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Existence of Nonoscillatory Solution of Third Order Linear Neutral Delay Difference Equation with Positive and Negative Coefficients

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Abstract: In this paper, by using fixed point theorem, the problem of existence of the nonoscillatory solution for a class of neutral delay difference equations with both positive and negative coefficients has been investigated. Under the assumption of third order, a sufficient condition is proposed for the existence of the nonoscillatory solution. Further studies on the underlying problem have also been conducted.

Keywords: *Neutral delay difference equation; oscillation; positive and negative coefficients.*

Mathematics Subject Classification (2000): 35D05, 35E05.

1 Introduction

In recent years, there are many scholars who have devoted their researches to the differential equations with positive and negative coefficients and obtained some interesting results, see for example, [1 − 8] and the references therein. At the same time, the research on difference equations with positive and negative coefficients is getting people's attention and is becoming a new field of research [9 − 12]. In [12] the existence of positive solution of the second order difference equation with positive and negative coefficients was studied. In this paper we consider the third order equation

$$\Delta^3[x(n) + px(n-\tau)] + R_1(n)x(n-\delta_1) - R_2(n)x(n-\delta_2) = 0 \qquad (1)$$

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where $p \in R$, $\tau \in \{1, 2, \dots\}$, $\delta_1, \delta_2 \in \{0, 1, 2, \dots\}$, $\{R_1(n)\}$, $\{R_2(n)\}$ are positive real sequences, and satisfy

$$\sum_{n=1}^{+\infty} n^2 R_i(n) < +\infty, \quad i = 1, 2. \tag{2}$$

The related conclusion in [12] is generalized in this paper to the case of third order equations. A sufficient condition for the existence of the positive solution of the equation (1) is obtained.

For simplicity, we list basic conceptions and symbols as follows:

Δ symbols for the forward difference operator, say $\Delta y(n) = y(n+1) - y(n)$;

Z symbols for the integer set and R for the real numbers set.

Assume $a \in Z$ and let $N(a) = \{a, a+1, \dots\}$, $N = N(0)$. For any given $a, b \in Z$ and $a \leq b$, let $N(a, b) = \{a, a+1, \dots, b\}$.

The solution of the difference equation (1) is called eventually positive if there exists a positive integer M such that $x(n) > 0$ for $n \in N(M)$. If there exists a positive integer M such that $x(n) < 0$ for $n \in N(M)$, then is called eventually negative.

The solution of the difference equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

2 Main Results and Proofs

Theorem 2.1 *Suppose*

(i) $\sum\limits_{n=1}^{+\infty} n^2 R_i(n) < +\infty, \quad i = 1, 2, \quad n \in N(n_0);$ (3)

(ii) *there exists a positive integer T_1 which is sufficiently large such that, for any $\alpha > 0$, when $n > T_1$, we have*

$$\alpha R_1(n) - R_2(n) \geq 0; \tag{4}$$

(iii) $p \neq \pm 1.$ (5)

Then the equation (1) has an eventually positive solution.

Proof Let L_∞ denote the set of all the bounded real sequences $x = \{x(n)\}$ on $N(n_0)$, define the norm $\|x\| = \sup x(n)$, then L_∞ forms a Banach space. There are four situations to be contemplated:

Case 1: $0 \leq p < 1$.

From (3) and (4), we select a positive integer $n_1 \geq \max\{T_1, n_0 + \delta\}$ which is large enough, where $\delta = \max\{\tau, \delta_1, \delta_2\}$, such that

$$\sum_{n=n_1}^{+\infty} n^2 [R_1(n) + R_2(n)] < 1 - p, \tag{6}$$

$$0 \leq \sum_{n=n_1}^{+\infty} n^2 [M_2 R_1(n) - M_1 R_2(n)] \leq p - 1 + M_2, \tag{7}$$

where M_1, M_2 are positive constants and satisfy

$$1 - M_2 < p \le \frac{1 - M_1}{1 + M_2}. \tag{8}$$

Let

$$A = \{x \in L_\infty \colon\; M_1 \le x(n) \le M_2, \;\; n \in N(n_0)\}. \tag{9}$$

It is clear that A is a bounded closed convex subset on L_∞.

Define a mapping $T \colon A \to L_\infty$ as following:

$$Tx(n) = \begin{cases} 1 - p - px(n-\tau) + \\ \sum\limits_{s=n}^{+\infty} C_{s+2-n}^2 (R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \ge n_1, \\ Tx(n_1), & n_0 \le n < n_1. \end{cases} \tag{10}$$

Now we shall prove that T is a self-mapping on A where there are two situations to be contemplated:

Case 1-a: $n \ge n_1$.

For any $x \in A,$ from (9), (10), we find that

$$\begin{aligned} Tx(n) &= 1 - p - px(n-\tau) + \sum_{s=n}^{+\infty} C_{s+2-n}^2 (R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)) \\ &\le 1 - p + \sum_{s=n}^{+\infty} s^2 (R_1(s)M_2 - R_2(s)M_1), \end{aligned}$$

therefore from (7), we have

$$Tx(n) \le 1 - p + p - 1 + M_2 = M_2. \tag{11}$$

From (4) we have

$$\begin{aligned} &\sum_{s=n}^{+\infty} C_{s+2-n}^2 (R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)) \\ &\qquad = \sum_{s=n}^{+\infty} C_{s+2-n}^2 x(s-\delta_2) \left[\frac{x(s-\delta_1)}{x(s-\delta_2)} R_1(s) - R_2(s) \right] \ge 0. \end{aligned}$$

Hence we also have

$$Tx(n) \ge 1 - p - px(n-\tau).$$

Since $0 \le p < 1$, from (8) and (9), we get

$$Tx(n) \ge 1 - p - pM_2 \ge M_1. \tag{12}$$

Case 1-b: $n_0 \le n < n_1$.

For any $x \in A,$ from (10) we know that

$$Tx(n) = Tx(n_1)$$

and from (11) and (12) we obtain

$$M_1 \le Tx(n_1) \le M_2.$$

Hence we have

$$M_1 \le Tx(n) \le M_2.$$

Considering the two cases of a and b, for any $x \in A$, we have

$$M_1 \le Tx \le M_2.$$

Hence, $Tx \in A$, namely T is a self-mapping on A.

In what follows, we shall prove that T is a contraction mapping on A where there are also two situations to be contemplated:

Proof of Case 1-a. $n \ge n_1$.

For any $x_1, x_2 \in A$, we have

$$\begin{aligned} |Tx_1(n) - Tx_2(n)| = \Bigg| &- px_1(n-\tau) + \sum_{s=n}^{+\infty} C_{s+2-n}^2 (R_1(s)x_1(s-\delta_1) - R_2(s)x_1(s-\delta_2)) \\ &+ px_2(n-\tau) - \sum_{s=n}^{+\infty} C_{s+2-n}^2 (R_1(s)x_2(s-\delta_1) - R_2(s)x_2(s-\delta_2))\Bigg| \\ &\le |-px_1(n-\tau) + px_2(n-\tau)| + \sum_{s=n}^{+\infty} C_{s+2-n}^2 R_1(s)|x_1(s-\delta_1) - x_2(s-\delta_1)| \\ &+ \sum_{s=n}^{+\infty} C_{s+2-n}^2 R_2(s)|x_1(s-\delta_2) - x_2(s-\delta_2)|. \end{aligned}$$

Hence from (5), we have

$$\begin{aligned} |Tx_1(n) - Tx_2(n)| &\le \left[p + \sum_{s=n}^{+\infty} C_{s+2-n}^2 (R_1(s) + R_2(s))\right] \|x_1 - x_2\| \\ &\le \left[p + \sum_{s=n}^{+\infty} s^2 (R_1(s) + R_2(s))\right] \|x_1 - x_2\|. \end{aligned}$$

Then from (6), there exists $0 < q_1 < 1$, such that

$$|Tx_1(n) - Tx_2(n)| \le q_1 \|x_1 - x_2\|.$$

Proof of Case 1-b. $n_0 \le n < n_1$.

From (10), we also have

$$|Tx_1(n) - Tx_2(n)| = |Tx_1(n_1) - Tx_2(n_1)| \le q_1 \|x_1 - x_2\|.$$

In both the cases of a and b, for any $x_1, x_2 \in A$, $n \ge n_0$, we have

$$|Tx_1(n) - Tx_2(n)| \le q_1 \|x_1 - x_2\|.$$

So T is a contraction mapping on A.

On summarizing the above cases we can conclude from the Banach contraction mapping principle that there exist a fixed point x of T on A, namely $Tx = x$, where $x = x(n)$ satisfies

$$Tx(n) = \begin{cases} 1 - p - px(n-\tau) + \\ \sum\limits_{s=n}^{+\infty} C_{s+2-n}^{2}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \ge n_1, \\ Tx(n_1), & n_0 \le n \le n_1. \end{cases}$$

From this, the fixed point $x(n)$ is a positive sequence. Differentiating three times the above expression, we get

$$\Delta^3[x(n) + px(n-\tau)] + R_1(n)x(n-\delta_1) - R_2(n)x(n-\delta_2) = 0.$$

Hence this fixed point $x(n)$ is a positive solution of the equation (1).

Case 2: $1 < p$.

From (3) and (4), we select a positive integer $n_2 > t_1 > n_0$ which is large enough and satisfies

$$n_2 + \tau = n_0 + \max\{\delta_1, \delta_2\}$$

such that

$$\frac{1}{p}\sum_{n=n_2}^{+\infty} n^2[R_1(n) + R_2(n)] < 1 - \frac{1}{p}, \tag{13}$$

$$\sum_{n=n_2}^{+\infty} n^2[M_4R_1(n) - M_3R_2(n)] \le 1 - p + pM_4, \tag{14}$$

where M_1, M_2 are positive constants and satisfy

$$(1 - M_3)p \ge 1 + M_4, \quad p(1 - M_4) < 1. \tag{15}$$

Let

$$A = \{x \in L_\infty\colon\ M_3 \le x(n) \le M_4,\ n \in N(n_0)\}. \tag{16}$$

It is clear that A is a bounded closed convex subset on L_∞.

Define a mapping $T\colon A \to L_\infty$ as follow:

$$Tx(n) = \begin{cases} 1 - \dfrac{1}{p} - \dfrac{1}{p}x(n+\tau) + \\ \dfrac{1}{p}\sum\limits_{s=n+\tau}^{+\infty} C_{s+2-n-r}^{2}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \ge n_2, \\ Tx(n_2), & n_0 \le n \le n_2. \end{cases} \tag{17}$$

In the following, we shall prove that T is a self-mapping on A. Here there are still two situations to be discussed:

Case 2-a: $n \ge n_2$.

For any $x \in A,$ from (16), (17) and $p > 1$ we find that

$$Tx(n) = 1 - \frac{1}{p} - \frac{1}{p}x(n+\tau) + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2))$$

$$\leq 1 - \frac{1}{p} + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2(R_1(s)M_4 - R_2(s)M_3)$$

and therefore, from (14) ,we have

$$Tx(n) \leq 1 - \frac{1}{p} + \frac{1}{p}(1 - p + M_4) = M_4. \tag{18}$$

Since from (4), we have

$$\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2))$$

$$= \sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}x(s-\delta_2)\left[\frac{x(s-\delta_1)}{x(s-\delta_2)}R_1(s) - R_2(s)\right] \geq 0.$$

Hence we also have

$$Tx(n) \geq 1 - \frac{1}{p} - \frac{1}{p}x(n+\tau).$$

Since $p > 1,$ from (15) and (16), we get

$$Tx(n) \geq 1 - \frac{1}{p} - \frac{1}{p}M_4 \geq M_3. \tag{19}$$

Case 2-b: $n_0 \leq n < n_2.$
For any $x \in A,$ from (17) we find that

$$Tx(n) = Tx(n_2).$$

Then, from (18) and (19) we obtain

$$M_3 \leq Tx(n_2) \leq M_4.$$

Hence

$$M_3 \leq Tx(n) \leq M_4.$$

Based on the two cases of a and b, for any $x \in A,$ we have

$$M_3 \leq Tx \leq M_4.$$

Hence $Tx \in$A, namely, T is a self-mapping on A.

In what follows, we shall prove that T is a contraction mapping on A where following two situations need to be discussed.

Proof of Case 2-a. $n \geq n_2.$

For any $x_1, x_2 \in A$, we have

$$
\begin{aligned}
&|Tx_1(n) - Tx_2(n)| \\
&= \Big| -\frac{1}{p}x_1(n+\tau) + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x_1(s-\delta_1) - R_2(s)x_1(s-\delta_2)) \\
&\quad + \frac{1}{p}x_2(n+\tau) - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x_2(s-\delta_1) - R_2(s)x_2(s-\delta_2))\Big| \\
&\le \Big| -\frac{1}{p}x_1(n+\tau) + \frac{1}{p}x_2(n+\tau)\Big| + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}R_1(s)|x_1(s-\delta_1) \\
&\quad - x_2(s-\delta_1)| + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}R_2(s)|x_1(s-\delta_2) - x_2(s-\delta_2)|.
\end{aligned}
$$

So from (5) we have

$$
\begin{aligned}
|Tx_1(n) - Tx_2(n)| &\le \left[\frac{1}{p} + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s) + R_2(s))\right]\|x_1 - x_2\| \\
&\le \left[\frac{1}{p} + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2(R_1(s) + R_2(s))\right]\|x_1 - x_2\|.
\end{aligned}
$$

Then, from (13), there exists $0 < q_2 < 1$, such that

$$|Tx_1(n) - Tx_2(n)| \le q_2\|x_1 - x_2\|.$$

Proof of Case 2-b. $n_0 \le n < n_2$.
From (17), we also have

$$|Tx_1(n) - Tx_2(n)| = |Tx_1(n_2) - Tx_2(n_2)| \le q_2\|x_1 - x_2\|.$$

Considering the cases of a and b, for any $x_1, x_2 \in A$, $n \ge n_0$, we have

$$|Tx_1(n) - Tx_2(n)| \le q_2\|x_1 - x_2\|.$$

So T is a contraction mapping on A.Based on the above discussion we can conclude from the Banach contraction mapping principle that there exist a fixed point x of T on A, namely, $Tx = x$, where $x = \{x(n)\}$ satisfies

$$
Tx(n) = \begin{cases}
1 - \dfrac{1}{p} - \dfrac{1}{p}x(n+\tau) + & \\
\dfrac{1}{p}\sum\limits_{s=n+\tau}^{+\infty} C^2_{s+2-n-r}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \ge n_2, \\
Tx(n_2), & n_0 \le n < n_2.
\end{cases}
$$

Therefore this fixed point $\{x(n)\}$ is a positive sequence. Differentiating three times the above expression, we get

$$\Delta^3[x(n) + px(n-\tau)] + R_1(n)x(n-\delta_1) - R_2(n)x(n-\delta_2) = 0.$$

Hence this fixed point $\{x(n)\}$ is a positive solution of the equation (1).

Case 3: $-1 < p < 0$.

From (3) and (4), we select a positive integer $n_3 \geq \max\{T_1, n_0 + \delta\}$ where $\delta = \max\{\tau, \delta_1, \delta_2\}$, such that

$$\sum_{n=n_3}^{+\infty} n^2[R_1(n) + R_2(n)] < p + 1, \tag{20}$$

$$0 \leq \sum_{n=n_3}^{+\infty} n^2[M_6R_1(n) - M_5R_2(n)] \leq (p+1)(M_6 - 1), \tag{21}$$

where M_5 and M_6 are positive constants and satisfy

$$0 < M_5 \leq 1 < M_6. \tag{22}$$

Let

$$A = \{x \in L_\infty :\ M_5 \leq x(n) \leq M_6,\ \ n \in N(n_0)\}. \tag{23}$$

It is obvious that A is a bounded closed convex subset on L_∞.

Define a mapping $T\colon\ A \to L_\infty$ as follow:

$$Tx(n) = \begin{cases} 1 + p - px(n-\tau) + & \\ \sum\limits_{s=n}^{+\infty} C_{s+2-n}^2(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \geq n_3, \\ Tx(n_3), & n_0 \leq n < n_3. \end{cases} \tag{24}$$

We shall prove that T is a self-mapping on A where the following two situations are to be discussed.

Case 3-a: $n \geq n_3$.

For any $x \in A$, from (23), (24), we find that

$$\begin{aligned} Tx(n) &= 1 + p - px(n-\tau) + \sum_{s=n}^{+\infty} C_{s+2-n}^2(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)) \\ &\leq 1 + p - pM_6 + \sum_{s=n}^{+\infty} s^2(R_1(s)M_6 - R_2(s)M_5) \end{aligned}$$

and therefore, from (21), we have

$$Tx(n) \leq 1 + p - pM_6 + (1+p)(M_6 - 1) = M_6. \tag{25}$$

From (4), we have

$$\sum_{s=n}^{+\infty} s^2(R_1(s)M_6 - R_2(s)M_5) \geq 0.$$

Hence, from (22) and (23), we have

$$\begin{aligned} Tx(n) &\geq 1 + p - pM_5 = (1+p) - (1+p)M_5 + M_5 \\ &= (1+p)(1-M_5) + M_5 = M_5. \end{aligned} \tag{26}$$

Case 3-b: $n_0 \leq n < n_3$.
For any $x \in A$, from (24) we find that

$$Tx(n) = Tx(n_3).$$

Then, from (25) and (26) we obtain

$$M_5 \leq Tx(n_3) \leq M_6.$$

Hence

$$M_5 \leq Tx(n) \leq M_6.$$

In both cases of a and b, for any $x \in A$, we have

$$M_5 \leq Tx \leq M_6,$$

namely, $Tx \in A$. Hence, T is a self-mapping on A. Now we shall prove that T is a contraction mapping on A under the two situations bellow.

Proof of Case 3-a. $n \geq n_3$.
For any $x_1, x_2 \in A$, we have

$$\begin{aligned} &|Tx_1(n) - Tx_2(n)| \\ &\quad = \Big| - px_1(n-\tau) + \sum_{s=n}^{+\infty} C^2_{s+2-n}(R_1(s)x_1(s-\delta_1) - R_2(s)x_1(s-\delta_2)) \\ &\quad + px_2(n-\tau) - \sum_{s=n}^{+\infty} C^2_{s+2-n}(R_1(s)x_2(s-\delta_1) - R_2(s)x_2(s-\delta_2))| \\ &\quad \leq |-px_1(n-\tau) + px_2(n-\tau)| + \sum_{s=n}^{+\infty} C^2_{s+2-n}R_1(s)|x_1(s-\delta_1) \\ &\quad - x_2(s-\delta_1)| + \sum_{s=n}^{+\infty} C^2_{s+2-n}R_2(s)|x_1(s-\delta_2) - x_2(s-\delta_2)|. \end{aligned}$$

Hence from (5), the following inequality is hold

$$\begin{aligned} |Tx_1(n) - Tx_2(n)| &\leq \left[p + \sum_{s=n}^{+\infty} C^2_{s+2-n}(R_1(s) - R_2(s))\right] \|x_1 - x_2\| \\ &\leq \left[p + \sum_{s=n}^{+\infty} s^2(R_1(s) - R_2(s))\right] \|x_1 - x_2\|. \end{aligned}$$

According to (20), there exists $0 < q_3 < 1$, such that

$$|Tx_1(n) - Tx_2(n)| \leq q_3 \|x_1 - x_2\|.$$

Proof of Case 3-b. $n_0 \leq n < n_3$.

From (20), (24) we have

$$|Tx_1(n) - Tx_2(n)| = |Tx_1(n_3) - Tx_2(n_3)| \leq q_3 \|x_1 - x_2\|.$$

In both cases of a and b, for any $x_1, x_2 \in A$, when $n \geq n_0$, we have

$$|Tx_1(n) - Tx_2(n)| \leq q_3 \|x_1 - x_2\|.$$

So T is a contraction mapping on A. Based on the Banach contraction mapping principle we know that there exist a fixed point x of T on A, say, $Tx = x$, where $x = \{x(n)\}$ satisfies

$$Tx(n) = \begin{cases} 1 + p - px(n-\tau) + \\ \sum\limits_{s=n}^{+\infty} C_{s+2-n}^{2} (R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \geq n_3, \\ Tx(n_3), & n_0 \leq n < n_3. \end{cases}$$

Thus this fixed point $\{x(n)\}$ is a positive sequence. Differentiating three times the above expression, we get

$$\Delta^3[x(n) + px(n-\tau)] + R_1(n)x(n-\delta_1) - R_2(n)x(n-\delta_2).$$

Hence, this fixed point $\{x(n)\}$ is a positive solution of the equation (1).

Case 4: $p < -1$

From (3) and (4), we select a positive integer $n_4 > T_1 > n_0$ which is large enough to satisfy

$$n_4 + \tau \geq n_0 + \max\{\delta_1, \delta_2\}$$

such that

$$\sum_{n=n_4}^{+\infty} n^2 [R_1(n) + R_2(n)] < -p - 1, \tag{27}$$

$$\sum_{n=n_4}^{+\infty} n^2 [M_8 R_1(n) - M_7 R_2(n)] \leq (p+1)(M_7 - 1) \tag{28}$$

where M_7, M_8 are positive constants and satisfy

$$0 < M_7 < 1 < M_8. \tag{29}$$

Let

$$A = \{x \in L_\infty : \ M_7 \leq x(n) \leq M_8, \ n \in N(n_0)\}. \tag{30}$$

It is obvious that A is a bounded closed convex subset on L_∞.

Define a mapping $T\colon\ A \to L_\infty$ as follow:

$$Tx(n) = \begin{cases} 1 + \dfrac{1}{p} - \dfrac{1}{p}x(n+\tau) + \\ \dfrac{1}{p}\sum\limits_{s=n+r}^{+\infty} C^2_{s+2-n-r}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \ge n_4, \\ Tx(n_4), & n_0 \le n < n_4. \end{cases} \tag{31}$$

We shall prove that T is a self-mapping on A under the following two situations.

Case 4-a: $n \ge n_4$.

For any $x \in A$, from (28), (30), (31) and $p < -1$ we find that

$$\begin{aligned} Tx(n) &= 1 + \frac{1}{p} - \frac{1}{p}x(n+\tau) + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)) \\ &\ge 1 + \frac{1}{p} - \frac{1}{p}M_7 + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)M_8 - R_2(s)M_7) \\ &\ge 1 + \frac{1}{p} - \frac{1}{p}M_7 + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2(R_1(s)M_8 - R_2(s)M_7) \\ &\ge 1 + \frac{1}{p} - \frac{1}{p}M_7 + \frac{1}{p}(p+1)(M_7 - 1) = M_7. \end{aligned} \tag{32}$$

Since from (4), we have

$$\begin{aligned} &\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)) \\ &\qquad = \sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}x(s-\delta_2)\left[\frac{x(s-\delta_1)}{x(s-\delta_2)}R_1(s) - R_2(s)\right] \ge 0. \end{aligned}$$

Hence, from (29), (30), we have

$$\begin{aligned} Tx(n) &= 1 + \frac{1}{p} - \frac{1}{p}x(n+\tau) + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C^2_{s+2-n-\tau}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)) \\ &\le 1 + \frac{1}{p} - \frac{1}{p}M_8 \le M_8. \end{aligned} \tag{33}$$

Case 4-b: $n_0 \le n < n_4$.

For any $x \in A$, from (31) we find that

$$Tx(n) = Tx(n_4).$$

Then, from (32) and (33) we obtain

$$M_7 \le Tx(n_4) \le M_8.$$

Hence, we have

$$M_7 \leq Tx(n) \leq M_8.$$

In both two cases of a and b, for any $x \in A,$ we have

$$M_7 \leq Tx \leq M_8.$$

Hence $Tx \in A,$ namely, T is a self-mapping on $A.$ We shall prove that T is a contraction mapping on A as bellow.

Proof of Case 4-a. $n \geq n_4.$
For any $x_1, x_2 \in A,$ we have

$$\begin{aligned}
&|Tx_1(n) - Tx_2(n)| \\
&\quad = \Bigg| -\frac{1}{p}x_1(n+\tau) + \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C_{s+2-n-\tau}^2 (R_1(s)x_1(s-\delta_1) - R_2(s)x_1(s-\delta_2)) \\
&\quad + \frac{1}{p}x_2(n+\tau) - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} C_{s+2-n-\tau}^2 (R_1(s)x_2(s-\delta_1) - R_2(s)x_2(s-\delta_2))\Bigg| \\
&\quad \leq \Bigg| -\frac{1}{p}x_1(n+\tau) + \frac{1}{p}x_2(n+\tau)\Bigg| - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2 R_1(s)|x_1(s-\delta_1) - x_2(s-\delta_1)| \\
&\quad - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2 R_2(s)|x_1(s-\delta_2) - x_2(s-\delta_2)| \\
&\quad \leq \Bigg[-\frac{1}{p} - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2 (R_1(s) + R_2(s))\Bigg] \|x_1 - x_2\|
\end{aligned}$$

and from (27), we know

$$\frac{1}{p} - \frac{1}{p}\sum_{s=n+\tau}^{+\infty} s^2 (R_1(s) + R_2(s)) < 1.$$

Then, there exists $0 < q_4 < 1,$ such that

$$|Tx_1(n) - Tx_2(n)| \leq q_4\|x_1 - x_2\|.$$

Proof of Case 4-b. $n_0 \leq n < n_4.$
From (31), we also have

$$|Tx_1(n) - Tx_2(n)| = |Tx_1(n_4) - Tx_2(n_4)| \leq q_4\|x_1 - x_2\|.$$

In both cases of a and b, for any $x_1, x_2 \in A,$ $n \geq n_0,$ we have

$$|Tx_1 - Tx_2| \leq q_4\|x_1 - x_2\|.$$

So T is a contraction mapping on A. Based on the above analysis, we can conclude from the Banach contraction mapping principle that there exist a fixed point x of T on A, namely, $Tx = x$, where $x = \{x(n)\}$ satisfies

$$Tx(n) = \begin{cases} 1 + \dfrac{1}{p} - \dfrac{1}{p}x(n+\tau) + \\ \dfrac{1}{p}\sum\limits_{s=n+r}^{+\infty} C^2_{s+2-n-r}(R_1(s)x(s-\delta_1) - R_2(s)x(s-\delta_2)), & n \ge n_4, \\ Tx(n_4), & n_0 \le n < n_4. \end{cases}$$

From this, the fixed point $\{x(n)\}$ is a positive sequence. Differentiating three times the above expression, we get

$$\Delta^3[x(n) + px(n-\tau)] + R_1(n)x(n-\delta_1) - R_2x(n-\delta_2) = 0.$$

Hence, this fixed point $\{x(n)\}$ is a positive solution of the equation (1). Therefore the theorem is proved.

3 Conclusions

Under the conditions of third order, this paper studies the existence of the nonoscillatory solution of the neutral delay difference equation with positive and negative coefficients and gets a sufficient condition for the existence of the nonoscillatory solution. We can find that the similar results of the second order difference equation in the literature [12] have been successfully extended to the third order one. This will naturally urge us to consider whether the high order one has the similar results. When we study this problem, the way applied in this paper can be helpful to us.

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