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Tracking Control of General Nonlinear Systems by a Direct Gradient Descent Method

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Abstract: This paper is concerned with tracking control of nonlinear multivariable systems whose relative degree is more than one. The control method is based on a direct steepest descent method using the gradient of a performance index. Simulation results demonstrate the usefulness of the proposed method.

Keywords: *Nonlinear control; state feedback; output feedback; tracking system; gradients; steepest descent; Lyapunov stability.*

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1 Introduction

Studies on nonlinear feedback control have been extensively made in recent years. Needless to say, stabilization and optimization are central concerns. Lyapunov stability theory and the Hamilton–Jacobi–Bellman equation for optimal control appear to be main tools for designing stabilizing feedback control laws. For affine nonlinear systems, lots of researches have been done based on feedback linearization, nonlinear optimal regulator [13, 5], the Hamilton–Jacobi–Bellman equation and inverse optimality theory [4], control Lyapunov function stabilization [20], back stepping technique [11], nonlinear H^∞ control and passivity-based control theory [22], etc. For general nonlinear systems, receding horizon control [12] is known as one of the few studies on on-line nonlinear optimal control. Besides, there are many studies on neuro-controllers [19, 15] based on the error back propagation method, but their stability and generalization ability remain unsolved questions.

In this paper the following general nonlinear system is considered as a controlled object:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^r$ is the control input, $y(t) \in R^m$ is the measured output, and we study the problem of output tracking so that $y(t)$ tracks a desired output $y_d(t)$ (the reference signal). Output tracking control (output regulation or servo mechanism) for nonlinear systems has intensively been investigated [23, 9, 21, 7, 10, 6, 1]. Among them, Vidyasagar [23] and Tsinias [21] showed that if the system (1) is stabilizable and weakly detectable by means of a continuous state feedback $u(t) = \alpha(x(t))$, then the system is also stabilized by $\alpha(z(t))$, where $z(t)$ is the output of a weak detector for the state $x(t)$. More precisely, when

$$\dot{z}(t) = g(z(t), y(t), u(t))$$

is an observer (i.e., $z(t) - x(t) \to 0$ as $t \to \infty$ for every $x(0)$ and $z(0)$) and $u(t) = \alpha(z(t))$ is an asymptotically stabilizing control law, then the closed-loop system

$$\begin{aligned} \dot{x}(t) &= f(x(t), \alpha(z(t))), \qquad y(t) = h(x(t)), \\ \dot{z}(t) &= g(z(t), y(t), \alpha(z(t))) \end{aligned}$$

is asymptotically stable in a neighborhood of $(x, z) = (0, 0)$. Note that, however, it is another hard task to obtain the state feedback law $u = \alpha(x)$.

In the pioneering work of Isidori and Byrnes [9], nonlinear output regulation problem has been formulated and solved, in which the objective is to design a dynamic controller such that the closed-loop system is stable and the error approaches zero asymptotically. Supposing the reference signal $y_d(t)$ to be generated by the exosystem

$$\dot{w}(t) = s(w(t)), \quad y_d(t) = q(w(t))$$

and considering the extended system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \qquad y(t) = h(x(t)), \\ \dot{w}(t) &= s(w(t)), \qquad\qquad y_d(t) = q(w(t)), \\ e(t) &= y_d(t) - y(t) \end{aligned} \tag{2}$$

they solved the output regulation problem by means of an error feedback controller (a dynamic controller)

$$\begin{aligned} \dot{z}(t) &= \eta(z(t), e(t)), \\ u(t) &= \alpha(z(t)). \end{aligned} \tag{3}$$

More precisely, the output regulation means that the unforced closed-loop system with $w = 0$ is exponentially stable and that the forced closed-loop system (2)–(3) satisfies $\lim_{t\to\infty} e(t) \to 0$ for any initial condition $(x(0), z(0), w(0))$ in a neighborhood of the origin $(0, 0, 0)$. Isidori and Byrnes [9] derived a necessary condition for the output regulation, called the nonlinear regulator equation, using the center manifold theorem. Though the Isidori–Byrnes theory is precise and sophisticated, it requires many assumptions and, in order to synthesize a solution numerically, one has to solve the nonlinear regulator equation described by a system of nonlinear partial differential equations, which is difficult to solve as in the Hamilton–Jacobi–Bellman equation. The nonlinear output regulation can achieve asymptotic disturbance rejection based on the exosystem as well as asymptotic output tracking. Actually, in the work of Isidori and Byrnes [9] trajectory tracking and/or disturbance rejection are unificatively formulated as the problem of output regulation. Furthermore, structurally stable and robust output regulation under parametric uncertainties has been investigated by Khalil [10], Huang [6] and Byrnes, *et al.* [1]. Huang and Rugh [7] also proposed an approximation method of finding a power series expansion of the solution to the nonlinear regulator equation.

Direct gradient descent control was proposed by Shimizu, *et al.* [18], which directly manipulates control inputs so as to decrease a performance index such as the squared error from a desired equilibrium state based on the gradient of the performance index with respect to the control inputs. The gradient is derived from sensitivity equations. A similar method called "speed gradient control" was also proposed by Fradkov, *et al.* [2, 3]. In their method, however, the performance function F contains only x (not both x and u). $F(x)$ and $F(x,u)$ makes a big difference in application. Further, their derivation is not based on the sensitivity equations but on the Lyapunov direct method.

In this paper we investigate output tracking control of nonlinear multivariable systems by use of the direct gradient descent method. Our main concern is the control of plants with relative degrees of more than one. The proposed method is an on-line implementation and can be executed in a very simple and practical manner. Our simulation results for various plants showed remarkably good performance, one of which will be demonstrated in the last section.

2 Direct Gradient Descent Control of Nonlinear Systems

The aim of our control is to modify $u(t)$ so that a performance index $F(y(t), u(t))$ decreases. The problem is written as

$$\underset{u(t)}{\text{decrease}}\, F(y(t), u(t)), \tag{4a}$$

$$\text{subj. to} \quad \dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0 \tag{4b}$$

$$y(t) = h(x(t)) \tag{4c}$$

where we make the following assumption:

Assumption 1 *Plant* (4b), (4c) is *locally controllable and observable.*

To solve this problem, we prepare some fundamental results concerning the gradient of the performance index. We confine our attention in this section to the basic state feedback regulation case:

$$\underset{u(t)}{\text{decrease}}\, F(x(t), u(t)), \tag{5a}$$

$$\text{subj. to} \quad \dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0, \tag{5b}$$

where we assume the following:

Assumption 2 *Function f is continuously differentiable*; f_u, F_x and F_u are *Lipschitz continuous.*

For any continuous $u\colon\ u(t),\ t \geq t_0$, system (4b) has a unique smooth solution $x\colon\ x(t)$, $t \geq t_0$. We denote the state trajectory x associated with a given u by $x(u)$, whose value at t will be denoted by $x(t;u)$. Then, for an arbitrarily fixed t, let us define a functional ϕ^t by

$$\phi^t[u] \triangleq F(x(t;u), u(t)). \tag{6}$$

The derivative of the objective $F(x(t;u), u(t))$ with respect to $u(t)$ can be conceptually given as

$$F_x(x(t;u), u(t)) \frac{dx(t;u)}{du(t)} + F_u(x(t;u), u(t)). \tag{7}$$

Here the notion $dx(t;u)/du(t)$ denotes the effect on $x(t;u)$ caused by the change of $u(t)$, but it is impossible and impractical to change $u(t)$ freely without any reference to the past trajectory of u. So we consider a time interval $[t', t]$, where t' is an arbitrarily given time such that $t_0 \leq t' < t$, and see the effect on the state at time t caused by the change of u as a function on the interval.

As a class of admissible control for the fixed interval $[t', t]$, we consider the space $U_{[t',t]}$ consisting of r-dimensional vector-valued continuous functions and define the inner product:

$$\langle u, v \rangle_{U_{[t',t]}} \triangleq \int_{t'}^{t} u(\tau)^{\mathrm{T}} v(\tau)\, d\tau. \tag{8}$$

Then the following theorem holds.

Theorem 1 *The operator $x(t;\cdot)\colon U_{[t',t]} \to R^n$ is Gâteaux differentiable, and its Jacobian is given, at time t, as follows:*

$$\nabla x(t;u)(t) = f_u(x(t;u), u(t))^{\mathrm{T}}. \tag{9}$$

Proof We show that the functional $x(t;\cdot)\colon U_{[t',t]} \to R^n$ is Gâteaux differentiable, and calculate the Gâteaux differential

$$\delta x(t;u;s) \triangleq \frac{d}{d\varepsilon} x(t; u + \varepsilon s)\Big|_{\varepsilon=0}.$$

Integrating (5b) from t' to t with $u + \varepsilon s$, we have

$$x(t;\, u + \varepsilon s) = x(t') + \int_{t'}^{t} f(x(\tau;\, u + \varepsilon s),\, u(\tau) + \varepsilon s(\tau))\, d\tau. \tag{10}$$

Differentiating (10) w.r.t. ε, letting $\varepsilon = 0$, and differentiating it w.r.t. t, we finally obtain

$$\frac{d}{dt}\frac{d}{d\varepsilon} x(t; u + \varepsilon s)\Big|_{\varepsilon=0} = f_x(x(t;u), u(t))\, \frac{d}{d\varepsilon} x(t; u + \varepsilon s)\Big|_{\varepsilon=0} + f_u(x(t;u), u(t)) s(t)$$

with

$$\frac{d}{d\varepsilon}\, x(t'; u + \varepsilon s)\Big|_{\varepsilon=0} = 0.$$

Since this is a time-variant linear differential equation w.r.t.

$$\delta x(t;u;s) = \frac{d}{d\varepsilon} x(t;u+\varepsilon s)\Big|_{\varepsilon=0},$$

its solution exists and is given by

$$\delta x(t;u;s) = \int_{t'}^{t} \Phi(t,\tau) f_u(x(\tau;u),u(\tau)) s(\tau)\, d\tau$$

where Φ is a continuous transition-matrix function defined on $\{(t,\tau)\colon\ t' \le \tau \le t\}$ by

$$\frac{\partial}{\partial t}\Phi(t,\tau) = f_x(x(t;u),u(t))\Phi(t,\tau), \quad \Phi(\tau,\tau) = I \tag{11}$$

(see, e.g., Pontryagin [14]). The Gâteaux differential of each component $x_i(t;\cdot)\colon\ U_{[t',t]} \to R$ is then expressed as

$$\delta x_i(t;u;s) = \int_{t'}^{t} \Phi_i(t,\tau) f_u(x(\tau;u),u(\tau)) s(\tau)\, d\tau,$$

where $\Phi_i(t,\tau)$ denotes the i-th row of $\Phi(t,\tau)$. Comparing this with definition (8), we can see that there exists $\nabla x_i(t;u) \in U_{[t',t]}$ satisfying

$$\delta x_i(t;u;s) = \langle \nabla x_i(t;u),\ s\rangle_{U_{[t',t]}} \quad \forall\, s \in U_{[t',t]}$$

and it is given by

$$\nabla x_i(t;u)(\tau) = f_u(x(\tau;u),u(\tau))^{\mathrm{T}} \Phi_i(t,\tau)^{\mathrm{T}}, \quad \tau \in [t',t].$$

Each $\nabla x_i(t;u)$ is an r-dimensional vector-valued function, and here we define an $(r\times n)$-matrix-valued function $\nabla x(t;u)$ by

$$\nabla x(t;u)(\tau) \triangleq (\nabla x_1(t;u)(\tau), \dots, \nabla x_n(t;u)(\tau)).$$

In other words, $\nabla x(t;u)$ is given by

$$\nabla x(t;u)(\tau) = f_u(x(\tau;u),u(\tau))^{\mathrm{T}} \Phi(t,\tau)^{\mathrm{T}}, \quad \tau \in [t',t]. \tag{12}$$

It follows from (11) and (12) that

$$\begin{aligned} \frac{d}{dt}\nabla x(t;u)(\tau) &= \nabla x(t;u)(\tau) f_x(x(t;u),u(t))^{\mathrm{T}}, \\ \nabla x(\tau;u)(\tau) &= f_u(x(\tau;u),u(\tau))^{\mathrm{T}} \end{aligned} \tag{13}$$

on the region $\{(t,\tau)\colon\ t' \le \tau \le t\}$, from which we obtain (9). It is noted that equation (13) represents the sensitivity equation of the state x with respect to the input u.

Since (9) does not depend on t', we regard it as the effect $dx(t;u)/du(t)$ in (7) and consider the transpose of (7), i.e.,

$$f_u(x(t;u),u(t))^{\mathrm{T}}F_x(x(t;u),u(t))^{\mathrm{T}}+F_u(x(t;u),u(t))^{\mathrm{T}},$$

as the *gradient* of the objective $F(x(t;u),u(t))$ with respect to $u(t)$. We denote it by $\nabla\phi^t[u](t)$, i.e.,

$$\nabla\phi^t[u](t)=f_u(x(t;u),u(t))^{\mathrm{T}}F_x(x(t;u),u(t))^{\mathrm{T}}+F_u(x(t;u),u(t))^{\mathrm{T}}. \tag{14}$$

As an on-line control law for problem (5), we apply the steepest descent method at each time $t\in[t_0,\infty)$ by using $\nabla\phi^t[u](t)$. Namely, $u(t)$ is modified by the direct gradient descent control algorithm

$$\dot{u}(t)=-\mathcal{L}\nabla\phi^t[u](t) \tag{15}$$

where $\mathcal{L}=\mathrm{diag}[\alpha_1,\alpha_2,\ldots,\alpha_r]$, $\alpha_i>0$, is a proportional constant. Substituting (14) into (15) yields

$$\dot{u}(t)=-\mathcal{L}\big\{f_u(x(t;u),u(t))^{\mathrm{T}}F_x(x(t;u),u(t))^{\mathrm{T}}+F_u(x(t;u),u(t))^{\mathrm{T}}\big\}. \tag{16}$$

Assumption 2 is a sufficient condition for systems (5b) and (16) to be solvable for a unique smooth pair (x,u). Furthermore, in order to realize this control, we set F to satisfy the following assumption:

Assumption 3 For every i,

$$F_{x_i}(x,u)\neq 0 \quad \forall\,(x,u)\neq(x_d,u_d)$$

where x_d is a desired stationary state and u_d is the corresponding control.

Let us set the performance index F in problem (5) as a quadratic form. Our purpose of control is then to transfer the state $x(t)$ to a desired stationary state x_d. At the stationary state, it must hold that $0=f(x_d,u_d)$. In general, we can arbitrarily specify r of n components of x_d, but the remaining $(n-r)$ components and u_d are dependently determined. We consider

$$F(x(t),u(t))\triangleq(x_d-x(t))^{\mathrm{T}}Q(x_d-x(t))+(u_d-u(t))^{\mathrm{T}}R(u_d-u(t)) \tag{17}$$

as a performance index to be decreased, where Q and R are (normally diagonal) positive definite matrices. Then the gradient is written as

$$\nabla\phi^t[u](t)=-2f_u(x(t;u),u(t))^{\mathrm{T}}Q(x_d-x(t;u))-2R(u_d-u(t)) \tag{18}$$

and hence the direct gradient descent control formula (16) is given by

$$\dot{u}(t)=2\mathcal{L}\big\{f_u(x(t;u),u(t))^{\mathrm{T}}Q(x_d-x(t;u))+R(u_d-u(t))\big\}. \tag{19}$$

The stability of direct gradient descent control is proved in Appendix by use of Lyapunov's direct method.

3 Output Tracking via Direct Gradient Descent Control

We define the inverse dynamics of nonlinear systems using the concept of relative degree of nonlinear dynamical systems (see, e.g., [8]). Let us consider each component $y_i(t)$ of $y(t)$, and denote by $y_i^{(j)}(t)$ the j-th order derivative of $y_i(t)$ with respect to t, which generally represents a function of x, u, $\dot{u}$, $\ddot{u},\ldots,u^{(j-1)}$. Then the relative degree of $y_i(t)$ is defined as follows.

Definition 1 The integer q_i satisfying

$$\frac{\partial y_i^{(j)}(t)}{\partial u} = 0, \quad j = 1, 2, \dots, q_i - 1, \tag{20a}$$

$$\frac{\partial y_i^{(q_i)}(t)}{\partial u} \neq 0 \tag{20b}$$

is called the *relative degree* of component $y_i(t)$.

Let us denote by $\alpha_i^j(x(t))$ and $\beta_i^{q_i}(x(t), u(t))$ the j-th derivative of $y_i(t)$ as $j = 1, 2, \dots, q_i - 1$ and as $j = q_i$, respectively. Then we have the following system of equations:

$$\begin{aligned} &y_1^{(q_1)}(t) = \beta_1^{q_1}(x(t), u(t)) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &y_m^{(q_m)}(t) = \beta_m^{q_m}(x(t), u(t)). \end{aligned} \tag{21}$$

Assumption 4

$$\operatorname{rank} \begin{bmatrix} \dfrac{\partial \beta_1^{q_1}(x(t), u(t))}{\partial u} \\ \cdots\cdots\cdots\cdots \\ \dfrac{\partial \beta_m^{q_m}(x(t), u(t))}{\partial u} \end{bmatrix} = r.$$

Then, by the implicit function theorem, there exists an inverse mapping of (21) in regard to $u(t)$. Hence $u(t)$ can be expressed as

$$u(t) = \eta\big(x(t), y_1^{(q_1)}(t), \dots, y_i^{(q_i)}(t), \dots, y_m^{(q_m)}(t)\big) \tag{22}$$

where q_i denotes the relative degree of $y_i(t)$. Let us call system (22) the inverse dynamics or the inverse system. The control $u(t)$ represented by (22) can be regarded as an input by which the q_i-th order derivative of $y_i(t)$ becomes equal to $y_i^{(q_i)}(t)$ when $x(t)$ is the present state.

Now we investigate an on-line tracking control for problem (4) based on the preliminary knowledge on state feedback regulation. Let us consider the case where the performance index is given in the quadratic form

$$F(y(t), u(t)) \triangleq (y_d(t) - y(t))^{\mathrm{T}} Q (y_d(t) - y(t)) + (u_d(t) - u(t))^{\mathrm{T}} R (u_d(t) - u(t)), \tag{23}$$

where $y_d(t)$ is a desired output, $u_d(t)$ is the corresponding control input, and Q, R are diagonal positive definite matrices. In what follows, we consider the functional

$$\phi^t[u] \triangleq F(y(t; u), u(t)) \tag{24}$$

where $y(t; u) \triangleq h(x(t; u))$. We assume sufficiently higher order continuous differentiability of f and h for a while. Precise description of required assumptions will be given at the end of the next section. Applying Theorem 1, we obtain an expression of the gradient needed for the gradient descent tracking control as follows:

$$
\begin{aligned}
\nabla\phi^t[u](t) &= \nabla x(t;u)(t)\,\frac{\partial y(t;u)}{\partial x}^{\mathrm{T}} F_y(y(t;u),u(t))^{\mathrm{T}} + F_u(y(t;u),u(t))^{\mathrm{T}} \\
&= -2f_u(x(t;u),u(t))^{\mathrm{T}}\,\frac{\partial h(x(t;u))}{\partial x}^{\mathrm{T}} Q(y_d(t)-y(t;u)) \\
&\quad - 2R(u_d(t)-u(t)).
\end{aligned}
\tag{25}
$$

From this we realize that, for $y_i(t)$ with relative degree of more than 1, the error $(y_{id}(t)-y_i(t))$ cannot be evaluated at all in the calculation of $\nabla\phi^t[u](t)$ since

$$
f_u(x(t),u(t))^{\mathrm{T}}\,\frac{\partial h_i(x(t))}{\partial x}^{\mathrm{T}} = 0.
$$

Therefore the error information on $y_i(t)$ is not used in modifying $u(t)$ by the direct gradient descent control with (25), which implies that it is not always possible to accomplish the tracking control.

In order to control those plants with higher relative degrees, it is essential to incorporate some information on higher order derivatives into the algorithm, and we consider the following performance index:

$$
\begin{aligned}
F\big(y_1^{(q_1-1)}(t),\dots,y_m^{(q_m-1)}(t),u(t)\big) \triangleq \sum_{i=1}^{m}&\omega_i\big(y_{id}^{(q_i-1)}(t)-y_i^{(q_i-1)}(t)\big)^2 \\
&+(u_d(t)-u(t))^{\mathrm{T}}R(u_d(t)-u(t)),
\end{aligned}
\tag{26}
$$

where $y_{id}^{(q_i-1)}(t)$ denotes the (q_i-1)-th order derivative of the i-th component $y_{id}(t)$ of the desired output (q_i is the relative degree of $y_i(t)$). Taking account of the inverse dynamics given by (22), it seems that we need the q_i-th order derivative for each output component, but, actually, the (q_i-1)-th order derivative turns out to be enough by the nature of the direct gradient descent control and by the definition of relative degree. We again use the same notation

$$
\phi^t[u] \triangleq F\big(y_1^{(q_1-1)}(t;u),\dots,y_m^{(q_m-1)}(t;u),u(t)\big). \tag{27}
$$

The gradient $\nabla\phi^t[u](t)$ is then given by

$$
\begin{aligned}
\nabla\phi^t[u](t) &= \sum_{i=1}^{m}\nabla x(t;u)(t)\,\frac{\partial y_i^{(q_i-1)}(t;u)}{\partial x}^{\mathrm{T}} F_{y_i^{(q_i-1)}} + F_u^{\mathrm{T}} \\
&= -2\sum_{i=1}^{m}\omega_i f_u(x(t;u),u(t))^{\mathrm{T}}\,\frac{\partial \alpha_i^{q_i-1}(x(t;u))}{\partial x}^{\mathrm{T}}\big(y_{id}^{(q_i-1)}(t)-y_i^{(q_i-1)}(t;u)\big) \\
&\quad - 2R(u_d(t)-u(t))
\end{aligned}
\tag{28}
$$

where we eliminated the arguments of F for simplicity. The direct gradient descent control is given as follows:

$$\begin{aligned}\dot{u}(t) &= -\alpha\nabla\phi^t[u](t)\\ &= 2\alpha\Big[\sum_{i=1}^{m}\omega_i f_u(x(t;u),u(t))^{\mathrm{T}}\,\frac{\partial\alpha_i^{q_i-1}(x(t;u))}{\partial x}^{\mathrm{T}}\big(y_{id}^{(q_i-1)}(t)-y_i^{(q_i-1)}(t;u)\big)\\ &\qquad + R(u_d(t)-u(t))\Big].\end{aligned} \tag{29}$$

4 Convergence of the Output Error

Execution of (29) can enforce $y_i^{(q_i-1)}(t)\to y_{id}^{(q_i-1)}(t)$ for each i, but this does not guarantee that $y_i(t)\to y_{id}(t)$ when $q_i>1$. In this section we shall utilize some device so that $y(t)$ can asymptotically converge to $y_d(t)$ whenever $y_i^{(q_i-1)}(t)\to y_{id}^{(q_i-1)}(t)$ for all i's.

Let us first consider a component $y_i(t)$ whose relative degree is 2. If we use $\dot{\tilde{y}}_{id}(t)\triangleq \dot{y}_{id}(t)+a_{i,0}(y_{id}(t)-y_i(t))$ instead of $\dot{y}_{id}(t)$ in (26) or (29), we obtain $\dot{y}_i(t)\to\dot{\tilde{y}}_{id}(t)$ and hence $\dot{y}_i(t)=\dot{y}_{id}(t)+a_{i,0}(y_{id}(t)-y_i(t))$, i.e., $\dot{y}_{id}(t)-\dot{y}_i(t)=-a_{i,0}(y_{id}(t)-y_i(t))$ for sufficiently large t. The tracking error $e_i(t)=y_{id}(t)-y_i(t)$ then satisfies $\dot{e}_i(t)=-a_{i,0}\,e_i(t)$, and hence, if $a_{i,0}>0$, we can expect that $e_i(t)\to 0$ (i.e., $y_i(t)\to y_{id}(t)$) as $t\to\infty$.

In a similar manner, let us consider the general case where the relative degree is q_i. If we use

$$\begin{aligned}\tilde{y}_{id}^{(q_i-1)}(t)\triangleq y_{id}^{(q_i-1)}(t)&+a_{i,q_i-2}\big(y_{id}^{(q_i-2)}(t)-y_i^{(q_i-2)}(t)\big)+\cdots\\ &+a_{i,1}(\dot{y}_{id}(t)-\dot{y}_i(t))+a_{i,0}(y_{id}(t)-y_i(t))\end{aligned} \tag{30}$$

instead of $y_{id}^{(q_i-1)}(t)$, we can expect $y_i^{(q_i-1)}(t)\to\tilde{y}_{id}^{(q_i-1)}(t)$, and hence the tracking error $e_i(t)=y_{id}(t)-y_i(t)$ asymptotically satisfies

$$e_i^{(q_i-1)}(t)+a_{i,q_i-2}e_i^{(q_i-2)}(t)+\cdots+a_{i,1}\dot{e}_i(t)+a_{i,0}e_i(t)=0.$$

If $a_{i,j},\ j=0,1,\dots,\,q_i-2,$ are chosen so that every root of the characteristic equation

$$\lambda^{q_i-1}+a_{i,q_i-2}\lambda^{q_i-2}+\cdots+a_{i,1}\lambda+a_{i,0}=0$$

is real negative, then we have $e_i(t)\to 0$ (i.e., $y_i(t)\to y_{id}(t)$) as $t\to\infty$. The output $y(t)$ can thus track the desired output $y_d(t)$ asymptotically. (Such an idea was also suggested in [24] and [15] for the case with relative degree 1.) If we consider $y_d(t)$ as the output of a reference model, this method can also be regarded as a model reference tracking control in which $y(t)$ asymptotically follows $y_d(t)$.

Hence we modify the performance index as follows:

$$\begin{aligned}F\big(y_1(t),\dot{y}_1(t),\dots,\,&y_1^{(q_1-1)}(t),\dots,\,y_m(t),\dot{y}_m(t),\dots,y_m^{(q_m-1)}(t),u(t)\big)\\ &\triangleq\sum_{i=1}^{m}w_i\big(\tilde{y}_{id}^{(q_i-1)}(t)-y_i^{(q_i-1)}(t)\big)^2+(u_d(t)-u(t))^{\mathrm{T}}R(u_d(t)-u(t))\end{aligned} \tag{31}$$

where $\tilde{y}_{id}^{(q_i-1)}(t)$ is defined by (30). Letting $\phi^t[u]$ denote the performance index (31), we have

$$\nabla\phi^t[u](t) = \sum_{i=1}^{m}\sum_{j=0}^{q_i-1} \nabla x(t;u)(t)\, \frac{\partial y_i^{(j)}(t;u)}{\partial x}^{\mathrm{T}} F_{y_i^{(j)}} + F_u^{\mathrm{T}}.$$

Noting (9) and (20a) and substituting (31), we obtain the gradient for the quadratic case:

$$\begin{aligned}\nabla\phi^t[u](t) = {} & -2\sum_{i=1}^{m} w_i f_u(x(t;u),u(t))^{\mathrm{T}}\, \frac{\partial \alpha_i^{q_i-1}(x(t;u))}{\partial x}^{\mathrm{T}} \\ & \times \left\{ \sum_{k=0}^{q_i-1} a_{i,k}\big(y_{id}^{(k)}(t) - y_i^{(k)}(t;u)\big)\right\} - 2R(u_d(t)-u(t)).\end{aligned} \tag{32}$$

Finally from (15) and (32) we have the following direct gradient descent control for output tracking:

$$\begin{aligned}\dot{u}(t) = 2\alpha\Bigg[& \sum_{i=1}^{m} w_i f_u(x(t;u),u(t))^{\mathrm{T}}\, \frac{\partial \alpha_i^{q_i-1}(x(t;u))}{\partial x}^{\mathrm{T}} \\ & \times \left\{ \sum_{k=0}^{q_i-1} a_{i,k}\big(y_{id}^{(k)}(t) - y_i^{(k)}(t;u)\big)\right\} + R(u_d(t)-u(t))\Bigg].\end{aligned} \tag{33}$$

Remark Let f and f_u be Lipschitz continuous; let f be $(\max_i q_i - 1)$-times continuously differentiable in x with Lipschitz continuous derivatives; let h_i, $i = 1, 2, \ldots, m$, be q_i-times continuously differentiable with Lipschitz continuous derivatives; let y_{id}, $i = 1,2,\ldots,m$, be q_i-times continuously differentiable and u_d be continuously differentiable. Then a simultaneous system of (1) and (33) has a unique smooth solution for arbitrarily given initial condition.

5 Simulation Results

Let us consider a link of length $2l$ and weight m, at one end of which a torque $\tau(t)$ is added as a control input. The single-link manipulator system is then described by $I\ddot{\theta}(t) + D\dot{\theta}(t) - mlg\sin\theta(t) = \tau(t)$, where θ is the angle of rotation, I is the moment of inertia of the link, and D is the viscous friction coefficient at the other end of the link. Letting $\theta(t) = x_1(t)$, $\dot{\theta}(t) = x_2(t)$, $\tau(t) = u(t)$, we have

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -\frac{D}{I}\,x_2(t) + \frac{mlg}{I}\,\sin x_1(t) + \frac{1}{I}\,u(t).\end{aligned}$$

We consider this nonlinear plant with output $y(t) = x_1(t)$, whose relative degree is 2. The gradient descent control formula is then given by

$$\begin{aligned}\dot{u}(t) &= 2\alpha\left[\frac{w}{I}\big\{(\dot{y}_d(t) - \dot{y}(t)) + a_0(y_d(t) - y(t))\big\} + R(u_d(t)-u(t))\right]\\ &= 2\alpha\left[\frac{w}{I}\big\{(x_{2d}(t) - x_2(t)) + a_0(x_{1d}(t) - x_1(t))\big\} + R(u_d(t)-u(t))\right].\end{aligned}$$

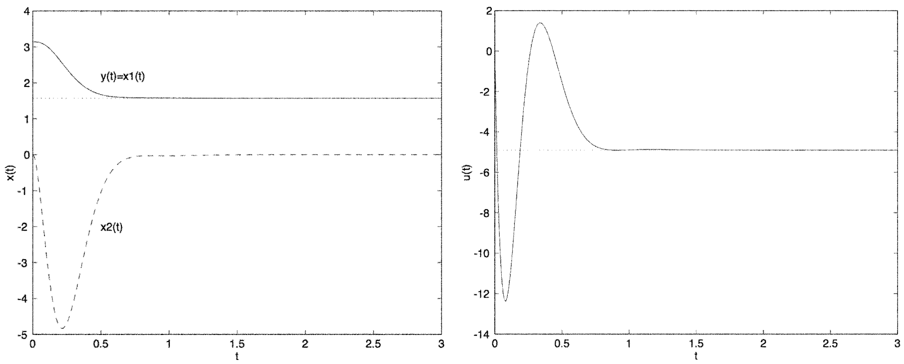

Figure 5.1. $y_d(t) = \pi/2$.

Case 1: $y_d(t) = \pi/2$**.**

Any equilibrium point (x_{1d}, x_{2d}, u_d) must satisfy $0 = x_{2d}$ and $0 = mlg \sin x_{1d} + u_d$. We set the system parameters as $l = 0.5$, $m = 1$, $I = 1/3$, $D = 0.00198$, and applied the direct steepest descent control with $\alpha = 10$, $a_0 = 3$, $R = w = 1$. The result is shown in Figure 5.1 for initial values $x(0) = (\pi, 0)^{\mathrm{T}}$, $u(0) = 0$, and desired values $(x_{1d}(t), x_{2d}(t), u_d(t)) = (\pi/2, 0, -mlg)$.

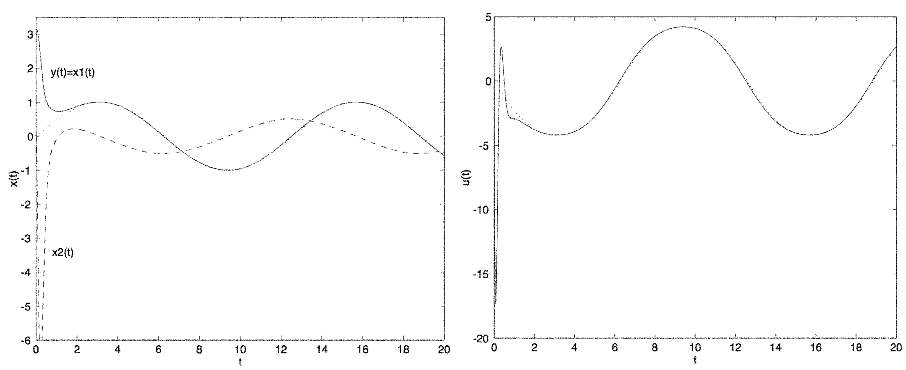

Figure 5.2. $y_d(t) = \sin 0.5t$.

Case 2: $y_d(t) = \sin 0.5t$**.**

The corresponding desired states $(x_{1d}(t), x_{2d}(t))$ and control $u_d(t)$ must satisfy

$$\dot{x}_{1d}(t) = x_{2d}(t),$$

$$\dot{x}_{2d}(t) = -\frac{D}{I}\, x_{2d}(t) + \frac{mlg}{I}\, \sin x_{1d}(t) + \frac{1}{I}\, u_d(t).$$

By substituting $x_{1d}(t) = \sin 0.5t$ here, we obtain

$$(x_{1d}(t), x_{2d}(t), u_d(t)) = (\sin 0.5\, t,\ 0.5 \cos 0.5\, t,\\ -0.25 I \sin 0.5\, t + 0.5 D \cos 0.5\, t - mlg \sin(\sin 0.5\, t)).$$

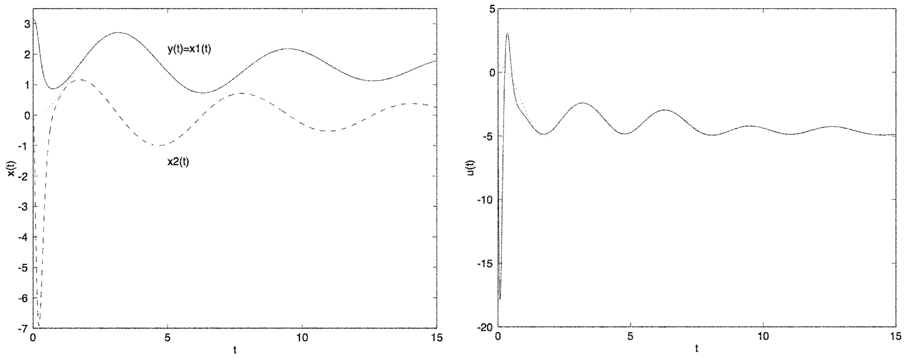

Figure 5.3. $y_d(t) = \dfrac{\pi}{2}\left\{1 - \dfrac{1}{\omega}\, e^{-\zeta t}(\zeta \sin \omega t + \omega \cos \omega t)\right\}$, where $\omega = \sqrt{1-\zeta^2}$, $\zeta = 0.1$.

Figure 5.2 shows the result for the same initial values by the same control parameters as in the previous case except $\alpha = 20$.

Case 3: $y_d(t) = \dfrac{\pi}{2}\left\{1 - \dfrac{1}{\omega}\, e^{-\zeta t}(\zeta \sin \omega t + \omega \cos \omega t)\right\}$**, where** $\omega = \sqrt{1-\zeta^2}$**,** $\zeta = 0.1$**.**

This reference $y_d(t)$ corresponds to the response of a second-order linear system with damping ratio ζ, zero initial states, and forced input $\pi/2$. For $0 < \zeta < 1$, each $y_d(t)$ generates an oscillating signal converging to $\pi/2$. A result for $\zeta = 0.1$ is shown in Figure 5.3 for the same initial states and control parameters as in Case 1.

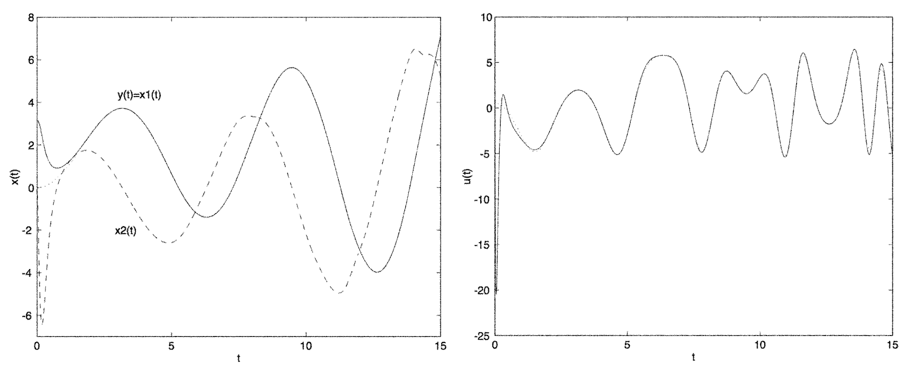

Figure 5.4. $y_d(t) = \dfrac{\pi}{2}\left\{1 - \dfrac{1}{\omega}\, e^{-\zeta t}(\zeta \sin \omega t + \omega \cos \omega t)\right\}$, where $\omega = \sqrt{1-\zeta^2}$, $\zeta = -0.1$.

Case 4: $y_d(t) = \dfrac{\pi}{2}\left\{1 - \dfrac{1}{\omega}\, e^{-\zeta t}(\zeta \sin \omega t + \omega \cos \omega t)\right\}$**, where** $\omega = \sqrt{1-\zeta^2}$**,** $\zeta = -0.1$**.**

For $\zeta < 0$, the reference $y_d(t)$ gives a divergent signal oscillating around $\pi/2$. Figure 5.4 shows a result for $\zeta = -0.1$ when the same initial states and control parameters are applied except $\alpha = 20$.

6 Concluding Remarks

We proposed the direct gradient descent control for tracking of general nonlinear systems with relative degrees of more than one. The effectiveness of the control method was confirmed by computer simulation, and very good performance was observed with various examples. Results for a Rayleigh model are given in [17]. In regard to stability, the direct gradient descent control is considered fairly stable since the resultant control inputs are always manipulated so as to decrease the squared error of outputs. We also observed that the choice of the proportional coefficient $\mathcal{L}$ did not seriously affect the stability, but the direct gradient descent control does not guarantee the monotone decrease of performance index. It is difficult to theoretically verify the stability of the proposed method in general. For individual plants, however, we can find some asymptotically stable region in a neighborhood of the desired equilibrium by constructing a Lyapunov function via Zubov's successive approximation method [25] as shown in [18]. Stability is guaranteed as long as the plant is controlled within that region.

Appendix: Stability

In this appendix, we establish the stability of the direct gradient descent control for the state feedback case, in which the control law is given by

$$\dot{x}(t) = f(x(t), u(t)), \tag{34}$$

$$\dot{u}(t) = -\alpha\big\{f_u(x(t), u(t))^{\mathrm{T}} F_x(x(t), u(t))^{\mathrm{T}} + F_u(x(t), u(t))^{\mathrm{T}}\big\}, \quad \alpha > 0. \tag{35}$$

As a performance index F, we consider the most practical quadratic error function

$$F(x(t), u(t)) = \frac{1}{2}(x_d - x(t))^{\mathrm{T}} Q (x_d - x(t)) + \frac{1}{2}(u_d - u(t))^{\mathrm{T}}(u_d - u(t)), \tag{36}$$

where $Q > 0$, and (x_d, u_d) is a desired equilibrium point, and assume:

Assumption 5 Plant (34) is Lyapunov asymptotically stable for the fixed u_d. That is, for a Lyapunov function

$$V_1(x) = \frac{1}{2}(x_d - x)^{\mathrm{T}} Q (x_d - x),$$

there exists a positive definite function σ such that

$$V_{1x}(x) f(x, u_d) = -(x_d - x)^{\mathrm{T}} Q f(x, u_d) \le -\sigma(\|x_d - x\|).$$

Assumption 6 The function $V_{1x}(x) f(x, u) = -(x_d - x)^{\mathrm{T}} Q f(x, u)$ is convex with respect to u. (This always holds for affine nonlinear systems.)

Assumption 5 is a sufficient condition for the internal stability of plant (34) when the input is fixed to u_d, and this implies that the equilibrium point x_d of the plant $\dot{x}(t) = f(x(t), u_d)$ is asymptotically stable.

Under these assumptions, one can show the asymptotical stability of extended system (34) and (35) by means of Lyapunov's direct method in a similar way to [2]. Let us consider a Lyapunov function candidate

$$V(x,u) = \frac{1}{2}\,\alpha(x_d - x)^{\mathrm{T}} Q (x_d - x) + \frac{1}{2}(u_d - u)^{\mathrm{T}}(u_d - u) > 0 \qquad \forall\,(x,u) \neq (x_d, u_d). \tag{37}$$

For (34) and (35), the time derivative of $V(x,u)$ is given by

$$\frac{dV(x,u)}{dt} = V_x(x,u)f(x,u) - V_u(x,u)\alpha\{f_u(x,u)^{\mathrm{T}}F_x(x,u)^{\mathrm{T}} + F_u(x,u)^{\mathrm{T}}\}$$
$$= -\alpha(x_d - x)^{\mathrm{T}}Qf(x,u) - \alpha(x_d - x)^{\mathrm{T}}Qf_u(x,u)(u_d - u) - \alpha(u_d - u)^{\mathrm{T}}(u_d - u).$$

On the other hand, since, by Assumption 6, $V_x(x,u)f(x,u) = \alpha V_{1x}(x)f(x,u) = -\alpha(x_d - x)^{\mathrm{T}}Qf(x,u)$ is convex with respect to u, we have

$$-\alpha(x_d - x)^{\mathrm{T}}Qf(x,u_d) \geq -\alpha(x_d - x)^{\mathrm{T}}Qf(x,u) - \alpha(x_d - x)^{\mathrm{T}}Qf_u(x,u)(u_d - u).$$

We thus obtain

$$\frac{dV(x,u)}{dt} \leq -\alpha(x_d - x)^{\mathrm{T}}Qf(x,u_d) - \alpha(u_d - u)^{\mathrm{T}}(u_d - u).$$

Since the first term of the right-hand side is negative definite by Assumption 5, we have $dV(x,u)/dt < 0$ for all $(x,u) \neq (x_d,u_d)$. The system (34) and (35) is hence asymptotically stable by the Lyapunov's theorem, i.e., $x(t) \to x_d$ and $u(t) \to u_d$ as $t \to \infty$.

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