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Feedback Stabilization of the Extended Nonholonomic Double Integrator

Fazal-ur-Rehman*

Faculty of Electronic Engineering,
GIK Institute of Engineering Sciences and Technology,
Topi, 23460, Swabi, Pakistan

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Abstract: This paper presents a simple control strategy of feedback stabilization for the extended nonholonomic double integrator. The strategy presents a time-varying feedback law based on the model reference approach, where the trajectory of the extended system is chosen as the model reference trajectory. The controllers are designed in such a way that after each time period , the trajectory of the nonholonomic double integrator intersects the trajectory of the model reference, which can be made asymptotically stable. The proposed feedback law is as a composition of a standard stabilizing feedback control for a Lie bracket extension of the original system and a periodic continuation of a specific solution to an open loop control problem stated for an abstract equation on a Lie group. This approach does not rely on a specific choice of a Lyapunov function, and does not require transformations of the model to chained forms.

Keywords: *Feedback stabilization; systems with drift; nonholonomic systems; controllability; Lie algebra; Lyapunov function.*

Mathematics Subject Classification (2000): 34K10, 34B15, 34K25.

1 Introduction

There has been much interest over the last few years in the problem of stabilization of nonholonomic systems. From practical point of view, nonholonomic systems often arise in the form of robot manipulators, mobile robots, and space and marine robots that are either designed with fewer actuators than degrees of freedom or must be able to function in the presence of actuator failures. From a theoretical stand point, there

*Corresponding author: frahman@giki.edu.pk

is considerable challenge in the synthesis of control laws for the nonholonomic systems since as pointed out in a famous paper of Brockett [6], they cannot be stabilized by continuously differentiable, time invariant, state feedback control laws. To overcome the limitations imposed by the Brockett's result, a number of approaches have been proposed for the stabilization of nonholonomic control systems to equilibrium points, see [11] for a comprehensive survey of the field. Among the proposed solutions are smooth time varying controllers [16, 17, 8, 12, 13, 15, 4], discontinues or piecewise smooth control laws [3, 5, 7, 9, 19], and hybrid controllers [5, 10, 20].

Despite the vast amount of papers published on the stabilization of nonholonomic systems, the majority has concentrated on the kinematics models of mechanical systems controlled directly by velocity inputs. Although in certain circumstances this can be acceptable, many physical systems (where forces and torques are actual inputs) will not perform well if their dynamics are neglected.

As a contribution to overcome this limitation, this paper derives a time-varying control law for the so-called the extended nonholonomic double integrator (ENDI) system. The extended nonholonomic double integrator (ENDI) system can be viewed as an extension of the so-called nonholonomic integrator [6]. Its importance stems from the fact that it captures the dynamics and kinematics of a nonholonomic system with three states and two first-order dynamics control inputs, (e.g., the dynamics of a wheeled robot subject to force and torque inputs).

This article presents a feedback stabilization control strategy based on model reference approach for ENDI. The trajectory of the extended system for ENDI model is chosen as the model reference trajectory. The extended system has equal number of inputs and state variables i.e. $m = n$ therefore can be made asymptotically stable by choosing an arbitrary Lyapunov function. This classical state feedback is then combined with a periodic continuation of a parameterized solution to an open loop steering problem for the comparison of flows of the original and extended systems. In combination with the time invariant state feedback for the extended system, the solution to this open loop problem delivers a time varying control, which provides for periodic intersection of the trajectories of the controlled extended system and the original system. For stabilizing the original system, the extended system trajectory serves as a reference. The time-invariant feedback for the extended system dictates the speed of convergence of the system trajectory to the desired terminal point, the open loop solution serves the averaging purpose in that it ensures that the "average motion" of the original system is that of the controlled extended system. The construction proposed here demonstrates that synthesis of time varying feedback stabilizers for ENDI with two control input can be viewed as a procedure of combining static feedback laws for a Lie bracket extension of the system with a solution of an open loop trajectory interception control problem.

2 The Kinematics Model of the Extended Nonholonomic Double Integrator

In [6], Brockett introduced the nonholonomic integrator system

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_2 u_1 - x_1 u_2,$$

where $(x_1, x_2, x_3)^{\mathrm{T}} \in R^3$ is the state vector and $(u_1, u_2)^{\mathrm{T}} \in R^2$ is a two-dimensional input. This system displays all basic properties of nonholonomic systems and is often quoted in the literature as a benchmark for control system design [3, 10, 14].

The nonholonomic integrator captures (under suitable state and control transformations) the kinematics of a wheeled robot. However, the nonholonomic integrator model fails to capture the case where both the kinematics and dynamics of a wheeled robot must be taken into account. To tackle this realistic case, the nonholonomic integrator model must be extended. It is shown in [2] that the dynamic equations of motion of a mobile robot of the unicycle type can be transformed into the system

$$x_1 = u_1, \quad x_2 = u_2, \quad x_3 = x_1\dot{x}_2 - x_2\dot{x}_1. \tag{1}$$

By defining the state variables $x = (x_1, x_2, x_3, x_4, x_5)^{\mathrm{T}} = (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2)^{\mathrm{T}}$, system (1) becomes as $\dot{x}_1 = x_4$, $\dot{x}_2 = x_5$, $\dot{x}_3 = x_1x_5 - x_2x_4$, $\dot{x}_4 = u_1$, $\dot{x}_5 = u_2$, which can be written in the following standard form:

$$\dot{x} = g_0(x) + g_1(x)u_1 + g_2(x)u_2, \tag{2}$$

where

$$g_0(x) = (x_4,\, x_5,\, x_1x_5 - x_2x_4,\, 0,\, 0)^{\mathrm{T}}, \quad g_1(x) = (0,0,0,1,0)^{\mathrm{T}}, \quad g_2(x) = (0,0,0,0,1)^{\mathrm{T}}.$$

As in [1], the system (2) will be referred to as the extended nonholonomic double integrator (ENDI).

The ENDI system (2) satisfies the following properties:

H1. The vector fields g_0, g_1, g_2 are real analytic and complete and, additionally,

$$g_0(0) = 0.$$

H2. The ENDI system is locally strongly accessible for any $x \in R^5$ as this satisfies the LARC (Lie algebra rank condition) for accessibility (see [18]), namely that $L(g_0, g_1, g_2)$, the Lie algebra of vector fields generated by $g_0(x)$, $g_1(x)$ and $g_2(x)$, spans R^5 at each point $x \in R^5$ that is

$$\operatorname{span}\{g_1, g_2, g_3, g_4, g_5\}(x) = R^5 \quad \text{for all } \ x \in R^5, \tag{3}$$

where

$$g_3(x) = [g_0(x),\, g_1(x)] = (1, 0, -x_2, 0, 0)^{\mathrm{T}}, \quad g_4(x) = [g_0(x),\, g_2(x)] = (0, 1, x_1, 0, 0)^{\mathrm{T}},$$
$$g_5(x) = [[g_0(x),\, g_1(x)],\, [g_0(x),\, g_2(x)]] = [g_3(x),\, g_4(x)] = (0, 0, 2, 0, 0)^{\mathrm{T}}.$$

H3. The controllability Lie algebra $L(g_0, g_1, g_2)$ is locally nilpotent i.e. all other Lie brackets which are not involve in accessibility rank condition are zero when evaluated at zero.

3 The Control Problem

(SP) Given a desired set point $x_{\mathrm{des}} \in R^5$, construct a feedback strategy in terms of the controls $u_i\colon R^5 \to R$, $i = 1, 2$, such that the desired set point x_{des} is an attractive set for (2), so that there exists an $\varepsilon > 0$, such that $x(t; t_0, x_0) \to x_{\mathrm{des}}$, as $t \to \infty$ for any initial condition $(t_0, x_0) \in R^+ \times B(x_{\mathrm{des}}; \varepsilon)$.

Without the loss of generality, it is assumed that $x_{\text{des}} = 0$, which can be achieved by a suitable translation of the coordinate system.

4 Basic Approach of Designing Stabilizing Control Law for ENDI

4.1 Extended system

The construction of the stabilizing feedback, presented in the next section, employs as its base a Lie bracket extension for the original system (2). This extension is a new system whose right hand side is a linear combination of the vector fields, which locally span the state space. The "coefficients" of this linear combination are regarded as "extended" controls. The extended system can be written as:

$$\dot{x} = g_0(x) + g_1(x)\nu_1 + g_2(x)\nu_2 + g_3(x)\nu_3 + g_4(x)\nu_4 + g_5(x)\nu_5. \tag{4}$$

Henceforth, equations (2) and (4) are referred to as the "original system", and the "extended system", respectively. The importance of the extended system for the purpose of control synthesis lies in the fact that, unlike the original system, it permits instantaneous motion in the "missing" Lie bracket directions g_3, g_4 and g_5.

4.2 Stabilization of the extended system

The extended system (4) can be made globally asymptotically stable if we define the following control inputs

$$\begin{aligned} \nu(x) &= (\nu_1(x),\, \nu_2(x),\, \nu_3(x),\, \nu_4(x),\, \nu_5(x))^{\mathrm{T}} \\ &= \{G(x)\}^{-1}(-x - g_0(x)) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -x_2 & x_1 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -x_1 - x_4 \\ -x_2 - x_5 \\ -x_3 - x_1x_5 + x_2x_4 \\ -x_4 \\ -x_5 \end{bmatrix} \end{aligned}$$

or

$$\begin{aligned} \nu(x) &= (\nu_1(x),\, \nu_2(x),\, \nu_3(x),\, \nu_4(x),\, \nu_5(x))^{\mathrm{T}} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5x_2 & -0.5x_1 & 0.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} -x_1 - x_4 \\ -x_2 - x_5 \\ -x_3 - x_1x_5 + x_2x_4 \\ -x_4 \\ -x_5 \end{bmatrix} = \begin{bmatrix} -x_4 \\ -x_5 \\ -x_1 - x_4 \\ -x_2 - x_5 \\ \nu_5 \end{bmatrix}, \end{aligned} \tag{5}$$

where

$$\begin{gathered} \nu_5 = -\frac{1}{2}x_2(x_1 + x_4) + \frac{1}{2}x_1(x_2 + x_5) - \frac{1}{2}(x_3 + x_1x_3 - x_2x_4) \\ G(x) = (g_1(x,)\, g_2(x,)\, g_3(x,)\, g_4(x,)\, g_5(x,)). \end{gathered}$$

The existence of $\{G(x)\}^{-1}$ is guaranteed by the LARC condition.

Lemma *The extended system (4) can be made asymptotically stable by using the feedback control as given in (5).*

Proof By considering a Lyapunov function $V(x) = \frac{1}{2}\,x^{\mathrm{T}}Qx$, where Q is some symmetric and positive definite matrix, it follows that, along the controlled extended system trajectories,

$$\frac{d}{dt}V(x) = x^{\mathrm{T}}Q(g_0(x) + G(x)\{G(x)\}^{-1}(-x - g_0(*x))) = -x^{\mathrm{T}}Qx = -2V(x) < 0,$$
$$\forall\, x \in R^5 \setminus \{0\}.$$

Confirming the asymptotic stability of (4) with feedback controls (5).

The discretization of the above control in time, with sufficiently high sampling frequency $1/T$, does not prejudice stabilization in that if the feedback control (5) is substituted by the discretized control

$$\nu_i^T(x(t)) \triangleq \nu_i(x(nT)), \quad t \in [nT,\, (n+1)T), \quad n = 0, 1, 2, \dots, \quad i = 1, 2, \dots, 5.$$

This leads to a parameterized extended system

$$\dot{x} = g_0(x) + g_1(x)a_1 + g_2(x)a_2 + g_3(x)a_3 + g_4(x)a_4 + g_5(x)a_5, \tag{6}$$

where $a_i = \nu_i^T(x(t))$, $i = 1, 2, \dots, 5$, (which are constant over each interval $[nT,$ $(n{+}1)T)$). For a sufficiently small T, the discretization of the extended controls preserves their stabilizing properties.

4.3 The trajectory interception problem

(TIP) Find control functions $m_i(a, t)$, $i = 1, 2$, in the class of functions which are continuous in $a = (a_1, a_2, a_3, a_4, a_5)$ and piece-wise continuous and locally bounded in t, such that for any initial condition $x(0) = x_0$ the trajectory $x^a(t; x_0, 0)$ of the extended, parameterized system (6) intersects the trajectory $x^m(t; x_0, 0)$ of the system (2) with controls m_i, $i = 1, 2$, i.e. the trajectory of the system

$$\dot{x} = g_0(x) + g_1(x)m_1(a, t) + g_2(x)m_2(a, t) \tag{7}$$

intercept with the trajectory of

$$\dot{x} = g_0(x) + g_1(x)a_1 + g_2(x)a_2 + g_3(x)a_3 + g_4(x)a_4 + g_5(x)a_5$$

precisely at time T, so that

$$x^a(T; x_0, 0) = x^m(T; x_0, 0). \tag{8}$$

4.4 The TIP in logarithmic coordinates of flows

To solve the TIP, we employ the formalism of [21] by considering a formal equation for the evolution of flows for the system (6)

$$\dot{U}(t) = U(t)\bigg(\sum_{i=0}^{5} g_i w_i\bigg), \quad w_0 = 1, \quad U(0) = I, \tag{9}$$

and its solution can be expressed locally as

$$U(t) = \prod_{i=0}^{5} e^{\gamma_i(t) g_i}, \tag{10}$$

where the functions $\gamma_i,\ i = 0, 1, \ldots, 5,$ are the logarithmic coordinates for this flow and can be computed approximately as follows.

Equation (10) is first substituted into (9) which yields

$$\begin{aligned} g_0 w_0 + g_1 w_1 + \cdots + g_5 w_5 = \dot{\gamma}_0 g_0 + \dot{\gamma}_1 (e^{\gamma_0 Adg_0}) g_1 + \dot{\gamma}_2 (e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1}) g_2 + \ldots \\ + \dot{\gamma}_5 (e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2} e^{\gamma_3 Adg_3} e^{\gamma_4 Adg_4}) g_5, \end{aligned} \tag{11}$$

where $(e^{AdX})Y = e^X Y e^{-X}$ and $(AdX)Y = [X,\, Y]$.

Employing the Campbell-Baker-Hausdorff formula

$$(e^{AdX})Y = e^X Y e^{-X} = Y + [X, Y] + [X,\, [X, Y]]/2! + \ldots,$$

and ignoring all other Lie brackets which are not involved in LARC equation (3). This gives

$$\begin{aligned} (e^{\gamma_0 Adg_0}) g_1 = e^{\gamma_0 g_0} g_1 e^{-\gamma_0 g_0} &= g_1 + (\gamma_0/1!)[g_0, g_1] + (\gamma_0^2/2!)[g_0,\, [g_0, g_1]] + \ldots \\ &\approx g_1 + \gamma_0 g_3. \end{aligned} \tag{12}$$

Similarly

$$\begin{aligned} (e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1}) g_2 = e^{\gamma_0 Adg_0} (e^{\gamma_1 Adg_1} g_2) = e^{\gamma_0 Afg_0}(g_2) \\ \approx g_2 + \gamma_0 [g_0, g_2] = g_2 + \gamma_0 g_4, \end{aligned} \tag{13}$$

$$\begin{aligned} (e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2}) g_3 &= e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} (e^{\gamma_2 Adg_2} g_3) \\ &\approx e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} (g_3 + \gamma_2 [g_2, g_3]) = e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} (g_3) \\ &\approx e^{\gamma_0 Adg_0}(g_3) \approx g_3. \end{aligned} \tag{14}$$

In a similar way we can obtain

$$(e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2} e^{\gamma_3 Adg_3}) g_4 \approx g_4 + \gamma_3 g_5, \tag{15}$$

$$(e^{\gamma_0 Adg_0} e^{\gamma_1 Adg_1} e^{\gamma_2 Adg_2} e^{\gamma_3 Adg_3} e^{\gamma_4 Adg_4}) g_5 \approx g_5. \tag{16}$$

Substituting (12)−(15) into equation (11) and comparing the coefficients of g_i, $i = 0, 1, \dots, 5$, yields the following approximate equations for the evolution of the logarithmic coordinates γ_i, $i = 0, 1, \dots, 5$,

$$
\begin{aligned}
&\dot\gamma_0 = 1,\\
&\dot\gamma_1 = w_1,\\
&\dot\gamma_2 = w_2,\\
&\dot\gamma_3 = -\gamma_0 w_1 + w_3,\\
&\dot\gamma_4 = -\gamma_0 w_2 + w_4,\\
&\dot\gamma_5 = \gamma_0\gamma_3 w_2 - \gamma_3 w_4 + w_5 \quad \text{with} \quad \gamma_i(0) = 0, \quad i = 0, 1, \dots, 5.
\end{aligned}
\tag{17}
$$

The TIP problem can thus be recast in the logarithmic coordinates as follows.

[TIP in LC:] On a given time horizon $T > 0$, find control functions $m_i(a,t)$, $i = 1, 2$, in the class of functions which are continuous in $a = [a_1, a_2, a_3, a_4, a_5]$, and piecewise continuous, and locally bounded in t, such that the trajectory $t \mapsto \gamma^a(t)$ of

$$\dot\gamma = M(\gamma)a, \quad \gamma(0) = 0, \tag{18}$$

intersects the trajectory $t \mapsto \gamma^m(t)$ of

$$\dot\gamma = M(\gamma)m(a,t), \quad \gamma(0) = 0, \tag{19}$$

in which $m(a,t) = [m_1(a,t),\, m_2(a,t),\, 0, 0, 0]$ at time T, so that

$$\gamma^a(T) = \gamma^m(T). \tag{20}$$

The TIP in logarithmic coordinates now takes the form of a trajectory interception problem for the following two control systems

$$
\begin{aligned}
&\dot\gamma_0 = 1,\\
&\dot\gamma_1 = w_1,\\
&\dot\gamma_2 = w_2,\\
&\dot\gamma_3 = -\gamma_0 w_1 + w_3,\\
&\dot\gamma_4 = -\gamma_0 w_2 + w_4,\\
&\dot\gamma_5 = \gamma_0\gamma_3 w_2 - \gamma_3 w_4 + w_5 \quad \text{with} \quad \gamma_i(0) = 0, \quad i = 0, 1, \dots, 5.
\end{aligned}
$$

$$
\begin{array}{llll}
\text{CS1}: & \dot\gamma_0 = 1, & \text{CS2}: & \dot\gamma_0 = 1,\\
& \dot\gamma_1 = m_1, & & \dot\gamma_1 = a_1,\\
& \dot\gamma_2 = m_2, & & \dot\gamma_2 = a_2,\\
& \dot\gamma_3 = -\gamma_0 m_1, & & \dot\gamma_3 = -\gamma_0 a_1 + a_3,\\
& \dot\gamma_4 = -\gamma_0 m_2, & & \dot\gamma_4 = -\gamma_0 a_2 + a_4,\\
& \dot\gamma_5 = \gamma_0\gamma_3 m_2, & & \dot\gamma_5 = \gamma_0\gamma_3 a_2 - \gamma_3 a_4 + a_5
\end{array}
\tag{21}
$$

with initial conditions $\gamma_i(0) = 0$, $i = 0, 1, \dots, 5$.

A solution to TIP is calculated by approximating the flow of $\dot x = g_0 + [g_0, g_1]$ by the flow of $\dot y = g_0 + k g_1 \sin \dfrac{2\pi t}{T}$, and the flow of $\dot x = g_0 + [[g_0, g_1], [g_0, g_2]]$ by $\dot y =$

$g_0 + kg_1 \sin \frac{2\pi t}{T} + kg_2 \cos \frac{2\pi t}{T}$, where k is some constant. Therefore we adopt the following parameterizations of $m_i,\ i = 1, 2$:

$$m_1 = c_1 + (c_3 + c_5) \sin \frac{2\pi t}{T} \quad \text{and} \quad m_2 = c_2 + c_4 \sin \frac{2\pi t}{T} + c_5 \cos \frac{2\pi t}{T}, \tag{22}$$

where $c_i,\ i = 1, 2, \ldots, 5$, are found as $c_1 = a_1,\ c_2 = a_2,\ c_3 = 6.28319\, a_3/T,\ c_4 = 6.28319\, a_4/T$ and $c_5 = 6.28319\, a_5/T$, or $c_1 = a_1,\ c_2 = a_2,\ c_3 = ka_3,\ c_4 = ka_4$ and $c_5 = ka_5$, where $k = 6.28319/T$.

The time varying stabilizing controls for model (2), are thus given by

$$\begin{aligned} u_1 &= c_1 + c_3 \sin \frac{2\pi t}{T} + c_5 \cos \frac{2\pi t}{T}, \\ u_2 &= c_2 + c_4 \sin \frac{2\pi t}{T} + c_5 \cos \frac{2\pi t}{T}. \end{aligned} \tag{23}$$

Theorem 4.1 *Suppose that a solution to the TIP problem can be found. Then, there exists an admissible time horizon $T_{\max}$ and a neighborhood of the origin R such that for any $T < T_{\max}$ the time-varying feedback controls given in (23) are asymptotically stabilizing the system (2) with the region of attraction R.*

Proof By considering a trivial Lyapunov function $V(x) = \frac{1}{2} x^{\mathrm{T}} x,\ x \in \mathbb{R}^5$ it follows that along the controlled system trajectories,

$$\begin{aligned} \frac{d}{dt} V(x) &= x^{\mathrm{T}} \dot{x} = x^{\mathrm{T}} (g_0(x) + g_1(x) u_1 + g_2(x) u_2) \\ &= x^{\mathrm{T}} \Big(g_0(x) + g_1(x) \Big(c_1 + c_3 \sin \frac{2\pi t}{T} + c_5 \cos \frac{2\pi t}{T} \Big) \\ &\quad + g_2(x) \Big(c_2 + c_4 \sin \frac{2\pi t}{T} + c_5 \cos \frac{2\pi t}{T} \Big) \Big) \\ &= x^{\mathrm{T}} \Big(g_0(x) + g_1(x) a_1 + k_3 a_3 g_1(x) \sin \frac{2\pi t}{T} + k_5 a_5 g_1(x) \sin \frac{2\pi t}{T} \\ &\quad + g_2(x) a_2 + k_4 a_4 g_2(x) \sin \frac{2\pi t}{T} + k_5 a_5 g_2(x) \cos \frac{2\pi t}{T} \Big) \\ &\approx x^{\mathrm{T}} (g_0(x) + g_1(x) a_1 + g_2(x) a_2 + g_3(x) a_3 + g_4(x) a_4 + g_5(x) a_5) \\ &= x^{\mathrm{T}} (g_0(x) + G\nu) = -x^{\mathrm{T}} x < 0, \end{aligned}$$

where $G = [g_1\ g_2\ g_3\ g_4\ g_5](x),\ \nu = G^{-1}\{-x - g_0(x)\}$ for all $x \in R^5 \setminus \{0\}$.

Confirming the asymptotic stability of (2) with feedback controls (23).

The simulation results employing the above controls are depicted in Figures 4.1 – 4.6. In first simulation we choose $x(0) = [0.9\ \ 0.7\ \ 0.4\ \ 0.8\ \ 0.6]^{\mathrm{T}}$ and $T = 0.9$. The results are shown in Figures 4.1 – 4.4. In 2nd simulation we choose $x(0) = [0.5\ \ 0.5\ \ 0.5\ \ 0.5\ \ 0.5]^{\mathrm{T}}$ and $T = 0.9$. The results are shown in Figures 4.5 – 4.8.

5 Conclusion

A time varying control law is derived for the extended nonholonomic double integrator (ENDI) system that captures any kinematics completely nonholonomic model with three states and two first order dynamic control inputs, e.g., the dynamics of a wheeled robot subject to force and torque inputs. The controller yields asymptotic stability and convergence of the closed loop system to an arbitrarily small neighborhood of the origin. Simulation results captured some of the features of the proposed control laws and their performance.

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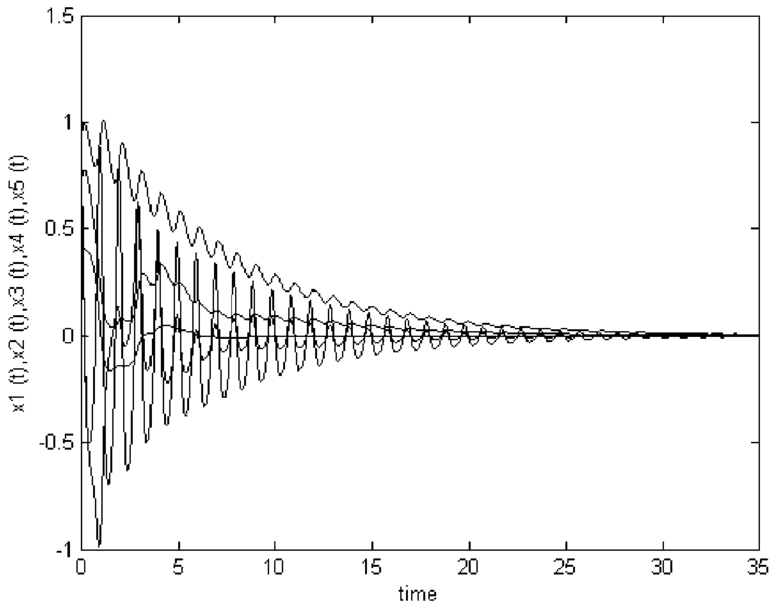

Figure 4.1. Collective Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), \ldots, x_5(t))$ versus time.

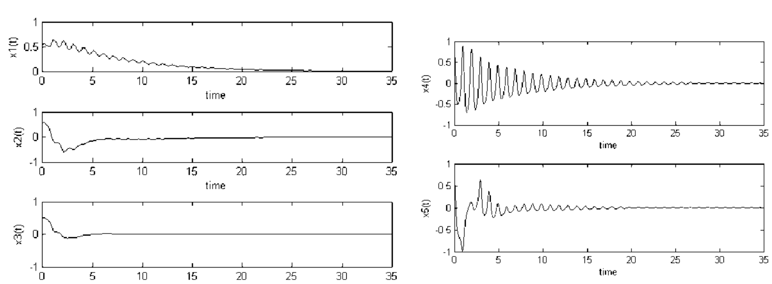

Figure 4.2. Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), \ldots, x_5(t))$ versus time.

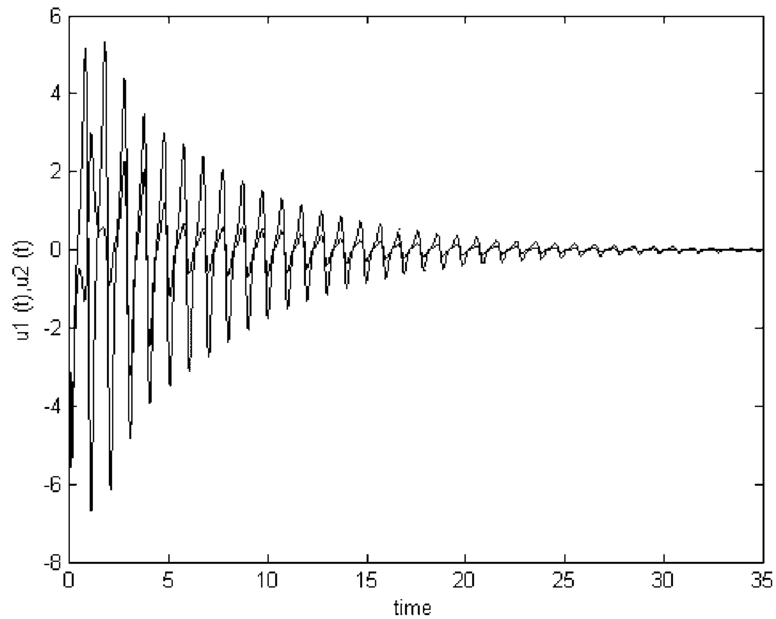

Figure 4.3. Plots of the control input $t \mapsto (u_1(t), u_2(t))$ versus time.

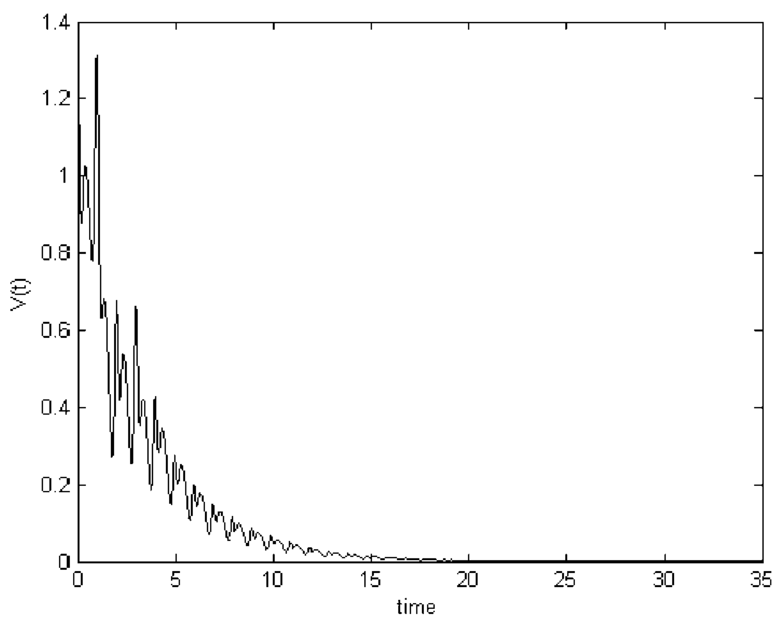

Figure 4.4. Plot of the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{5} x_i^2(t)$ along the controlled state trajectories versus time.

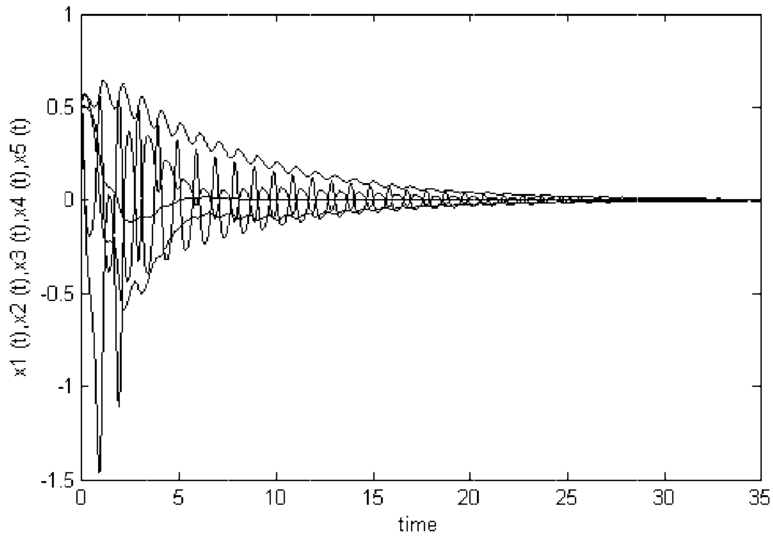

Figure 4.5. Collective plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), \ldots, x_5(t))$ versus time.

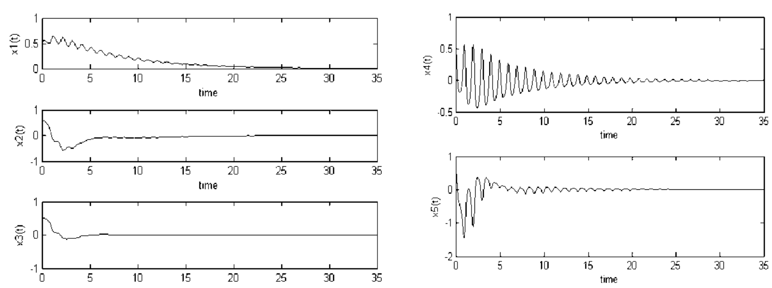

Figure 4.6. Plots of the controlled state trajectories $t \mapsto (x_1(t), x_2(t), \ldots, x_5(t))$ versus time.

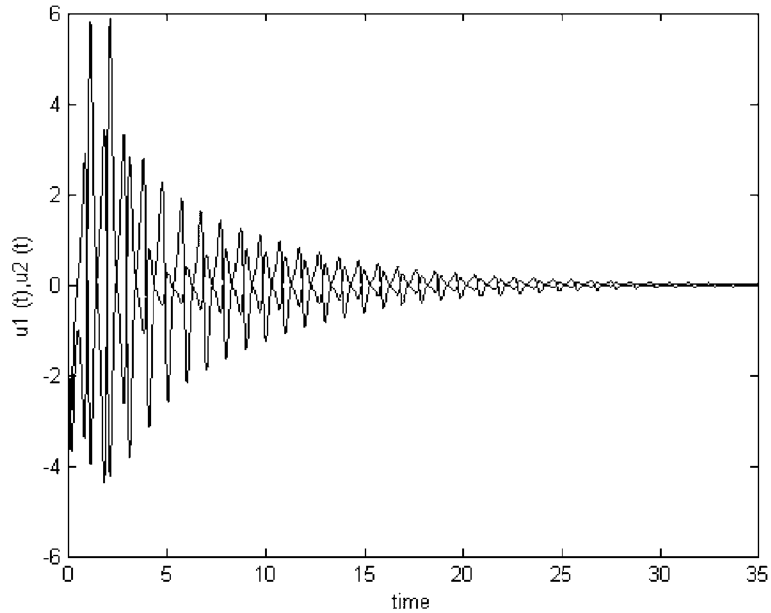

Figure 4.7. Plots of the control input $t \mapsto (u_1(t), u_2(t))$ versus time.

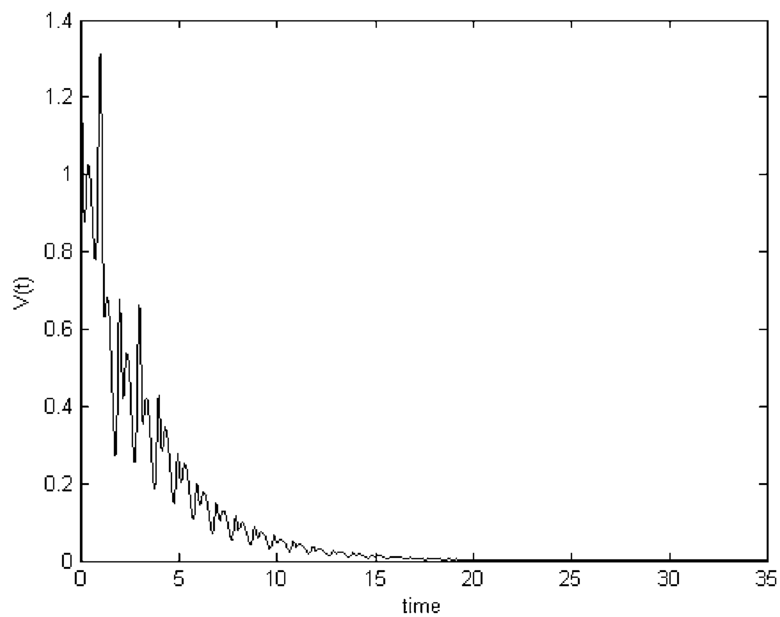

Figure 4.8. Plot of the Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i=1}^{5} x_i^2(t)$ along the controlled state trajectories versus time.