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Exponential Stability of Perturbed Nonlinear Systems

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Abstract: In this paper, we deal with the stability analysis problem of perturbed nonautonomous nonlinear systems. Uniform exponential stability is studied by using Lyapunov techniques. The question addressed is related to the restriction about the perturbed term under the assumption that the origin of the nominal system is globally exponentially stable. A new Lyapunov function is used to obtain a large class of stable dynamical systems.

Keywords: *Nonlinear systems; Lyapunov function; exponential stability.*

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1 Introduction

Consider the nonautonomous perturbed system

$$\dot{x} = A(t)x + g(t,x), \tag{1}$$

where $A(n \times n)$, g are piecewise continuous in t and g is locally Lipschitz in x such that

$$g(t,0) = 0, \quad \forall\, t \geq 0.$$

It is known [3] that, if the linearization of the nonlinear system (1) about the origin has an exponentially stable equilibrium point then the origin is an exponentially stable equilibrium for the perturbed nonlinear system and it turns out that exponential stability of the linearization is a necessary and sufficient condition for exponential stability of the origin of (1). For the global case, the stability analysis problem is to find sufficient conditions under which the perturbed system (1) is globally asymptotically or exponentially

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stable with the assumption that the nominal system is globally exponentially stable. Therefore, to obtain stability of the whole system, we shall make some restrictions on the perturbed term. Suppose that the origin of the nominal system

$$\dot{x} = A(t)x \tag{2}$$

is globally exponentially stable with

$$W(t,x) = x^T P(t)x$$

as an associate Lyapunov function, where $P(t)$ is a continuous differentiable symmetric and bounded positive definite matrix, such that

$$0 < c_1 I \le P(t) \le c_2 I, \quad \forall\, t \ge 0, \tag{3}$$

which satisfies the matrix differential equation

$$\dot{P}(t) + P(t)A(t) + A(t)^T P(t) = -Q(t)$$

with $Q(t)$ is continuous, symmetric and positive definite that is

$$Q(t) \ge c_3 I > 0, \quad \forall\, t \ge 0.$$

Here the constants c_1, c_2, $c_3 > 0$ and I is identical matrix.

Then calculating the derivative of W along the trajectories of the system (1) one can obtain the definiteness of $\dot{W}$ by imposing some conditions on $g(t,x)$.

For the case when

$$\|g(t,x)\| \le \eta(t)\|x\|,$$

where $\eta(t)$ is a continuous function, we obtain after taking the derivative of W along the trajectories of the whole system,

$$\dot{W}(t,x) \le -x^T Q(t)x + 2x^T P(t)g(t,x).$$

Then, one gets the following estimation on the derivative of W,

$$\dot{W}(t,x) \le (-c_3 + 2c_2\eta(t))\|x\|^2$$

which implies the global exponential stability of the equilibrium point of (1) under the condition

$$\eta(t) \le k < \frac{1}{2}\frac{c_3}{c_2}$$

with $k > 0$.

Moreover, one can obtain exponential convergence to zero for system (1) especially, where

$$g(t,x) = B(t)x$$

under the conditions $B(t)$ is continuous and

$$B(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Similar conclusions can be obtained (see [5]), where

$$\int_0^{+\infty} \|B(t)\| < \infty$$

or

$$\int_0^{+\infty} \|B(t)\|^2 < \infty.$$

Actually, the synthesis of stability of perturbed systems is based on the stability of the nominal system with $W(t,x)$ as a Lyapunov function candidate for the whole system provided that the size of the perturbation is known (see $[1,2,4-7,11,12]$). Panteley and Loria [8, 9] studied this problem for cascaded time-varying nonlinear systems, which can be regarded as perturbed systems, where growth conditions are given to ensure the global uniform asymptotic stability of some classes of time-varying nonlinear systems.

Our approach is to find more general classes of perturbed systems which can be globally exponentially stable by considering a new Lyapunov function which has the following form

$$V(t,x) = x^T P(t)x + \Psi(t,x),$$

where $\Psi(t,x)$ is a C^1-function which will be chosen, for some classes of systems, in such a way that $V(t,x)$ is positive definite radially unbounded and its derivative along the trajectories of (1) is negative definite. We use a cross term in the Lyapunov function, as in [10] introduced for cascade nonlinear systems, to obtain a large class of stable perturbed systems. The proposed new method is based on the non uniqueness of Lyapunov functions with a stable nominal system, which guarantees exponential stability with the requirement on the upper bound of the perturbed term. We prove that the system can be globally uniformly exponentially stable. The perturbation term is a known function which could result in general from errors in modelling, aging of parameters or disturbances. Naturally, the choice of the function $\Psi(t,x)$ depends on the perturbation term $g(t,x)$ and its smoothness is given under some restrictions on the dynamics of the system. Furthermore, we give an illustrative example in dimensional one and we show for a certain class of perturbed systems that the proposed method gives better result than the classical method.

2 Stability

In this paper the solution of a differential time-varying equation

$$\dot{x} = A(t)x + g(t,x)$$

with initial conditions $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, $x(t_0) = x_0$ is denoted $\phi(t, t_0, x_0)$.

$\dot{V}_{(\star)}(t,x)$ is the derivative of Lyapunov function $V(t,x)$ along the trajectories represented by the differential equation $(\star)$.

According to [3, 5], the equilibrium point $x = 0$ of (1) is uniformly stable if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ independent of t_0, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\, t \geq t_0 \geq 0.$$

The equilibrium point $x = 0$ of (1) is globally uniformly asymptotically stable if it is uniformly stable and for any initial state $x(t_0)$, one has

$$x(t) \to 0 \quad \text{as} \quad t \to +\infty$$

uniformly in t_0, that is there exists $T = T(\varepsilon) > 0$, such that

$$\|x(t)\| < \varepsilon, \quad \forall\, t \geq t_0 + T(\varepsilon), \quad \forall\, x(t_0).$$

The equilibrium point $x = 0$ of (1) is globally exponentially stable if the following estimation holds for any initial state $x(t_0)$,

$$\|x(t)\| < \lambda_1 e^{-\lambda_2(t-t_0)}, \quad \forall\, t \geq t_0 \geq 0,$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$.

Throughout this paper, we suppose that

(A_1). There exists a continuous differentiable, symmetric, bounded, positive definite matrix $P(t)$ which satisfies (3).

(A_2). There exist a continuous function $\rho\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $k > 0$, such that

$$\forall\, t \geq 0, \quad \forall\, x \in \mathbb{R}^n, \quad \|g(t,x)\| \leq \rho(t)\|x\|$$

with

$$\rho(t) \leq k, \quad \forall\, t \geq 0$$

and

$$\int_0^{+\infty} \rho(t)dt < +\infty.$$

Note that, the quadratic function

$$W(t,x) = x^T P(t)x$$

implies by the assumption (A_1) the two following inequalities,

$$\begin{aligned} c_1\|x\|^2 \leq W(t,x) \leq c_2\|x\|^2, \\ \dot{W}_{(2)}(t,x) \leq -c_3\|x\|^2. \end{aligned} \tag{4}$$

Our goal is to seek a suitable function Ψ which is of class C^1 to compensate the perturbed term which is not always possible only for some restrictive dynamical systems. Thus, we will consider a Lyapunov function for system (1) of the form $V(t,x) = x^T P(t)x + \Psi(t,x)$, where Ψ is a C^1-function which will be chosen later such that V is definite positive function and $\dot{V}$ definite negative for some restriction on g. Notice that, continuity of the partial derivatives of the cross term can be proven for some classes of system of the form (1). Thus, if we consider the derivative of $V(t,x)$ along the trajectories of the system (1) we get

$$\dot{V}_{(1)}(t,x) = \dot{W}_{(2)}(t,x) + 2x^T P(t) \cdot g(t,x) + \dot{\Psi}(t,x).$$

The first term of the right-hand side constitute the derivative of $V(t,x)$ along the trajectories of the nominal system, which is negative definite and satisfies (4). The second term is the effect of the perturbation while the third one is the derivative of the cross term. We choose

$$\Psi(t,x) = \int_t^{+\infty} 2\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))\,ds.$$

Thus, one can verify the following statement

$$2x^T P(t)\cdot g(t,x) + \dot{\Psi}(t,x) = 0$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

It follows with this choice, that

$$\dot{V}_{(1)}(t,x) = \dot{W}_{(2)}(t,x) \le -c_3\|x\|^2.$$

This yields by (A_1), the exponential stability of (1) provided that $\Psi(t,x)$ exists and it is a C^1-function or simply uniformly continuous rending $V(t,x)$ definite positive for a given perturbed function $g(t,x)$.

First, one can state the following proposition which provides a stability result.

Proposition 2.1 *If (A_1) and (A_2) are satisfied, then the origin of the system (1) is uniformly stable.*

Proof Let $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ be an initial condition. The derivative of W along the trajectories of (1) is given by

$$\begin{aligned}\dot{W}_{(1)} &= \frac{d}{ds}(W(s,\phi(s,t,x)))\\ &= \frac{\partial W}{\partial s}(s,\phi(s,t,x)) + 2\phi^T(s,t,x)P(s)A(s)\phi(s,t,x)\\ &\quad + 2\phi^T(s,t,x)P(s).g(s,\phi(s,t,x)).\end{aligned}$$

Thus,

$$\begin{aligned}\dot{W}_{(1)} &\le 2\phi^T(s,t,x)P(s).g(s,\phi(s,t,x))\\ &\le 2c_2\rho(s)\|\phi(s,t,x)\|^2\\ &\le 2\frac{c_2}{c_1}\,\rho(s)W(s,\phi(s,t,x))\end{aligned}$$

which implies that

$$W(s,\phi(s,t,x)) \le MW(t,x),$$

where

$$M = \exp\left\{2\,\frac{c_2}{c_1}\left(\int_0^{+\infty}\rho(u)\,du\right)\right\}.$$

We conclude that

$$\|\phi(s,t,x)\| \le \sqrt{\frac{c_2}{c_1}\,M}\,\|x\|, \qquad \forall\, s \ge t.$$

Then the equilibrium point of the system (1) is uniformly stable.

The above proposition is conceptually important because it shows the stability of the origin for all perturbations satisfying the condition (A_2).

Now, concerning the cross term, we have the following lemma.

Lemma 2.1 *Under assumptions* (A_1) *and* (A_2)*, the function* $\Psi(t,x)$ *exists and is continuous on* $\mathbb{R}_+ \times \mathbb{R}^n$*.*

Proof Observe that, using the above proposition and the fact that for all (t,x) the function $\psi(t,x)$ exists, we have each solution of (1) which starts at (t,x) is bounded for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and for all $s \geq t$.

Indeed, on the one hand

$$|\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))| \leq c_2\rho(s)\|\phi(s,t,x)\|^2$$

which gives

$$|\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))| \leq M_1\rho(s)\|x\|^2$$

which belongs to $L^1(\mathbb{R}_+)$, where $M_1 = M\,\frac{c_2^2}{c_1}$.

Thus, the integral exists for all $(t,x) \in \mathbb{R}+\times\mathbb{R}^n$ and then $\psi(t,x)$ exists.

On the other hand, the continuity of Ψ can be shown by observing that, for all $s \geq t$, the function

$$(t,x) \longmapsto \phi(s,t,x)^T P(s)g(s,\phi(s,t,x))$$

is continuous on $\mathbb{R}_+ \times \mathbb{R}^n$ and the fact that for all $(t,x) \in \mathbb{R}_+ \times K$, $s \geq t$, where K is a compact set in $\mathbb{R}^n$, we have

$$|\phi^T(s,t,x)P(s)g(s,\phi(s,t,x)| \leq M_K\rho(s).$$

The upper bound $M_K\rho(s)$ is in $L^1(\mathbb{R}_+)$ where M_K is a positive constant which depends only on K.

Next, the proposed Lyapunov function candidate for (1) must be definite positive and we will use this fact to show the exponential stability of the origin of system (1).

Theorem 2.1 *If the assumptions* (A_1) *and* (A_2) *hold, then there exist some positive constants* d_1*,* d_2 *such that*

$$d_1\|x\|^2 \leq V(t,x) \leq d_2\|x\|^2.$$

It means that, the Lyapunov function $V(t,x)$ is a decreascent function.

Proof Observe that,

$$\int_t^s \dot{W}_{(1)}(u,\phi(u,t,x))\,du = W(s,\phi(s,t,x)) - W(t,x).$$

Then, we obtain

$$W(s,\phi(s,t,x)) - W(t,x) = \int_t^s \dot{W}_{(2)}(u,\phi(u,t,x))\,du$$

$$+ \int_t^s 2\phi^T(u,t,x)P(u)g(u,\phi(u,t,x))\,du.$$

Because $W(s,\phi(s,t,x))$ is bounded and $\Psi(t,x)$ exists, it means that the integral

$$\int_t^{+\infty} 2\phi^T(u,t,x)P(u)g(u,\phi(u,t,x))\,du$$

exists.

Then

$$\lim_{s\to+\infty} W(s,\phi(s,t,x)) = W_\infty(t,x)$$

exists.

It follows that,

$$\begin{aligned} V(t,x) &= W_\infty(t,x) - \int_t^{+\infty} \dot{W}_{(2)}(u,\phi(u,t,x))\,du, \\ V(t,x) &\geq -\int_t^{+\infty} \dot{W}_{(2)}(u,\phi(u,t,x))\,du \\ V(t,x) &\geq \int_t^{+\infty} c_3\|\phi(s,t,x)\|^2 ds. \end{aligned} \tag{5}$$

Remark also that

$$\phi(s,t,x) = x + \int_t^s A(u)\phi(u,t,x) + g(u,\phi(u,t,x))\,du$$

which gives

$$\|\phi(s,t,x)\| \geq \|x\| - \int_t^s (L\|\phi(u,t,x)\| + \rho(u)\|\phi(u,t,x)\|)du$$

Thus,

$$\begin{aligned} \|\phi(s,t,x)\| &\geq \|x\| - \int_t^s (L+k)\|\phi(u,t,x)\|du \\ &\geq \|x\| - \lambda(s-t)\|x\| \\ &\geq \frac{\|x\|}{2}, \quad \text{for} \quad s \in \left[t, t+\frac{1}{2\lambda}\right], \end{aligned}$$

where

$$\lambda = (L+k)\sqrt{\frac{Mc_2}{c_1}}.$$

Hence from (5), we obtain

$$V(t,x) \geq d_1\|x\|^2.$$

Still to prove the existence of d_2, which implies in conjunction with the above expression that $V(t,x)$ is a decreascent function.

For any (t,x), we have

$$V(t,x) = W(t,x) + \Psi(t,x).$$

Thus,

$$V(t,x) \leq c_2\|x\|^2 + \int_t^{+\infty} 2|\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))|ds.$$

It follows that,

$$V(t,x) \leq c_2\|x\|^2 + \int_t^{+\infty} M_1\rho(s)\|x\|^2 ds \leq c_2\|x\|^2 + M_2\|x\|^2$$
$$\leq d_2\|x\|^2.$$

Theorem 2.2 *Suppose that the assumptions (A_1), (A_2) hold and the function g is chosen in such a way that*

$$\Psi(t,x) = \int_t^{+\infty} 2\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))\,ds$$

is a C^1-function, then $x=0$ is globally exponentially stable equilibrium point for (1).

Proof Still to prove that

$$\dot{\Psi}(t,x) = -2x^T P(t)g(t,x).$$

We have

$$\dot{\Psi}(t,x) = \frac{d}{ds}\left(\Psi(s,\phi(s,t,x))\right)\Bigg|_{s=t},$$

$$\dot{\Psi}(t,x) = \frac{d}{ds}\left(\int_t^{+\infty} 2\phi^T(u,t,x)P(u)g(u,\phi(u,t,x))ds\right)\Bigg|_{s=t}.$$

Since the solutions of (1)

$$u \longmapsto \Phi(u,t,x)$$

and

$$u \longmapsto \Phi(u,s,\Phi(s,t,x))$$

are equal for $u=s$, this implies that, for all $u \geq s \geq t \geq 0$,

$$\Phi(u,t,x) = \Phi(u,s,\Phi(s,t,x)).$$

Thus,

$$\dot{\Psi}(t,x) = \frac{d}{ds}\left(\int_t^{+\infty} 2\phi^T(u,t,x)P(u)g(u,\phi(u,t,x))ds \right)\Bigg|_{s=t}.$$

So,

$$\dot{\Psi}(t,x) = -(2\phi^T(s,t,x)P(s)g(s,\phi(s,t,x))ds)\Big|_{s=t}.$$

Hence,

$$\dot{\Psi}(t,x) = -2\phi^T(s,t,x)P(s)g(t,x).$$

Using the fact that V is a decreascent function in conjunction with the above expression yields the global exponential stability of (1).

Finally, we give an example to illustrate the applicability of the result of this paper. Moreover, we will compare in the next section our approach with the classical one for a certain class of nonlinear system.

Example As a simple example, to compute the cross term, we consider the following scalar linear equation

$$\dot{x} = -ax + \rho(t)x, \qquad a > 0, \tag{6}$$

with $\rho(t)$ satisfies (A_2). If we choose

$$W(x) = x^2$$

as a Lyapunov function of

$$\dot{x} = -ax$$

we obtain

$$\phi(s,t,x) = \exp\left(-a(s-t) + \int_t^s \rho(u)\,du \right)x.$$

Thus,

$$\Psi(t,x) = x^2 \int_t^{+\infty} \rho(s)\exp\left(2\int_t^s \rho(u)\,du\right)e^{-2a(s-t)}ds.$$

So,

$$\Psi(t,x) = -x^2 + 2ax^2 \int_t^{+\infty} \exp\left(2\int_t^s \rho(u)\,du\right)e^{-2a(s-t)}ds.$$

It follows that, Ψ is a C^1-function and then $x = 0$ is an exponentially stable equilibrium point for (4).

3 Stability of a Certain Class of Perturbed Systems

Consider the following system

$$\dot{x} = Ax + \rho(t)B(x)x, \tag{7}$$

where $x \in \mathbb{R}^n$, $t \geq 0$, $A(n \times n)$ is a constant matrix which is supposed Hurwitz and $\rho(t)$ satisfies (A_2).

Moreover, We assume that

(A_3). $B(\cdot)$ is a C^1-function and there exists a positive constant M, such that

$$\forall\, x \in \mathbb{R}^n \quad \|B(x)\| \leq M.$$

We have the following result of stability for system (7).

Proposition 3.1 *If (A_1), (A_2) and (A_3) are satisfied, then Ψ is C^1 in $\mathbb{R}_+ \times \mathbb{R}^n$ and $x = 0$ is a globally exponentially stable equilibrium point for (7).*

Proof We denote

$$X(s) = \frac{\partial}{\partial x}\,(\Phi(s,t,x))$$

and

$$Y(s) = \frac{\partial}{\partial t}\,(\Phi(s,t,x)), \qquad s \geq t.$$

Thus, X and Y satisfies the following two statements

$$\dot{X} = \left(A + \rho(s)\frac{\partial B}{\partial x}\,(\Phi(s,t,x))\Phi(s,t,x) + \rho(s)B(\Phi(s,t,x))\right)X$$

with

$$X(t) = I$$

and

$$\dot{Y} = \left(A + \rho(s)\frac{\partial B}{\partial x}\,(\Phi(s,t,x))\Phi(s,t,x) + \rho(s)B(\Phi(s,t,x))\right)Y$$

with

$$Y(t) = 0.$$

Let K be a compact set of $\mathbb{R}^n$. Because $\Phi(s,t,x)$ is uniformly bounded and $B(\cdot)$ is a C^1-function, then there exists $M_K > 0$, such that $\forall\, s \geq t \geq 0$, $\forall\, x \in K$,

$$\left\|\rho(s)\,\frac{\partial B}{\partial x}\,(\Phi(s,t,x))\Phi(s,t,x) + \rho(s)B(\Phi(s,t,x))\right\| \leq M_K\rho(s).$$

Note that Lemma 2.1 implies that $X(s,t,x)$ and $Y(s,t,x)$ are bounded when x leaves in K.

Thus, we have

$$\Psi(t,x) = \int_t^{+\infty} \frac{\partial W}{\partial x}\,(\phi(s,t,x))\rho(s)B(\phi(s,t,x))\phi(s,t,x)\,ds.$$

Because X and Y are bounded when $x \in K$, then there exist M_1 and M_2, such that

$$\left\|\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial x}\,(\phi(s,t,x))\rho(s)B(\phi(s,t,x))\phi(s,t,x)\right)\right\| \leq M_1\rho(s)$$

and

$$\left\|\frac{\partial}{\partial t}\left(\frac{\partial W}{\partial x}\,(\phi(s,t,x))\rho(s)B(\phi(s,t,x))\phi(s,t,x)\right)\right\| \leq M_2\rho(s)$$

for all $s \geq t \geq 0$ and $x \in K$.

Hence, we conclude that Ψ is a C^1-function on $\mathbb{R}_+ \times \mathbb{R}^n$ and then $x = 0$ is globally exponentially stable equilibrium point of system (7).

Remark To compare the result given in this paper with the usual techniques of stability for perturbed systems, we shall consider the Lyapunov function of the nominal system as a Lyapunov function for the whole system. Let $V(t,x) = x^T Px$, where $P > 0$ is symmetric and positive definite so that

$$A^T P + PA = -Q$$

with Q symmetric and positive definite matrix. Then the derivative of $V(t,x)$ along the solutions of system (7) gives

$$\dot{V}_{(2)}(t,x) = -x^T Qx + \rho(t)x^T \Big(B^T(x)P + PB(x)\Big)x.$$

It follows that,

$$\dot{V}_{(2)}(t,x) \le \Big(-\lambda_{min}(Q) + 2\lambda_{max}(P)\rho(t)\|B(x)\|\Big)\|x\|^2,$$

$$\dot{V}_{(2)}(t,x) \le -\Big(\lambda_{min}(Q) - 2kM\lambda_{max}(P)\Big)\|x\|^2.$$

Then, if we choose

$$\lambda_{min}(Q) - 2kM\lambda_{max}(P) > 0$$

which implies that k must satisfy the following inequality

$$k < \frac{\lambda_{min}(Q)}{2M\lambda_{max}(P)}. \tag{8}$$

Hence, the system (7) is globally exponentially stable. Notice that, with our choice of Lyapunov function we don't need that the upper bound of $\rho(t)$ is limited as in (8). So, we obtain a class of stable differential system more large than by using the classical method.

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