{\beta^*}}|s|^{\beta^*}$;

(11) *there exists* $k_\epsilon > 0$ *such that* $\forall\, (p,q) \in R^{2N}$, $H_\epsilon(p,q) \leq k_\epsilon(|p|^\alpha + |q|^\beta)$;

(12) $\forall\, (r,s) \in R^{2N}$ $H_\epsilon^*(r,s) \geq \frac{1}{\alpha^* c_\epsilon^{\alpha^*}}|r|^{\alpha^*} + \frac{1}{\beta^* d_\epsilon^{\beta^*}}|s|^{\beta^*}$, *where* c_ϵ *and* d_ϵ *are given by* $\frac{c_\epsilon^\alpha}{\alpha} = \frac{d_\epsilon^\beta}{\beta} = k_\epsilon$.

Proof (9) Set $S = \{(p,q) \in R^{2N} \colon\ |p|^\alpha + |q|^\beta = 1\}$. For $(p,q) \in R^{2N}$ such that $|p|^\alpha + |q|^\beta \geq 1$, we set $s = |p|^\alpha + |q|^\beta$, so $(s^{-1/\alpha}p,\, s^{-1/\beta}q) \in S$ and by (H_6) we have

$$H_\epsilon(p,q) \geq s \min_{(p,q)\in S}\{H_\epsilon(p,q)\}.$$

For $|p|^\alpha + |q|^\beta \leq 1$, we have $H_\epsilon(p,q) \geq 0 \geq \frac{a_\epsilon^\alpha}{\alpha}(|p|^\alpha + |q|^\beta - 1)$. This is the desired result.

(7) By the inequality (9) we have for $|p|^\alpha + |q|^\beta \geq 1$

$$\frac{H_\epsilon(p,q)}{|p|+|q|} \geq \frac{a_\epsilon^\alpha}{\alpha}\left(\frac{|p|^\alpha}{|p|+|q|} + \frac{|q|^\beta}{|p|+|q|}\right)$$

and since $\alpha, \beta > 1$, so

$$\lim_{|(p,q)|\to\infty} \frac{H_\epsilon(p,q)}{|p|+|q|} = +\infty.$$

Since H_ϵ is strictly convex and of class C^1, so by a result of convex analysis the conjugate H_ϵ^* of H_ϵ is of class C^1.

(8) Let $(r,s) \in R^{2N}$ and $(p,q) = H_\epsilon^{*\prime}(r,s) = (({H_\epsilon}^*)'_r(r,s), ({H_\epsilon}^*)'_s(r,s))$, so by the Fenchel reciprocity and the hypothesis (H_3) we obtain

$$ {H_\epsilon}^*(r,s) \geq \left(1 - \frac{1}{\alpha}\right)({H_\epsilon}^*)'_r(r,s)r + \left(1 - \frac{1}{\beta}\right)({H_\epsilon}^*)'_s(r,s)s, $$

hence the result.

(10) Let $(r,s) \in R^{2N}$, we have

$$ H_\epsilon^*(r,s) = \sup_{(p,q)\in R^{2N}} \{pr + sq - H_\epsilon(p,q)\}, $$

thus by the inequality (9)

$$ \begin{aligned} {H_\epsilon}^*(r,s) &\leq \sup_{(p,q)\in R^{2N}} \left\{pr + sq - \frac{a_\epsilon^\alpha}{\alpha}|p|^\alpha - \frac{b_\epsilon^\beta}{\beta}|q|^\beta + \frac{a_\epsilon^\alpha}{\alpha}\right\} \\ &\leq \frac{1}{\alpha^* a_\epsilon^{\alpha^*}}|r|^{\alpha^*} + \frac{1}{\beta^* b_\epsilon^{\beta^*}}|s|^{\beta^*} + \frac{a_\epsilon^\alpha}{\alpha}. \end{aligned} $$

(11) For $(p,q) \in R^{2N}$ such that $|p|^\alpha + |q|^\beta \geq 1$, there exists $\theta \in]0,1[$ such that

$$ \begin{aligned} H_\epsilon(p,q) &= \frac{\partial H_\epsilon}{\partial p}(\theta(p,q))p + \frac{\partial H_\epsilon}{\partial q}(\theta(p,q))q \\ &\leq \left|\frac{\partial H_\epsilon}{\partial p}(\theta(p,q))\right||p| + \left|\frac{\partial H_\epsilon}{\partial q}(\theta(p,q))\right||q| \end{aligned} $$

so by the hypothesis (H_4)

$$ \begin{aligned} H_\epsilon(p,q) &\leq l_\epsilon\left(|p| + |p|^\alpha + |p||q|^{\beta\frac{(\alpha-1)}{\alpha}} + |q| + |q|^\beta + |q||p|^{\alpha\frac{(\beta-1)}{\beta}}\right) \\ &\leq l_\epsilon\left(|p| + |q| + |p|^\alpha + |q|^\beta + \frac{|p|^\alpha}{\alpha} + \frac{|q|^\beta}{\alpha^*} + \frac{|q|^\beta}{\beta} + \frac{|p|^\alpha}{\beta^*}\right). \end{aligned} $$

So there exists $\tilde{k}_\epsilon > 0$ such that

$$ H_\epsilon(p,q) \leq \tilde{k}_\epsilon(|p|^\alpha + |q|^\beta). $$

For $(p,q) \in R^{2N}$ such that , $s = |p|^\alpha + |q|^\beta \leq 1$, we have by ($H_6$)

$$ \begin{aligned} H(p,q) &\leq sH(s^{-1/\alpha},\, s^{-1/\beta}) \\ &\leq s \max_{(p,q)\in S}\{H(p,q)\} \leq k(|p|^\alpha + |q|^\beta), \end{aligned} $$

where $k = \max_{(p,q)\in S}\{H(p,q)\}$.

Hence, by picking $k_\epsilon = \max(\tilde{k}_\epsilon,\, k + \epsilon)$, we obtain the result.

(12) Let $(r,s)\in R^{2N}$,

$$\begin{aligned} H^*_\epsilon(r,s) &= \sup_{(p,q)\in R^{2N}}\{pr+sq-H_\epsilon(p,q)\}\\ &\geq \sup_{(p,q)\in R^{2N}}\{pr+sq-k_\epsilon(|p|^\alpha+|q|^\beta)\}\\ &\geq \frac{1}{\alpha^* c_\epsilon^{\alpha^*}}|r|^{\alpha^*}+\frac{1}{\beta^* d_\epsilon^{\beta^*}}|s|^{\beta^*}. \end{aligned}$$

Denote for μ a real ≥ 1

$$L_0^\mu=\left\{p\in L^\mu(0,T;R^N)/\int_0^T p(t)dt=0\right\}.---???$$

We define on $L_0^{\alpha^*}\times L_0^{\beta^*}$ the dual action functional f_ϵ by

$$f_\epsilon(p,q)=\frac{1}{2}\int_0^T\langle J(p,q),\pi(p,q)\rangle dt+\int_0^T H^*_\epsilon(p,q)\,dt,$$

where

$$(\pi y)(t)=\int_0^t y(s)\,ds-\frac{1}{T}\int_0^T dt\int_0^t y(s)\,ds$$

is the primitive of y with zero mean.

We are interested in the search of a non trivial critical point of f_ϵ, by using the Ambrosetti–Rabinowitz theorem.

Lemma 3.2 *f_ϵ is of class C^1 and for all $(p,q)\in L_0^{\alpha^*}\times L_0^{\beta^*}$, there exists $(\xi_\epsilon,\mu_\epsilon)\in R^N\times R^N$ such that*

$$f'_\epsilon(p,q)=-J\pi(p,q)+H^{*'}_\epsilon(p,q)+(\xi_\epsilon,\mu_\epsilon).$$

The proof is a simple application of the version of the theorem of Krasnoselskii.

Lemma 3.3 *There exist $\rho>0$ and $\gamma>0$ such that*

$$\|(p,q)\|_{L_0^{\alpha^*}\times L_0^{\beta^*}}=\rho\Rightarrow f_\epsilon(p,q)\geq\gamma.$$
$$0<\|(p,q)\|_{L_0^{\alpha^*}\times L_0^{\beta^*}}\leq\rho\Rightarrow f_\epsilon(p,q)>f_\epsilon(0,0)=0$$

Proof It's easy to verify that for all $(p,q)\in L_0^{\alpha^*}\times L_0^{\beta^*}$ we have

$$\left|\int_0^T\prec p(t),\pi q(t)\succ dt\right|=\left|-\int_0^T\prec \pi p(t),q(t)\succ dt\right|\leq T^{\frac{1}{\alpha}+\frac{1}{\beta}}(|p|^2_{L^{\alpha^*}}+|q|^2_{L^{\beta^*}}).$$

So, by the inequality (12) for all $\epsilon\in]0,\epsilon_0]$ and $(p,q)\in L_0^{\alpha^*}\times L_0^{\beta^*}$,

$$\begin{aligned} f_\epsilon(p,q) &\geq -T^{\frac{1}{\alpha}+\frac{1}{\beta}}(|p|^2_{L^{\alpha^*}}+|q|^2_{L^{\beta^*}})+\frac{1}{\alpha^* c_{\epsilon_0}^{\alpha^*}}|p|^{\alpha^*}_{L^{\alpha^*}}+\frac{1}{\beta^* d_{\epsilon_0}^{\beta^*}}|q|^{\beta^*}_{L^{\beta^*}}\\ &\geq -T^{\frac{1}{\alpha}+\frac{1}{\beta}}|p|^2_{L^{\alpha^*}}+\frac{1}{\alpha^* c_{\epsilon_0}^{\alpha^*}}|p|^{\alpha^*}_{L^{\alpha^*}}-T^{\frac{1}{\alpha}+\frac{1}{\beta}}|q|^2_{L^{\beta^*}}+\frac{1}{\beta^* d_{\epsilon_0}^{\beta^*}}|q|^{\beta^*}_{L^{\beta^*}} \end{aligned}$$

hence, since $\alpha^*<2$, $\beta^*<2$, the desired result is obtained.

Lemma 3.4 *There exists* $(p_0, q_0) \in (L_0^{\alpha^*} \times L_0^{\beta^*}) \setminus \{(0,0)\}$ *such that* $f_\epsilon(p_0, q_0) = 0$.

Proof Let $Z = (p,q) \in R^{2N}$, setting $\omega(t) = Z \sin\left(-\frac{2\pi}{T}t\right) + JZ\cos\left(\frac{2\pi}{T}t\right)$, we have

$$\forall\, t \in [0,T], \quad |\omega(t)|^2 = |Z|^2 = |p|^2 + |q|^2.$$

Thus

$$\frac{1}{2}\int_0^T \prec -J\pi\omega, \omega \succ dt = -\frac{T}{4\pi}\int_0^T |\omega(t)|^2 dt = -\frac{T^2}{4\pi}|Z|^2.$$

So, it follows by the inequality (10), that for all $s \geq 0$ we have

$$f_\epsilon(s\omega) \leq -\frac{T^2}{4\pi}s^2|Z|^2 + \frac{T}{\alpha^* {a_\epsilon}^{\alpha^*}} s^{\alpha^*}|Z|^{\alpha^*} + \frac{T}{\beta^* b_\epsilon^{\beta^*}} s^{\beta^*}|Z|^{\beta^*} + \frac{a_\epsilon^\alpha}{\alpha}T.$$

Since $\alpha^* < 2$ and $\beta^* < 2$, we obtain the result by applying the Lemma 3.3.

Lemma 3.5 f_ϵ *verifies the Palais–Smale condition.*

Proof Let $(\omega_n)_{n\in N} = ((p_n, q_n))_{n\in N}$ a sequence of $L_0^{\alpha^*} \times L_0^{\beta^*}$ verifying $(f_\epsilon(\omega_n))_n$ is bounded and $f'_\epsilon(\omega_n)$ converges to zero as n goes to infinity. So, there exist two constants A and B such that

$$A \leq -\frac{1}{2}\int_0^T \prec J\pi\omega_n(t), \omega(t) \succ dt + \int_0^T H_\epsilon^*(\omega_n(t))\, dt \leq B, \tag{13}$$

and

$$(-\pi q_n, \pi p_n) + (({H_\epsilon}^*)'_p(\omega_n),\, ({H_\epsilon}^*)'_q(\omega_n)) + (\xi_{\epsilon,n}, \mu_{\epsilon,n}) = (\lambda_n, \eta_n) \tag{14}$$

converges to zero in $L_0^{\alpha^*} \times L_0^{\beta^*}$ as n goes to infinity.

By taking πp_n and πq_n from the expression (14) and substituting it into (13), we obtain:

$$\begin{aligned}
&\left(\frac{1}{\alpha^*} + \frac{1}{\beta^*}\right)\int_0^T H_\epsilon^*(\omega_n(t))\, dt + \frac{1}{\beta^*}\int_0^T [\prec \eta_n, q_n \succ - \prec \mu_{\epsilon,n}, q_n \succ]\, dt \\
&- \frac{1}{\beta^*}\int_0^T \prec ({H_\epsilon}^*)'_q(\omega_n), q_n \succ dt + \frac{1}{\alpha^*}\int_0^T [\prec \lambda_n, p_n \succ - \prec \xi_{\epsilon,n}, p_n \succ]\, dt \\
&- \frac{1}{\alpha^*}\int_0^T \prec ({H_\epsilon}^*)'_p(\omega_n), p_n \succ dt \leq \left(\frac{1}{\alpha^*} + \frac{1}{\beta^*}\right)B,
\end{aligned}$$

thus

$$\begin{aligned}
&\left(\frac{1}{\alpha^*} + \frac{1}{\beta^*}\right)\int_0^T H_\epsilon^*(\omega_n(t))\, dt - \int_0^T \left[\frac{1}{\alpha^*} \prec ({H_\epsilon}^*)'_p(\omega_n), p_n \succ + \frac{1}{\beta^*} \prec ({H_\epsilon}^*)'_q(\omega_n), q_n \succ\right] dt \\
&+ \frac{1}{\alpha^*}\int_0^T \prec \lambda_n, p_n \succ dt + \frac{1}{\beta^*}\int_0^T \prec \eta_n, q_n \succ dt \leq \left(\frac{1}{\alpha^*} + \frac{1}{\beta^*}\right)B.
\end{aligned}$$

We deduce by the inequality (8) that:

$$\left(\frac{1}{\alpha^*}+\frac{1}{\beta^*}-1\right)\int_0^T H_\epsilon^*(\omega_n(t))\,dt-\frac{1}{\alpha^*}|\lambda_n|_{L^\alpha}|p_n|_{L^{\alpha^*}}-\frac{1}{\beta^*}|\eta_n|_{L^\beta}|q_n|_{L^{\beta^*}}$$
$$\le\left(\frac{1}{\alpha^*}+\frac{1}{\beta^*}\right)B.$$

Hence, since $\frac{1}{\alpha^*}+\frac{1}{\beta^*}-1>0$ and by the inequality (12), we have

$$\left(\frac{1}{\alpha^*}+\frac{1}{\beta^*}-1\right)\left[\frac{1}{\alpha^* c_\epsilon^*}|p|_{L^{\alpha^*}}^{\alpha^*}+\frac{1}{\beta^* d_\epsilon^{\beta^*}}|q|_{L^{\beta^*}}^{\beta^*}\right]-\frac{1}{\alpha^*}|\lambda_n|_{L^\alpha}|p|_{L^{\alpha^*}}-\frac{1}{\beta^*}|\eta_n|_{L^\beta}|q_n|_{L^{\beta^*}}$$
$$\le\left(\frac{1}{\alpha^*}+\frac{1}{\beta^*}\right)B.$$

Since $\alpha^*,\beta^*<2$ and $|\lambda_n|_{L^\alpha}\to 0$, $|\eta_n|_{L^\beta}\to 0$ as $n\to\infty$, we deduce that there exists a constant $d>0$ such that for all $n\in N$ $|p_n|_{L^{\alpha^*}},|q_n|_{L^{\beta^*}}\le d$ and up to a subsequence, we may assume that (p_n,q_n) is weakly convergent to $\omega=(p,q)$ in $L_0^{\alpha^*}\times L_0^{\beta^*}$.

Consider the set

$$D=\{-J\pi(p_n,q_n),\ n\in N\}\subset C([0,T],\,R^{2N}).$$

By (H_5), we verify that $({H_\epsilon^*}'(p_n,q_n))$ is bounded in $L_0^\alpha\times L_0^\beta$ and since (λ_n,η_n) goes to zero in $L_0^\alpha\times L_0^\beta$ as n goes to infinity, so by the formula (14), $(\xi_{\epsilon,n_k},\mu_{\epsilon,n_k})$ is bounded in R^{2N} and therefore we can suppose that $(\xi_{\epsilon,n_k},\mu_{\epsilon,n_k})$ converges to (ξ,μ).

Finally, since

$$H_\epsilon^{*'}(p_{n_k},q_{n_k})=(\lambda_{n_k},\eta_{n_k})+J\pi(p_{n_k},q_{n_k})-(\xi_{\epsilon,n_k},\mu_{\epsilon,n_k}),$$

we have by the Fenchel reciprocity:

$$(p_{n_k},q_{n_k})=H_\epsilon'((\lambda_{n_k},\eta_{n_k})+J\pi(p_{n_k},q_{n_k})-(\xi_{\epsilon,n_k},\mu_{\epsilon,n_k})),$$

By (H_4) and the version of the theorem of Krasnoselskii, the map $(u,v)\longmapsto H_\epsilon'(u,v)$ defined on $L_0^\alpha\times L_0^\beta$ into $L^{\alpha^*}\times L^{\beta^*}$ is continuous. Thus the sequence $(p_{n_k},q_{n_k})=H_\epsilon'((\lambda_{n_k},\eta_{n_k})+J\pi(p_{n_k},q_{n_k})-(\xi_{\epsilon,n_k},\mu_{\epsilon,n_k}))$ is convergent in $L^{\alpha^*}\times L^{\beta^*}$ and the lemma is proved.

The functional f_ϵ verifies all the hypotheses of the Ambrosetti–Rabinowitz theorem, consequently there exists $\bar y_\epsilon=(\bar p_\epsilon,\bar q_\epsilon)\in L_0^{\alpha^*}\times L_0^{\beta^*}$ such that

$$f_\epsilon'(\bar y_\epsilon)=0$$

and

$$f_\epsilon(\bar y_\epsilon)\ge\gamma.$$

By the Lemma 3.2, there exists $(\xi_\epsilon,\mu_\epsilon)\in R^{2N}$ such that

$$0=-J\pi(\bar y_\epsilon)+H_\epsilon^{*'}(\bar y_\epsilon)+(\xi_\epsilon,\mu_\epsilon),$$

which gives by the Fenchel reciprocity

$$\bar{y}_\epsilon = H'_\epsilon(J\pi(\bar{y}_\epsilon) - (\xi_\epsilon, \mu_\epsilon)).$$

Setting $\bar{x}_\epsilon = (\bar{u}_\epsilon, \bar{v}_\epsilon) = J\pi(\bar{y}_\epsilon) - (\xi_\epsilon, \mu_\epsilon)$, we have

$$\dot{\bar{x}}_\epsilon = (\dot{\bar{u}}_\epsilon, \dot{\bar{v}}_\epsilon) = J(\bar{y}_\epsilon) = JH'_\epsilon(\bar{u}_\epsilon, \bar{v}_\epsilon) = JH'_\epsilon(\bar{x}_\epsilon).$$

Thus the Hamiltonian system

$$(\mathcal{H}_\epsilon) \qquad \dot{x} = JH'_\epsilon(x)$$

possesses a T-periodic solution.

Lemma 3.6 *Let h_ϵ be the energy of the found solution $\bar{x}_\epsilon$. Then*

$$h_\epsilon \le \frac{\alpha+\beta}{\alpha\beta-\alpha-\beta}\left[\left(\frac{1}{2}-\frac{1}{\alpha}\right)\left[\frac{4\pi}{Ta_\epsilon^2}\right]^{\frac{\alpha}{\alpha-2}} + \left(\frac{1}{2}-\frac{1}{\beta}\right)\left[\frac{4\pi}{Tb_\epsilon^2}\right]^{\frac{\beta}{\beta-2}}\right] + \frac{\alpha+\beta}{\alpha\beta-\alpha-\beta}\,\frac{a_\epsilon^\alpha}{\alpha}. \quad (15)$$

Proof We have

$$\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)f_\epsilon(\bar{y}_\epsilon) = \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\left[\int_0^T \frac{1}{2} \prec H'_\epsilon(\bar{x}_\epsilon), \bar{x}_\epsilon \succ dt - \int_0^T H_\epsilon(\bar{x}_\epsilon)\,dt\right]$$

$$= \frac{1}{\alpha}\int_0^T \prec (H_\epsilon)'_u(\bar{x}_\epsilon), \bar{u}_\epsilon \succ dt + \frac{1}{\beta}\int_0^T \prec (H_\epsilon)'_v(\bar{x}_\epsilon), \bar{v}_\epsilon \succ dt - (\frac{1}{\alpha}+\frac{1}{\beta})\int_0^T H_\epsilon(\bar{x}_\epsilon)\,dt$$

and by (H_3) we obtain

$$\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)f_\epsilon(\bar{y}_\epsilon) \ge \left(1-\frac{1}{\alpha}-\frac{1}{\beta}\right)\int_0^T H_\epsilon(\bar{x}_\epsilon)\,dt,$$

which implies that

$$f_\epsilon(\bar{y}_\epsilon) \ge \frac{\alpha\beta-\alpha-\beta}{\alpha+\beta}\,h_\epsilon T.$$

On the other hand, by the Ambrosetti–Rabinowitz theorem we have

$$f_\epsilon(\bar{y}_\epsilon) = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} f_\epsilon(\gamma_\epsilon(s)),$$

where $\Gamma = \{\gamma \in C([0,T],\, L_0^{\alpha^*} \times L_0^{\beta^*})/\gamma(0) = 0 \text{ and } \gamma(1) = (p_0, q_0)\}$.

For $s \in R^+$, we set $\omega_s(t) = s\omega(t)$ where ω is defined in the proof of Lemma 3.4. We have

$$\begin{aligned} f_\epsilon(\bar{y}_\epsilon) &\le \sup_{s\in[0,1]} f_\epsilon(s(p_0,q_0)) \le \sup_{s\ge 0} f_\epsilon(s\omega) \\ &\le \sup_{s\ge 0}\left\{-\frac{T^2}{4\pi}s^2|Z|^2 + \frac{T}{\alpha^* a_\epsilon^{\alpha^*}}s^{\alpha^*}|Z|^{\alpha^*} + \frac{T}{\beta^* b_\epsilon^{\beta^*}}s^{\beta^*}|Z|^{\beta^*} + \frac{a_\epsilon^\alpha}{\alpha}T\right\} \\ &\le \sup_{s\ge 0}\left\{\frac{-T^2}{8\pi}s^2|Z|^2 + \frac{T}{\alpha^* a_\epsilon^{\alpha^*}}s^{\alpha^*}|Z|^{\alpha^*}\right\} + \frac{a_\epsilon^\alpha}{\alpha}T \\ &\quad + \sup_{s\ge 0}\left\{-\frac{T^2}{8\pi}s^2|Z|^2 + \frac{T}{\beta^* b_\epsilon^{\beta^*}}s^{\beta^*}|Z|^{\beta^*}\right\}. \end{aligned}$$

Setting

$$\varphi(s) = -\frac{T^2}{8\pi}s^2|Z|^2 + \frac{T}{\alpha^* a_\epsilon^{\alpha^*}} s^{\alpha^*}|Z|^{\alpha^*}, \quad \psi(s) = -\frac{T^2}{8\pi}s^2|Z|^2 + \frac{T}{\beta^* b_\epsilon^{\beta^*}} s^{\beta^*}|Z|^{\beta^*}.$$

So φ attains its maximum at

$$\bar{s} = \left[\frac{4\pi}{a_\epsilon^{\alpha^*}T}\right]^{\frac{1}{2-\alpha^*}} \frac{1}{|Z|},$$

and ψ attains its maximum at

$$\bar{\bar{s}} = \left[\frac{4\pi}{b_\epsilon^{\beta^*}T}\right]^{\frac{1}{2-\beta^*}} \frac{1}{|Z|}.$$

A simple computation gives

$$\varphi(\bar{s}) = T\left(\frac{1}{2} - \frac{1}{\alpha}\right)\left[\frac{4\pi}{a_\epsilon^{\alpha^*}T}\right]^{\frac{\alpha}{\alpha-2}}$$

and

$$\psi(\bar{\bar{s}}) = T\left(\frac{1}{2} - \frac{1}{\beta}\right)\left[\frac{4\pi}{b_\epsilon^{\beta^*}T}\right]^{\frac{\beta}{\beta-2}},$$

so

$$f_\epsilon(\bar{y}_\epsilon) \le T\left[\left(\frac{1}{2} - \frac{1}{\alpha}\right)\left[\frac{4\pi}{a_\epsilon^{\alpha^*}T}\right]^{\frac{\alpha}{\alpha-2}} + \left(\frac{1}{2} - \frac{1}{\beta}\right)\left[\frac{4\pi}{b_\epsilon^{\beta^*}T}\right]^{\frac{\beta}{\beta-2}}\right] + \frac{a_\epsilon^\alpha}{\alpha}T.$$

Consequently

$$h_\epsilon \le \frac{\alpha+\beta}{\alpha\beta-\alpha-\beta}\left[\left(\frac{1}{2} - \frac{1}{\alpha}\right)\left[\frac{4\pi}{a_\epsilon^{\alpha^*}T}\right]^{\frac{\alpha}{\alpha-2}} + \left(\frac{1}{2} - \frac{1}{\beta}\right)\left[\frac{4\pi}{b_\epsilon^{\beta^*}T}\right]^{\frac{\beta}{\beta-2}}\right] + \frac{\alpha+\beta}{\alpha\beta-\alpha-\beta}\frac{a_\epsilon^\alpha}{\alpha}$$

and the Lemma 3.6 is proved.

Lemma 3.7 *The set* $E = \{\bar{x}_\epsilon :\ 0 < \epsilon \le \epsilon_0\}$ *is relatively compact in* $C([0,T], R^{2N})$.

Proof We have for all $\epsilon \in]0, \epsilon_0]$,

$$0 < \frac{a^\alpha}{\alpha} < \frac{a_\epsilon^\alpha}{\alpha} < \frac{a^\alpha}{\alpha} + \epsilon_0.$$

Thus, by (15), there exists $R \in R_+^*$ such that

$$H(\bar{x}_\epsilon(t)) \le R$$

for all $t \in [0,T]$ and $\epsilon \in]0, \epsilon_0]$.

Since $\lim_{|x|\to\infty} H(x) = +\infty$, so there exists $\lambda \in R_+^*$ such that for all $t \in [0,T]$ and $\epsilon \in]0, \epsilon_0]$ $\bar{x}_\epsilon(t) \in B(0, \lambda)$. Consequently, for all $t \in [0,T]$, the set $E(t)$ is relatively compact in R^{2N}.

On the other hand, since H' is continuous, there exists $\eta > 0$ independent of ϵ such that for all $\epsilon \in]0, \epsilon_0]$ and $t, t' \in [0, T]$, $\|\bar{x}_\epsilon(t) - \bar{x}_\epsilon(t')\| \leq \eta |t - t'|^{1/2}$. Thus E is equicontinuous. Hence, by the theorem of Ascoli, E is relatively compact in $C([0, T], R^{2N})$.

So, we may extract from E a subsequence $(\bar{x}_{\epsilon_n})$, $\epsilon_n \to 0$, which is convergent uniformly in $[0, T]$. Let $\bar{x} = (\bar{u}, \bar{v})$ be its limit; we have

$$\begin{aligned}\dot{\bar{x}}_{\epsilon_n} = JH'_{\epsilon_n}(\bar{x}_{\epsilon_n}) &= J(H'(\bar{x}_{\epsilon_n}) + \epsilon_n(\alpha|\bar{u}_{\epsilon_n}|^{\alpha-2}\bar{u}_{\epsilon_n},\, \beta|\bar{v}_{\epsilon_n}|^{\beta-2}\bar{v}_{\epsilon_n}))\\ &\to JH'(\bar{u}, \bar{v}) \quad \text{uniformly},\end{aligned}$$

which implies that

$$\dot{\bar{x}} = JH'(\bar{x}).$$

So it's clear that $H^*_{\epsilon_n}(\bar{p}_{\epsilon_n}, \bar{q}_{\epsilon_n})$ is convergent uniformly to $H^*(\dot{\bar{x}})$ and

$$0 < \gamma \leq \lim_{n\to\infty} f_{\epsilon_n}(\bar{y}_{\epsilon_n}) = f(\dot{\bar{x}}).$$

Since $f(0,0) = 0$, so $\dot{\bar{x}} \neq 0$ and $\bar{x}$ is not constant.

Finally, we have $\lim_{n\to\infty} a_{\epsilon_n} = a$ and $\lim_{n\to\infty} b_{\epsilon_n} = b$, thus $\lim_{n\to\infty} \bar{h}_{\epsilon_n} = \bar{h}$ and so $h = H(\bar{x}) \leq \bar{h}$.

References

[1] Aubin, J.P. and Ekelend, I. *Applied Nonlinear Analysis.* Wiley, 1986.

[2] Aze, D. and Blot, J. *Systèmes Hamiltoniens: Leurs Solutions Périodiques.* Cedic, 1982.

[3] Ekeland, I. and Temam, R. *Analyse Convexe et Problèmes Variationnels.* Paris, 1974.

[4] Felmer, P.L. Periodic solutions of super-quadratic Hamiltonian systems. *J. Diff. Eqns* **102** (1993) 188 – 207.

[5] Krasnosel'skii, M.A. *Topological Methods in the Theory of Nonlinear Integral Equations.* Pergamon Press, 1964.

[6] Kunze, M. Periodic solutions of a singular Lagrangian system related to dispersion-managed fiber communication devices. *Nonlinear Dynamics and Systems Theory* **1**(2) (2001) 159–168.

[7] Mawhin, J. and Willem, M. *Critical Point Theory and Hamiltonian System.* Springer–Verlag, 1987.

[8] Rabinowitz, P.H. Minimax methods in critical point theory with application to differential equations. *CBMS Reg. Conf. Ser. in Math.* 65, Amer. Math. Soc., Providence, R.I., 1986.

[9] Timoumi, M. Oscillations de systèmes Hamiltoniens surquadratiques non coercitifs. *Demonstratio Mathématica* **XXVII**(2) (1994) 293 – 300.