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Partial Functional Differential Equations and Applications to Population Dynamics

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Abstract: In this paper we consider a semilinear partial functional differential equation with a nonlocal history condition arising in the study of problems in population dynamics. We reformulate it as a functional differential equation in a Banach space. Using the theory of strongly continuous and analytic semigroups we analyze the existence, uniqueness of mild, strong and classical solutions. Finally, we study the finite dimensional approximation of solutions.

Keywords: *Partial functional differential equation; strongly continuous and analytic semigroups; mild, strong and classical solutions; projections; finite dimensional approximations.*

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1 Introduction

Of concern is the following nonlocal history-valued boundary value problem for a partial functional differential equation,

$$\begin{aligned} \frac{\partial w}{\partial t}(x,t) &= a\,\frac{\partial^2 w}{\partial x^2}(x,t) + f(w(x,t), w(x,t-\tau)), \\ & t > 0, \quad 0 < x < \pi, \\ w(0,t) &= w(\pi,t) = 0, \quad t > 0, \\ h(w|_{[-\tau,0]})(x,t) &= \phi(x,t), \quad -\tau \le t \le 0,\ \tau > 0, \quad 0 \le x \le \pi, \end{aligned} \tag{1.1}$$

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where w is the sought-for function in the space $\mathcal{C}_{[-\tau,T]} = C([0,\pi]\times[-\tau,T])$, for arbitrarily fixed $0 < T < \infty$, of all continuous functions endowed the supremum norm, h is a function defined from the space $\mathcal{C}_{[-\tau,0]}$ into itself, $\phi \in \mathcal{C}_{[-\tau,0]}$, the function $w|_{[-\tau,0]} \in \mathcal{C}_{[-\tau,0]}$ is the restriction of $w \in \mathcal{C}_{[-\tau,T]}$ on $[0,\pi]\times[-\tau,0]$, $a > 0$ is a constant.

Let X be the Banach space $C[0,\pi]$ of all real-valued continuous functions on $[0,\pi]$ endowed with the supremum norm

$$\|\xi\|_X = \sup_{0\le x\le\pi} |\xi(x)|, \quad \xi \in X,$$

and for $t \in [0,T]$, $0 < T < \infty$, let $\mathcal{C}_t = C([-\tau,t];\, X)$, $0 < \tau < \infty$, be the Banach space of all continuous functions from $[-\tau,t]$ into X endowed with the supremum norm

$$\|\psi\|_t = \sup_{-\tau\le\theta\le t} \|\psi(\theta)\|_X, \quad \psi \in \mathcal{C}_t.$$

Let $\mathcal{C}_0(\chi) = \{\bar\chi \in \mathcal{C}_0\colon\ \bar\chi(0) = \chi(0)\}$. Define a function F from $\mathcal{C}_0(\chi)$ into X by

$$F(\chi) = f(\chi(0), \chi(-\tau)), \quad \chi \in \mathcal{C}_0.$$

Then (1.1) can be written as the following nonlocal history-valued functional differential equation

$$\begin{aligned} u'(t) + Au(t) &= F(u_t), \quad t \in (0,T], \\ H(u_0) &= \phi \quad \text{on} \quad [-\tau,0], \end{aligned} \tag{1.2}$$

where A is a linear operator defined on $D(A) = \{\xi \in C[0,\pi]\colon \xi'' \in C[0,\pi],\ \xi(0) = \xi(\pi) = 0\}$ with $A\xi = -a\xi''$ for $\xi \in D(A)$, for $u \in \mathcal{C}_T$ and $t \in [0,T]$, $u_t \in \mathcal{C}_0$ given by $u_t(\theta) = u(t+\theta)$, $\theta \in [-\tau,0]$, the map H is defined from $\mathcal{C}_0$ into itself and $\phi \in \mathcal{C}_0$.

For the earlier works on existence, uniqueness and stability of various types of solutions of differential and functional differential equations with nonlocal conditions we refer to Byszewski and Akca [2], Byszewski and Lakshmikantham [4], Byszewski [5], Balachandran and Chandrasekaran [3], Lin and Liu [7] and references cited in these papers.

Our main aim is to consider various types of nonlocal history conditions H and their applications. We use the ideas and techniques used by Bahuguna [1] to study such conditions and their applications.

A few examples of H are the following. Let g be map from $\mathcal{C}_0$ into X be a map given by one of the following.

(I) Let $k \in L^1(0,\tau)$ such that $\kappa = \int_0^\tau k(s)\, ds \neq 0$. Let

$$g(\xi) = \int_{-\tau}^{0} k(-s)\xi(s)\, ds, \quad \xi \in \mathcal{C}_0.$$

(II) Let $-\tau \le t_1 < t_2 < \cdots < t_l \le 0$, $c_i \ge 0$ with $C = \sum_{i=1}^{l} c_i \neq 0$. Let

$$g(\xi) = \sum_{i=1}^{l} c_i \xi(t_i), \quad \xi \in \mathcal{C}_0.$$

(III) Let t_i and c_i be as in (II) and let $\epsilon_i > 0,\ i = 1, 2, \dots, l$. Let

$$g(\xi) = \sum_{i=1}^{l} \frac{c_i}{\epsilon_i} \int_{t_i-\epsilon_i}^{t_i} \xi(s)ds, \quad \xi \in \mathcal{C}_0.$$

If we define $\phi \in \mathcal{C}_0$ given by $\phi(\theta) \equiv x$ for all $\theta \in [-\tau, 0]$ and $H\colon \mathcal{C}_0 \to \mathcal{C}_0$ given by $H(\xi)(\theta) \equiv g(\xi)$ for all $\theta \in [-\tau, 0]$ and all $\xi \in \mathcal{C}_0$, then the condition $g(\xi) = x$ is equivalent to the condition $H(\xi) = \phi$.

Let $\chi \in \mathcal{C}_0$ be such that $H(\chi) = \phi$. The function $u \in \mathcal{C}_{\widetilde{T}},\ 0 < \widetilde{T} \le T$, such that

$$u(t) = \begin{cases} \chi(t) & t \in [-\tau, 0] \\ S(t)\chi(0) + \int\limits_0^t S(t-s)F(u_s)\,ds, & t \in [0, \widetilde{T}], \end{cases} \tag{1.3}$$

is called a *mild solution* of (1.2) on $[-\tau, \widetilde{T}]$. If a mild solution u of (1.2) on $[-\tau, \widetilde{T}]$ is such that $u(t) \in D(A)$ for a.e. $t \in [0, \widetilde{T}]$, u is differentiable a.e. on $[0, \widetilde{T}]$ and

$$u'(t) + Au(t) = F(u_t), \quad \text{a.e. on } [0, \widetilde{T}],$$

it is called a *strong solution* of (1.2) on $[-\tau, \widetilde{T}]$. If a mild solution u of (1.2) on $[-\tau, \widetilde{T}]$ is such that $u \in C^1((0, \widetilde{T}]; X)$, $u(t) \in D(A)$ for $t \in (0, \widetilde{T}]$ and satisfies

$$u'(t) + Au(t) = F(u_t), \quad t \in (0, \widetilde{T}],$$

then it is called a *classical solution* of (1.2) on $[-\tau, \widetilde{T}]$.

We first establish the existence of a mild solution $u \in \mathcal{C}_{\widetilde{T}}$ of (1.2) for some $0 < \widetilde{T} \le T$ and its continuation to the whole of $[-\tau, \infty)$. Under the additional assumption of Lipschitz continuity on ψ on $[-\tau, 0]$, we show that the mild solution u is a strong solution of (1.2) on the interval of existence and it is Lipschitz continuous. Under further additional assumption that $S(t)$ is analytic, we show that u is a classical solution of (1.2) on the interval of existence. We also show that u is unique if and only if χ satisfying $H(\chi) = \phi$ is unique. Next, we establish a global existence result. Finally, we study the finite dimensional approximation of solutions in a Hilbert space.

2 Local Existence of Mild Solutions

We first prove the following result establishing the local existence and uniqueness of a mild solution of (1.2).

Theorem 2.1 *Suppose that $-A$ is the infinitesimal generator of a C_0-semigroup $S(t),\ t \ge 0$ of bounded linear operators in X. Let $H\colon \mathcal{C}_0 \to \mathcal{C}_0$ be such that there exists a function $\chi \in \mathcal{C}_0$ such that $H(\chi) = \phi$. Let $\mathcal{C}_0(\chi) = \{\bar{\chi} \in \mathcal{C}_0\colon \bar{\chi}(0) = \chi(0)\}$. Let $F\colon \mathcal{C}_0(\chi) \to X$ satisfy a Lipschitz condition*

$$\|F(\chi_1) - F(\chi_2)\|_X \le L_F \|\chi_1 - \chi_2\|_0,$$

for all $\chi_i \in \mathcal{C}_0(\chi)$, $i = 1, 2$, *where* L_F *is a non-negative constant. Then there exists a mild solution* u *of (1.2) on* $[-\tau, T_0]$ *for some* $0 < T_0 \le T$. *Moreover, the mild solution* u *is unique if and only if* χ *is unique.*

Proof Let $M \ge 1$ and $\omega \ge 0$ be such that $\|S(t)\|_{B(X)} \le Me^{\omega t}$ for $t \ge 0$. Here $B(X)$ is the space of all bounded linear operators from X into itself. We choose $0 < T_0 \le T$ be such that

$$T_0 M e^{\omega T} L_F \le 3/4.$$

Define a map $\mathcal{F}\colon \mathcal{C}_{T_0}(\chi) \to \mathcal{C}_{T_0}(\chi)$ by

$$\mathcal{F}w(t) = \begin{cases} \chi(t) & t \in [-\tau, 0], \\ S(t)\chi(0) + \int_0^t S(t-s)F(w_s)\,ds, & t \in [0, T_0]. \end{cases} \tag{2.1}$$

Here and subsequently, any function in $\mathcal{C}_T(\chi) = \{\psi \in \mathcal{C}_T\colon \psi(0) = \chi(0)\}$ is also in $\mathcal{C}_{\widetilde{T}}(\chi)$, $0 \le \widetilde{T} \le T$, as its restriction on the subinterval. Also, for $w_i \in \mathcal{C}_{T_0}(\chi)$, $i = 1, 2$, we have

$$\|\mathcal{F}w_1(t) - \mathcal{F}w_2(t)\|_X \le T_0 M e^{\omega T} L_F \|w_1 - w_2\|_{T_0}.$$

Since $T_0 M e^{\omega T} L_F \le 3/4$, $\mathcal{F}$ is a strict contraction on $\mathcal{C}_{T_0}(\chi)$ and hence has a unique fixed point $u \in \mathcal{C}_{T_0}(\chi)$.

Clearly, if $\chi \in \mathcal{C}_T$ satisfying $H(\chi) = \phi$ on $[-\tau, 0]$ is unique on $[-\tau, 0]$, then u is unique. If there are two χ and $\tilde{\chi}$ in $\mathcal{C}_0$ satisfying $H(\chi) = H(\tilde{\chi}) = \phi$ on $[-\tau, 0]$, with $\chi \ne \tilde{\chi}$ on $[-\tau, 0]$, then the corresponding solutions u and $\tilde{u}$ of (1.2) belonging to $\mathcal{C}_{T_0}(\chi)$ and $\mathcal{C}_{\tilde{T}_0}(\tilde{\chi})$ are different. This completes the proof of Theorem 2.1.

3 Global Existence of Solutions

Theorem 3.1 *Assume the hypotheses of Theorem 2.1. Then the local mild solution* u *of (1.2) exists on the whole interval* $[-\tau, \infty)$.

Proof Let $0 < T < \infty$ be arbitrarily fixed. If $T_0 < T$, consider the functional differential equation

$$\begin{aligned} v'(t) + Av(t) &= F(v_t), \quad 0 < t \le T - T_0, \\ \widetilde{H}(v_0) &= \tilde{\phi}, \end{aligned} \tag{3.1}$$

where $\widetilde{H}\colon \mathcal{C}_0(\chi) \to \mathcal{C}_0(\chi)$ given by $\widetilde{H}\chi = \chi$ for $\chi \in \mathcal{C}_0(\chi)$ and $\tilde{\phi}(\theta) = u(T_0 + \theta)$ for $\theta \in [-\tau, 0]$. Since all the hypotheses of Theorem 2.1 are satisfied for problem (3.1), we have the existence of a mild solution $w \in \mathcal{C}_{T_1}(\chi)$, $0 < T_1 \le T - T_0$ of (3.1). This mild solution w is unique as $\widetilde{H}$ in (3.1) is the identity map on $\mathcal{C}_0(\chi)$. We define

$$\bar{u}(t) = \begin{cases} u(t), & t \in [-\tau, T_0] \\ w(t - T_0), & t \in [T_0, T_0 + T_1]. \end{cases} \tag{3.2}$$

Then $\bar{u}$ is a mild solution of (1.2) on $[-\tau, T_0 + T_1]$, unique for fixed χ. Continuing this way, we get the existence of a mild solution u either on the whole interval $[-\tau, T]$

or on the maximal interval $[-\tau, t_{\max})$ of existence. In the later case we may use the arguments similar in the proof of Theorem 6.2.2 in Pazy [9, P.193–194], to conclude that $\lim\limits_{t\to t_{\max}-} \|u(t)\|_X = \infty$.

In order to show the global existence, we show that $\|u(t)\|_X \le C$ for $t \ge 0$. Let $M_1 = \max\{M, e^{\omega\tau}, (M/\omega)\|F(0)\|_X, \|\chi\|_0\}$. For $t \in [-\tau, 0]$, $e^{-\omega t}\|u(t)\|_X \le M_1$ and for $t \in [0, T)$, we have

$$e^{-\omega t}\|u(t)\|_X \le M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds. \tag{3.3}$$

From (3.3), for any $0 \le r \le t$, we have

$$e^{-\omega t}\|u(r)\|_X \le M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds. \tag{3.4}$$

Putting $r = t + \eta$, $-t \le \eta \le 0$, in (3.4), we get

$$e^{-\omega t}\|u(t+\eta)\|_X \le M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds. \tag{3.5}$$

Now, if $-\tau \le -t$, then

$$\begin{aligned} e^{-\omega t} \sup_{-\tau\le\eta\le 0} \|u(t+\eta)\|_X &\le e^{-\omega t} \sup_{-\tau\le\eta\le -t} \|u(t+\eta)\|_X + e^{-\omega t} \sup_{-t\le\eta\le 0} \|u(t+\eta)\|_X \\ &\le 2M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds, \end{aligned}$$

and for the case $-t \le -\tau$, we have

$$\begin{aligned} e^{-\omega t} \sup_{-\tau\le\eta\le 0} \|u(t+\eta)\|_X &\le e^{-\omega t} \sup_{-t\le\eta\le 0} \|u(t+\eta)\|_X \\ &\le 2M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds. \end{aligned}$$

Thus,

$$e^{-\omega t} \sup_{-\tau\le\eta\le 0} \|u(t+\eta)\|_X \le 2M_1 + ML_F \int_0^t \big[e^{-\omega s} \sup_{-\tau\le\theta\le 0} \|u(s+\theta)\|_X\big] ds.$$

Gronwall's inequality implies that

$$e^{-\omega t} \sup_{-\tau\le\eta\le 0} \|u(t+\eta)\|_X \le 2M_1 + ML_F \int_0^t f(s) \exp\{2M\|F(0)\|_X(t-s)\}\, ds. \tag{3.6}$$

Inequality (3.6) implies that $\|u(t)\|_X$ is bounded by a continuous function and therefore $\|u(t)\|_X$ is bounded on every compact interval $[-\tau, T]$, $0 < T < \infty$. Since T is arbitrary, the global existence follows.

4 Regularity of Solutions

Theorem 4.1 *Assume the hypotheses of Theorem 2.1. If, in addition,* $\chi \in \mathcal{C}_0$ *satisfying* $H(\chi) = \phi$ *is Lipschitz continuous on* $[-\tau, 0]$ *and* $\chi(0) \in D(A)$*, then the solution* u *corresponding to* χ *is Lipschitz continuous on every compact subinterval of existence. If, in addition,* X *is reflexive, then* u *is a strong solution of (1.2) on the interval of existence and this strong solution is a classical solution of (1.2) provided* $S(t)$ *is an analytic semigroup.*

Proof We shall prove the result for the first case when the mild solution u exists on the whole interval. The proof can be modified easily for the second case.

We need to show the Lipschitz continuity of u only on $[0, T]$. In what follows, C_i's are positive constants depending only on R, T and $\|\chi\|_0$. Let $t \in [0, T]$ and $h \geq 0$. Then

$$\begin{aligned} \|u(t+h) - u(t)\|_X &\leq \|(S(h) - I)S(t)\chi(0)\|_X + \int_{-h}^{0} \|S(t-s)F(u_{s+h})\|_X \, ds \\ &\quad + \int_0^t \|s(t-s)[F(u_{s+h}) - F(u_s)]\|_X ds \\ &\leq C_1 \left[h + \int_0^t [\|u_{s+h} - u_s\|_{\mathcal{C}_0}] \, ds \right] \\ &\leq C_1 \left[h + \int_0^t \sup_{-\tau \leq \theta \leq 0} \|u(s+h+\theta) - u(s+\theta)\|_X \right] ds. \end{aligned} \tag{4.1}$$

For the case when $-\tau \leq t < 0$ and $0 \leq t + h$ (clearly, $t + h \leq h$ in this case), we have

$$\begin{aligned} \|u(t+h) - u(t)\|_X &\leq \|(S(t+h) - I)\chi(0)\|_X + \|\chi(t) - \chi(0)\|_X \\ &\quad + \int_0^h \|S(t+h-s)F(u_s)\|_X ds \leq C_2 h. \end{aligned} \tag{4.2}$$

Combining the inequalities (4.1) and (4.2), we have for $-\tau \leq \bar{t} \leq t$,

$$\|u(\bar{t}+h) - u(\bar{t})\|_X \leq C_3 \left[h + \int_0^t \sup_{-\tau \leq \theta \leq 0} \|u(s+h+\theta) - u(s+\theta)\|_X ds \right]. \tag{4.3}$$

Putting $\bar{t} = t + \bar{\theta}$, $-t - \tau \leq \bar{\theta} \leq 0$, in (4.3), and taking supremum over $\bar{\theta}$ on $[-\tau, 0]$, we get

$$\sup_{-\tau \leq \theta \leq 0} \|u(t+h+\theta) - u(t+\theta)\|_X \leq 2C_3 \left[h + \int_0^t \sup_{-\tau \leq \theta \leq 0} \|u(s+h+\theta) - u(s+\theta)\|_X ds \right]. \tag{4.4}$$

Applying Gronwall's inequality in (4.4), we obtain

$$\|u(t+h)-u(t)\|_X \le \sup_{-\tau\le\theta\le 0}\|u(t+h+\theta)-u(t+\theta)\|_X \le C_4 h.$$

Thus, u is Lipschitz continuous on $[-\tau, T]$.

The function $\overline{F}\colon [0,T] \to X$ given by $\overline{F}(t) = F(u_t)$, is Lipschitz continuous and therefore differentiable a.e. on $[0,T]$ and $\overline{F}'$ is in $L^1((0,T); X)$. Consider the Cauchy problem

$$\begin{aligned} v'(t) + Av(t) &= \overline{F}(t), \quad t \in (0,T], \\ v(0) &= u(0), \end{aligned} \tag{4.5}$$

By the Corollary 2.10 on page 109 in Pazy [9], there exists a unique strong solution v of (4.5) on $[0,T]$. Clearly, $\bar{v}$ defined by

$$\bar{v}(t) = \begin{cases} u(t), & t \in [-\tau, 0] \\ v(t), & t \in [0,T], \end{cases}$$

is a strong solution of (1.2) on $[-\tau, T]$. But this strong solution is also a mild solution of (1.2) and $\bar{v} \in \mathcal{C}_T(\chi)$. By the uniqueness of such a function in $\mathcal{C}_T(\chi)$, we get $\bar{v}(t) = u(t)$ on $[-\tau, T]$. Thus u is a strong solution of (1.2). If $S(t)$ is analytic semigroup in X then we may use Corollary 3.3 on page 113 in Pazy [9] to obtain that u is a classical solution of (1.2). This completes the proof of Theorem 4.1.

5 Finite Dimensional Approximations

In this section we assume that X is a separable Hilbert space. Furthermore, we assume that in (1.2), the linear operator A satisfies the following hypothesis.

(H1) A is a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subset X$ into X such that $D(A)$ is dense in X, A has the pure point spectrum

$$0 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$$

and a corresponding complete orthonormal system of eigenfunctions $\{u_i\}$, i.e.,

$$Au_i = \lambda_i u_i \quad \text{and} \quad (u_i, u_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If (H1) is satisfied then the semigroup $S(t)$ generated by $-A$ is analytic in X. It follows that the fractional powers A^α of A for $0 \le \alpha \le 1$ are well defined from $D(A^\alpha) \subseteq X$ into X (cf. Pazy [9], pp. 69−75). $D(A^\alpha)$ is a Banach space endowed with the norm

$$\|x\|_\alpha = \|A^\alpha x\|_X, \quad x \in D(A^\alpha). \tag{5.1}$$

For $t \in [0,T]$, we denote by $\mathcal{C}_t^\alpha = C([-r,t]; D(A^\alpha))$ endowed with the norm

$$\|\zeta\|_{t,\alpha} = \sup_{-r\le\eta\le t}\|\zeta(\eta)\|_\alpha, \quad \zeta \in \mathcal{C}_t^\alpha.$$

In addition, we assume the following hypotheses.

(H2) There exists a function $\chi \in \mathcal{C}_0^\alpha$ satisfying $H(\chi) = \phi$.

(H3) The map F is defined from $\mathcal{C}_0^\alpha(\chi) = \{\bar{\chi} \in \mathcal{C}_0^\alpha \colon \bar{\chi}(0) = \chi(0)\}$ into $D(A^\beta)$ for $0 < \beta \le \alpha < 1$ and there exists a non-negative constant L_F such that

$$\|F(\zeta_1) - f(\zeta_2)\|_X \le L_F \|\zeta_1 - \zeta_2\|_{0,\alpha},$$

for $\zeta_i \in \mathcal{C}_0^\alpha(\chi)$, for $i = 1, 2$.

Let X_n denote the finite dimensional subspace of X spanned by $\{u_0, u_1, \dots, u_n\}$ and let $P^n \colon X \to X_n$ be the corresponding projection operator for $n = 0, 1, 2, \dots$. Let $\chi \in \mathcal{C}_0$ be such that $H(\chi) = \phi$. Let $\bar{\chi}$ be the extension of χ by the constant value $\chi(0)$ on $[0, T]$. We set

$$T_0 = \min\left\{T, \left(\frac{3(1-\alpha)}{8L_F C_\alpha}\right)^{1-\alpha}\right\},$$

where C_α is a positive constant such that $\|A^\alpha S(t)\| \le C_\alpha t^{-\alpha}$ for $t > 0$.

We define

$$F_n \colon \mathcal{C}_0(\chi) \to X_n,$$

given by

$$F_n(\zeta) = P^n F(P^n \zeta), \quad \zeta \in \mathcal{C}_0(\chi),$$

where $(P^n\zeta)(\theta) = P^n\zeta(\theta)$, $-\tau \le \theta \le 0$. We denote $\psi_n = P^n\psi$ for any $\psi \in \mathcal{C}_T$.

Let $A^\alpha \colon \mathcal{C}_t^\alpha \to \mathcal{C}_t$ be given by $(A^\alpha\psi)(s) = A^\alpha(\psi(s))$, $s \in [-r, t]$, $t \in [0, T_0]$. We define a map $\mathcal{F}_n \colon \mathcal{C}_{T_0}(\chi) \to \mathcal{C}_{T_0}(\chi)$ as follows:

$$(\mathcal{F}_n\xi)(t) = \begin{cases} A^\alpha \chi_n(t), & t \in [-\tau, 0], \\ S(t)A^\alpha\chi_n(0) + \int_0^t A^\alpha S(t-s) F_n(A^{-\alpha}\xi_s)\, ds, & t \in [0, T_0], \end{cases} \tag{5.2}$$

for $\xi \in \mathcal{C}_{T_0}(\chi)$.

Proposition 5.1 *There exists a unique $w_n \in \mathcal{C}_{T_0}(\chi)$ such that $F_n w_n = w_n$ on $[-r, T_0]$.*

Proof For $\xi_1, \xi_2 \in \mathcal{C}_{T_0}(\chi)$, $(\mathcal{F}_n\xi_1)(t) - (\mathcal{F}_n\xi_2)(t) = 0$ on $[-\tau, 0]$ and for $t \in [0, T_0]$, we have

$$\|(\mathcal{F}_n\xi_1)(t) - (\mathcal{F}_n\xi_2)(t)\|_X \le 2L_F C_\alpha \frac{T_0^{1-\alpha}}{1-\alpha} \|\xi_1 - \xi_2\|_{T_0} \le \frac{3}{4} \|\xi_1 - \xi_2\|_{T_0}.$$

Taking the supremum over $[-\tau, T_0]$, it follows that $\mathcal{F}_n$ is a strict contraction on $\mathcal{C}_{T_0}(\chi)$ and hence there exits a unique $w_n \in \mathcal{C}_{T_0}(\chi)$ with $w_n = \mathcal{F}_n w_n$ on $[-\tau, T_0]$. This completes the proof of Proposition 5.1.

Let $u_n = A^{-\alpha} w_n$. Then $u_n \in \mathcal{C}_{T_0}^\alpha$ and satisfies

$$u_n(t) = \begin{cases} \chi_n(t), & t \in [-\tau, 0], \\ S(t)\chi_n(0) + \int_0^t S(t-s) F_n(u_s)\, ds, & t \in [0, T_0]. \end{cases} \tag{5.3}$$

Proposition 5.2 *The sequence* $\{u_n\} \subset \mathcal{C}_{T_0}(\chi)$ *is a Cauchy sequence and therefore converges to a function* $u \in \mathcal{C}_{T_0}(\chi)$.

Proof For $n, m \in N,\ n \geq m,\ t \in [-\tau, 0]$, we have

$$\|u_n(t) - u_m(t)\|_\alpha \leq \|A^\alpha(\chi_n(t) - \chi_m(t))\|_X \leq \|(P^n - P^m)A^\alpha \chi(t)\|_X \to 0 \quad \text{as} \quad m \to \infty.$$

For $t \in (0, T_0]$ and n, m as above, we have

$$\|u_n(t) - u_m(t)\|_\alpha \leq \|(P^n - P^m)S(t)A^\alpha \chi(0)\|_X + \int_0^t \|A^\alpha S(t-s)[F_n((u_n)_s) - F_m((u_m)_s))]\|_X \, ds.$$

Now, using the fact that $F((u_m)_s) \in D(A^\beta),\ m \geq n_0$ and $0 < \alpha < \beta < 1$, we have

$$\begin{aligned}\|F_n((u_n)_s) - F_m((u_m)_s)\|_X &\leq \|(P^n - P^m)F(P^m(u_m)_s)\|_X \\ &\quad + L_F[\|(P^m - P^m)A^\alpha(u_m)_s\|_0] + L_F\|u_n - u_m\|_{s,\alpha} \\ &\leq C_1 \frac{1}{\lambda_m^\beta} + C_2\|u_n - u_m\|_{s,\alpha},\end{aligned}$$

for some positive constants C_1 and C_2 independent of n and m. Thus, we have the following estimate

$$\begin{aligned}\|u_n(t) - u_m(t)\|_\alpha &\leq C_0\|(P^n - P^m)A^\alpha \chi(0)\|_X \\ &\quad + \frac{C_1 T^\gamma}{\lambda_m^\beta} + C_2 \int_0^t (t-s)^\alpha \|u_n - u_m\|_{s,\alpha} \, ds,\end{aligned} \tag{5.4}$$

where $C_0 = Me^{\omega T}$. Since $u_n - u_m = \chi_n - \chi_m$ on $[-\tau, 0]$, we have for $0 \leq \bar{t} \leq t$,

$$\begin{aligned}\|u_n(\bar{t}) - u_m(\bar{t})\|_\alpha &\leq \|\chi_n - \chi_m\|_{0,\alpha} + C_0\|(P^n - P^m)A^\alpha \chi(0)\|_X \\ &\quad + \frac{C_1 T}{\lambda_m^\beta} + C_2 \int_0^{\bar{t}} (\bar{t}-s)^\alpha \|u_n - u_m\|_{s,\alpha} \, ds.\end{aligned} \tag{5.5}$$

We put $\bar{t} = t + \eta,\ -t \leq \eta \leq 0$, to obtain

$$\begin{aligned}\|u_n(t+\eta) - u_m(t+\eta)\|_\alpha &\leq \|\chi_n - \chi_m\|_{0,\alpha} + C_0\|(P^n - P^m)A^\alpha \chi(0)\|_X + \\ &\quad \frac{C_1 T}{\lambda_m^\beta} + C_2 \int_0^{t+\eta} (t+\eta-s)^\alpha \|u_n - u_m\|_{s,\alpha} \, ds.\end{aligned} \tag{5.6}$$

Now, we put $s - \eta = \bar{s}$ to get

$$\begin{aligned}\|u_n(t+\eta) - u_m(t+\eta)\|_\alpha &\leq \|\chi_n - \chi_m\|_{0,\alpha} + C_0\|(P^n - P^m)A^\alpha \chi(0)\|_X \\ &\quad + \frac{C_1 T}{\lambda_m^\beta} + C_2 \int_{-\eta}^{t} (t-\bar{s})^\alpha \|u_n - u_m\|_{\bar{s}+\eta,\alpha} \, d\bar{s} \\ &\leq \|\chi_n - \chi_m\|_{\alpha,0} + C_0\|(P^n - P^m)A^\alpha \chi(0)\| \\ &\quad + \frac{C_1 T}{\lambda_m^\beta} + C_2 \int_0^t (t-\bar{s})^\alpha \|u_n - u_m\|_{\bar{s},\alpha} \, d\bar{s}.\end{aligned} \tag{5.7}$$

For $t \geq \tau$, we have

$$\sup_{-\tau\leq\eta\leq 0}\|u_n(t+\eta)-u_m(t+\eta)\|_\alpha \leq \sup_{-t\leq\eta\leq 0}\|u_n(t+\eta)-u_m(t+\eta)\|_\alpha$$

$$\leq \|\chi_n-\chi_m\|_{\alpha,0}+C_0\|(P^n-P^m)A^\alpha\chi(0)\|+\frac{C_1T}{\lambda_m^\beta}+C_2\int_0^t(t-\bar{s})^\alpha\|u_n-u_m\|_{\bar{s},\alpha}\,d\bar{s}. \tag{5.8}$$

Since $u_n(t+\eta)=\chi(t+\eta)$ for $t+\eta\leq 0$ for all $n\geq n_0$, for $0\leq t\leq\tau$, we have

$$\sup_{-\tau\leq\eta\leq 0}\|u_n(t+\eta)-u_m(t+\eta)\|_\alpha$$

$$\leq \sup_{-\tau\leq\eta\leq -t}\|u_n(t+\eta)-u_m(t+\eta)\|_\alpha+\sup_{-t\leq\eta\leq 0}\|u_n(t+\eta)-u_m(t+\eta)\|_\alpha$$

$$\leq \|\chi_n-\chi_m\|_{\alpha,0}+C_0\|(P^n-P^m)A^\alpha\chi(0)\|+\frac{C_1T}{\lambda_m^\beta}+C_2\int_0^t(t-\bar{s})^\alpha\|u_n-u_m\|_{\bar{s},\alpha}\,d\bar{s}. \tag{5.9}$$

Combining (5.8) and (5.9), we have

$$\begin{aligned}\|u_n-u_m\|_{t,\alpha}\leq{}&\|\chi_n-\chi_m\|_{\alpha,0}+C_0\|(P^n-P^m)A^\alpha\chi(0)\|\\ &+\frac{C_1T}{\lambda_m^\beta}+C_2\int_0^t(t-\bar{s})^\alpha\|u_n-u_m\|_{\bar{s},\alpha}\,d\bar{s}.\end{aligned} \tag{5.10}$$

Application of Lemma 5.6.7 on page 159 in Pazy [9] gives the required result. This completes the proof of Proposition 5.2.

With the help of Propositions 5.1 and 5.2, we may state the following existence, uniqueness and convergence result.

Theorem 5.3 *Suppose that assumptions (H1) – (H3) hold. Then there exist functions* $u_n\in([-\tau,T_0];X_n)$, $n\in N$, *and* $u\in\mathcal{C}_{T_0}$ $(0<T_0\leq T)$ *unique for a given* $\chi\in\mathcal{C}_0$ *with* $H(\chi)=\phi$, *such that*

$$u_n(t)=\begin{cases}\chi_n(t), & t\in[-\tau,0],\\ S(t)\chi_n(0)+\int_0^t S(t-s)F_n((u_n)_s)\,ds, & t\in[0,T_0],\end{cases} \tag{5.11}$$

and

$$u(t)=\begin{cases}\chi(t), & t\in[-\tau,0],\\ S(t)\chi(0)+\int_0^t S(t-s)F(u_s)\,ds, & t\in[0,T_0],\end{cases} \tag{5.12}$$

such that $u_n\to u$ *in* $\mathcal{C}_{T_0}$ *as* $n\to\infty$, *where* $\psi_n(t)=P^n\psi(t)$ *for* $\psi\in\mathcal{C}_{T_0}$ *and* $F_n(\zeta)=P^nF(P^n\zeta)$, $\zeta\in\mathcal{C}_0$.

6 Applications

As an applicability of the theory developed in previous sections, we cite two examples of partial differential equation with retarded arguments and a nonlocal history condition. These problems are closely related to a mathematical model for population density with a time delay and self regulation (cf. [6, 10]).

Example 6.1

$$
\begin{aligned}
\frac{\partial w}{\partial t}(x,t) &= a\,\frac{\partial^2 w}{\partial x^2}(x,t) + b\,w(x,t-\tau)(1-w(x,t)),\\
&\qquad\qquad t>0, \quad 0<x<\pi,\\
w(0,t) &= w(\pi,t) = 0, \quad t>0,\\
h(w|_{[-\tau,0]})(x,t) &= \phi(x,t), \quad -\tau\le t\le 0, \quad \tau>0, \quad 0\le x\le \pi,
\end{aligned}
\tag{6.1}
$$

where $w(\cdot,t)$ is the population density at time t, b is the constant rate of growth for the species. τ is a fixed positive constant and $\phi \in \mathcal{C}_{[-\tau,0]} = C([0,\pi]\times[-\tau,0])$. Let $X = C[0,\pi]$. For each t, define an operator A by

$$Au = -au'',$$

for $u \in D(A) = \{u \in C([0,\pi])\colon\ u'' \in C([0,\pi]),\ u(0)=u(\pi)=0\}$. It follows that $-A$ generates an analytic semigroup in X. The nonlinear map H can be defined as mentioned in the first section.

Let $\mathcal{C}_0(\chi)$ be the set consisting of all continuous function $\bar\chi\colon [-\tau,0]\to X$ such that $\bar\chi(0)=\chi(0)$ and define $F\colon \mathcal{C}_0(\chi)\to X$ by

$$F(\chi) = b\chi(-\tau)(1-\chi(0)), \quad \chi\in\mathcal{C}_0(\chi).$$

It is easily verified that F satisfies Lipschitz condition. The problem (6.1) now take the abstract form

$$
\begin{aligned}
u'(t) + Au(t) &= F(u_t), \quad t\in(0,T],\\
H(u_0) &= \phi, \quad \text{on } [-\tau,0],
\end{aligned}
\tag{6.2}
$$

Then the theorems ensure the existence of a unique solution of the problem (6.2) (hence a unique solution of the problem (6.1)).

Example 6.2

$$
\begin{aligned}
\frac{\partial w}{\partial t}(x,t) &= a\,\frac{\partial^2 w}{\partial x^2}(x,t) + b\,w(x,t)\left[1-\int_{-\tau}^{0} w_t(x,s)\,d\eta(s)\right],\\
&\qquad\qquad t>0, \quad 0<x<\pi,\\
w(0,t) &= w(\pi,t) = 0, \quad t>0,\\
h(w|_{[-\tau,0]})(x,t) &= \phi(x,t), \quad -\tau\le t\le 0, \quad \tau>0, \quad 0\le x\le \pi,
\end{aligned}
\tag{6.3}
$$

which is a population model when diffusion occurs within the population. Here $\eta(\cdot)$ is bounded, nondecreasing function on $[-\tau,0]$, $\tau\ge 0$. All other functions and maps are as described in Example 6.1.

Let $X = C([0,\pi])$. The linear operator A is defined as in the previous example. Also we define $F\colon \mathcal{C}_0(\chi)\to X$ by

$$F(\chi) = b\chi(0)\left[1-\int_{-\tau}^{0}\chi(s)\,d\eta(s)\right], \quad \chi\in\mathcal{C}_0(\chi).$$

Then clearly F satisfies Lipschitz condition and problem (6.3) transforms into the abstract form (6.2).

Since all the assumptions taken into account for establishing the existence and uniqueness results are satisfied, we can apply these results to considered problem which shows that there exists a unique solution of (6.3).

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