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Adaptive Output Control of a Class of Time-Varying Uncertain Nonlinear Systems

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Abstract: In this paper, we present a new scheme to design adaptive controllers for single-input single-output uncertain time-varying systems in the presence of unknown bounded disturbances. No knowledge is assumed on the sign of the term multiplying the control. The control design is achieved by introducing certain well defined functions, estimating variation rates of parameters and incorporating a Nussbaum gain. To overcome the problem of overparametrization, tuning functions, which are different from the standard ones due to the use of projection operations, are employed. It is shown that the proposed controller can guarantee global uniform ultimate boundedness.

Keywords: *Adaptive control; backstepping; time-varying systems; tuning functions; Nussbaum gain.*

Mathematics Subject Classification (2000): 93C40.

1 Introduction

Adaptive control has seen significant development since the appearance of a Lyapunov-based recursive design procedure known as backstepping [7]. A great deal of attention has been paid to tackle both linear and nonlinear systems with unknown parameters and a number of results have been obtained in [1 – 6]. However, only limited number of results are available for nonlinear systems with time-varying parameters and/or without the knowledge on the sign of the term multiplying the control, i.e. high frequency gain in the case of linear systems, in the presence of external disturbances. In this paper, we shall also call this term the high frequency gain for nonlinear systems for simplicity.

In [9], output feedback control was considered for linear time-varying systems when the sign of high-frequency gain is known. In [11], the problem of adaptive control with

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unknown sign of high-frequency gain for linear time invariant systems was studied. In [2], Nussbaum gain incorporated with the backstepping technique was used to design adaptive output stabilizer for high order uncertain time invariant nonlinear systems with unknown sign of high-frequency gain in the absence of external disturbances. The nonlinearities considered should satisfy sector conditions. In [3], disturbance decoupling was addressed for nonlinear time invariant systems with known sign of the high frequency gain. The result obtained is critically depending on a function of the system output y and the reference trajectory y_r. It should be noted that such a function is undefined at the time instants when $y = y_r$. Therefore, the control signal is undefined at these time instants. In [4], a flat zone was used to handle the problem of nonlinear time invariant systems with unknown sign of high frequency gain in the presence of disturbances. The bound of the disturbance and all the unknown parameters need to be estimated at every step in the backstepping process. This results in the problem of overparametrization and makes the implementation complicated. In [6] state-feedback control was considered for a class of uncertain time-varying nonlinear systems in the presence of disturbances. Due to state feedback, no filter is required for state estimation. Thus the derivatives of the time varying parameters and the term of the disturbance need not to be considered in controller design. This also makes the stability analysis greatly simplified. Again, parameters are required to be estimated at every step, which results in overparametrization. In the case of output feedback control of nonlinear time-varying systems in the presence of disturbances, no result is available. In this case, filters similar to [7] are required to estimate system states and the state equations of the state estimation error will be used in the design and analysis. In these equations, the external disturbances and derivatives of time-varying parameters will appear and have great impact on the errors. This makes the design and analysis quite difficult, especially when the sign of high frequency gain is unknown and tuning functions are used.

In this paper, we consider such a case and propose a new control design scheme to solve the problem. The nonlinearities considered are not required to satisfy the sector type of conditions like [2]. To handle the disturbances, well defined functions are introduced to eliminate their effects in the Lyapunov functions employed in the recursive design steps. To deal with the time variation problem, an estimator is used to estimate the bound of the variation rates. Furthermore, the overparametrization problem is also solved by using the concept of tuning functions. As projection operation is used, the design of tuning functions are different from existing schemes as in [7]. With our proposed controller, global system stability is ensured.

2 System Model and Problem Formulation

Consider the following class of single-input-single-output (SISO) nonlinear time-varying systems in the feedback form

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \theta_{a1}(t)\psi_{a1}(y) + d_1(t)\phi_{a1}(y) + \psi_{01}(y), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{x}_{\rho-1} &= x_\rho + \theta_{a\rho-1}(t)\psi_{a\rho-1}(y) + d_{\rho-1}(t)\phi_{a\rho-1}(y) + \psi_{0\rho-1}(y), \\
\dot{x}_\rho &= x_{\rho+1} + \theta_{a\rho}(t)\psi_{a\rho}(y) + d_\rho(t)\phi_{a\rho}(y) + \psi_{0\rho}(y) + b_m(t)u, \qquad (1) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

$$\dot{x}_n = \theta_{an}(t)\psi_{an}(y) + d_n(t)\phi_{an}(y) + \psi_{0n}(y) + b_0(t)u,$$
$$y = e_1^{\mathrm{T}} x,$$

where $x = [x_1, \cdots, x_n]^{\mathrm{T}} \in R^n$, $u \in R$ and $y \in R$ are system states, input and output respectively, $b_i(t)$, $i = m, \ldots, 0$, are bounded uncertain time-varying piecewise continuous high-frequency gains, $\theta_{ai}(t) \in R^{p_i}$ are uncertain time-varying parameters, $d_i(t)$, $i = 1, \ldots, n$, denote unknown time-varying bounded disturbances, ψ_{ai} and ϕ_{ai} are known smooth nonlinear functions in R^n. Similar class of systems was analyzed in [8].

In order to cope with the unknown sign of high-frequency gain, the Nussbaum gain technique is employed in this paper. A function $N(\chi)$ is called a Nussbaum-type function if it has the following properties [10]

$$\lim_{s\to\infty} \sup \frac{1}{s} \int_0^s N(\chi)d\chi = \infty, \tag{2}$$

$$\lim_{s\to\infty} \inf \frac{1}{s} \int_0^s N(\chi)d\chi = -\infty. \tag{3}$$

In this paper, the even Nussbaum function $\exp(\chi^2)\cos(\frac{\pi}{2}\chi)$ is exploited. As in [6] the following Lemma will be employed in later analysis.

Lemma 1 *Let $V(t)$ and $\chi(t)$ be a smooth function defined on $[0, t_f)$ with $V(t) \geq 0$, $\forall\, t \in [0, t_f)$, and $N(\chi) = \exp(\chi^2)\cos(\frac{\pi}{2}\chi)$ be an even smooth Nussbaum-type function. If the following inequality holds:*

$$V(t) \leq f_0 + e^{-f_1 t} \int_0^t g_1 N(\chi)\dot{\chi}d\tau + e^{-f_1 t} \int_0^t \dot{\chi}(t)e^{f_1 \tau}d\tau \tag{4}$$

where constant $f_1 > 0$, g_1 is a parameter which takes values in the unknown closed intervals $I_1 = [l_1^-, l_1^+]$ with $0 \notin I_1$, and f_0 represents some suitable constant, then $V(t)$, $\chi(t)$ and $\int_0^t g_1 N(\chi)\dot{\chi}d\tau$ must be bounded on $[0, t_f)$.

For the considered system (1), the following assumptions are imposed.

Assumption 1 The uncertain parameter vector θ is inside a compact set Ω_θ, where $\theta = [b_m(t), \ldots, b_0(t), \theta_{a1}(t), \ldots, \theta_{an}(t)]^{\mathrm{T}}$. In addition, there exists an unknown bounded positive constant q so that $q \geq \|\dot{\theta}\|$. Also q is inside a compact intervals $\Omega_q = [I^-, I^+]$ and $b_m(t) \neq 0, \forall\, t$.

Assumption 2 The relative degree ρ is fixed and known. This is ensured by Assumption 1.

Assumption 3 The reference signal y_r and its $(\rho - 1)$-th order derivatives are also assumed to be known and bounded.

Assumption 4 The system is minimum phase in the sense defined in [8].

In order to design the desired adaptive control law with output via backstepping procedures, we now transform system (1) into the following form

$$\dot{x} = Ax + F(y,u)^{\mathrm{T}}\theta + \Phi_a(y)d(t)^{\mathrm{T}} + \psi_0(y) \tag{5}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{6}$$

$$F(y,u)^{\mathrm{T}} = \left[\begin{bmatrix} 0_{(\rho-1)\times(m+1)} \\ I_{m+1} \end{bmatrix} u, \quad \Psi_a(y) \right], \tag{7}$$

$$\Psi_a(y) = \begin{bmatrix} \psi_{a1}^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & \psi_{a2}^{\mathrm{T}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \psi_{an}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \Psi_{a1}(y) \\ \vdots \\ \Psi_{an}(y) \end{bmatrix}, \tag{8}$$

$$\Phi_a(y) = \begin{bmatrix} \phi_{a1}^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & \phi_{a2}^{\mathrm{T}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \phi_{an}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \Phi_{a1}^{\mathrm{T}}(y) \\ \vdots \\ \Phi_{an}^{\mathrm{T}}(y) \end{bmatrix}, \tag{9}$$

$$\theta = [b_m(t), \dots, b_0(t), \theta_{a1}(t), \dots, \theta_{an}(t)]^{\mathrm{T}}, \tag{10}$$

$$d(t) = [d_1(t), \dots, d_n(t)], \tag{11}$$

$$\psi_0(y) = [\psi_{01}(y), \dots, \psi_{0n}(y)]^{\mathrm{T}}. \tag{12}$$

We employ the filters similar to those in [7], i.e.

$$\dot{\xi} = A_0\xi + ky + \psi_0(y) \tag{13}$$

$$\dot{\Omega}^{\mathrm{T}} = A_0\Omega^{\mathrm{T}} + F(y,u)^{\mathrm{T}} \tag{14}$$

where

$$k \triangleq [k_1, k_2, \dots, k_n]^{\mathrm{T}}, \tag{15}$$

$$A_0 = A - ke_1^{\mathrm{T}}. \tag{16}$$

The vector k in (15) is chosen such that the matrix A_0 is strictly stable. It can be shown that Ω obtained from (14) satisfies the following equations

$$\Omega^{\mathrm{T}} = [v_m, \dots, v_1, v_0, \Xi], \tag{17}$$

$$\dot{\Xi} = A_0\Xi + \Psi_a(y), \tag{18}$$

$$\dot{\lambda} = A_0\lambda + e_n u, \tag{19}$$

$$v_j = A_0^j\lambda. \tag{20}$$

From our designed filters, system (1) can be represented as

$$\dot{y} = b_m v_{m,2} + \beta + \bar{\omega}^{\mathrm{T}}\theta + \epsilon_2 + d(t)\Phi_{a1}(y), \tag{21}$$

$$\dot{v}_{m,i} = v_{m,i+1} - k_i v_{m,1}, \quad i = 2, 3, \dots, \rho - 1, \tag{22}$$

$$\dot{v}_{m,\rho} = v_{m,\rho+1} - k_\rho v_{m,1} + u, \tag{23}$$

where

$$\beta = \xi_2 + \psi_{01}, \tag{24}$$

$$\omega = [v_{m,2}, v_{m-1,2}, \dots, v_{0,2}, \Xi_2 + \Psi_{a1}]^{\mathrm{T}}, \tag{25}$$

$$\bar{\omega} = [0, v_{m-1,2}, \dots, v_{0,2}, \Xi_2 + \Psi_{a1}]^{\mathrm{T}}. \tag{26}$$

In the above equations, ϵ_2, $v_{i,2}$ and $\xi_{i,2}$ denote the second entries of ϵ, v_i and ξ_i respectively, ϵ is the estimation error defined in (28).

With the above filters, a state estimate is given by

$$\hat{x} = \xi + \Omega^{\mathrm{T}}\theta \tag{27}$$

and the estimation error ϵ is defined as

$$\epsilon = x - \hat{x} \tag{28}$$

From the equations (5), (13), (14), (27) and (28), the estimation error satisfies

$$\dot{\epsilon} = A_0\epsilon + \Phi_a(y)d(t)^{\mathrm{T}} - \Omega^{\mathrm{T}}\dot{\theta}. \tag{29}$$

Remark 1 The error ϵ will be used in our design and analysis given later. As the disturbances and derivatives of time-varying parameters appear in (29), their effects should be considered in controller design. However for the state-feedback control in [6], no filter is required for state estimation. Their effects may not be necessarily considered in controller design and this makes the problem much simpler.

We now divide the error ϵ into two parts, i.e. $\epsilon = \epsilon_a + \epsilon_b$, where ϵ_a satisfies

$$\dot{\epsilon}_a = A_0\epsilon_a + \Phi_a(y)d(t)^{\mathrm{T}} \tag{30}$$

with $\epsilon_a(0) = \epsilon(0)$, and $\epsilon_b = \int_0^t e^{A_0(t-\tau)}(-\Omega^{\mathrm{T}}\dot{\theta})d\tau$. It can be shown that

$$\begin{aligned} \|\epsilon_b\| &\le \int_0^t \|e^{A_0(t-\tau)}\|\,\|\Omega\|\,\|\dot{\theta}\|\,d\tau \\ &\le q\int_0^t \|e^{A_0(t-\tau)}\|\,\|\Omega\|\,d\tau \le q\int_0^t e^{-\lambda_\theta(t-\tau)}k_\theta\|\Omega\|\,d\tau, \end{aligned} \tag{31}$$

where λ_θ and k_θ are chosen positive parameters so that

$$k_\theta e^{-\lambda_\theta t} \ge \|e^{A_0 t}\|, \quad \forall\, t \ge 0. \tag{32}$$

Thus ϵ_b satisfies

$$|\epsilon_b| \leq h(t)q, \tag{33}$$

where $h(t)$ is generated by

$$\dot{h} = -\lambda_\theta h + k_\theta\left(\|\Omega\|^2 + \frac{1}{4}\right). \tag{34}$$

Suppose $P \in R^{n\times n}$ is a positive definite matrix, satisfying $PA_0 + A_0^{\mathrm{T}}P \leq -3I$ and let

$$V_\epsilon = \epsilon_a^{\mathrm{T}} P \epsilon_a. \tag{35}$$

It can be shown that

$$\begin{aligned}\dot{V}_\epsilon &= \epsilon_a^{\mathrm{T}}(PA_0 + A_0^{\mathrm{T}}P)\epsilon_a + 2\epsilon_a^{\mathrm{T}}P\Phi_a(y)d(t)^{\mathrm{T}} \\ &\leq -2\|\epsilon_a\|^2 + \|P\Phi_a(y)d(t)^{\mathrm{T}}\|^2.\end{aligned} \tag{36}$$

The problem of this paper is to design an adaptive controller to make system (1) BIBO stable.

3 Control Design

In this section, we present the adaptive control design using the backstepping technique with tuning functions in ρ steps. In order to avoid using the sign of the high frequency gain, we take the change of coordinates

$$z_1 = y - y_r, \tag{37}$$

$$z_i = v_{m,i} - \alpha_{i-1}, \ i = 2, 3, \ldots, \rho, \tag{38}$$

where α_{i-1} is the virtual control at each step and will be determined in later discussions. Before presenting the detail, a useful function is introduced. Firstly we define $s(x)$ as

$$s(x) = \begin{cases} x^2 & |x| \geq \delta, \\ (\delta^2 - x^2)^\rho + x^2 & |x| < \delta, \end{cases} \tag{39}$$

where δ is a positive design parameter. It can be shown that $s(x)$ is $(\rho-1)$-th order differentiable and bounded below for $|x| < \delta$. Based on $s(x)$, a function $H(z_1)$ is defined as follows

$$H(z_1) = \frac{\Phi_a(y)}{s(z_1)} = \begin{cases} \dfrac{\Phi_a(y)}{z_1^2} & |z_1| \geq \delta, \\ \dfrac{\Phi_a(y)}{(\delta^2 - z_1^2)^\rho + z_1^2} & |z_1| < \delta. \end{cases} \tag{40}$$

Clearly H is well defined and for $|z_1| < \delta$, H is bounded as $s(z_1)$ is bounded below.

Remark 2 In [3], a similar function to (40) was used to design controllers for disturbance decoupling. However, the function is undefined at the time instants when $y = y_r$. Thus, the controller presented is undefined at these time instants.

From (36) and (40) it can be shown that

$$\dot{V}_\epsilon \le -2\|\epsilon_a\|^2 + \frac{1}{2}s^4\|PH\|^4 + \frac{1}{2}\|d(t)\|^4. \tag{41}$$

We now illustrate the backstepping design procedures using Nussbaum gain with details given for the first two steps.

Step 1 It follows from (21) and (37) that

$$\dot{z}_1 = b_m v_{m,2} + \beta + \bar{\omega}^{\mathrm{T}}\theta + \epsilon_2 + d(t)\Phi_{a1}(y) - \dot{y}_r. \tag{42}$$

Without using the sign of b_m, the following virtual control law α_1 is designed

$$\alpha_1 = N(\chi)\bar{\alpha}_1 e^{-ft}, \tag{43}$$

$$N(\chi) = \exp(\chi^2)\cos\frac{\pi}{2}\chi, \tag{44}$$

where f is a positive real design parameter, χ is generated by

$$\dot{\chi} = z_1\bar{\alpha}_1 \tag{45}$$

and $\bar{\alpha}_1$ is chosen to be

$$\begin{aligned}\bar{\alpha}_1 = {} & \big(c_1 + l_1 + (e_1^{\mathrm{T}}\hat{\theta})^2\big)z_1 + \beta + \bar{\omega}^{\mathrm{T}}\hat{\theta} - \dot{y}_r \\ & + z_1 h^2\hat{q} + \frac{1}{4}z_1\|\Phi_{a1}(y)\|^2 + \sum_{i=1}^{\rho}\frac{1}{8l_i}z_1 s^3(z_1)\|PH\|^4,\end{aligned} \tag{46}$$

where c_1 and l_1 are two positive real design parameters, $\hat{\theta}$ and $\hat{q}$ denote the estimates of θ and q. Notice that

$$b_m v_{m,2} = b_m(z_2 + \alpha_1) = \hat{b}_m z_2 + b_m\alpha_1 + \tilde{b}_m z_2, \tag{47}$$

where $\tilde{b}_m = b_m - \hat{b}_m$, $\hat{b}_m$ is the first element of $\hat{\theta}$, i.e. $\hat{b}_m = e_1^{\mathrm{T}}\hat{\theta}$. Then from (42) and (46) we have

$$\begin{aligned}\dot{z}_1 - \bar{\alpha}_1 = {} & -(c_1 + l_1 + \hat{b}_m^2)z_1 + (\bar{\omega}^{\mathrm{T}} + z_2 e_1^{\mathrm{T}})\tilde{\theta} + \epsilon_{a,2} + \epsilon_{b,2} - z_1 h^2\hat{q} + \hat{b}_m z_2 + b_m\alpha_1 \\ & + d(t)\Phi_{a1}(y) - \frac{1}{4}z_1\|\Phi_{a1}(y)\|^2 - \sum_{i=1}^{\rho}\frac{1}{8l_i}z_1 s^3\|PH\|^4,\end{aligned} \tag{48}$$

where $\tilde{\theta} = \theta - \hat{\theta}$, $\epsilon_{a,2}$ and $\epsilon_{b,2}$ represent the second entry of ϵ_a and ϵ_b. To proceed, we define the Lyapunov function

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \frac{1}{2}\tilde{q}^2 + \frac{1}{4l_1}V_\epsilon, \tag{49}$$

where Γ is a positive definite matrix of $R^{(n+2)\times(n+2)}$. Then the derivative of V_1 along with (41), (43) and (48) is given by

$$\begin{aligned}\dot{V}_1 &= z_1(\dot{z}_1-\bar{\alpha}_1)+z_1\bar{\alpha}_1+\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\dot{\theta}-\dot{\hat{\theta}})+\tilde{q}\dot{\tilde{q}}+\frac{1}{4l_1}\dot{V}_\epsilon \\ &\le -(c_1+\hat{b}_m^2)z_1^2+\hat{b}_m z_1z_2+\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\tau_1-\dot{\hat{\theta}})-l_1z_1^2+\epsilon_{a,2}z_1-\frac{1}{2l_1}\|\epsilon_a\|^2 \\ &\quad+\epsilon_{b,2}z_1-\tilde{q}\dot{\hat{q}}-h^2\hat{q}z_1^2+d(t)\Phi_{a1}(y)z_1-\frac{1}{4}z_1^2\|\Phi_{a1}(y)\|^2+b_m\alpha_1z_1+\bar{\alpha}_1z_1 \\ &\quad+\frac{1}{8l_1}s^4\|PH\|^4-\sum_{i=1}^{\rho}\frac{1}{8l_i}z_1^2s^3\|PH\|^4+\frac{1}{8l_1}\|d(t)\|^4+\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\theta},\end{aligned} \tag{50}$$

where

$$\tau_1=\Gamma z_1(\bar{\omega}+z_2e_1). \tag{51}$$

Here we know that

$$\epsilon_{b,2}z_1-h^2\hat{q}z_1^2\le hq|z_1|-h^2\hat{q}z_1^2\le q(h^2z_1^2+1/4)-h^2\hat{q}z_1^2=h^2\tilde{q}z_1^2+\frac{q}{4}.$$

Then we can get

$$\begin{aligned}\dot{V}_1 &\le (b_mN(\chi)e^{-ft}+1)\dot{\chi}-c_1z_1^2+\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\tau_1-\dot{\hat{\theta}}) \\ &\quad+\tilde{q}(\iota_1-\dot{\hat{q}})-\frac{1}{4l_1}\|\epsilon_a\|^2+\frac{1}{4}z_2^2+M_1,\end{aligned} \tag{52}$$

where

$$\iota_1=h^2z_1^2, \tag{53}$$

$$M_1=\|d(t)\|^2+\frac{1}{8l_1}\|d(t)\|^4-\sum_{i=2}^{\rho}\frac{1}{8l_i}s^4\|PH\|^4+\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\theta}+\frac{1}{4}q+\overline{N}, \tag{54}$$

$$\overline{N}=\begin{cases}0, & |z_1|\ge\delta,\\ \sum\limits_{i=1}^{\rho}\dfrac{1}{8l_i}(\delta^2-z_1^2)^\rho s^3\|PH\|^4, & |z_1|<\delta.\end{cases} \tag{55}$$

From (40) we know that $\overline{N}$ is bounded.

Step 2 Now, we evaluate the dynamics of the second state z_2. Differentiating (38) for $i=2$ and using (22), we have

$$\dot{z}_2=v_{m,3}-k_2v_{m,1}-\dot{\alpha}_1. \tag{56}$$

Note that α_1 is a function of y, $\hat{\theta}$, $\hat{q}$, ξ, Ξ, λ, χ and y_r and following from similar analysis to [7] by substituting (38) with $i=3$ into (56), we get

$$\dot{z}_2=\alpha_2-\beta_2-\frac{\partial\alpha_1}{\partial y}\Big(\epsilon_2+\omega^{\mathrm{T}}\tilde{\theta}+d(t)\Phi_{a1}(y)\Big)+z_3-\frac{\partial\alpha_1}{\partial y}\omega^{\mathrm{T}}\hat{\theta}-\frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}-\frac{\partial\alpha_1}{\partial\hat{q}}\dot{\hat{q}}, \tag{57}$$

where

$$\beta_2 \triangleq k_2 v_{m,1} + \frac{\partial \alpha_1}{\partial y}\beta + \frac{\partial \alpha_1}{\partial \Pi}\dot{\Pi} + \sum_{j=1}^{m+1} \frac{\partial \alpha_1}{\partial \lambda_j}(-k_j\lambda_1 + \lambda_{j+1}) + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r + \frac{\partial \alpha_1}{\partial \chi}\dot{\chi} \tag{58}$$

where $\Pi = [\xi^{\mathrm{T}}, \mathrm{Vec}\,(\Xi)^{\mathrm{T}}]^{\mathrm{T}}$. Define the Lyapunov function and choose the virtual control for this step as

$$V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{4l_2}V_\epsilon, \tag{59}$$

$$\begin{aligned}\alpha_2 = &-\left(c_2 + \frac{1}{4}\right)z_2 + \frac{\partial \alpha_1}{\partial y}\omega^{\mathrm{T}}\hat{\theta} - z_2\left\|\frac{\partial \alpha_1}{\partial \hat{\theta}}\right\|^2\|\tau_2\|^2 - z_2 h^2\hat{q}\left\|\frac{\partial \alpha_1}{\partial y}\right\|^2 \\ &- z_2\left\|\frac{\partial \alpha_1}{\partial \hat{q}}\right\|^2\iota_2^2 - l_2\left\|\frac{\partial \alpha_1}{\partial y}\right\|^2 z_2 + \beta_2 - \frac{z_2}{4}\left\|\frac{\partial \alpha_1}{\partial y}\Phi_{a1}(y)\right\|^2,\end{aligned} \tag{60}$$

$$\tau_2 = \tau_1 - \Gamma\,\frac{\partial \alpha_1}{\partial y}\omega z_2, \tag{61}$$

$$\iota_2 = \iota_1 + h^2\left\|\frac{\partial \alpha_1}{\partial 21}\right\|^2 z_2^2. \tag{62}$$

Using (52), (59) and (60), we have that

$$\begin{aligned}\dot{V}_2 &\le \dot{V}_1 + z_2\dot{z}_2 + \frac{1}{4l_2}\,\dot{V}_\epsilon \\ &\le -\sum_{i=1}^{2} c_i z_i^2 + (b_m N(\chi)e^{-ft} + 1)\dot{\chi} + z_2 z_3 - \sum_{i=1}^{2}\frac{1}{4l_i}\|\epsilon_a\|^2 + M_2 \\ &\quad + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\tau_1 - \dot{\hat{\theta}}) - z_2\,\frac{\partial \alpha_1}{\partial y}\,\omega^{\mathrm{T}}\tilde{\theta} + z_2^2\left\|\frac{\partial \alpha_1}{\partial \hat{\theta}}\right\|^2\|\dot{\hat{\theta}}\|^2 - z_2^2\left\|\frac{\partial \alpha_1}{\partial \hat{\theta}}\right\|^2\|\tau_2\|^2 \\ &\quad + \tilde{q}(\iota_1 - \dot{\hat{q}}) + h^2\tilde{q}\left\|\frac{\partial \alpha_1}{\partial y}\right\|^2 z_2^2 + z_2^2\left\|\frac{\partial \alpha_1}{\partial \hat{q}}\right\|^2\dot{\hat{q}}^2 - z_2^2\left\|\frac{\partial \alpha_1}{\partial \hat{q}}\right\|^2\iota^2 \\ &\le -\sum_{i=1}^{2} c_i z_i^2 + (b_m N(\chi)e^{-ft} + 1)\dot{\chi} + z_2 z_3 + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\tau_2 - \dot{\hat{\theta}}) + \tilde{q}(\iota_2 - \dot{\hat{q}}) + M_2 \\ &\quad + z_2^2\left\|\frac{\partial \alpha_1}{\partial \hat{\theta}}\right\|^2(\|\dot{\hat{\theta}}\|^2 - \|\tau_2\|^2) + z_2^2\left(\frac{\partial \alpha_1}{\partial \hat{q}}\right)^2(\dot{\hat{q}}^2 - \iota_2^2) - \sum_{i=1}^{2}\frac{1}{4l_i}\|\epsilon_a\|^2,\end{aligned} \tag{63}$$

where

$$M_2 = \sum_{i=1}^{2}\frac{1}{8l_i}\|d(t)\|^4 + 2\|d(t)\|^2 - \sum_{i=3}^{\rho}\frac{1}{8l_i}\,s^4\|PH\|^4 + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\theta} + \frac{1}{2} + \frac{1}{2}\,q + \overline{N}. \tag{64}$$

Remark 3 Note that M_2 contains $s^4\|PH\|^4$ and this term may not be bounded. As seen from our analysis, $\frac{1}{8l_2}s^4\|PH\|^4$ disappears in M_2 due to the use of V_ϵ at step 2. If we use V_ϵ at each step, this term will disappear in M_ρ on the last step.

Step i $(i = 3, \ldots, \rho)$ These steps are similar to those in [7]. Define

$$V_i = V_{i-1} + \frac{1}{2} z_i^2 + \frac{1}{4l_i} V_\epsilon, \tag{65}$$

$$\begin{aligned} \alpha_i = & -c_i z_i - l_i \left\| \frac{\partial \alpha_{i-1}}{\partial y} \right\|^2 z_i - z_{i-1} + \beta_i + \frac{\partial \alpha_{i-1}}{\partial y} \omega^{\mathrm{T}} \hat{\theta} - \frac{z_i}{4} \left\| \frac{\partial \alpha_{i-1}}{\partial y} \Phi_{a1}(y) \right\|^2 \\ & - z_i \left\| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right\|^2 \|\tau_i\|^2 + \left(\sum_{k=2}^{i-1} z_k^2 \left\| \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \right\|^2 \right) (\tau_i + \tau_{i-1})^{\mathrm{T}} \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega \\ & - z_i \left\| \frac{\partial \alpha_{i-1}}{\partial \hat{q}} \right\|^2 \iota_i^2 - \left(\sum_{k=2}^{i-1} z_k^2 \left\| \frac{\partial \alpha_{k-1}}{\partial \hat{q}} \right\|^2 \right) (\iota_i + \iota_{i-1})^{\mathrm{T}} h^2 \left\| \frac{\partial \alpha_{i-1}}{\partial y} \right\|^2 z_i \\ & - z_i h^2 \hat{q} \left\| \frac{\partial \alpha_{i-1}}{\partial y} \right\|^2, \end{aligned} \tag{66}$$

$$\tau_i = \tau_{i-1} - \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega z_i, \tag{67}$$

$$\iota_i = \iota_{i-1} + h^2 \left\| \frac{\partial \alpha_{i-1}}{\partial y} \right\|^2 z_i^2, \tag{68}$$

where

$$\beta_i \triangleq k_i v_{m,1} + \frac{\partial \alpha_{i-1}}{\partial y} \beta + \frac{\partial \alpha_{i-1}}{\partial \Pi} \dot{\Pi} + \frac{\partial \alpha_{i-1}}{\partial y_r} \dot{y}_r + \sum_{j=1}^{m+1} \frac{\partial \alpha_{i-1}}{\partial \lambda_j} (-k_j \lambda_1 + \lambda_{j+1}) + \frac{\partial \alpha_{i-1}}{\partial \chi} \dot{\chi}. \tag{69}$$

Also note that

$$\begin{aligned} \|\tau_i\|^2 &= \tau_i^{\mathrm{T}} \tau_i = \tau_i^{\mathrm{T}} \tau_i - \tau_{i-1}^{\mathrm{T}} \tau_{i-1} + \tau_{i-1}^{\mathrm{T}} \tau_{i-1} = (\tau_i + \tau_{i-1})^{\mathrm{T}} (\tau_i - \tau_{i-1}) + \tau_{i-1}^{\mathrm{T}} \tau_{i-1} \\ &= -(\tau_i + \tau_{i-1})^{\mathrm{T}} \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega z_i + \tau_{i-1}^{\mathrm{T}} \tau_{i-1}, \\ \iota_i^2 &= (\iota_i + \iota_{i-1})^{\mathrm{T}} h^2 \left\| \frac{\partial \alpha_{i-1}}{\partial y} \right\|^2 z_i^2 + \iota_{i-1}^2. \end{aligned} \tag{70}$$

Then the actual adaptive controller is obtained and given by

$$u(t) = \alpha_\rho - v_{m,\rho+1}, \tag{71}$$

$$\dot{\hat{\theta}} = \mathrm{Proj}\,(\tau_\rho), \tag{72}$$

$$\dot{\hat{q}} = \mathrm{Proj}\,(\iota_\rho), \tag{73}$$

where $\mathrm{Proj}\,(\cdot)$ is a smooth projection operation to ensure the estimates belong to compact sets for all time. Such an operation can be found in [7].

Remark 4 Note that the designed tuning functions are different from existing schemes in [7] as the projection operations are used in the parameter estimators.

By using the properties that $-\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\operatorname{Proj}(\tau) \le -\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tau$ and $\operatorname{Proj}(\tau)^{\mathrm{T}}\operatorname{Proj}(\tau) \le \tau^{\mathrm{T}}\tau$ the final Lyapunov function V_ρ satisfies

$$
\begin{aligned}
\dot{V}_\rho \le & -\sum_{k=1}^{\rho} c_k z_k^2 + (b_m N(\chi)e^{-ft} + 1)\dot{\chi} + M_\rho - \sum_{i=1}^{\rho}\frac{1}{4l_i}\|\epsilon_a\|^2 \\
& + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\tau_\rho - \operatorname{Proj}(\tau_\rho)) + \left(\sum_{k=2}^{\rho} z_k^2\left\|\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\right\|^2\right)(\operatorname{Proj}(\tau_\rho)^{\mathrm{T}}\operatorname{Proj}(\tau_\rho) - \|\tau_\rho\|^2) \\
& + \tilde{q}(\iota_\rho - \operatorname{Proj}(\iota_\rho)) + \left(\sum_{k=2}^{\rho} z_k^2\left(\frac{\partial\alpha_{k-1}}{\partial\hat{q}}\right)^2\right)(\operatorname{Proj}(\iota_\rho)^2 - \iota_\rho^2) \\
\le & -\sum_{k=1}^{\rho} c_k z_k^2 + b_m N(\chi)e^{-ft}\dot{\chi} + \dot{\chi} + M_\rho - \sum_{i=1}^{\rho}\frac{1}{4l_i}\|\epsilon_a\|^2,
\end{aligned}
\tag{74}
$$

where

$$
M_\rho = \sum_{i=1}^{\rho}\frac{1}{8l_i}\|d(t)\|^4 + \rho\|d(t)\|^2 + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\theta} + \frac{\rho-1}{2} + \frac{\rho}{4}q + \overline{N} \tag{75}
$$

Integrating both sides of (74) over the interval $[0,t]$ gives

$$
\begin{aligned}
\int_0^t \dot{V}_\rho e^{f\tau} d\tau \le & -\int_0^t \sum_{k=1}^{\rho} c_k z_k^2 e^{f\tau} d\tau + \int_0^t b_m N(\chi)\dot{\chi}\, d\tau + \int_0^t \dot{\chi} e^{f\tau} d\tau \\
& + \int_0^t M_\rho e^{f\tau} d\tau - \int_0^t \sum_{i=1}^{\rho}\frac{1}{4l_i}\|\epsilon_a\|^2 e^{f\tau} d\tau.
\end{aligned}
\tag{76}
$$

Note that $V_\epsilon \le \|P\|\|\epsilon_a\|^2$. Then

$$
\begin{aligned}
V_\rho &= \sum_{k=1}^{\rho}\frac{1}{2}z_k^2 + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \frac{1}{2}\tilde{q}^2 + \sum_{i=1}^{\rho}\frac{1}{4l_i}V_\epsilon \\
&\le \sum_{k=1}^{\rho}\frac{1}{2}z_k^2 + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \frac{1}{2}\tilde{q}^2 + \sum_{i=1}^{\rho}\frac{1}{4l_i}\|P\|\|\epsilon_a\|^2.
\end{aligned}
\tag{77}
$$

This yields

$$
\begin{aligned}
0 \le V_\rho(t) \le V_\rho(0) + e^{-ft}\int_0^t b_m N(\chi)\dot{\chi}\, d\tau + \int_0^t \dot{\chi} e^{-f(t-\tau)}\, d\tau \\
+ \int_0^t \frac{f}{2}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}) + \tilde{q}^2)e^{-f(t-\tau)}\, d\tau + \int_0^t M_\rho e^{-f(t-\tau)}\, d\tau
\end{aligned}
\tag{78}
$$

where $f = \min\left\{\frac{1}{\|P\|_2}, 2c_1, 2c_2, \ldots, 2c_\rho,\right\} > 0$. Due to the utilization of projection operations for $\hat{\theta}$ and $\hat{q}$, the boundedness of $\tilde{\theta}$ and $\tilde{q}$ can be guaranteed. Together with the boundedness of $d(t)$, q and $\dot{\theta}$, the boundedness of M_ρ and

$$\int_0^t \frac{f}{2}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \tilde{q}^2)e^{-f(t-\tau)}d\tau + \int_0^t M_\rho e^{-f(t-\tau)}d\tau$$

can be guaranteed. Thus by comparing with (4), f_0 is selected as the upper bound of

$$V_\rho(0) + \int_0^t \frac{f}{2}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \tilde{q}^2)e^{-f(t-\tau)}d\tau + \int_0^t M_\rho e^{-f(t-\tau)}d\tau, g_1 = b_m$$

and $f_1 = f$. Using Lemma 1, we can conclude that $V_\rho(t)$ and $\chi(t)$, hence z_i, $(i = 1, \ldots, \rho)$ are bounded. Finally, the stability of the whole system can be established as in [7].

To conclude this section, the results established are presented in the following theorem.

Theorem 1 *Consider the uncertain time-varying nonlinear system (1) satisfying Assumptions 1–4. With the application of the controller (71) and the parameter updating laws (72) and (73), the resulting closed loop system is BIBO stable.*

4 A Simulation Example

In this section, the proposed design method is illustrated on the following simple linear system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) + \theta_1(t)y(t),\\ \dot{x}_2(t) &= b(t)u(t) + d(t),\\ y(t) &= x_1(t),\end{aligned} \tag{79}$$

where $\theta_1(t) = 1 + \sin(t)$, $b(t) = 1 + \exp(-t)$, $d(t) = \cos(t)$ are unknown timevarying parameters in the controller design. The objective is to control the system output $y(t)$ to follow a desired trajectory $y_r(t) = \sin(t) + \sin(2t)$. The filters are implemented as

$$\dot{\xi} = A_0\xi + ky, \tag{80}$$

$$\dot{\lambda} = A_0\lambda + e_2u, \tag{81}$$

$$\dot{\Xi} = A_0\Xi + \Psi, \quad \Psi = [y\ 0]^{\mathrm{T}}, \tag{82}$$

$$A_0 = \begin{bmatrix} -k_1 & 1\\ -k_2 & 0\end{bmatrix}. \tag{83}$$

The control law α_1 in (43), $u(t)$ in (71), and the parameter update law $\hat{\theta}$ in (72) are used with $\theta = [b\ \ \theta_1]^{\mathrm{T}}$. The design parameters are chosen as $c_1 = c_2 = 5$, $\Gamma = I_2$, $l_1 = l_2 = 2$, $k_1 = 6$, $k_2 = 8$. The initials $y(0) = 0.1$, $\hat{\theta}(0) = [0.2\ \ 0.5]^{\mathrm{T}}$ and others are set to zero. The simulation results presented in the Figure 4.1 show the system output $y(t)$ and the

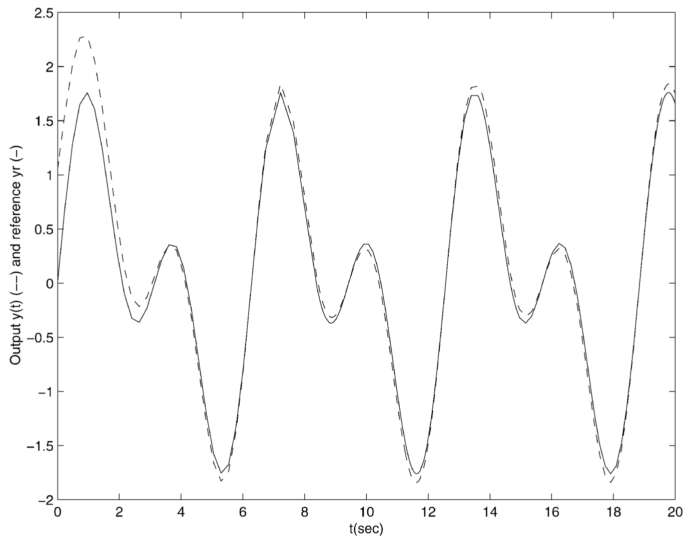

Figure 4.1. Output $y(--)$ and trajectory $y_r(-)$.

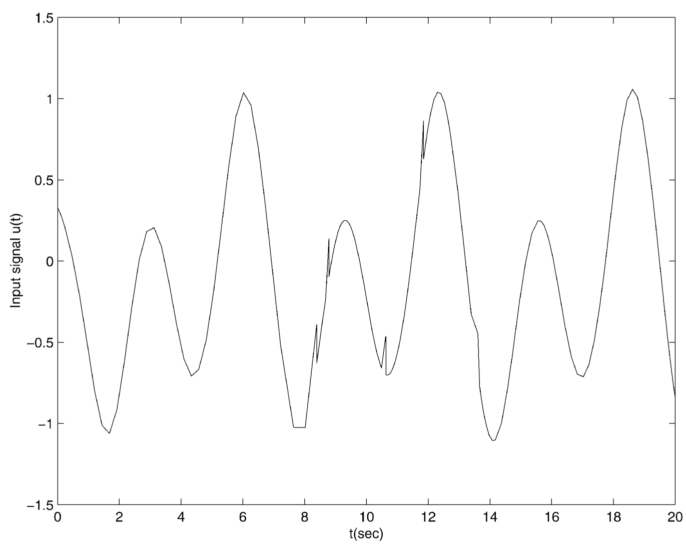

Figure 4.2. Control signal $u(t)$.

desired trajectory signal $y_r(t)$. Figure 4.2 shows the control signal $u(t)$. Clearly, these simulation results verify that our proposed scheme is effective.

5 Conclusion

In this paper, a scheme is proposed to design an adaptive output-feedback controller for uncertain time-varying nonlinear systems with unknown sign of high-frequency gains in the presence of disturbances. No growth conditions on system nonlinearities are imposed. In the design, certain well defined functions are used to cancel the effects of disturbances.

To deal with the time variation problem, an estimator is used to estimate the bound of the variation rates. Furthermore, the overparametrization problem is also solved by using the concept of tuning functions. It is shown that the controller obtained by the proposed design scheme can make the whole adaptive control system stable. Simulations performed on a simple system also verify the effectiveness of the proposed scheme.

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