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A Parametrization Approach for Solving the Hamilton–Jacobi Equation and Application to the $\mathcal{A}_2$-Toda Lattice

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Abstract: Hamilton–Jacobi (HJ)-theory is an extension of Lagrangian mechanics and concerns itself with a directed search for a coordinate transformation in which the equations of motion can be easily integrated. Hamilton (1838) has developed the method for obtaining the desired transformation equations by finding a smooth function S called a *generating function* or *Hamilton's principal function*, which satisfies a certain nonlinear first-order partial-differential equation (PDE) also known as the *Hamilton–Jacobi equation* (HJE).

Unfortunately, the HJE being nonlinear is very difficult to solve; and thus, except for the case in which the variables in the equation are separable, its application remains limited. It is thus our aim in this paper to present a new approach for solving the Hamilton–Jacobi equation for a fairly large class of Hamiltonian systems and to apply it in particular to the $\mathcal{A}_2$-Toda lattice.

Keywords: *Lagrangian mechanics; Hamiltonian system; contact transformation; generating function; Hamilton–Jacobi equation.*

Mathematics Subject Classification (2000): 70H20.

1 Introduction to Hamilton–Jacobi Theory

Hamilton–Jacobi (HJ)-theory is an extension of Lagrangian mechanics and concerns itself with a directed search for a coordinate transformation in which the equations of motion can be easily integrated. The equations of motion of a given mechanical system can often be simplified considerably by a suitable transformation of variables such that all the new

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position and momentum coordinates are constants. A particular type of transformation is chosen in such a way that the new equations of motion retain the same form as in the former coordinates; such a transformation is called *canonical* or *contact* and can greatly simplify the solution to the equations of motion. Hamilton (1838) has developed the method for obtaining the desired transformation equations using what is today known as *Hamilton's principle*. It turns out that the required transformation can be obtained by finding a smooth function S called a *generating function* or *Hamilton's principal function*, which satisfies a certain nonlinear first-order partial-differential equation (PDE) also known as the *Hamilton–Jacobi equation* (HJE).

Unfortunately, the HJE being nonlinear, is very difficult to solve; and thus, it might appear that little practical advantage has been gained in the application of the HJ-theory. Nonetheless, under certain conditions, and when the Hamiltonian is independent of time, it is possible to separate the variables in the HJE, and the solution can then always be reduced to quadratures. Thus, the HJE becomes a useful computational tool only when such a separation of variables can be achieved.

The aim of this paper is two-fold. First, to give an overview of the essentials of Hamilton–Jacobi theory, namely; (i) the Hamiltonian reformulation of the equations of motion of a mechanical system; and (ii) the Hamiltonian transformation of the equations of motion. Secondly, to present an approach for solving the HJE for a fairly large class of Hamiltonian systems in which the variables in the equation may not be separable and/or the Hamiltonian is not time-independent. We apply the approach to a class of integrable Hamiltonian systems known as the Toda lattice. Computational results are presented to show the usefulness of the method.

The rest of the paper in organized as follows. In the remainder of this section, we introduce notations. In Section 2, we discuss the Hamiltonian formulation of the equations of motion of a natural mechanical system. Then we discuss Hamiltonian coordinate transformations and generating functions of the transformations in Section 3. In Section 4, we discuss the Hamilton–Jacobi equation which is the central focus of the paper. In Section 5, we review the Toda lattice as a Hamiltonian system, and discuss the method of Lax for solving the system. Then in Section 6, we discuss the main results of the paper, which is a parametrization approach for solving the HJE. We also apply the results to the $\mathcal{A}_2$-Toda lattice. Finally, in Section 7, we give conclusions.

Notation The notation is fairly standard except where otherwise stated. Moreover, R, R^n will denote respectively, the real line and the n-dimensional real vector space, $t \in R$ will denote the time parameter. Let M^n, N^n, $\ldots$ denote Riemannian manifolds with dimension n, which are compact. Let $TM = \bigcup_{x\in M} T_xM$, $T^\star M = \bigcup_{x\in M} T_x^\star M$ respectively denote the tangent and cotangent bundles of M with dimensions $2n$. Moreover, π_M and $\pi_M^\star$ will denote the natural projections $TM \to M$ and $T^\star M \to M$ respectively. $SO(n, M)$ and $sl(n, M)$ will denote the special orthogonal group and the lie-algebra of the special linear group of matrices over M respectively. A $C^\infty(M)$ vector-field is a mapping $f\colon M \to TM$ such that $\pi \circ f = I_M$ (the identity on M), and f has continuously differentiable partial derivatives of arbitrary order. A vector field f also defines a differential equation (or a dynamic system) $\dot{x}(t) = f(x),\ x \in M,\ x(t_0) = x_0$.

A differential k-form $\omega_x^k,\ k = 1, 2, \ldots,$ at a point $x \in M$ is an exterior product of k-vectors from T_xM to R i.e. $\omega_x^k\colon T_xM \times \ldots \times T_xM$ (k copies) $\to R$, which is a k-linear skew-symmetric function of k-vectors on T_xM. The space of all smooth k-forms on M is denoted by $\Omega^k(M)$. The $\mathcal{F}$-derivative (Frèchet derivative) of a real-valued function

$U\colon R^n \to R$ is defined as any ϱ such that $\lim\limits_{v\to 0}\dfrac{1}{\|v\|}[U(x+v)-U(x)-\langle\varrho,v\rangle]=0$, for any $v\in R^n$. For a smooth function $f\colon R^n\to R$, $f_x=\dfrac{\partial f}{\partial x}=(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n})$. Further, let $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty\colon M\to R$ denote respectively, 1, 2, and ∞ norms on M, where

$$\|v(q)\|_1=\sum_i^n |v_i(q)|,\quad \|v(q)\|_2=\sum_{i=1}^n |v_i(q)|^2$$

and $\|v(q)\|_\infty=\max_i\{v_i(q)\colon i=1,\dots,n\}$ for any vector $v\colon M_q\to T_qM$. Also, if $f\colon[0,1]\to R$, then

$$\|f(s)\|_{L_p}=\left(\int_0^1|f(s)|^p\right)^{\frac{1}{p}},\quad 1\le p<\infty,$$

while $\|f(s)\|_{L_\infty}=\sup_{s\in[0,1]}|f(s)|$.

2 The Hamiltonian Formulation of Mechanics

To review the approach, let the configuration space of the system be defined by a smooth n-dimensional Riemannian manifold M. If (φ,U) is a coordinate chart, we write $\varphi=q=(q_1,\dots,q_n)$ for the local coordinates and $\dot q_i=\dfrac{\partial}{\partial q_i}$ in the tangent bundle $TM|_U=TU$. We shall be considering *natural mechanical systems* which are defined as follows.

Definition 2.1 A Lagrangian mechanical system on a Riemannian manifold is called *natural* if the Lagrangian function $L\colon T\mathcal{U}\times R\to R$, with $\mathcal{U}\subset M$ open, is equal to the difference between the kinetic energy and the potential energy of the system as

$$L(q,\dot q,t)=T(q,\dot q,t)-V(q,t), \tag{2.1}$$

where $T\colon\mathcal{U}\to R$ is the kinetic energy which is given by the quadratic form

$$T=\frac{1}{2}\langle v,v\rangle,\quad v\in T_qU$$

and $V\colon M\times R\to R$ is the potential energy of the system (which may be independent of time).

For natural mechanical systems, the kinetic energy is a positive-definite symmetric quadratic form of the generalized velocities,

$$T(q,\dot q,t)=\frac{1}{2}\,\dot q^{\mathrm T}\Psi(q,t)\dot q. \tag{2.2}$$

It is further known from Lagrangian mechanics and as can be derived using the D'Alembert's *principle of virtual work* or Hamilton's *principle of least action* [3, 7, 8], that the motion of a *holonomic conservative*[1] mechanical system satisfies Lagrange's equations of motion given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n. \tag{2.3}$$

Then the above equation (2.3) may always be written in the form

$$q = g(q, \dot{q}, t), \tag{2.4}$$

for some function $g\colon T\mathcal{U} \times R \to R^n$.

On the other hand, in the Hamiltonian formulation, we choose to replace all the $\dot{q}_i$ by independent coordinates, p_i, in such a way that

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \dots, n. \tag{2.5}$$

If we let

$$p_i = h_i(q, \dot{q}), \quad i = 1, \dots, n, \tag{2.6}$$

then the Jacobian of h with respect to $\dot{q}$, using (2.1), (2.2) and (2.5), is given by $\Psi(q)$ which is positive definite, and hence equation (2.5) can be inverted to yield

$$\dot{q}_i = g_i(q_1, \dots, q_n, p_1, \dots, p_n, t), \quad i = 1, \dots, n, \tag{2.7}$$

for some continuous functions $g_1, \dots, g_n$. The coordinates $q = (q_1, q_2, \dots, q_n)^{\mathrm{T}}$, in this framework, are referred to as the *generalized coordinates* and $p = (p_1, p_2, \dots, p_n)^{\mathrm{T}}$ are the *generalized momenta*. Together, these variables form a new system of coordinates for the system known as the *phase space* of the system. If (U, φ) where $\varphi = (q_1, q_2, \dots, q_n)$ is a chart on M, then since $p_i\colon TU \to R$, they are elements of $T^\star U$, and together with the q_i's form a system of $2n$ local coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$, where $p_i(q) \in T_q^\star M$, $i = 1, \dots, n$, for the phase-space.

We now define the Hamiltonian function of the system $H\colon T^\star M \times R \to R$ as the *Legendre transform* [3, 5] of the Lagrangian function with respect to $\dot{q}$ by

$$H(q, p, t) = p^{\mathrm{T}}\dot{q} - L(q, \dot{q}, t). \tag{2.8}$$

Consider now the differential of H with respect to q, p and t as

$$dH = \left(\frac{\partial H}{\partial p}\right)^{\mathrm{T}} dp + \left(\frac{\partial H}{\partial q}\right)^{\mathrm{T}} dq + \frac{\partial H}{\partial t} dt. \tag{2.9}$$

The above expression must be equal to the total differential of $H = p\dot{q} - L$ for $p = \dfrac{\partial L}{\partial \dot{q}}$:

$$dH = \dot{q}^{\mathrm{T}} dp - \left(\frac{\partial L}{\partial q}\right)^{\mathrm{T}} dq - \left(\frac{\partial L}{\partial t}\right)^{\mathrm{T}} dt. \tag{2.10}$$

[1] Holonomic if the constraints on the system are expressible as equality constraints. Conservative if there exists a time-dependent potential.

Thus, in view of the independent nature of the coordinates, we obtain a set of three relationships:

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q}, \qquad \text{and} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}.$$

Finally, applying Lagrange's equation (2.3) together with (2.5) and the preceding results, one obtains the expression for $\dot{p}$. Since we used Lagrange's equation, $\dot{q} = \frac{dq}{dt}$ and $\dot{p} = \frac{dp}{dt}$. The resulting Hamiltonian canonical equations of motion are then given by

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}(q,p,t), \tag{2.11}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}(q,p,t). \tag{2.12}$$

Thus, we have proven the following theorem.

Theorem 2.1 [3] *The system of Lagrange's equations (2.3) is equivalent to the system of* $2n$ *first-order Hamilton's equations (2.11), (2.12).*

In addition, for time-independent conservative systems, $H(q,p)$ has a simple physical interpretation. From (2.8) and using (2.5), we have

$$\begin{aligned} H(q,p,t) &= p^{\mathrm{T}}\dot{q} - L(q,\dot{q},t) = \dot{q}^{\mathrm{T}}\frac{\partial L}{\partial \dot{q}} - (T(q,\dot{q},t) - U(q,t)) \\ &= \dot{q}^{\mathrm{T}}\frac{\partial T}{\partial \dot{q}} - T(q,\dot{q},t) + U(q,t) \\ &= 2T(q,\dot{q},t) - T(q,\dot{q},t) + U(q,t) = T(q,\dot{q},t) + U(q,t), \end{aligned} \tag{2.13}$$

i.e., the total energy of the system. This completes the Hamiltonian formulation of the equations of motion, and can be seen as an off-shoot of the Lagrangian formulation. It can also be seen that, while the Lagrangian formulation involves n second-order equations, the Hamiltonian description sets up a system of $2n$ first-order equations in terms of the $2n$ variables p and q. This remarkably new system of coordinates gives new insight and physical meaning to the equations. However, the system of Lagrange's equations and Hamilton's equations are completely equivalent as the above theorem asserts.

Furthermore, because of the symmetry of Hamilton's equations (2.11), (2.12) and the even dimension of the system, a new structure emerges on the phase space $T^\star M$ of the system. This structure is defined by a nondegenerate closed differential 2-form $\omega^2 \in \Omega^2(M)$ which in the above local coordinates is defined as

$$\omega^2 = dp \wedge dq = \sum_{i=1}^{n} dp_i \wedge dq_i. \tag{2.14}$$

Thus, the pair $(T^\star M, \omega^2)$ form a *symplectic manifold* [1, 3, 11], and together with a C^r Hamiltonian function $H\colon T^\star M \to R$ define a Hamiltonian mechanical system. With this notation we have the following representation of a Hamiltonian system.

Definition 2.2 Let $(T^\star M, \omega^2)$ be a symplectic manifold and $H\colon T^\star M \to R$ the Hamiltonian function. Then the vector field X_H determined by the condition

$$\omega^2(X_H, Y) = dH(Y) \tag{2.15}$$

for all vector fields Y, is called the *Hamiltonian vector field* with energy function H. The tuple $(T^\star M,\, \omega^2,\, X_H)$ is called a Hamiltonian system.

Remark 2.1 It is important to note that, the nondegeneracy of ω^2 guarantees that X_H exists, and is a C^{r-1} vector field. Moreover, on a connected symplectic manifold, any two Hamiltonians for the same vector field X_H have the same differential (2.15), so differ by a constant only.

We also have the following proposition [1].

Proposition 2.1 *Let $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ be canonical coordinates so that ω^2 is given by (2.14). Then, in these coordinates*

$$X_H = \left(\frac{\partial H}{\partial p_1},\ \ldots,\ \frac{\partial H}{\partial p_n},\ -\frac{\partial H}{\partial q_1},\ \ldots,\ -\frac{\partial H}{\partial q_n}\right) = J\cdot\nabla H$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Thus, $(q(t), p(t))$ is an integral curve of X_H if and only if Hamilton's equations (2.11), (2.12) hold.

Now suppose that a transformation of coordinates is introduced $q_i \to Q_i,\ p_i \to P_i$, $i = 1, \ldots, n,$ defined by

$$q_i = \phi_i(Q, P, t), \tag{2.16}$$
$$p_i = \psi_i(Q, P, t) \tag{2.17}$$

such that every Hamiltonian function transforms as

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \to K(Q_1, \ldots, Q_n, P_1, \ldots, P_n, t)$$

in such a way that the new equations of motion retain the same form as in the former coordinates, i.e.,

$$\frac{dQ}{dt} = \frac{\partial K}{\partial p}(Q, P, t), \tag{2.18}$$
$$\frac{dP}{dt} = -\frac{\partial K}{\partial q}(Q, P, t). \tag{2.19}$$

Such a transformation is called *canonical* or *contact* and can greatly simplify the solution to the equation of motion, especially if Q, P are selected such that $K(\cdot,\cdot,\cdot)$ is a constant independent of Q and P. Should this happen, then Q and P will also be constants and the solution to the equations of motion are immediately at hand (given the transformation). We simply transform back to the original coordinates; under the assumption that the transformation is univalent and invertible. Hamilton (1838) has developed a method for obtaining the desired transformation equations using what is today known as *Hamilton's principle* [3, 7, 8, 10].

3 The Transformation Generating Function

A given Hamiltonian system can often be simplified considerably by a suitable transformation of variables such that all the new position and momentum coordinates (Q_i, P_i) are constants. A particular type of transformation is discussed in this section.

Accordingly, define the Lagrangian function of the system $L\colon T\mathcal{U} \times R \to R$ as the Legendre transform [3] of the Hamiltonian function by

$$L(q, \dot{q}, t) = p^{\mathrm{T}}\dot{q} - H(q, p, t). \tag{3.1}$$

Then, in the new coordinates, the new Lagrangian function is

$$\bar{L}(Q, \dot{Q}, t) = P^{\mathrm{T}}\dot{Q} - K(Q, P, t). \tag{3.2}$$

Since both $L(\cdot,\cdot,\cdot)$ and $\bar{L}(\cdot,\cdot,\cdot)$ are conserved, each must separately satisfy Hamilton's principle. However, $L(\cdot,\cdot,\cdot)$ and $\bar{L}(\cdot,\cdot,\cdot)$ need not be equal in order to satisfy the above requirement. Indeed we can write [8]

$$L(q, \dot{q}, t) = \bar{L}(Q, \dot{Q}, t) + \frac{dS}{dt}(q, p, Q, P, t) \tag{3.3}$$

for some arbitrary function $S\colon \mathcal{X} \times \mathcal{X} \times R \to R$, where $\mathcal{X} \subset T^\star M$ is open.

The next step is to show that, first, if such a function is known, then the transformation we seek follows directly. Secondly, that the function can be obtained by solving a certain partial differential equation.

The generating function S relates the old to the new coordinates via the equation

$$S = \int (L - \bar{L})\, dt = \sigma(q, p, Q, P, t) \tag{3.4}$$

for some function $\sigma\colon \mathcal{X}\times\mathcal{X}\times R \to R$. Thus, S is a function of $4n+1$ variables, and hence no more than four independent sets of relationships among the dependent coordinates can exist. Two such relationships expressing the old sets of coordinates in terms of the new set are given by (2.16), (2.17). Hence only two independent sets of relationships among the coordinates remain for defining S and no more than two of the four sets of coordinates may be involved. Therefore, there are four possibilities

$$S_1 = f_1(q, Q, t); \qquad S_2 = f_2(q, P, t), \tag{3.5}$$

$$S_3 = f_3(p, Q, t); \qquad S_4 = f_4(p, P, t). \tag{3.6}$$

Any one of the above four types of generating functions may be selected, and a transformation obtained from it. For example, if we consider the generating function S_1, taking its differential, we have

$$dS_1 = \sum_{i=1}^{n} \frac{\partial S_1}{\partial q_i}\, dq_i + \sum_{i=1}^{n} \frac{\partial S_1}{\partial Q_i}\, dQ_i + \frac{\partial S_1}{\partial t}\, dt. \tag{3.7}$$

Again, taking the differential as defined by (3.1), (3.2) (3.3), we have

$$dS_1 = \sum_{i=1}^{n} p_i \, dq_i - \sum_{i=1}^{n} P_i \, dQ_i + (K - H) \, dt. \tag{3.8}$$

Finally, using the independence of coordinates, we equate coefficients, and obtain the desired transformation equations

$$\begin{aligned} p_i &= \frac{\partial S_1}{\partial q_i}(q, Q, t) \\ P_i &= -\frac{\partial S_1}{\partial Q_i}(q, Q, t) \\ K - H &= \frac{\partial S_1}{\partial t}(q, Q, t), \quad i = 1, \dots, n. \end{aligned} \tag{3.9}$$

Similar derivation can be applied to the remaining three types of generating functions.

4 The Hamilton–Jacobi Equation

In this section, we turn our attention to the last missing link in the Hamiltonian transformation theory; an approach for determining the transformation generating function, S. There is only one equation available for this purpose

$$H(q, p, t) + \frac{\partial S}{\partial t} = K(P, Q, t). \tag{4.1}$$

However, there are two unknown functions in this equation: S and K. Thus, the best we can do is to assume a solution for one and then solve for the other. In this regard, suppose we arbitrarily introduce the condition that K is to be *identically zero*? Under this condition, $\dot{Q}$ and $\dot{P}$ vanish; resulting in $Q = \alpha$, and $P = \beta$, constants. The inverse transformation then yields the motion $q(\alpha, \beta, t)$, $p(\alpha, \beta, t)$ in terms of these constants of integration, α and β.

Consider now generating functions of the first type. Having forced a solution on K, we must now solve the partial differential equation (PDE)

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{4.2}$$

for S, where $\frac{\partial S}{\partial q} = \left(\frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}\right)^{\mathrm{T}}$. This equation is known as the *Hamilton–Jacobi equation (HJE)*, and was improved and modified by Jacobi in 1838. For a given function $H(q, p, t)$, this is a first-order PDE in the unknown function $S(q, \alpha, t)$ which is customarily called *Hamilton's principal function*. We need a solution for this equation which depends on n arbitrary constants $\alpha_1, \alpha_2, \dots, \alpha_n$ in such a way that the Jacobian determinant of $\frac{\partial S}{\partial q_i}$ with respect to (wrt) the α_j satisfies

$$\left| \frac{\partial^2 S}{\partial q_i \partial \alpha_j} \right| \neq 0. \tag{4.3}$$

The above condition excludes the possibility in which one of the n constants α_j is additive; that is, one must have

$$S(q,\alpha,t) \neq \overline{S}(q,\alpha_1,\alpha_2,\dots,\alpha_{n-1},t)+\alpha_n. \tag{4.4}$$

A solution $S(q,\alpha,t)$ satisfying (4.3) is called a "complete solution" of the HJE (4.2), and solving the HJE is equivalent to finding the solutions of the equations of motion (2.11), (2.12). Conversely, the solution of (4.2) is nothing more than a solution of the equations (2.11), (2.12) using the method of characteristics [5, 6]. However, it is generally not simpler to solve (4.2) instead of (2.11), (2.12).

If a complete solution $S(q,\alpha,t)$ of (4.2) is known, then one has

$$\frac{\partial S}{\partial q_i} = p_i, \tag{4.5}$$

$$\frac{\partial S}{\partial \alpha_i} = -\beta_i, \quad i=1,\dots,n. \tag{4.6}$$

Since the condition (4.3) is satisfied, the second algebraic equation above may be solved for q and the first solved for $p(\alpha,\beta,t)$. One thus has a canonical transformation from (α,β) to (q,p). And it follows from the definition of canonical transformation that the inverse transformation $\alpha=\alpha(q,p,t)$, $\beta=\beta(q,p,t)$ also is canonical.

On the other hand, if the Hamiltonian is not explicitly a function of time or is independent of time, which arises in many dynamical systems of practical interest, then the solution to (4.2) can then be formulated in the form

$$S(q,\alpha,t) = -ht + W(q,\alpha) \tag{4.7}$$

with $h=h(\alpha)$. Consequently, the use of (4.7) in (4.2) yields the following PDE in W

$$H\left(q,\frac{\partial W}{\partial q}\right) = h, \tag{4.8}$$

where h is the energy constant (if the kinetic energy of the system is homogeneous quadratic, the constant equals the total energy, E). Moreover, since W does not involve time, the new and the old Hamiltonians are equal, and it follows that $K=h$. The function W, known as *Hamilton's characteristic function*, thus generates a canonical transformation in which all the new coordinates are cyclic. Further, one may choose $h=\alpha_n$ for example, so that

$$W = W(q,\alpha_1,\dots,\alpha_{n-1},h) \tag{4.9}$$

depends on $n-1$ additional arbitrary constants besides h. Noting that the Jacobian determinant of S wrt the n arbitrary coordinates, and the n constants $\alpha_1,\dots,\alpha_{n-1}$, h may not vanish, then from (4.5), (4.6) and (4.7), we have the following system

$$\begin{aligned} &\frac{\partial W}{\partial \alpha_i} = -\beta_i, \quad i=1,2\dots,n-1,\\ &\frac{\partial W}{\partial h} = t-\beta_n,\\ &\frac{\partial W}{\partial q} = p. \end{aligned} \tag{4.10}$$

where the term $t-\beta_n$ in the preceding equation follows directly from the fact that the system is autonomous. The above system of equations may be solved for $n-1$ components of q, say, for $q_1, q_2, \ldots, q_{n-1}$ resulting in

$$\begin{aligned} q_1 &= q_1(\alpha_1, \alpha_2, \ldots, \alpha_n, h, \beta_1, \beta_2, \ldots, \beta_{n-1}, q_n), \\ q_2 &= q_2(\alpha_1, \alpha_2, \ldots, \alpha_n, h, \beta_1, \beta_2, \ldots, \beta_{n-1}, q_n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ q_{n-1} &= q_{n-1}(\alpha_1, \alpha_2, \ldots, \alpha_n, h, \beta_1, \beta_2, \ldots, \beta_{n-1}, q_n), \end{aligned} \tag{4.11}$$

where the time t is replaced as the parameter q_n. These equations are then the solution for the system.

5 The Toda Lattice

The Toda lattice as a Hamiltonian system describes the motion of n particles moving in a straight line with "exponential interaction" between them. Mathematically, it is equivalent to a problem in which a single particle moves in R^n. Accordingly, let the positions of the particles at time t (in R) be $q_1(t), \ldots, q_n(t)$, respectively. We assume also that each particle has mass 1, and therefore the momentum of the i-th particle at time t is $p_i = \dot{q}_i$. Consequently, the Hamiltonian function for the *finite (or non-periodic)* lattice is defined by

$$H(q,p) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{2(q_j - q_{j+1})}. \tag{5.1}$$

Thus the canonical equations for the system are given by

$$\begin{aligned} \frac{dq_j}{dt} &= p_j \quad j = 1, \ldots, n, \\ \frac{dp_1}{dt} &= -2e^{2(q_1-q_2)}, \\ \frac{dp_j}{dt} &= -2e^{2(q_j-q_{j+1})} + 2e^{2(q_{j-1}-q_j)}, \quad j = 2, \ldots, n-1, \\ \frac{dp_n}{dt} &= 2e^{2(q_{n-1}-q_n)}. \end{aligned} \tag{5.2}$$

It may be assumed in addition that $\sum_{j=1}^{n} q_j = \sum_{j=1}^{n} p_j = 0$, and the coordinates $q_1, \ldots, q_n$ can be chosen so that this condition is satisfied. While for the periodic lattice in which the first particle interacts with the last, the Hamiltonian function is defined by

$$\widetilde{H}(q,p) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{2(q_j - q_{j+1})} + e^{2(q_n - q_1)}. \tag{5.3}$$

We may also consider the infinite lattice, in which there are infinitely many particles.

Using the inverse scattering method of solving the initial value problem for the Korteweg-de Vries equation (KdV) formulated by Lax [13], the solution for the lattice can be derived using matrix formalism which led to a simplification of the equations of motion. To introduce this formalism, define the following $(n \times n)$ matrices

$$L = \begin{pmatrix} p_1 & Q_{1,2} & 0 & \cdots & 0 & 0 \\ Q_{1,2} & p_2 & Q_{2,3} & \cdots & 0 & 0 \\ 0 & Q_{2,3} & p_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & Q_{n-1,n} \\ 0 & 0 & 0 & \cdots & Q_{n-1,n} & p_n \end{pmatrix}, \tag{5.4}$$

$$M = \begin{pmatrix} 0 & Q_{1,2} & 0 & \cdots & 0 & 0 \\ -Q_{1,2} & 0 & Q_{2,3} & \cdots & 0 & 0 \\ 0 & -Q_{2,3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & Q_{n-1,n} \\ 0 & 0 & 0 & \cdots & -Q_{n-1,n} & 0 \end{pmatrix}, \tag{5.5}$$

where $Q_{ij} = e^{(q_i - q_j)}$. We then have the following proposition [9].

Proposition 5.1 *The Hamiltonian system for the non-periodic Toda lattice (5.2) is equivalent to the Lax equation $\dot{L} = [L, M]$, where the function L, M take values in $sl(n, R)$ and $[cdot, \cdot]$ is the Lie bracket operation in $sl(n, R)$.*

Using the above matrix formalism, the solution of the Toda system (5.2) can be derived [9, 13].

Theorem 5.1 *The solution of the Hamiltonian system for the Toda lattice is given by $L(t) = Ad(\exp tV)_I^{-1} V$, where $V = L(0)$ and I represents the identity matrix.*

The can explicitly write the solution for the case of $n = 2$. Letting $q_1 = -q$, $q_2 = q$, $p_1 = -p$ and $p_2 = p$, we have

$$L = \begin{pmatrix} p & Q \\ Q & -p \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & Q \\ -Q & 0 \end{pmatrix}, \tag{5.6}$$

where $Q = c^{-2q}$. The solution of $\dot{L} = [L,\, M]$ with

$$L(0) = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix},$$

is

$$L(t) = Ad\left[\exp\, t \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}\right]_I^{-1} \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}.$$

Now

$$\exp\, t \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} = \begin{pmatrix} \cosh tv & \sinh tv \\ \sinh tv & \cosh tv \end{pmatrix},$$

and hence,

$$\left[\exp\, t \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}\right]_I^{-1} = \frac{1}{\sqrt{\sinh^2 tv + \cosh^2 tv}} \begin{pmatrix} \cosh tv & \sinh tv \\ \sinh tv & \cosh tv \end{pmatrix}.$$

Therefore,

$$L(t) = \frac{v}{\sinh^2 tv + \cosh^2 tv} \begin{pmatrix} -2\sinh\, tv \cosh\, tv & 1 \\ 1 & 2\sinh\, tv \cosh\, tv \end{pmatrix},$$

which means that

$$p(t) = -v\,\frac{\sinh\, 2tv}{\cosh\, 2tv}, \qquad Q(t) = \frac{v}{\cosh\, 2tv}.$$

Furthermore, if we recall that $Q(t) = e^{-2q(t)}$, it follows that

$$q(t) = -\frac{1}{2}\log\left(\frac{v}{\cosh\, 2tv}\right) = -\frac{1}{2}\log\, v + \frac{1}{2}\log\cosh\, 2vt. \tag{5.7}$$

6 Solving the Hamilton–Jacobi Equation

It is clear from the preceding discussion that the success of the Hamiltonian approach to mechanics depends heavily on the ability to solve the HJE. Because the prospects of success are limited by the inadequate state of the mathematical art in solving nonlinear PDEs. At present, the only technique of general utility is the method of *separation of variables*. If the Hamiltonian is explicitly a function of time, then separation of variables is not readily achieved for the HJE. However, if on the other hand, the Hamiltonian is not explicitly a function of time or is independent of time, which arises in many dynamical systems of practical interest, then the HJE (4.2) degenerates to the HJE (4.8). Nevertheless, solving this resulting HJE still remains a very difficult problem in general.

In this section we propose a parametrization approach for solving the Hamilton–Jacobi equation for a fairly large class of Hamiltonian systems, and then apply the approach to the $\mathcal{A}_2$-Toda lattice as special cases. To present the approach, let the configuration space of the class of Hamiltonian systems be a smooth n-dimensional manifold M with local coordinates $q = (q_1, \dots, q_n)$, i.e. if (φ, U) is a coordinate chart, we write $\varphi = q$ and $\dot{q}_i = \dfrac{\partial}{\partial q_i}$ in the tangent bundle $TM|_U = TU$. Further, let the class of systems under consideration be represented by Hamiltonian functions $H\colon T^\star M \to R$ of the form:

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q), \tag{6.1}$$

where $(p_1(q), \dots, p_n(q)) \in T_q^\star M$, and together with $(q_1, \dots, q_n)$ form the $2n$ symplectic coordinates for the phase-space $T^\star M$ of any system in the class, while $V\colon M \to R_+$ is the potential function which we assume to be nonseparable in the variables q_i, $i = 1, \dots, n$. The time-independent HJE corresponding to the above Hamiltonian function is given by

$$\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial W}{\partial q_i}\right)^2 + V(q) = h, \tag{6.2}$$

where $W\colon M \to R$ is the Hamilton's characteristic function for the system.

We then have the following theorem concerning the solution of this HJE.

Theorem 6.1 *Let M be an open subset of R^n which is simply connected and let $q = (q_1, \dots, q_n)$ be the coordinates on M. Suppose ρ, $\theta_i \colon M \to R$ for $i = 1, \dots, \lfloor \frac{n+1}{2} \rfloor$; $\theta = (\theta_1, \cdots, \theta_{\lfloor \frac{n+1}{2} \rfloor})$; and $\zeta_i \colon R \times R^{\lfloor \frac{n+1}{2} \rfloor} \to R$ are C^2 functions such that*

$$\frac{\partial \zeta_i}{\partial q_j}(\rho(q), \theta(q)) = \frac{\partial \zeta_j}{\partial q_i}(\rho(q), \theta(q)), \qquad \forall\, i, j = 1, \dots, n, \tag{6.3}$$

and

$$\frac{1}{2}\sum_{i=1}^{n} \zeta_i^2(\rho(q), \theta(q)) + V(q) = h \tag{6.4}$$

is solvable for the functions ρ, θ. Let

$$\omega^1 = \sum_{i=1}^{n} \zeta_i(\rho(q), \theta(q)) dq_i,$$

$\omega^1 \in \Omega^p rime(M)$, and suppose C is a path in M from an initial point q_0 to an arbitrary point $q \in M$. Then

(i) *ω^1 is closed;*
(ii) *ω^1 is exact;*
(iii) *if $W(q) = \int_C \omega^1$, then W satisfies the HJE (6.2).*

Proof (i)

$$d\omega^1 = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial}{\partial q_j} \zeta_i(\rho(q), \theta(q)) dq_j \wedge dq_i,$$

which by (6.3) implies $d\omega^1 = 0$; hence, ω^1 is closed.

(ii) Since by (i) ω^1 is closed, by the simple connectedness of M (Poincaré's lemma [1]), ω^1 is also exact.

(iii) By (ii) ω^1 is exact, therefore the integral $W(q) = \int_C \omega^1$ is independent of the path C. Therefore, W corresponds to a scalar function. Furthermore, $dW = \omega^1$ and $\dfrac{\partial W}{\partial q_i} = \zeta_i(\rho(q), \theta(q))$, and thus substituting in the HJE (6.2) and if (6.4) holds, then W satisfies the HJE.

In the next corollary we shall construct explicitly the functions ζ_i, $i = 1, \dots, n$, in the above theorem.

Corollary 6.1 *Assume the dimension n of the system is 2, and M, ρ, θ are as in the hypotheses of Theorem 6.1, and that conditions (6.3), (6.4) are solvable for θ and ρ. Also, define the functions ζ_i, $i = 1, 2$, postulated in the theorem by $\zeta_1(q) = \rho(q)\cos\theta(q)$, $\zeta_2(q) = \rho(q)\sin\theta(q)$. Then, if*

$$\omega^1 = \sum_{i=1}^{2} \zeta_i(\rho(q), \theta(q))\, dq_i, \qquad W = \int_C \omega^1,$$

and $q\colon [0,1] \to M$ *is a parametrization of* C *such that* $q(0) = q_0$, $q(1) = q$, *then*

(i) W *is given by*

$$W(q,h) = \gamma \int_0^1 \sqrt{(h - V(q(s))}\left[\cos\theta(q(s))q_1^{'}(s) + \sin\theta(q(s))q_2^{'}(s)\right] ds \qquad (6.5)$$

where $\gamma = \pm\sqrt{2}$ *and* $q_i' = \dfrac{dq_i(s)}{ds}$;

(ii) W *satisfies the HJE (6.2).*

Proof (i) If (6.3) is solvable for the function θ, then substituting the functions $\zeta_i(\rho(q), \theta(q))$, $i = 1, 2$ as defined above in (6.4), we get immediately

$$\rho(q) = \pm\sqrt{2(h - V(q))}.$$

Further, by Theorem 6.1, ω^1 given above is exact, and $W = \int_C \omega^1 dq$ is independent of the path C. Therefore, if we parametrize the path C by s, then the above line integral can be performed coordinate-wise with W given by (6.5) and $\gamma = \pm\sqrt{2}$.

(ii) follows from Theorem 6.1.

Remark 6.1 The above corollary constructs one explicit parametrization that may be used. However, because of the number of parameters available in the parametrization are limited, the above parametrization is only suitable for systems with $n = 2$. Other types of parametrizations that are suitable could also be employed.

If however the dimension n of the system is 3, then the following corollary gives a procedure for solving the HJEs.

Corollary 6.2 *Assume the dimension* n *of the system is 3, and* M, ρ, *are as in the hypotheses of Theorem 6.1. Let* $\zeta_i\colon R \times R \times R \to R$, $i = 1, 2, 3$, *be defined by* $\zeta_1(q) = \rho(q)\sin\theta(q)\cos\varphi(q)$, $\zeta_2(q) = \rho(q)\sin\theta(q)\sin\varphi(q)$, $\zeta_3(q) = \rho(q)\cos\theta(q)$, *and assume (6.3) are solvable for* θ *and* φ, *while (6.4) is solvable for* ρ. *Then, if*

$$\omega^1 = \sum_{i=1}^{3} \zeta_i(\rho(q), \theta, \varphi) dq_i,$$

$W = \int_C \omega^1$, *and* $q\colon [0,1] \to M$ *is a parametrization of* C *such that* $q(0) = q_0$, $q(1) = q$, *then*

(i) W *is given by*

$$W(q,h) = \gamma \int_0^1 \sqrt{(h - V(q(s)))}\Big\{\sin\theta(q(s))\cos\varphi(q(s))q_1^{'}(s) + \sin\theta(q(s))\sin\varphi(q(s))q_2^{'}(s) + \cos\theta(q(s))q_3^{'}(s)\Big\} ds, \qquad (6.6)$$

where $\gamma = \pm\sqrt{2}$;

(ii) W *satisfies the HJE (6.2).*

Proof Proof follows along the same lines as Corollary 6.1.

Remark 6.2 Notice that, the parametrization employed in the above corollary is now of a spherical nature.

The following theorem gives bounds on the solution W and its derivatives.

Theorem 6.2 *Let $N \subset M$ be the region in which the solution W of the HJE given in Corollaries 6.1 and 6.2 exists. Then if C is a path $q\colon [0,1] \to N$ in N parametrized by $s \in [0,1]$ such that $q(0) = q_0$, $q(1) = q$ we have the following bounds on the solution and its derivatives:*

(i) $\|W(q(s),h)\|_\infty \le |\gamma|\sqrt{h}\,\|q(s)\|_{\mathcal{L}_1}$;

(ii) $\left\|\dfrac{\partial W}{\partial q}\right\|_2 = |\sqrt{2}\rho(q)/\gamma|$;

(iii) $\left\|\dfrac{\partial W}{\partial q}\right\|_\infty = |\gamma|\sqrt{h}$.

Proof (i) From (6.5) or (6.6),

$$\begin{aligned}\|W(q,h)\|_\infty &\le |\gamma|\sum_{i=1}^{n}\int_0^1 \sup_{q(s)\in N}\left|\sqrt{(h-V(q))}\right|\,|q_i'(s)\,dq_i(s)|\\ &\le |\gamma|\sqrt{h}\int_0^1 (|q_1'(s)\,ds| + |q_2'(s)\,ds| + \ldots + |q_n'(s)\,ds|)\\ &\le |\gamma|\sqrt{h}\|q(s)\|_{L_1}.\end{aligned}$$

(ii) Using the definition of $\partial W/\partial q_i$ given in Corollaries 6.1 and 6.2, we have

$$\left\|\frac{\partial W}{\partial q_i}\right\|_2^2 = \sum_{i=1}^{n}\left|\frac{\partial W}{\partial q_i}\right|^2 = |\sqrt{2}\rho(q)/\gamma|^2,$$

hence the result.

(iii) Follows by taking the sup over $q \in M$ of $\partial W/\partial q_i$, $i = 1,\ldots,n$.

Furthermore, the following proposition gives regularity of the solution.

Proposition 6.1 *If the functions ρ, θ_i, $i = 1,\ldots,\lfloor\frac{n+1}{2}\rfloor$ in Theorem 6.1 and Corollaries 6.1 and 6.2, $n = 1,2$, or 3 exist and the HJE (6.2) is solvable for W, then if θ_i, $i = 1,\ldots,\lfloor\frac{n+1}{2}\rfloor$, are C^1, then W is C^2, and consequently if θ_i, $i = 1,\ldots,\lfloor\frac{n+1}{2}\rfloor$, are C^r, $r \ge 1$, then W is C^{r+1}.*

Proof From the expressions (6.5), (6.6) for W, we see that ρ is a smooth function, since V is smooth. Hence, the differentiability of W depends on the differentiability of the θ_i, $i = 1,2$, or 3. Further, it is clear that, the integration increases the differentiability of W by 1 over that of the θ_i, $i = 1,2$, or 3.

We can combine Corollaries 6.1 and 6.2 for any n in the following proposition.

Proposition 6.2 *Let M be an open subset of R^n which is simply connected and let q_0 be a fixed point in M. Suppose there exists a C^1 matrix function $\mathcal{R}\colon R^l \to SO(n,R)$ for some smooth vector function $\theta = (\theta_1,\ldots,\theta_l)$, $\theta_i\colon M \to R$, $i = 1,\ldots,l$, and a C^1 vector function $\varrho(q) = [\rho(q),\ldots,\rho(q)]$, $\rho\colon M \to R$, such that the Jacobian matrix*

$$\frac{\partial}{\partial q}\mathcal{R}(\theta(q))\varrho(q) \tag{6.7}$$

is symmetric and

$$\frac{1}{2}\langle\varrho(q)|\varrho(q)\rangle + V(q) = h. \tag{6.8}$$

Let

$$\tilde{\omega}^1 = \sum_{i=1}^{n}[\mathcal{R}(\theta(q))\varrho(q)]_i dq_i$$

and suppose C is a path from q_0 to an arbitrary point $q \in M$. Then,

(i) $\tilde{\omega}^1$ *is closed;*
(ii) $\tilde{\omega}^1$ *is exact;*
(iii) *if* $\widetilde{W}(q) = \int\limits_C \tilde{\omega}^1$, *then* $\widetilde{W}$ *satisfies the HJE (6.2).*

Proof (i)

$$d\tilde{\omega}^1 = \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial}{\partial q_j}[\mathcal{R}(\theta(q))\varrho(q)]_i dq_j \wedge dq_i$$

which by (6.7) implies that $d\tilde{\omega}^1 = 0$; hence, $\tilde{\omega}^1$ is closed.

(ii) Again by simple-connectedness of M, (i) implies (ii).

(iii) By (ii) the integral $\widetilde{W}(q) = \int_C \tilde{\omega}^1$ is independent of the path, and W corresponds to a scalar function. Moreover, if $dW = \tilde{\omega}^1$ and $\partial W/\partial q_i = [\mathcal{R}(\theta(q))\varrho(q)]_i$, then substituting in the HJE (6.2) and if (6.8) holds, then W satisfies the HJE (6.2).

If the HJE (6.2) is solvable, then the dynamics of the system evolves on the n-dimensional *Lagrangian submanifold* [1, 11] $\widetilde{N}$ which is an immersed submanifold of maximal dimension, and can be locally parametrized as the graph of the function W, i.e.,

$$\widetilde{N} = \left\{\left(q, \frac{\partial W}{\partial q}\right) : \; q \in N \subset M, \;\; W \;\; \text{is a solution of HJE (6.2)}\right\}$$

as described in Section 1. Moreover, for any other solution W' of the HJE, the volume enclosed by this surface is invariant. This is stated in the following proposition.

Proposition 6.3 *Let $N \subset M$ be the region in M where the solution W of the HJE (6.2) exists. Then, for any orientation of M, the volume form of $\widetilde{N}$*

$$\omega^n = \left(\sqrt{1 + \sum_{j=1}^{n}\left(\frac{\partial W}{\partial q_j}\right)^2}\right) dq_1 \wedge dq_2 \ldots \wedge dq_n$$

is given by

$$\omega^n = (\sqrt{1 + 2(h - V(q))})dq_1 \wedge dq_2 \ldots \wedge dq_n.$$

Proof From the HJE (6.2), we have

$$\sqrt{1+\sum_{j=1}^{n}\left(\frac{\partial W}{\partial q_j}\right)^2} = \sqrt{1+2(h-V(q)}, \quad \forall\, q \in N$$

$$\Updownarrow$$

$$\omega^n = \left(\sqrt{1+\sum_{j=1}^{n}\left(\frac{\partial W}{\partial q_j}\right)^2}\right) dq_1 \wedge \ldots \wedge dq_n = \left(\sqrt{1+2(h-V(q)}\right) dq_1 \wedge \ldots \wedge dq_n$$

$$\forall\, q \in N.$$

We now apply the above ideas to solve the HJE for the two-particle $\mathcal{A}_2$-Toda lattice. We consider the nonperiodic system described in Section 5.

6.1 Solution of the Hamilton–Jacobi equation for the $\mathcal{A}_2$-Toda system

Consider the two-particle nonperiodic Toda system (or $\mathcal{A}_2$ system) given by the Hamiltonian (5.1)

$$H(q_1,q_2,p_1,p_2) = \frac{1}{2}\,(p_1^2+p_2^2) + e^{2(q_1-q_2)}. \tag{6.9}$$

Then, the HJE corresponding to the system is given by

$$\frac{1}{2}\left\{\left(\frac{\partial W}{\partial q_1}\right)^2 + \left(\frac{\partial W}{\partial q_2}\right)^2\right\} + e^{2(q_1-q_2)} = h_2. \tag{6.10}$$

The following proposition gives the solution of the above HJE corresponding to $\mathcal{A}_2$-Toda lattice.

Proposition 6.4 *Consider the HJE (6.10) corresponding to the $\mathcal{A}_2$-Toda lattice. Then a solution to the HJE is given by*

$$\begin{aligned} W(q_1',q_2',h_2) &= \cos\frac{\pi}{4}\int_1^{q_1'} \rho(q)\,dq_1 + m\sin\frac{\pi}{4}\int_1^{q_1'} \rho(q)\,dq_1 \\ &= (1+m)\left\{\frac{\sqrt{h_2-e^{-2(b+m-1)}}-\sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2-e^{-2(b+m-1)}}}{\sqrt{h_2}}\right]}{m-1}\right. \\ &\quad\left. -\frac{\sqrt{h_2-e^{-2b-2(m-1)q_1'}}-\sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2-e^{-2b-2(m-1)q_1'}}}{\sqrt{h_2}}\right]}{m-1}\right\}, \quad q_1>q_2, \end{aligned}$$

and

$$W(q_1', q_2', h_2) = \cos\frac{\pi}{4}\int_1^{q_1'} \rho(q)\, dq_1 + m\sin\frac{\pi}{4}\int_1^{q_1'} \rho(q)\, dq_1$$

$$= (1+m)\Bigg\{\frac{\sqrt{h_2 - e^{-2(b-m+1)}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2 - e^{-2(b-m+1)}}}{\sqrt{h_2}}\right]}{m-1}$$

$$- \frac{\sqrt{h_2 - e^{-2b+2(1-m)q_1'}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2 - e^{-2b+2(1-m)q_1'}}}{\sqrt{h_2}}\right]}{m-1}\Bigg\}, \quad q_2 > q_1.$$

Furthermore, a solution for the system equations (5.2) for the $\mathcal{A}_2$ *with the symmetric initial conditions* $q_1(0) = -q_2(0)$ *and* $\dot{q}_1(0) = \dot{q}_2(0) = 0$ *is*

$$q(t) = -\frac{1}{2}\log\sqrt{h_2} + \frac{1}{2}\log[\cosh 2\sqrt{h_2}(\beta - t)] \tag{6.11}$$

where h_2 *is the energy and*

$$\beta = \frac{1}{2\sqrt{h_2}}\tanh^{-1}\left(\frac{2\dot{q}_1^2(0)}{\sqrt{2h_2}}\right).$$

Proof Applying the results of Theorem 6.1 we have

$$\frac{\partial W}{\partial q_1} = \rho(q)\cos\theta(q), \qquad \frac{\partial W}{\partial q_2} = \rho(q)\sin\theta(q)$$

and substituting in the HJE (6.10) we immediately get

$$\rho(q) = \pm\sqrt{2(h_2 - e^{2(q_1 - q_2)})}$$

and

$$\rho_{q_2}(q)\cos\theta(q) - \theta_{q_2}\rho(q)\sin\theta(q) = \rho_{q_1}(q)\sin\theta(q) + \theta_{q_1}\rho(q)\cos\theta(q). \tag{6.12}$$

The above equation (6.12) is a first-order PDE in θ and can be solved by the method of *characteristics* [5, 6]. However, the geometry of the system allows for a simpler solution. We make the simplifying assumption that θ is a constant function. Consequently, equation (6.12) becomes

$$\rho_{q_2}(q)\cos\theta = \rho_{q_1}(q)\sin\theta \implies \tan\theta = \frac{\rho_{q_2}(q)}{\rho_{q_1}(q)} = -1 \;\Rightarrow \theta = -\frac{\pi}{4}.$$

Thus,

$$p_1 = \rho(q)\cos\frac{\pi}{4}, \qquad p_2 = -\rho(q)\sin\frac{\pi}{4},$$

and integrating dW along the straightline path from $(1,-1)$ on the line

$$L\colon \quad q_2 = \frac{q_2'+1}{q_1'-1} q_1 + \left(1 + \frac{q_2'+1}{q_1'-1}\right) \stackrel{\text{def}}{=} m q_1 + b$$

(this follows from the configuration of the lattice) to some arbitrary point (q_1', q_2') we get

$$\begin{aligned} W(q_1', q_2', h_2) &= \cos\frac{\pi}{4} \int_1^{q_1'} \rho(q)\, dq_1 + m \sin\frac{\pi}{4} \int_1^{q_1'} \rho(q)\, dq_1 \\ &= (1+m) \Bigg\{ \frac{\sqrt{h_2 - e^{-2(b+m-1)}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2 - e^{-2(b+m-1)}}}{\sqrt{h_2}}\right]}{m-1} \\ &\quad - \frac{\sqrt{h_2 - e^{-2b-2(m-1)q_1'}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2 - e^{-2b-2(m-1)q_1'}}}{\sqrt{h_2}}\right]}{m-1} \Bigg\}. \end{aligned}$$

Similarly, if we integrate from point $(-1,1)$ to (q_1', q_2'), we get

$$\begin{aligned} W(q_1', q_2', h_2) &= \cos\frac{\pi}{4} \int_{-1}^{q_1'} \rho(q)\, dq_1 + m \sin\frac{\pi}{4} \int_{-1}^{q_1'} \rho(q)\, dq_1 \\ &= (1+m) \Bigg\{ \frac{\sqrt{h_2 - e^{-2(b-m+1)}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2 - e^{-2(b-m+1)}}}{\sqrt{h_2}}\right]}{m-1} \\ &\quad - \frac{\sqrt{h_2 - e^{-2b+2(1-m)q_1'}} - \sqrt{h_2}\tanh^{-1}\left[\frac{\sqrt{h_2 - e^{-2b+2(1-m)q_1'}}}{\sqrt{h_2}}\right]}{m-1} \Bigg\}. \end{aligned}$$

Finally, from (2.11) and (6.9), we can write

$$\dot q_1 = p_1 = \rho(q)\cos\frac{\pi}{4}, \tag{6.13}$$

$$\dot q_2 = p_2 = -\rho(q)\sin\frac{\pi}{4}. \tag{6.14}$$

Then $\dot q_1 + \dot q_2 = 0$ which implies that $q_1 + q_2 = k$, a constant, and by our choice of initial conditions, $k = 0$. Now integrating the above equations from $t = 0$ to t we get

$$\frac{1}{2\sqrt{h_2}}\ \tanh^{-1}\frac{\rho(q)}{\sqrt{2h_2}} = \frac{1}{2\sqrt{h_2}}\ \tanh^{-1}\frac{\rho(q(0))}{\sqrt{2h_2}} - t,$$

$$\frac{1}{2\sqrt{h_2}}\ \tanh^{-1}\frac{\rho(q)}{\sqrt{2h_2}} = \frac{1}{2\sqrt{h_2}}\ \tanh^{-1}\frac{\rho(q(0))}{\sqrt{2h_2}} - t.$$

If we let

$$\beta = \frac{1}{2\sqrt{h_2}}\ \tanh^{-1}\frac{\rho(q(0))}{\sqrt{2h_2}},$$

then upon simplification we get

$$q_1 - q_2 = \frac{1}{2}\log\left[h_2\left(1 - \tanh^2 2\sqrt{h_2}(\beta - t)\right)\right]$$
$$= \frac{1}{2}\log\left[h_2 \operatorname{sech}^2 2\sqrt{h_2}(\beta - t)\right].$$

Since $k = 0$, then $q_1 = -q_2 = -q$, and we get

$$q(t) = -\frac{1}{2}\log\sqrt{h_2} - \frac{1}{2}\log[\operatorname{sech} 2\sqrt{h_2}(\beta - t)]$$
$$= -\frac{1}{2}\log\sqrt{h_2} + \frac{1}{2}\log[\cosh 2\sqrt{h_2}(\beta - t)].$$

Now, from (6.10) and (6.13), (6.14),

$$\rho(q(0)) = \dot{q}_1^2(0) + \dot{q}_2^2(0),$$

and in particular, if $\dot{q}_1(0) = \dot{q}_2(0) = 0$, then $\beta = 0$. Therefore,

$$q(t) = -\frac{1}{2}\log\sqrt{h_2} + \frac{1}{2}\log(\cosh 2\sqrt{h_2}t)$$

which is of the form (5.7) with $v = \sqrt{h}$.

Next, we consider a more general solution to the HJE for the $\mathcal{A}_2$-Toda lattice. We try to solve the equation (6.12) under the fact that

$$p_1 + p_2 = \alpha \tag{6.15}$$

a constant, which follows from (5.2). Then, from the proceeding, the above equation implies that

$$\rho(q)\cos\theta(q) + \rho(q)\sin\theta(q) = \alpha. \tag{6.16}$$

Now suppose we seek a solution to (6.12) and (6.16) for $\theta(q)$ such that

$$\frac{\partial\theta(q)}{\partial q_1} = \frac{\partial\theta(q)}{\partial q_2}. \tag{6.17}$$

The above condition is satisfied if

$$\theta(q_1, q_2) = f(q_1 + q_2) \tag{6.18}$$

for some smooth function $f\colon R \to R$ of one variable, and

$$\frac{\partial\theta(q)}{\partial q_1} = \frac{\partial\theta(q)}{\partial q_2} = f'(q_1 + q_2), \tag{6.19}$$

where $f'(\cdot)$ is the derivative of the function with respect to its argument. Then substituting in (6.12) and using (6.16), we get

$$\rho_{q_2}(q)\cos f(q_1 + q_2) - \rho_{q_1}(q)\sin f(q_1 + q_2) = \alpha f'(q_1 + q_2) \tag{6.20}$$

which after substituting for $\rho_{q_1}(q)$ and $\rho_{q_2}(q)$ and making the change of variables $x = q_1 + q_2$, $y = q_1 - q_2$ becomes

$$\frac{\sqrt{2}e^{2y}}{\sqrt{h_2 - e^{2y}}}(\cos f(x) + \sin f(x)) = \alpha f'(x). \tag{6.21}$$

The above equation represents a first-order nonlinear ODE in the function $f(x)$, and can be integrated in this way

$$\int_0^x \frac{\sqrt{2}e^{2y}}{\sqrt{h_2 - e^{2y}}}\,dx = \int_0^x \frac{\alpha f'(x)}{(\cos f(x) + \sin f(x))}\,dx \tag{6.22}$$

to yield

$$f(x) = 2\tan^{-1}\left[\tanh\left(\frac{\sqrt{2}e^{2y}}{\alpha\sqrt{h_2 - e^{2y}}}\right) + 1\right]. \tag{6.23}$$

This implies that

$$\theta(q_1, q_2) = 2\tan^{-1}\left[\tanh\left(\frac{\sqrt{2}e^{2(q_1-q_2)}}{\alpha\sqrt{h_2 - e^{2(q_1-q_2)}}}\right) + 1\right]. \tag{6.24}$$

We can now obtain W by taking the line integral of $p_1(q) = \rho(q)\cos\theta(q)$ and $p_2 = \rho(q)\sin\theta(q)$ along the straightline path from $(1, -1)$ on the line

$$L\colon \quad q_2 = \frac{q_2' + 1}{q_1' - 1}q_1 - \left(1 + \frac{q_2' + 1}{q_1' - 1}\right) \stackrel{\text{def}}{=} mq_1 + b$$

to some arbitrary point (q_1', q_2') for $q_1 > q_2$ and from $(-1, 1)$ to (q_1', q_2') for $q_2 > q_1$. Hence we have

$$W(q, \alpha, h_2) = \int_L [\rho(q)\cos\theta(q) + m\rho(q)\sin\theta(q)]\,dq_1. \tag{6.25}$$

Using the half-angle formula, we can write

$$T(q_1) \stackrel{\text{def}}{=} \tan\frac{\theta(q_1)}{2} = \tanh\left(\frac{\sqrt{2}e^{2q_1(1-m)-b}}{\alpha\sqrt{h_2 - e^{2q_1(1-m)-b}}}\right) + 1, \tag{6.26}$$

$$\cos\theta(q_1) = \frac{1 - T^2(q_1)}{1 + T^2(q_1)}, \tag{6.27}$$

$$\sin\theta(q_1) = \frac{2T(q_1)}{1 + T^2(q_1)}. \tag{6.28}$$

Therefore,

$$\begin{aligned} W(q, \alpha, h_2) &= \int_1^{q_1'} \sqrt{2(h_2 - e^{2x(1-m)-b})}\left(\frac{1 - T^2(x)}{1 + T^2(x)} + m\frac{2T(x)}{1 + T^2(x)}\right)dx \\ &= \int_1^{q_1'} \sqrt{2(h_2 - e^{2x(1-m)-b})}\left(\frac{1 - 2mT(x) - T^2(x)}{1 + T^2(x)}\right)dx \quad \text{for} \quad q_1 > q_2 \end{aligned}$$

and

$$W(q,\alpha,h_2) = \int_{-1}^{q_1'} \sqrt{2(h_2 - e^{2x(1-m)-b})}\left(\frac{1-2mT(x)-T^2(x)}{1+T^2(x)}\right)dx \quad \text{for} \quad q_2 > q_1.$$

Unfortunately the above integrals cannot be computed in closed-form.

7 Conclusion

In this paper, we have presented a review of Hamilton–Jacobi theory and a new approach for solving the HJE for a fairly large class of Hamiltonian systems in which the variables may not be separable. The approach can also be extended to the case in which the Hamiltonian is not time-independent, and relies on finding a parametrization that allows for the equation to be solved.

The approach has been applied to the $\mathcal{A}_2$-Toda lattice, and computational results have been presented to show the usefulness of the method. It has been shown that, for the two-particle non-periodic $\mathcal{A}_2$-Toda system, the HJE can be completely integrated as expected to obtain the characteristic function and subsequently a complete solution to the equations of motion. The approach can also be applied to a fairly large class of Hamiltonian systems.

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