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# Convergence of Solutions to a Class of Systems of Delay Differential Equations

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**Abstract:** This paper is concerned with a delay differential system which can be regarded as a mathematical model of compartmental system with pipes and time delays. It is shown that every solution of such a differential system tends to a constant vector as $t \to \infty$. The obtained results improve and extend some existing ones in the literature.

**Keywords:** *Convergence; delay differential equation; compartmental system.*

**Mathematics Subject Classification (2000):** 34C12, 39A11.

## 1 Introduction

Recently, there has been much attention in the study of the asymptotic behavior of solutions for the following scalar delay differential equation

$$\frac{dx(t)}{dt} = -F(x(t)) + F(x(t-r)), \tag{1.1}$$

where $r > 0$ is a constant, and $F\colon R \to R$ is continuous. System (1.1), which has been used to model a variety of phenomena such as some population growth, the spread of epidemics, the dynamics of capital stocks, *etc.* has been discussed extensively in the literature (see, for example, [2−5, 7, 8, 10, 12−14, 17]), in which various approaches including the first integral, invariance principle of Lyapunov–Razumikhin type, *etc.* have been applied to conclude that every solution of system (1.1) tends to a constant. However, most of the study deals with the problem of convergence of solutions of system (1.1) under the assumption that $F$ is either strictly increasing or locally Lipschitz continuous and nondecreasing. To the best of our knowledge, there exist no results for the asymptotic behavior of system (1.1) with $F$ only assumed to be nondecreasing. Meanwhile, we stress

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the fact that the aforementioned approaches seem to fail to be applicable to system (1.1) when $F$ is only assumed to be nondecreasing and hence a different analysis is needed in this case. This situation motivates us to study further system (1.1) or its more general case with new methods or techniques based on the assumption that $F$ is only required to be nondecreasing in this work.

More precisely, in this paper we are concerned with the following system of delay differential equations

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -F(x_1(t)) + F(x_2(t-r_2)), \\ \frac{dx_2(t)}{dt} &= -F(x_2(t)) + F(x_1(t-r_1)), \end{aligned} \tag{1.2}$$

where $r_1$ and $r_2$ are positive constants, $F$ is continuous and nondecreasing on $R$. System (1.2) can be used as a mathematical model of compartmental system with pipes and time delays, where $x_i(t)$ denotes the amount of the material in the $i$-th compartment at time $t$, $r_i$ denotes the transit time for the material flow to pass through the pipe, $F(x_i(t))$ denotes the rate of flow of material loss of the $i$ compartment, and $F(x_i(t-r_i))$ denotes the rate of material flows from the $i$-th compartment into the $j$-th compartment through a pipe, $i \neq j$, $i, j = 1, 2$. Compartmental models are frequently used in, e.g., theoretical epidemiology, physiology, population dynamics, the analysis of ecosystems, and chemical reaction kinetics. For more details, we refer to the work of Anderson [1], Györi [9, 10] and Györi and Wu [11]. The main goal of the present paper is to show that every solution of system (1.2) approaches a constant vector by using monotonicity arguments. To this end, we begin by describing some monotonicity properties possessed by system (1.2) with the help of the comparison principles for delay differential equations developed by Smith [15]. Then, we introduce the notion of the admitting closed interval with respect to $F$ and present some important properties of system (1.2) by making use of the notion. Finally, based on the above preparations, we prove our main results, which improve and extend the corresponding results in the aforementioned literature.

The paper is organized as follows. In Section 2, we introduce some necessary notations and establish some preliminary results, which are important in the proofs of our main results. Our main results are presented in Section 3.

## 2 Preliminary Results

In this section, some important properties of system (1.2) will be presented, which are of importance in proving our main results in Section 3.

Throughout this paper, we will use $R_+$ to denote the set of all nonnegative real numbers and $R_+^2$ denote the set of all nonnegative vectors in $R^2$. Define

$$C = C([-r_1, 0], R) \times C([-r_2, 0], R)$$

as the Banach space equipped with a supremum norm. Define

$$C_+ = C([-r_1, 0], R_+) \times C([-r_2, 0], R_+).$$

It follows that $C_+$ is an order cone in $C$ and hence, $C_+$ induces a closed partial ordered relation on $C$. For any $\varphi, \psi \in C$ and $A \subseteq C$, the following notations will be used:

$\varphi \le \psi$ iff $\psi - \varphi \in C_+$, $\varphi < \psi$ iff $\varphi \le \psi$ and $\varphi \ne \psi$, $\varphi \ll \psi$ iff $\psi - \varphi \in \operatorname{Int} C_+$, $\varphi \le A$ iff $\varphi \le \psi$ for any $\psi \in A$, $\varphi < A$ iff $\varphi < \psi$ for any $\psi \in A$, $\varphi \ll A$ iff $\varphi \ll \psi$ for any $\psi \in A$. Notations such as "$\ge$", "$>$" and "$\gg$" have the natural meanings.

In what follows, we assume that $\varphi \in C$ and use $x_t(\varphi)$ $(x(t, \varphi))$ to denote the solution of (1.2) with the initial data $x_0(\varphi) = \varphi$. For any $x \in R^2$, let us define $\widehat{x} = ((\widehat{x})_1, (\widehat{x})_2)$, where $(\widehat{x})_i(\theta) = x_i$, $\theta \in [-r_i, 0]$, $i = 1, 2$.

**Lemma 2.1** *For any constants $K$, $t_0$ and $x_0$, the initial value problem*

$$\begin{aligned} \frac{dx(t)}{dt} &= -F(x(t)) + K, \\ x(t_0) &= x_0 \end{aligned} \tag{2.1}$$

*exists a unique solution $x(t, t_0, x_0)$ on $[t_0, \infty)$.*

*Proof* From the Peano theorem, we know that solutions of the initial value problem (2.1) locally exist. Again, since $F$ is nondecreasing, it follows from [6] that right-hand solutions of the initial value problem (2.1) are also unique. Hence, $x(t, t_0, x_0)$ exists and is unique on $[t_0, \eta)$ for some positive constant $\eta$, where $[t_0, \eta)$ denotes the maximal right-interval of existence of $x(t, t_0, x_0)$. We will show that $\eta = +\infty$. Otherwise, $\overline{\lim\limits_{t \to \eta^-}} |x(t, t_0, x_0)| = +\infty$. We next distinguish several cases to finish the proof.

Case 1. There exists $t_1 \in [t_0, \eta)$ such that $-F(x(t_1, t_0, x_0)) + K = 0$. Then let

$$\widetilde{x}(t) = \begin{cases} x(t, t_0, x_0), & t_0 \le t \le t_1, \\ x(t_1, t_0, x_0), & t \ge t_1. \end{cases}$$

It follows that $\widetilde{x}(t)$ satisfies (2.1) and hence, $x(t, t_0, x_0) \equiv \widetilde{x}(t)$. But this contradicts $\eta < +\infty$.

Case 2. $-F(x(t, t_0, x_0)) + K < 0$ for $t \in [t_0, \eta)$. Then $x(t, t_0, x_0)$ is strictly decreasing on $[t_0, \eta)$ and thus, $x(t, t_0, x_0) \le x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. It follows that $-F(x(t, t_0, x_0)) + K \ge -F(x(t_0, t_0, x_0)) + K$ for all $t \in [t_0, \eta)$, and hence, $x(t, t_0, x_0) \ge (K - F(x(t_0, t_0, x_0)))t + x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. Therefore, $\lim\limits_{t \to \eta^-} |x(t, t_0, x_0)| < +\infty$, which yields a contradiction.

Case 3. $-F(x(t, t_0, x_0)) + K > 0$ for $t \in [t_0, \eta)$. Then $x(t, t_0, x_0)$ is strictly increasing on $[t_0, \eta)$ and thus, $x(t, t_0, x_0) \ge x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. It follows that $-F(x(t, t_0, x_0)) + K \le -F(x(t_0, t_0, x_0)) + K$ for all $t \in [t_0, \eta)$, and hence, $x(t, t_0, x_0) \le (K - F(x(t_0, t_0, x_0)))t + x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. Therefore, $\lim\limits_{t \to \eta^-} |x(t, t_0, x_0)| < +\infty$, which yields a contradiction.

The proof of the lemma is complete.

**Lemma 2.2** *Let $0 < r \in R$ be given and $d \in C([t_0, t_0 + r])$. Then, for any constant $x_0$, the initial value problem*

$$\begin{aligned} \frac{dx}{dt}(t) &= -F(x(t)) + d(t), \\ x(t_0) &= x_0, \end{aligned}$$

*exists a unique solution $x(t, t_0, x_0)$ on $[t_0, t_0 + r]$.*

*Proof* Lemma 2.2 follows by applying the standard technique of differential inequalities and Lemma 2.1.

**Lemma 2.3** *Let $\varphi \in C$. Then $x_t(\varphi)$ exists and is unique on $[0,+\infty)$.*

*Proof* Let $\tau = \min\{r_1, r_2\}$. We will show that $x_t(\varphi)$ exists and is unique on $[0,\tau]$. To see this, let $d_1(t) = F(\varphi_2(t-r_2))$ and $d_2(t) = F(\varphi_1(t-r_1))$ for any $t \in [0,\tau]$. Consider the following system

$$\begin{aligned} \frac{dx}{dt}_i(t) &= -F(x_i(t)) + d_i(t),\\ x_i(0) &= \varphi_i(0), \end{aligned}$$

where $i \in \{1,2\}$. By Lemma 2.2, $x_i(t)$ exists and is unique on $[0,\tau]$. Hence, $x_i(t,\varphi)$ exists and is unique on $[0,\tau]$, that is, $x_t(\varphi)$ exists and is unique on $[0,\tau]$. It follows from induction that $x_t(\varphi)$ exists and is unique on $[0,+\infty)$. The proof of the lemma is now complete.

Before continuing, it is convenient to introduce the following notations and establish some convention. Set

$$E_F = \{e \in R^2 \colon F(e_1) = F(e_2)\}.$$

Define $O(\varphi) = \{x_t(\varphi) \colon t \geq 0\}$. If $O(\varphi)$ is bounded, then $\overline{O(\varphi)}$ is compact in $C$, where $\overline{O(\varphi)}$ denotes the closure of $O(\varphi)$. If $O(\varphi)$ is bounded, define

$$\omega(\varphi) = \bigcap_{t\geq 0} \overline{O(x_t(\varphi))},$$

i. e., $\omega(\varphi) = \{\psi \in C \colon$ there exists a subsequence $t_k \to +\infty$ such that $x_{t_k}(\varphi) \to \psi\}$. It follows that $\omega(x)$ is nonempty, compact, invariant and connected.

We make the following key definition.

**Definition 2.1** Let

$$s(\alpha) = \sup\{\beta \in R \colon F(\beta) = F(\alpha)\} \quad \text{and} \quad i(\alpha) = \inf\{\beta \in R \colon F(\beta) = F(\alpha)\}.$$

$[a,b]$ is called *an admitting closed super-interval with respect to $F$* if $F(a) = F(b)$ and $a = i(a)$; $[a,b]$ is called *an admitting closed sub-interval with respect to $F$* if $F(a) = F(b)$ and $b = s(b)$.

**Lemma 2.4** *Let $\varphi, \psi \in C$ with $\psi \geq \varphi$. Then $x_t(\psi) \geq x_t(\varphi, F)$ for $t \in R_+$. Moreover, we have the following*

(1) *Assume that $e \in E$, $\varphi \geq \hat{e}$ and $i \in \{1,2\}$. If $e_i < s(e_i)$ and $\varphi_i(0) > e_i$, then $x_i(t,\varphi) > e_i$ for $t \geq 0$.*
(2) *Assume that $e \in E$, $\varphi \leq \hat{e}$ and $i \in \{1,2\}$. If $e_i > i(e_i)$ and $\varphi_i(0) < e_i$, then $x_i(t,\varphi) < e_i$ for $t \geq 0$.*

*Proof* The first assertion of the lemma follows from [15, Proposition 2.1].

We next will prove conclusion (1), conclusion (2) can be proved similarly. It follows that $x_t(\varphi) \geq \hat{e}$ for $t \geq 0$. Without loss of generality, we may assume that $i = 1$. From (1.2), we get

$$\begin{aligned} x_1'(t,\psi) &= -F(x_1(t,\psi)) + F(x_2(t-r_2,\, \psi))\\ &\geq -F(x_1(t,\psi)) + F(e_2) = -F(x_1(t,\psi)) + F(e_1). \end{aligned}$$

Consider the following auxiliary system

$$y'(t) = -F(y(t)) + F(e_1),$$
$$y(0) = \min\{x_1(0,\varphi), s(e_1)\}.$$

Then, from Lemma 2.1, we know that $y(t) \equiv y(0) > e_1$ for $t \geq 0$. Thus, by the comparison theorem for ordinary differential equations in Walter [16], we have

$$x_1(t,\varphi) \geq y(t) > e_1 \quad \text{for} \quad t \geq 0.$$

This completes the proof.

Now, we are in a position to present an important properties of system (1.2).

**Lemma 2.5** *Let $[a,b]$ be an admitting closed super-interval with respect to $F$. Then the following conclusions hold:*

(1) *if $e_1 \in [a,b)$ and $e_2 = a$, then for any $M > 0$, there exists $\varepsilon_M > 0$ such that $\lim\limits_{t\to+\infty} x_t(\varphi, F) > \widehat{\langle e_1, e_2\rangle}$ for any $\varphi \geq (e_1 + \widehat{M, e_2} - \varepsilon_M)$;*

(2) *if $e_1 = a$ and $e_2 \in [a,b)$, then for any $M > 0$, there exists $\varepsilon_M > 0$ such that $\lim\limits_{t\to+\infty} x_t(\varphi, F) > \widehat{\langle e_1, e_2\rangle}$ for any $\varphi \geq (e_1 - \widehat{\varepsilon_M, e_2} + M)$.*

*Proof* We only prove conclusion (1), the other conclusion can be proved similarly. Without loss of generality, we may assume that $F(a) = 0$. Let $M > 0$. In view of Lemma 2.4, we may assume that $M < b - e_1$. Let $r(x) = M - x + F(e_2 - x)r_2$. Then $\lim\limits_{x\to 0^+} r(x) = M > 0$. Hence, there exists $\varepsilon_M > 0$ such that $r(\varepsilon_M) > M/2$. Let $\xi = (e_1 + \widehat{M, e_2} - \varepsilon_M) \in C$. Then by Lemma 2.4, $x(t,\xi,F) \leq (e_1 + M, e_2)$ for $t \geq 0$. In what follows, define

$$t_1 = \inf\{t > 0\colon\, x_1(t,\xi,F) \leq e_1\} \quad \text{and} \quad t_2 = \inf\{t > 0\colon\, x_2(t,\xi,F) = e_2\}.$$

We will show that $t_1 \leq t_2 + r_2$. If not, then $t_1 > t_2 + r_2$. From (1.2), we obtain

$$\frac{dx_2(t)}{dt} = -F(x_2(t)) + F(x_1(t - r_1)).$$

Therefore,

$$x_2(t_2) = e_2 \quad \text{and} \quad \frac{dx_2(t)}{dt} = -F(x_2(t)) \quad \text{for} \quad t \in [t_2,\, t_1 + r_1].$$

By Lemma 2.1, we have $x_2(t) = e_2$ for any $t \in [t_2,\, t_1 + r_1]$. From (1.2) again, we obtain

$$\frac{dx_1(t)}{dt} = -F(x_1(t)) + F(x_2(t - r_2)).$$

Hence,

$$x_1(t_2 + r_2) > e_1 \quad \text{and} \quad \frac{dx_1(t)}{dt} = -F(x_1(t)) \quad \text{for} \quad t \in [t_2 + r_2,\, t_1 + r_1 + r_2].$$

Thus, by Lemma 2.1, we have

$$x_1(t) = x_1(t_2 + r_2) > e_1 \quad \text{for} \quad t \in [t_2 + r_2,\, t_1 + r_1 + r_2].$$

Therefore, $x_1(t_1) > e_1,$ a contradiction to the definition of $t_1$. We next distinguish two cases to finish the proof.

Case 1. $t_1 = +\infty$. It follows that $t_2 = +\infty$. Thus,

$$x_1(t) > e_1 \quad \text{and} \quad x_2(t) < e_2 \quad \text{for} \quad t \in R_+.$$

Therefore, from system (1.2), it follows that $x_1(t)$ is decreasing and $x_2(t)$ is increasing on $R_+$. Hence, there exist $e_1', e_2' \in R$ such that $x_1(t) \to e_1'$ and $x_2(t) \to e_2'$. So, we have $e_1' \geq e_1$ and $e_2' \leq e_2$. In view of Definition 2.1 and the fact that $e_2 = a,$ we obtain $e_2' = e_2$. We will show that $e_1' > e_1$. Otherwise, $e_1' = e_1$. From (1.2), it follows that

$$\begin{aligned} \frac{dx_1'(t)}{dt} &= -F(x_1(t)) + F(x_2(t - r_2)),\\ \frac{dx_2'(t)}{dt} &= -F(x_2(t)) + F(x_1(t - r_1)). \end{aligned}$$

Thus,

$$\begin{aligned} x_1(t) - (e_1 + M) &= \int_0^t F(x_2(s - r_2))\, ds,\\ x_2(t) - (e_2 - \varepsilon_M) &= -\int_0^t F(x_2(s))\, ds. \end{aligned}$$

Letting $t \to \infty,$ we have

$$\begin{aligned} -M &= \int_{-r_2}^0 F(x_2(s))ds + \int_0^\infty F(x_2(s))\, ds,\\ \varepsilon_M &= -\int_0^\infty F(x_2(s))\, ds. \end{aligned}$$

Therefore, $M + F(e_2 - \varepsilon_M)r_2 - \varepsilon_M = 0,$ a contradiction to the choice of $\varepsilon_M$.

Case 2. $t_1 < \infty$. Then, from (1.2), it follows that

$$\begin{aligned} \frac{dx_1'(t)}{dt} &= -F(x_1(t)) + F(x_2(t - r_2)),\\ \frac{dx_2'(t)}{dt} &= -F(x_2(t)) + F(x_1(t - r_1)). \end{aligned}$$

Thus,

$$\frac{dx_1(t)}{dt} = F(x_2(t - r_2)) \quad \text{for} \quad t \in [0, t_1], \quad \text{and} \quad \frac{dx_2(t)}{dt} = -F(x_2(t)) \quad \text{for} \quad t \in [0,\, t_1 + r_1].$$

Hence,

$$x_1(t_1) - x_1(0) = \int_0^{t_1} F(x_2(s - r_2))\,ds, \quad \text{and} \quad x_2(t_1 - r_2) - x_2(0) = -\int_0^{t_1 - r_2} F(x_2(s))\,ds.$$

Therefore, $x_1(t_1) - x_1(0) = \int\limits_0^{t_1-r_2} F(x_2(s))ds + F(e_2 - \varepsilon_M)r_2$. It follows that

$$x_1(t_1) - x_1(0) = x_2(0) - x_2(t_1 - r_2) + F(e_2 - \varepsilon_M)r_2.$$

Consequently,

$$e_1 \ge x_1(t_1) \ge e_1 + M + e_2 - \varepsilon_M - e_2 + F(e_2 - \varepsilon_M)r_2,$$

that is, $M - \varepsilon_M + F(e_2 - \varepsilon_M)r_2 \le 0$, a contradiction to the choice of $\varepsilon_M$. The proof of the lemma is now complete.

Arguing as in the proof of Lemma 2.5, we can get the following result.

**Lemma 2.6** *Let $[a, b]$ is an admitting closed sub-interval with respect to $F$. Then the following conclusions hold:*

(1) *if $e_1 \in [a, b)$ and $e_2 = a$, then for any $M > 0$, there exists $\varepsilon_M > 0$ such that $\lim\limits_{t\to+\infty} x_t(\varphi, F) > \widehat{\langle e_1, e_2\rangle}$ for any $\varphi \ge (e_1 + \widehat{M, e_2} - \varepsilon_M)$;*

(2) *if $e_1 = a$ and $e_2 \in [a, b)$, then for any $M > 0$, there exists $\varepsilon_M > 0$ such that $\lim\limits_{t\to+\infty} x_t(\varphi, F) > \widehat{\langle e_1, e_2\rangle}$ for any $\varphi \ge (e_1 - \widehat{\varepsilon_M, e_2} + M)$.*

In what follows, we assume that $\varphi \in C$. If $O(\varphi)$ is bounded, define

$$D_\varphi^+ = \{e \in E_F\colon\ \hat{e} \le \omega(\varphi)\} \quad \text{and} \quad D_\varphi^- = \{e \in E_F\colon\ \hat{e} \ge \omega(\varphi)\}.$$

We are now in a position to state another lemma.

**Lemma 2.7** *$D_\varphi^+$ contains the maximum element, that is, $\sup D_\varphi^+ \in D_\varphi^+$. Hence, there exists $e^* \in D_\varphi^+$ such that $e^* \ge D_\varphi^+$.*

*Proof* Since $O(\varphi)$ is bounded, $\omega(\varphi)$ is compact. Hence, there exists $\alpha \in R$ such that

$$\widehat{(\alpha, \alpha)} \le \omega(\varphi).$$

Let $D = \{e \in D_\varphi^+\colon\ (\alpha, \alpha) \le e\}$. Then $D$ is compact. It follows that $D$ contains the maximal element and we denote it by $e^* = (e_1^*, e_2^*)$. Next we will show that $\sup D = e^*$. If not, then there exist $e_1, e_2 \in R$ such that $(e_1, e_2) \in D$ and $(e^* - (e_1, e_2)) \notin R_+^2$. Without loss of generality, we may assume that $e_1^* > e_1$ and $e_2^* < e_2$. By the definition of $D$, we obtain

$$\widehat{(e_1^*, e_2)} \le \omega(\varphi) \quad \text{and} \quad F(e_1^*) = F(e_2).$$

Therefore,

$$\widehat{(e_1^*, e_2)} \in D \quad \text{and} \quad (e_1^*, e_2^*) < (e_1^*, e_2),$$

a contradiction to the definition of $e^*$. It follows that $\sup D_\varphi^+ = e^*$. This completes the proof.

Arguing as in the proof of Lemma 2.7, we can get the following result.

**Lemma 2.8** *$D_\varphi^-$ contains the minimum element, that is, $\inf D_\varphi^- \in D_\varphi^-$. Hence, there exists $e^* \in D_\varphi^-$ such that $e^* \leq D_\varphi^-$.*

## 3 Main Results

The purpose of this section is to show that every solution of (1.2) tends to a constant vector as $t \to \infty$, which is our main result in this paper.

**Lemma 3.1** *Assume that $\varphi \in C$. Then $O(\varphi)$ is bounded. Hence, $\omega(\varphi)$ is compact.*

*Proof* Lemma 3.1 follows immediately from Lemma 2.4 and system (1.2).

**Lemma 3.2** *Let $\varphi \in C$ and $e^* = \sup D_\varphi^+$. If $\omega(\varphi)\backslash\{\widehat{e^*}\} \neq \phi$, then $e_1^* = e_2^* = s(e_2^*)$.*

*Proof* By way of contradiction, if this is not true, then there exists $i \in \{1, 2\}$ such that $e_i < s(e_2^*)$. We next distinguish several cases to finish the proof.

Case 1. $e_1^* < s(e_2^*)$ and $e_2^* < s(e_2^*)$.

By the invariance of $\omega(\varphi)$, we may assume that there exists $\psi \in \omega(\varphi)$ such that $\psi_1(0) > e_1^*$. From the conclusion (1) of Lemma 2.4, it follows that

$$(x_{r_1}(\psi))_1(\theta) > e_1^*, \quad \theta \in [-r_1, 0].$$

Thus, we can choose $M > 0$ such that

$$e_1^* + 3M < s(e_2^*) \quad \text{and} \quad (x_{r_1}(\psi)) \geq (e_1^* + \widehat{3M,\, i}(e_2^*)).$$

Let $a = i(e_1^*)$, $b = s(e_1^*)$, $e_1 = M + e_1^*$ and $e_2 = a$. Then, for the above $M > 0$ and the admitting closed super-interval $[a, b]$, by Lemma 2.5 (1), there exists $\varepsilon_M > 0$ such that

$$\varliminf_{t\to\infty} x_t(\eta) \geq \widehat{(e_1, e_2)}, \quad \eta \geq (e_1 + \widehat{M,\, e_2} - \varepsilon_M).$$

From the choice of $M > 0$, it follows that $x_{r_1}(\psi) \gg (e_1 + \widehat{M, e_2} - \varepsilon_M)$. By the definition of $\omega(\varphi)$ again, there exists $t_1 > 0$ such that $x_{t_1}(\varphi) \geq (e_1 + \widehat{M,\, e_2} - \varepsilon_M)$. Hence, $\varliminf_{t\to\infty} x_t(\varphi) \geq \widehat{(e_1, e_2)}$. Thus,

$$\omega(\varphi) \geq \widehat{(e_1, e_2)} = (e_1^* \widehat{+ M,}\, e_2^*).$$

Again, from the choice of $M > 0$, it follows that $(e_1^* \widehat{+ M,}\, e_2^*) \in E_F$. But this contradicts the fact that $e^* = \sup D_\varphi^+$.

Case 2. $e_1^* < s(e_2^*)$ and $e_2^* = s(e_2^*)$.

We claim that for any $\psi \in \omega(\varphi)$, $\psi_1(\theta) = e_1^*$, $\theta \in [-r_1, 0]$. If not, then, by the invariance of $\omega(\varphi)$, there exists $\psi \in \omega(\varphi)$ such that $\psi_1(0) > e_1^*$. Arguing as in the proof of Case 1, we can prove that this is a contraction. Therefore, our claim is true. Since $\omega(\varphi) \setminus \{e^*\} \neq \phi$, it follows from the above claim and the invariance of $\omega(\varphi)$ that there exists $\psi \in \omega(\varphi)$ such that $\psi_2(0) > e_2^*$. From (1.2), we obtain

$$x_1'(t, \psi) = -F(x_1(t, \psi)) + F(x_2(t - r_2,\, \psi)).$$

Thus, $F(e_1^*) = F(x_2(t - r_2, \psi))$. Hence, $F(\psi_2(0)) = F(e_1^*)$. From the Definition 2.1, it follows that

$$\psi_2(0) \le s(e_1^*) = s(e_2^*) = e_2^*,$$

which yields a contradiction. (*Please, make a correction of this statement!*)

Case 3. $e_1^* = s(e_2^*)$ and $e_2^* < s(e_2^*)$.

Arguing as in the proof of Case 2, we can conclude that this is a contraction.

Therefore, $e_1^* = e_2^* = s(e_2^*)$. This completes the proof.

**Lemma 3.3** *Let $\varphi \in C$ and $e^{**} = \inf D_\varphi^-$. If $\omega(\varphi) \setminus \{\widehat{e^{**}}\} \ne \phi$, then $e_1^{**} = e_2^{**} = i(e_2^{**})$.*

*Proof* The proof of the lemma is similar to that of Lemma 3.2 and thus is omitted.

The main result of this paper is the next theorem.

**Theorem 3.1** *Let $\varphi \in C$. Then there exists $e^* \in E_F$ such that $\omega(\varphi) = \{\widehat{e^*}\}$.*

*Proof* Let $e^* = \sup D_\varphi^+$ and $e^{**} = \inf D_\varphi^-$. We will show that $\omega(\varphi) = \{\widehat{e^*}\}$. Otherwise, $\omega(\varphi) \setminus \{e^*\} \ne \phi$ and $\omega(\varphi) \setminus \{e^{**}\} \ne \phi$. Hence, by Lemmas 3.2 and 3.3, we obtain

$$e_1^* = e_2^* = s(e_1^*) \quad \text{and} \quad e_1^{**} = e_2^{**} = i(e_1^{**}).$$

Thus, $e_1^* < e_1^{**}$. Observe that for any $\psi \in \omega(\varphi)$, we know that $\widehat{e^{**}} - \psi,\ \psi - \widehat{e^*} \notin \operatorname{Int} C_+$. We next assume that $\psi \in \omega(\varphi)$. Then by the invariance of $\omega(\varphi)$, there exists a *full orbit* of the solution semiflow of (1.2) in $\omega(\varphi)$ through $\psi$, and we below will use $x_t(\psi)$ to denote such a *full orbit*. Hence, $x(t, \psi)$ is continuously differentiable in its first arguments $t \in R$. Let $x(t) = x(t, \psi)$, $t \in R$. We next distinguish several cases to finish the proof.

Case 1. There exist $t_1, t_2 \in [-r_1, 0]$ such that $x_1(t_1) = e_1^*$ and $x_1(t_2) = e_1^{**}$.

It follows that $\dfrac{dx_1(t_1)}{dt} = \dfrac{dx_1(t_2)}{dt} = 0$. From (1.2), we get

$$F(x_2(t_1 - r_2)) = F(e_1^*) \quad \text{and} \quad F(x_1(t_2 - r_2)) = F(e_1^{**}).$$

Thus, by the definition of 2.1, we have (*Please, make a correction of this statement!*)

$$x_2(t_1 - r_2) = e_1^* \quad \text{and} \quad x_2(t_2 - r_2) = e_1^{**}.$$

Similarly, we can get

$$x_1(t_1 - r_1 - r_2) = e_1^* \quad \text{and} \quad x_1(t_2 - r_1 - r_2) = e_1^{**}.$$

Therefore, by induction, we can get

$$\begin{aligned} x_1(t_1 - k(r_1 + r_2)) &= e_1^*, \\ x_1(t_2 - k(r_1 + r_2)) &= e_1^{**}, \\ x_2(t_1 - r_2 - k(r_1 + r_2)) &= e_1^*, \\ x_2(t_2 - r_2 - k(r_1 + r_2)) &= e_1^{**}. \end{aligned}$$

Without loss of generality, we may assume that $t_1 < t_2$. Let

$$a_k = \int\limits_{t_1-k(r_1+r_2)}^{t_2-k(r_1+r_2)} F(x_1(s))\,ds \quad \text{and} \quad b_k = \int\limits_{t_1-r_2-k(r_1+r_2)}^{t_2-r_2-k(r_1+r_2)} F(x_2(s))\,ds.$$

Integrating (1.2), we get

$$\int\limits_{t_1-k(r_1+r_2)}^{t_2-k(r_1+r_2)} \frac{dx_1(s)}{dt}\,ds = \int\limits_{t_1-k(r_1+r_2)}^{t_2-k(r_1+r_2)} (-F(x_1(s)) + F(x_2(s-r_2)))\,ds$$

and

$$\int\limits_{t_1-r_2-k(r_1+r_2)}^{t_2-r_2-k(r_1+r_2)} \frac{dx_2(s)}{dt}\,ds = \int\limits_{t_1-r_2-k(r_1+r_2)}^{t_2-r_2-k(r_1+r_2)} (-F(x_2(s)) + F(x_1(s-r_1)))\,ds.$$

Thus,

$$e_1^{**} - e_1^* = -a_k + b_k \quad \text{and} \quad e_1^{**} - e_1^* = -b_k + a_{k+1}.$$

That is, $2(e_1^{**} - e_1^*) = a_{k+1} - a_k$. Summarizing up in the above equation as $k$ goes from 1 to $n$, we get

$$\sum_{k=1}^{n} 2(e_1^{**} - e_1^*) = \sum_{k=1}^{n} (a_{k+1} - a_k).$$

Hence,

$$2n(e_1^{**} - e_1^*) = a_{n+1} - a_1 \leq 2r_1 F(e_1^{**}),$$

which yields a contradiction by letting $n \to +\infty$.

Case 2. There exist $t_1, t_2 \in [-r_2, 0]$ such that $x_2(t_1) = e_1^*$ and $x_2(t_2) = e_1^{**}$.

Using a similar argument as that of Case 1, we can show that this is also a contradiction.

Case 3. There exist $t_1 \in [-r_1, 0]$ and $t_2 \in [-r_2, 0]$ such that $x_1(t_1) = e_1^*$ and $x_2(t_2) = e_1^{**}$.

Then, from (1.2), we know that

$$F(x_2(t_1 - r_2)) = F(x_1(t_1)) \quad \text{and} \quad F(x_1(t_2 - r_1)) = F(x_2(t_2)).$$

Thus,

$$x_2(t_1 - r_2) = e_1^* \quad \text{and} \quad x_1(t_2 - r_1) = e_1^{**}.$$

Without loss of generality, we may assume that $t_1 < t_2$. Then,

$$0 \leq t_1 - (t_2 - r_1) = t_1 + r_1 - t_2 \leq r_1, \quad x_1(t_1) = e_1^* \quad \text{and} \quad x_1(t_2 - r_1) = e_1^{**}.$$

Using a similar argument as that of Case 1, we can show that this is also a contradiction.

Case 4. There exist $t_1 \in [-r_1, 0]$ and $t_2 \in [-r_2, 0]$ such that $x_1(t_1) = e_1^{**}$ and $x_2(t_2) = e_1^*$.

Likewise, by using a similar argument as that of Case 3, it is easily shown that this is a contradiction.

Therefore, we can now conclude that $\omega(\varphi) = \{\widehat{e^*}\}$. This completes the proof.

If $r_1 = r_2 = r$ and consider the synchronized solutions of (1.2) with $x(t) = y(t) = \varphi(t)$ for $t \in [-\max\{\tau_1, \tau_2\}, 0]$, then, as an application of Theorem 3.1, we get the following result for system (1.1).

**Corollary 3.1** *Every solution of system (1.1) tends to a constant as $t \to \infty$.*

*Remark 3.1* If $F$ in (1.1) is strictly increasing on $R$, then Corollary 3.1 has been proved by [13]. If, however, $F$ is only assumed to be nondecreasing on $R$, then the result of Corollary 3.1 is actually new. For example, consider the case where

$$F(t) = \begin{cases} x^3, & t > 0, \\ 0, & -1 \le t \le 0, \\ (x+1)^3, & t < -1, \end{cases}$$

or

$$F(t) = \begin{cases} x-1, & t > 1, \\ 0, & -1 \le t \le 1, \\ x+1, & t < -1, \end{cases}$$

in (1.1), Corollary 3.1 can be applied to (1.1) while the corresponding result of [13] fails since, in this case, $F$ is not strictly increasing on $R$.

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