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Stability of Nonautonomous Neutral Variable Delay Difference Equation

Hai-Long Xing[1], Xiao-Zhu Zhong[1], Yan Shi[2*],
Jing-Cui Liang[1] and Dong-Hua Wang[1]

[1] *School of Science, Yanshan University, Qinghuangdao 066004, China*
[2] *School of Information Science, Kyushu Tokai University, 9-1-1, Toroku, Kumamoto 862-8652, Japan*

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Abstract: This paper studies the stability of a class of nonautonomous neutral delay difference equation. The case of several variable delays is mainly considered, and the sufficient conditions of uniform stability and uniform asymptotical stability are obtained. Some results with a constant delay in the literature are extended and improved.

Keywords: *Nonautonomous; neutral difference equations; stability.*

Mathematics Subject Classification (2000): 35D05, 35E05.

1 Introduction

Consider the nonautonomous neutral variable delay difference equation

$$\Delta(x(n) - cx(n-k)) + f(n,\, x(n-l_1(n)),\, \dots\, x(n-l_m(n))) = 0, \quad n \in N, \tag{1}$$

where $c \in (-1,1)$; $k \in N$; $\{l_i(n)\}$ is a positive integer sequence and satisfies $l_i(n) \le l$, $i = 1,\dots,m$, $n \in N$; l is a given positive integer, $f(n, x_1, \dots x_m)\colon N \times R^m \to R$, and $f(n, 0, \dots 0)$ satisfies $f(n, x_1, \dots x_m) \equiv 0$, $n \in N$.

*Corresponding author: shi@ktmail.ktokai-u.ac.jp

In recent years there are lots of researches on stability of special-formed zero solution to the equation (1) (see [1−9]). In 1999, Z. Zhou and J.S. Yu studied the equation

$$\Delta(x(n) - cx(n-k)) + h(n, x(n-l)) = 0$$

where $c \in (-1, 1)$; $k \in N$; $l \in N$; $f(n, x)$: $N \times R \to R$ and $f(n, 0)$ satisfies $f(n, 0) \equiv 0$, $n \in N$, and obtained a sufficient condition of the stability and asymptotical stability for zero solution to this equation [7]. It will be more practical for the fact that if the function $f(n, x)$ is replaced by function $f(n, x_1, \dots x_m)$ and the constant delay is replaced by the variable delay. Based on the above-mentioned consideration, we studied the stability of equation (1) and discovered that the concerned conclusion can be extended to the more general equation (1) and obtained a sufficient condition of the stability and asymptotical stability of equation (1).

For simplicity, the basic conceptions and symbols which occur in the paper will be introduced as follows: "Δ" stands for the forward difference operator, say, $\Delta y(n) = y(n+1) - y(n)$; Z is the integer number set; R is the real number set. Suppose that $a \in Z$, let $N(a) = \{a, a+1, \dots\}$, $N = N(0)$. For any given $a, b \in Z$ and $a \le b$, let $N(a, b) = \{a, a+1, \dots, b\}$.

Definition 1.1 Sequence $\{x(n)\}$ is said to be *the solution of equation* (1) if for a certain $n_0 \in N$, the sequence is defined on the $N(n_0 - r)$, where $r = \max\{l, k\}$ and satisfies equation (1). Obviously, equation (1) has zero solution permanently.

Definition 1.2 If for any $\varepsilon > 0$ and $n_0 \in N$, there exists a $\delta(\varepsilon, n_0) > 0$, such that when $|x(n_0 + j)| < \delta$, $j \in N(-r, 0)$, the solution of equation (1) satisfies $|x(n)| < \varepsilon$, $n \in N(n_0)$, then the zero solution of equation (1) is said to be stable. If δ can be chosen independent of n_0, then the zero solution of equation (1) is said to be uniformly stable.

Definition 1.3 The *zero solution* of equation (1) is said to be *attractive*, if for any $n_0 \in N$, there exists a $\delta(\varepsilon, n_0) > 0$, such that when $\|x(n_0 + j)| < \delta$, $j \in N(-r, 0)$, the solution of equation (1) satisfies $\lim\limits_{n \to +\infty} x(n) = 0$, then the zero solution of equation (1) is said to be attractive. If δ can be chosen independent of n_0, the zero solution of equation (1) is said to be uniformly attractive.

Definition 1.4 The *zero solution* of equation (1) is said to be *uniformly asymptotically stable*, if its zero solution is uniformly stable and uniformly attractive.

Let

$$n - \alpha(n) = \min\{n - l_i(n) \colon x(n - l_i(n)) = \max\{x(n - l_1(n)), \dots, x(n - l_m(n))\}\}, \quad (2)$$

$$n - \beta(n) = \min\{n - l_i(n) \colon x(n - l_i(n)) = \min\{x(n - l_1(n)), \dots, x(n - l_m(n))\}\}, \quad (3)$$

S is a real number sequence, for any $x = \{x(1), \dots, x(n), \dots\} \in S$, let $\|x\| = \sup\{|x(i)|\}$, for a given $H > 0$, denote

$$S_H = \{x \in S \colon \|x\| < H\}. \quad (4)$$

If $m > n$, we assume that $C_n^m = 0$.

2 Main Results and Proofs

Theorem 2.1 *Suppose that there exists a nonnegative real number sequence* $\{p(n)\}$, *such that*

(1) *for positive constant* H *and any* $x \in S_H$, *when* $n \in N$, *we have*

$$p(n)x(n-\beta(n)) \le f(n,\, x(n-l_1(n)), \dots x(n-l_m(n))) \le p(n)x(n-\alpha(n)); \tag{5}$$

(2) *the following inequalities are satisfied*

$$2|c|(2-|c|) + \sum_{i=n-\alpha(n)}^{n} p(i) < \frac{3}{2} + \frac{(1-2|c|)^2}{2(l+1)}, \quad n \in N, \tag{6}$$

$$2|c|(2-|c|) + \sum_{i=n-\beta(n)}^{n} p(i) < \frac{3}{2} + \frac{(1-2|c|)^2}{2(l+1)}, \quad n \in N. \tag{7}$$

Then the zero solution of equation (1) is uniformly stable.

Theorem 2.2 *Suppose that there exists a nonnegative real number sequence* $\{p(n)\}$, *such that*

(1) *for positive constant* H *and any* $x \in S_H$, *when* $n \in N$, *we have*

$$p(n)x(n-\beta(n)) \le f(n, x(n-l_1(n)), \dots, x(n-l_m(n))) \le p(n)x(n-\alpha(n)); \tag{8}$$

(2)
$$\sum_{n=1}^{+\infty} p(n) = +\infty; \tag{9}$$

(3)
$$2|c|(2-|c|) + \sum_{i=n-\alpha(n)}^{n} p(i) < \frac{3}{2} + \frac{(1-2|c|)^2}{2(l+1)}, \quad n \in N, \tag{10}$$

$$2|c|(2-|c|) + \sum_{i=n-\beta(n)}^{n} p(i) < \frac{3}{2} + \frac{(1-2|c|)^2}{2(l+1)}, \quad n \in N. \tag{11}$$

Then the zero solution of equation (1) is uniformly asymptotically stable.

Proof of Theorem 2.1 For any $0 < \varepsilon < H$, $n_0 \in N$, there is a $\delta > 0$, when the solution $\{x(n)\}$ to the equation satisfies $|x(n_0+i)| < \delta$, $i = -r, -r+1, \dots, 0$, we get

$$|x(n)| < \varepsilon, \quad n \in N(n_0). \tag{12}$$

We select

$$\delta = \frac{(1-|c|)}{(1+|c|)(2|c|+3)^{3r}}\, \varepsilon.$$

In the following, we will prove that when $n \in N(n_0+1, n_0+3r)$, (12) holds. In fact, from(1), we can see that

$$\begin{aligned} |x(n_0+1)| &= |cx(n_0+1-k) - cx(n_0-k) + x(n_0) \\ &\quad - f(n_0, x(n_0-l_1(n_0)), \dots, x(n_0-l_m(n_0)))| \\ &< (1+2|c|+p(n_0))\delta \le (2|c|+3)\delta < \varepsilon < H. \end{aligned}$$

Generally, when $i \in N(1,3r)$, we have

$$|x(n_0+i)| < (2|c|+3)^i\delta < \varepsilon < H. \tag{13}$$

In the following, we will prove that when $n \in N(n_0+3r+1)$, (12) holds. In fact, otherwise, there must be a $n_1 \in N(n_0+3r+1)$ such that $|x(n_1)| \geq \varepsilon$ and when $n \in N(n_0, n_1-1)$, such that

$$|x(n)| < \varepsilon. \tag{14}$$

Suppose $x(n_1) > 0$, we then have $x(n_1) \geq \varepsilon$. Let

$$y(n) = x(n) - cx(n-k), \quad n \in N(n_0), \tag{15}$$

then

$$y(n_1) = x(n_1) - cx(n_1-k) \geq (1-|c|)\varepsilon. \tag{16}$$

Because

$$y(n_0+3r) \leq |x(n_0+3r)| + |c||x(n_0+3r-k)| < (1+|c|)(2|c|+3)^{3r}\delta = (1-|c|)\varepsilon$$

then there is a $n^* \in N(n_0+3r+1, n_1)$, such that

$$\begin{gathered} y(n^*-1) < (1-|c|)\varepsilon, \\ y(n^*) \geq (1-|c|)\varepsilon, \end{gathered} \tag{17}$$

and when $n \in N(n^*+1,\, n_1)$, we have $y(n) \geq (1-|c|)\varepsilon$, thus we get

$$\Delta y(n^*-1) > 0. \tag{18}$$

From (6) we can see that $|c| < \dfrac{1}{2}$, such that

$$x(n^*) = y(n^*) + cx(n^*-k) \geq y(n^*) - |c|\varepsilon \geq (1-2|c|)\varepsilon. \tag{19}$$

From (5) and (18) we can see that

$$\begin{aligned} &p(n^*-1)x(n^*-1-\beta(n^*-1)) \\ &\quad \leq f(n^*-1, x(n^*-1-l_1(n^*-1)), \ldots, x(n^*-1-l_m(n^*-1))) = -\Delta y(n^*-1) < 0, \end{aligned}$$

then we have

$$x(n^*-1-\beta(n^*-1)) < 0. \tag{20}$$

Therefore from (19) and (20) we can see that there exists $n_2 \in N(n^*-\beta(n^*-1), n^*)$ and $\xi \in (0,1)$, such that $x(n_2-1) < 0$. And when $n \in N(n_2, n^*)$, we have

$$x(n) > 0, \tag{21}$$

$$x(n_2-1) + \xi(x(n_2 - x(n_2-1)) = 0, \tag{22}$$

then from (22) and (15), we get

$$-[y(n_2-1) + \xi(y(n_2 - y(n_2-1))] = -[(1-\xi)x(n_2-k-1) + \xi x(n_2-k)]c \leq |c|\varepsilon \tag{23}$$

and

$$[y(n_2-1)+\xi(y(n_2-y(n_2-1)] = [(1-\xi)x(n_2-k-1)+\xi x(n_2-k)]c \le |c|\varepsilon$$

that is

$$y(n_2-1) \le |c|\varepsilon - \xi(y(n_2-y(n_2-1)). \tag{24}$$

In the following we will prove that when $n \in N(n_0+r,\, n^*-1)$, we have

$$-x(n) \le \left(2|c| + \sum_{i=n}^{n_2-1} p(i) + \xi p(n_2-1)\right)\varepsilon. \tag{25}$$

In fact, from (21) we can see that when $n \in N(n_2,\, n^*-1)$, obviously the above inequality holds.

In the following we will prove that when $n \in N(n_0+r,\, n_2-1)$, inequality (25) holds.

From (5) we can see that when $n \in N(n_0+r)$, we have

$$\Delta y(n) \le -p(n)x(n-\beta(n)), \tag{26}$$

thus when $n \in N(n_0+r,\, n_2-1)$, we get

$$\Delta y(n) \le p(n)\varepsilon. \tag{27}$$

Then when $n \in N(n_0+r,\, n_2-1)$, we have

$$\begin{aligned} -[y(n) - y(n_2-1) - \xi(y(n_2) - y(n_2-1))] \\ = \sum_{i=n}^{n_2-2} \Delta y(i) + \xi\Delta y(n_2-1) \le \left(\sum_{i=n}^{n_2-2} p(i) + \xi p(n_2-1)\right)\xi. \end{aligned}$$

From (14) and (15), when $n \in N(n_0+r,\, n_2-1)$, we have

$$\begin{aligned} -x(n) &= -(y(n) + cx(n-k)) = -[y(n) - y(n_2-1) - \xi(y(n_2) \\ &\quad - y(n_2-1))] - [y(n_2-1) + \xi(y(n_2) - y(n_2-1))] - cx(n-k) \\ &\le \left[\sum_{i=n}^{n_2-2} p(i) + \xi p(n_2-1)\right]\varepsilon + 2|c|\varepsilon. \end{aligned}$$

Therefore, inequality (25) holds.

Suppose

$$\beta = \frac{2}{3} + \frac{(1-2|c|)^2}{2(l+1)} - 2|c(2-|c|)|. \tag{28}$$

Then from (7), we have

$$\sum_{i=n-\beta(n)}^{n} p(i) < \beta, \quad n \in N. \tag{29}$$

Let

$$d = \sum_{i=n_2}^{n^*-1} p(i) + (1-\xi)p(n_2-1). \tag{30}$$

There are two situations needed to be contemplated.

Case 1 $d \le 1 - 2|c|$.

From (24), (25) and (26), we can see that

$$\begin{aligned}
y(n^*) &= y(n_2-1) + \sum_{n=n_2-1}^{n^*-1} \Delta y(n) \le |c|\varepsilon - \xi(y(n_2) - y(n_2-1)) + \sum_{n=n_2-1}^{n^*-1} \Delta y(n) \\
&= |c|\varepsilon + (1-\xi)\Delta y(n_2-1) + \sum_{n=n_2}^{n^*-1} \Delta y(n) \le |c|\varepsilon - (1-\xi)p(n_2-1) \\
&\times x(n_2-1-\beta(n_2-1)) - \sum_{n=n_2}^{n^*-1} p(n)x(n-\beta(n)) \\
&\le |c|\varepsilon + (1-\xi)p(n_2-1)\Bigg[\sum_{i=n_2-1-\beta(n_2-1)}^{n_2-2} p(i) + \xi p(n_2-1) + 2|c|\Bigg]\varepsilon \\
&+ \sum_{n=n_2}^{n^*-1} p(n)\Bigg[\sum_{i=n-\beta(n)}^{n_2-2} p(i) + \xi p(n_2-1) + 2|c|\Bigg].\varepsilon
\end{aligned}$$

From (29) we get

$$\begin{aligned}
y(n^*) &< |c|\varepsilon + (1-\xi)p(n_2-1)[\beta - (1-\xi)p(n_2-1) + 2|c|]\varepsilon \\
&\quad + \sum_{n=n_2}^{n^*-1} p(n)\Bigg[\sum_{i=n-\beta(n)}^{n} p(i) \sum_{i=n_2}^{n} p(i) - (1-\xi)p(n_2-1) + 2|c|\Bigg]\varepsilon \\
&< |c|\varepsilon + (1-\xi)p(n_2-1)[\beta - (1-\xi)p(n_2-1) \\
&\quad + 2|c|]\varepsilon + \sum_{n=n_2}^{n^*-1} p(n)\Bigg[\beta - \sum_{i=n_2}^{n} p(i) - (1-\xi)p(n_2-1) + 2|c|\Bigg]\varepsilon.
\end{aligned}$$

From (30), we have

$$\begin{aligned}
y(n^*) &< |c|\varepsilon + \Bigg[(\beta+2|c|)d - (1-\xi)^2 p^2(n_2-1) - \sum_{n=n_2}^{n^*-1} p(n) \sum_{i=n_2}^{n} p(i) \\
&\quad - (1-\xi)p(n_2-1)\sum_{n=n_2}^{n^*-1} p(n)\Bigg]\varepsilon = |c|\varepsilon + \Bigg[(\beta+2|c|)d - (1-\xi)^2 p^2(n_2-1) \\
&\quad - \frac{1}{2}\Bigg(\sum_{n=n_2}^{n^*-1} p(n)\Bigg)^2 - \frac{1}{2}\sum_{n=n_2}^{n^*-1} p^2(n) - (1-\xi)p(n_2-1)\sum_{n=n_2}^{n^*-1} p(n)\Bigg]\varepsilon.
\end{aligned}$$

Because

$$\begin{aligned}
\sum_{n=n_2}^{n^*-1} p(n)^2 + (1-\xi)^2 p^2(n_2-1) &\ge \frac{1}{n^*-n_2+1}\Bigg(\sum_{n=n_2}^{n^*-1} p(n) + (1-\xi)p(n_2-1)\Bigg)^2 \\
&= \frac{1}{n^*-n_2+1} d^2 \ge \frac{1}{l+1} d^2
\end{aligned}$$

we have

$$y(n^*) < \left[|c| + (\beta + 2|c|)d - \left(\frac{1}{2} + \frac{1}{2(l+1)}\right)d^2\right]\varepsilon. \tag{31}$$

Because the function $g(x) = |c| + (2|c| + \beta)x - \dfrac{l+2}{2(l+1)}x^2$ is monotonously increasing on the interval $[0,\, 1 - 2|c|]$, then we have

$$\begin{aligned} y(n^*) &< \left[|c| + (\beta + 2|c|)(1 - 2|c|) - \left(\frac{1}{2} + \frac{1}{2(l+1)}\right)(1 - 2|c|)^2\right]\varepsilon \\ &\le [1 - |c| - |c|(1 - 2|c|^2] \le (1 - |c|)\varepsilon \end{aligned}$$

which contradicts inequality (17). Therefore, Case 1 is impossible.

Case 2 $d > 1 - 2|c|$.

In this case there exists a positive integer $n_3 \in N(n_2, n^*)$, which satisfies

$$2|c| + \sum_{n=n_3}^{n^*-1} p(n) \le 1 \quad \text{and} \quad 2|c| + \sum_{n=n_3-1}^{n^*-1} p(n) > 1,$$

then there is a $\eta \in (0, 1]$, such that

$$2|c| + \sum_{n=n_3}^{n^*-1} p(n) + (1-\eta)p(n_3 - 1) = 1. \tag{32}$$

Because

$$y(n^*) = y(n_2 - 1) + \sum_{n=n_2-1}^{n_3-2} \Delta y(n) + \eta\Delta y(n_3 - 1) + (1-\eta)\Delta y(n_3 - 1) + \sum_{n=n_3}^{n^*-1} \Delta y(n)$$

and making use of (24), we get

$$y(n^*) \le |c|\varepsilon + \eta\Delta y(n_3 - 1) + (1-\xi)\Delta y(n_2 - 1) + \sum_{n=n_2}^{n_3-2} \Delta y(n) + (1-\eta)\Delta y(n_3 - 1) + \sum_{n=n_3}^{n^*-1} \Delta y(n).$$

From (27), we get

$$\eta\Delta y(n_3 - 1) + (1-\xi)\Delta y(n_2 - 1) + \sum_{n=n_2}^{n_3-2} \Delta y(n) < \left[(1-\xi)p(n_2 - 1) + \sum_{n=n_2}^{n_3-2} p(n) + \eta p(n_3 - 1)\right]\varepsilon$$

and from (25) and (26), we have

$$\begin{aligned} (1-\eta)\Delta y(n_3 - 1) + \sum_{n=n_3}^{n^*-1} \Delta y(n) &\le (1-\eta)p(n_3 - 1)\Bigg[2|c| \\ &+ \sum_{i=n_3-1-\beta(n_3-1)}^{n_2-2} p(i) + \xi p(n_2 - 1)\Bigg]\varepsilon + \sum_{n=n_3}^{n^*-1} p(n)\left[2|c| + \sum_{i=n-\beta(n)}^{n_2-2} p(i) + \xi p(n_2 - 1)\right]\varepsilon. \end{aligned}$$

We then have

$$y(n^*) \le |c|\varepsilon + \left[(1-\xi)p(n_2-1) + \sum_{n=n_2}^{n_3-2} p(n) + \eta p(n_3-1)\right]\varepsilon$$
$$+ (1-\eta)p(n_3-1)\left[2|c| + \sum_{i=n_3-1-\beta(n_3-1)}^{n_2-2} p(i) + \xi p(n_2-1)\right]\varepsilon$$
$$+ \sum_{n=n_3}^{n^*-1} p(n)\left[2|c| + \sum_{i=n-\beta(n)}^{n_2-2} p(i) + \xi p(n_2-1)\right]\varepsilon.$$

From (29) and (32), we get

$$y(n^*) \le |c|\varepsilon + \left[(1-\xi)p(n_2-1) + \sum_{n=n_2}^{n_3-2} p(n) + \eta p(n_3-1)\right]\varepsilon$$
$$+ (1-2|c|)[2|c| - (1-\xi)p(n_2-1)]\varepsilon + (1-\eta)p(n_3-1)\left[\sum_{\substack{i=n_3-1-\\ \beta(n_3-1)}}^{n_3-1} p(i) - \sum_{i=n_2}^{n_3-1} p(i)\right]\varepsilon$$
$$+ \sum_{n=n_3}^{n^*-1} p(n)\left[\sum_{i=n-\beta(n)}^{n} p(i) - \sum_{i=n_3}^{n} p(i) - \sum_{i=n_2}^{n_3-1} p(i)\right]\varepsilon$$
$$< |c|\varepsilon + 2|c|(1-2|c|)\varepsilon + 2|c|(1-\xi)p(n_2-1)\varepsilon + 2|c|\sum_{i=n_2}^{n_3-1} p(i)\varepsilon$$
$$- \varepsilon(1-\eta)p(n_3-1) + \varepsilon(1-\eta)\beta p(n_3-1) + \sum_{n=n_3}^{n^*-1} p(n)\left[\beta - \sum_{i=n_3}^{n} p(i)\right]\varepsilon$$
$$= |c|\varepsilon + 2|c|(1-2|c|)\varepsilon - 2|c|\xi p(n_2-1)\varepsilon + 2|c|\left[\sum_{i=n_2-1}^{n^*-1} p(i) - \sum_{i=n_3}^{n^*-1} p(i)\right]\varepsilon$$
$$- \varepsilon(1-\eta)p(n_3-1) + (1-2|c|)\beta\varepsilon - \frac{1}{2}\left[\sum_{i=n_3}^{n^*-1} p(i)\right]^2\varepsilon - \frac{1}{2}\sum_{i=n_3}^{n^*-1} p^2(i)\varepsilon.$$

Because

$$-2|c|\sum_{i=n_3}^{n^*-1} p(i) - (1-\eta)p(n_3-1) - \frac{1}{2}\left[\sum_{i=n_3}^{n^*-1} p(i)\right]^2 - \frac{1}{2}\sum_{i=n_3}^{n^*-1} p^2(i)$$
$$= -2|c|(1-2|c| - (1-\eta)p(n_3-1)) - (1-\eta)p(n_3-1)$$
$$- \frac{1}{2}[1-2|c| - (1-\eta)p(n_3-1)]^2 - \frac{1}{2}\sum_{i=n_3}^{n^*-1} p^2(i)$$
$$= -2|c|(1-2|c|) - \frac{1}{2}(1-2|c|)^2 - \frac{1}{2}\left[\sum_{i=n_3}^{n^*-1} p^2(i) + (1-\eta)^2 p^2(n_3-1)\right]$$

and

$$\sum_{i=n_3}^{n^*-1} p^2(i) + (1-\eta)^2 p^2(n_3-1) \geq \frac{1}{n^*-n_3}\left[\sum_{i=n_3}^{n^*-1} p(i) + (1-\eta)p(n_3-1)\right]^2$$
$$\geq \frac{1}{l+1}\left[\sum_{i=n_3}^{n^*-1} p(i) + (1-\eta)p(n_3-1)\right]^2$$

we get

$$y(n^*) < \varepsilon[|c| + 2|c|(1-2|c|) + 2|c|\beta + (1-2|c|)\beta - 2|c|(1-2|c|)$$
$$- \frac{1}{2}(1-2|c|)^2 - \frac{1}{2}\frac{(1-2|c|)^2}{(l+1)}] = (1-|c|)\varepsilon.$$

This inequality contradicts (17). Therefore Case 2 is also impossible.

Based on the above two cases, we see that (12) holds. Hence the zero solution of equation (1) is uniformly stable.

Proof of Theorem 2.2 From Theorem 2.1, we see that the zero solution of equation (1) is uniformly stable, thus we only need to prove that the zero solution of equation (1) is uniformly attractive.

Select

$$\delta = \frac{(1-|c|)}{(1+|c|)(2|c|+3)^{3r}}\,H.$$

In the following, we prove that for any $n_0 \in N$, if the solution $\{x(n)\}$ of the equation satisfies $|x(n_0+i)| < \delta$, $i = -r, -r+1, \dots, 0$, we have

$$\lim_{n\to+\infty} x(n) = 0. \tag{33}$$

The following proof is similar to that of Theorem 2.1, so we have

$$x(n)| < H, \quad n \in N(n_0). \tag{34}$$

Let

$$y(n) = x(n) - cx(n-k), \quad n \in N(n_0), \tag{35}$$

then

$$|y(n)| < (1+|c|)H, \quad n \in N(n_0). \tag{36}$$

There are two situations that needed to be contemplated.

Case 1 $\{y(n)\}$ is eventually monotonous.

Let

$$A = \lim_{n\to+\infty}\inf x(n), \quad B = \lim_{n\to+\infty}\sup x(n). \tag{37}$$

We will prove that $A = B = 0$ and $A \leq 0$.

In fact, if $A > 0$, then for any $0 < \varepsilon < A$, there is $n_1 \in N(n_0+l)$, such that

$$x(n_1 - l) > A - \varepsilon > 0.$$

Hence, when $n \in N(n_1 - l)$, we have

$$x(n) > A - \varepsilon. \tag{38}$$

Therefore from (35), we get

$$\begin{aligned} y(n_1) - y(n_1+1) &= f(n_1, x(n_1 - l_1(n)), \dots, x(n_1 - l_m(n))) \\ &\geq p(n_1)x(n_1 - \beta(n_1)) > p(n_1)(A - \varepsilon). \end{aligned}$$

In general, for $m = 0, 1, \dots,$ we have

$$y(n_1 + m) - y(n_1 + m + 1) > p(n_1 + m)(A - \varepsilon).$$

Then we have

$$y(n_1) - y(n_1 + m + 1) > \sum_{i=0}^{m} p(n_1 + i)(A - \varepsilon).$$

From (36) and $\{y(n)\}$ being eventually monotonous, we can see that the limit value of $\{y(n)\}$ exists. Therefore from (9), we know that the above inequality doesn't hold and hence $A \leq 0$.

In the following we will prove $A = 0$. Suppose

$$\lim_{n \to +\infty} y(n) = y^*.$$

We will prove that

$$y^* = 0. \tag{39}$$

In fact, if (39) doesn't hold, we assume that $y^* > 0$, from the definition of A. We can see that there is a positive integer sequence $\{n_j\}$, such that

$$\lim_{j \to +\infty} n_j = +\infty, \quad \lim_{n \to +\infty} x(n_j) = A,$$

then when $j \to +\infty$, we have

$$cx(n_j - k) = x(n_j) - y(n_j) \to A - y^*, \tag{40}$$

and since

$$\lim_{n \to +\infty} f(n_1, x(n_1 - l_1(n), \dots, x(n_1 - l_m(n))) = \lim_{n \to +\infty} (-\Delta y(n)) = 0 \tag{41}$$

from(40), we see that there must exist $c \neq 0$.

If $c = 0$, we must have

$$\lim_{j \to +\infty} cx(n_j - k) = 0 = A - y^*$$

that is $A = y^*$, which obviously doesn't hold.

Hence

$$\lim_{j \to +\infty} cx(n_j - k) = \frac{A - y^*}{c}. \tag{42}$$

From the definitions of A and B, we see that

$$A = \lim_{n\to+\infty} \inf x(n) \le \lim_{j\to+\infty} x(n_j - k) = \frac{A - y^*}{c} \le \lim_{n\to+\infty} \sup x(n) = B. \tag{43}$$

If $c > 0$, from $A < (A - y^*)/c$ we have $(1 - c)A > y^*$, then we see that the inequality doesn't hold.

If $c < 0$, from $x(n) = y(n) + cx(n - k), n \in N(n_0)$, we get

$$\lim_{n\to+\infty} \sup x(n) = \lim_{n\to+\infty} \sup(y(n) + cx(n - k)),$$

then $B = y^* + cB$.

From (43), we can see that $cA \ge cB$, then we have $B \le y^*+cA$. Since $B \ge (A-y^*)/c$, we have $(1 + c)y^* \le (1 - c^2)A$ which can not hold. Therefore (39) must hold. Hence, $A = y^* + cA = cA$, that is $(1 - c)A = 0$ or $A = 0$.

In the following we will prove $B = 0$.

In fact, according to the definition of B, we can see that there is a positive integer sequence $\{l_j\}$, such that

$$\lim_{j\to+\infty} l_j = +\infty \quad \text{and} \quad \lim_{j\to+\infty} x(l_j) = B.$$

If $c = 0$, obviously, we get $B = 0$. If $c < 0$, while $j \to +\infty$, we get

$$y(l_j) - y(l_j - k) = x(l_j) - (1 + c)x(l_j - k) + cx(l_j - 2k) \to 0,$$

then for $j \to +\infty$, we have

$$(1 + c)x(l_j - k) - cx(l_j - k) \to B.$$

Since the line $(1 + c)x - cy = B$, $c > 0$ and the region $0 \le x, y \le B$ only have one crossover point (B, B), so

$$\lim_{j\to+\infty} x(l_j - k) = \lim_{j\to+\infty} x(l_j - 2k) = B.$$

Therefore

$$\lim_{j\to+\infty} y(l_j - k) = (1 - c)B = 0,$$

that is $B = 0$.

If $c > 0$, we can similarly prove that $B = 0$.

In conclusion, if $\{y(n)\}$ is eventually monotonous, then

$$\lim_{n\to+\infty} \inf x(n) = \lim_{n\to+\infty} \sup x(n) = 0,$$

that is

$$\lim_{n\to+\infty} x(n) = 0.$$

Case 2 $\{y(n)\}$ is not eventually monotonous. Let

$$M = \lim_{n\to+\infty} \sup |x(n)|, \quad N = \lim_{n\to+\infty} \sup |y(n)|.$$

If (33) doesn't hold, we must have $M > 0$. Then for any $\varepsilon > 0$, and ε satisfies $\varepsilon < \dfrac{1-2|c|}{1+|c|}M$, $\varepsilon < I$, there must exist a $n_2 \in N(n_0 + r)$, such that when $n \in N(n_2 - r)$, we have

$$|x(n)| < M + \varepsilon. \tag{44}$$

Therefore, when $n \in N(n_2)$, we get

$$y(n) \geq |x(n)| - |c|(M + \varepsilon_1) \tag{45}$$

and we have $I \geq M - |c|(M + \varepsilon_1)$. Because of the arbitrariness of ε, we have

$$I \geq (1 - |c|)M. \tag{46}$$

Since $\{y(n)\}$ is not eventually monotonous, for the above ε, there must exist a $n^* \in N(n_2 + 2r + 1)$, which satisfies

$$y(n^*) > I - \varepsilon, \tag{47}$$

such that

$$y(n^*) > y(n^* + 1), \quad y(n^*) \geq y(n^* - 1). \tag{48}$$

Therefore, we have

$$x(n^*) = y(n^*) - cx(n^* - k) \geq I - \varepsilon - |c|(M + \varepsilon) \geq (1 - |c|)M - \varepsilon - |c|(M + \varepsilon) > 0$$

and

$$x(n^* - 1 - \beta(n^* - 1)) \leq 0. \tag{49}$$

Thus there must be a $n_3 \in N(n^* - \beta(n^* - 1), n^*)$ and a $\xi \in [0, 1)$, such that

$$x(n_3 - 1) \leq 0, \quad x(n) > 0, \quad \text{where} \quad n \in N(n_3, n^*), \tag{50}$$

$$x(n_3 - 1) + \xi(x(n_3) - x(n_3 - 1)) = 0. \tag{51}$$

Then from (35) and (44), we have

$$-[y(n_3 - 1) + (y(n_3) - y(n_3 - 1))] = c[(1 - \xi)x(n_3 - 1 - k) + \xi x(n_3 - 1)] < |c|(M + \varepsilon),$$
$$y(n_3 - 1) + \xi(y(n_3) - y(n_3 - 1)) = -c[(1 - \xi)x(n_3 - 1 - k) + \xi x(n_3 - 1)] < |c|(M + \varepsilon). \tag{52}$$

That is

$$y(n_3 - 1) < |c|(M + \varepsilon) - \xi \Delta y(n_3 - 1). \tag{53}$$

Now we will prove that, when $n \in N(n_2, n^*)$, we have

$$-x(n) \leq \left[2|c| + \sum_{i=n}^{n_3-2} p(i) + \xi p(n_3 - 1)\right](M + \varepsilon). \tag{54}$$

In fact, when $n \in N(n_3, n^*)$, from (50) we can see that the above equality holds. In the following, we will prove that, when $n \in N(n_2,\, n_3 - 1)$, (54) holds. From (1) and (6), we see that, when $n \in N(n_2)$, we have

$$\Delta y(n) < -p(n)x(n - \beta(n)). \tag{55}$$

Thus, from (44) we see that, when $n \in N(n_2)$, we have

$$\Delta y(n) < p(n)(M+\varepsilon). \tag{56}$$

Therefore, for any $n \in N(n_2, n_3-1)$, we have

$$\begin{aligned} -[y(n) - y(n_3-1) - \xi(y(n_3) - y(n_3-1))] &= \sum_{i=n}^{n_3-2} \Delta y(i) + \xi \Delta y(n_3-1) \\ &< -\sum_{i=n}^{n_3-2} p(i)x(i-\beta(i)) - \xi p(n_3-1)x(n_3-1-\beta(n_3-1)) \\ &\le \Big[\sum_{i=n}^{n_3-2} p(i) + \xi p(n_3-1)\Big](M+\varepsilon). \end{aligned} \tag{57}$$

Then from (35), (44), (52) and (57), we know that if $n \in N(n_2)$, we have

$$\begin{aligned} -x(n) = -(y(n) + cx(n-k)) &= -[y(n) - y(n_3-1) + \xi(y(n_3) - y(n_3-1))] \\ &\quad - y(n_3-1) - \xi(y(n_3) - y(n_3-1)) - cx(n-k) \\ &\le \Big[\sum_{i=n}^{n_3-2} p(i) + \xi p(n_3-1) + 2|c|\Big](M+\varepsilon). \end{aligned}$$

Therefore (54) holds.

Suppose

$$\beta = \frac{3}{2} + \frac{(1-2|c|)^2}{2(l+1)} - 2|c|(2-|c|).$$

Then from (11), we have

$$\sum_{i=n-\beta(n)}^{n} p(i) \le \beta, \quad n \in N. \tag{58}$$

Denote

$$d = \sum_{i=n_3}^{n^*-1} p(i) + (1-\xi)p(n_3-1). \tag{59}$$

In the following, we have two situations to contemplate.

Case 2-a $d \le 1 - 2|c|$.

From (53), we obtain

$$y(n^*) = y(n_3-1) + \sum_{i=n_3-1}^{n^*-1} \Delta y(i) \le |c|(M+\varepsilon) - \xi \Delta y(n_3-1) + \sum_{i=n_3-1}^{n^*-1} \Delta y(i).$$

From (54) and (55), we get

$$\begin{aligned} y(n^*) &\le |c|(M+\varepsilon) + (1-\xi)\Delta y(n_3-1) + \sum_{i=n_3}^{n^*-1} \Delta y(i) \\ &< |c|(M+\varepsilon) + (1-\xi)p(n_3-1)\Bigg[\sum_{\substack{i=n_3-1-\\ \beta(n_3-1)}}^{n_3-2} p(i) + \xi p(n_3-1) + 2|c|\Bigg](M+\varepsilon) \\ &+ \sum_{i=n_3}^{n^*-1} p(i)\Bigg[\sum_{j=i-\beta(i)}^{n_3-2} p(j) + \xi p(n_3-1) + 2|c|\Bigg](M+\varepsilon). \end{aligned}$$

The following proof is similar to Case 1 of Theorem 2.1, we have

$$y(n^*) < (1-|c|)(M+\varepsilon).$$

Case 2-b $d > 1-2|c|$.

Now there exists a positive integer $n_4 \in N(n_3, n^*)$, such that

$$2|c| + \sum_{i=n_4}^{n^*-1} p(i) < 1 \quad \text{and} \quad 2|c| + \sum_{i=n_4-1}^{n^*-1} p(i) > 1.$$

Therefore there is a $\eta \in (0,1)$, such that

$$2|c| + \sum_{i=n_4}^{n^*-1} p(i) + (1-\eta)p(n_4-1) = 1. \tag{60}$$

Since

$$y(n^*) = y(n_3+1) + \sum_{n=n_4-1}^{n4-2} \Delta y(n) + \eta\Delta y(n_4-1) + (1-\eta)\Delta y(n_4-1) + \sum_{n=n_4}^{n^*-1} \Delta y(n),$$

then from (53), (54) and (56), we obtain

$$\begin{aligned} y(n^*) &< |c|(M+\varepsilon)(1-\xi)\Delta y(n_3-1) + \sum_{n=n_3-2}^{n_4-2} \Delta y(n) + \eta\Delta y(n_4-1) \\ &+ (1+\eta)\Delta y(n_4-1) + \sum_{n=n_4}^{n^*-1} \Delta y(n) \\ &\le |c|(M+\varepsilon) + \Bigg[(1-\xi)p(n_3-1) + \sum_{n=n_3-2}^{n_4-2} p(n) + \eta p(n_4-1)\Bigg](M+\varepsilon) \\ &+ (1-\eta)p(n_4-1)\Bigg[2|c| + \sum_{i=n_4-1-\beta(n_4-1)}^{n_3-2} p(i) + \xi p(n_3-1)\Bigg](M+\varepsilon) \end{aligned}$$

$$+\sum_{n=n_4}^{n^*-1} p(n)\Big[2|c| + \sum_{i=n-\beta(n)}^{n_3-2} p(i) + \xi p(n_3-1)\Big](M+\varepsilon)$$

$$= |c|(M+\varepsilon) + \Big[(1-\xi)p(n_3-1) + \sum_{n=n_3-2}^{n_4-2} p(n) + \eta p(n_4-1)\Big](M+\varepsilon)$$

$$+ (1-2|c|)[2|c| - (1-\xi)p(n_3-1)](M+\varepsilon)$$

$$+ (1-\eta)p(n_4-1)\Big[\sum_{i=n_4-1-\beta(n_4-1)}^{n_4-1} p(i) - \sum_{i=n_3}^{n_4-1} p(i)\Big](M+\varepsilon)$$

$$+ \sum_{n=n_4}^{n^*-1} p(n)\Big[\sum_{i=n-\beta(n)}^{n} p(i) - \sum_{i=n_4}^{n} p(i) - \sum_{i=n_3}^{n_4-1} p(i)\Big](M+\varepsilon).$$

The following proof is similar to Theorem 2.1. We have

$$y(n^*) < (1-|c|)(M+\varepsilon).$$

Based on the two cases a and b, we have

$$y(n^*) < (1-|c|)(M+\varepsilon).$$

Hence, from (47), we have

$$I - \varepsilon < y(n^*) < (1-|c|)(M+\varepsilon).$$

From the arbitrariness of ε, we have

$$I < (1-|c|)M,$$

which contradicts (46). Therefore Case 2 is impossible. Thus when $\{y(n)\}$ is not eventually monotonous, (33) also holds.

Based on these two cases, we can see that (33) must hold. Thus the zero solution of the equation is uniformly attractive. Therefore the zero solution of equation (1) is said to be uniformly asymptotically stable.

3 Conclusions

According to the above analysis, in the cases of several variable delay, we have obtained the sufficient conditions of uniform stability and uniform asymptotical stability. These results extent and improve the relative theorem in the literature [7]. And the methods used in this paper can have important significances in the studies of the stabilities of difference equation with several variable delays.

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