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A Nonlinear Model for Dynamics of Delaminated Composite Beam with Account of Contact of the Delamination Crack Faces, Based on the First Order Shear Deformation Theory

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Abstract: In this work, a new approach is developed for dynamic analysis of a composite beam with an interplay crack, in which a physically impossible interpenetration of the crack faces is prevented by imposing a special constraint, leading to taking account of a force of contact interaction of the crack faces and to nonlinearity of the formulated boundary value problem. Longitudinal force resultants in the delaminated parts of the beam are taken into account also, which is another source of the nonlinearity. The shear deformation and rotary inertia terms are included into the formulation, to achieve better accuracy. The model is based on the first order shear deformation theory, i.e. the longitudinal displacement is assumed to vary linearly through the beam's thickness. A variational formulation of the problem, nonlinear partial differential equations of motion with boundary conditions, a weak form for the partial differential equations and a finite element formulation on the basis of the weak form are developed. An example problem of a clamped-free beam with a piezoelectric actuator is considered, and its finite element solution is obtained. A noticeable difference of forced vibrations of the delaminated and undelaminated beams due to the contact interaction of the crack faces is predicted by the developed model. Besides, linear eigenvalue analysis shows decrease of natural frequencies upon increase of the crack length, and crack opening and closing during the vibration in higher mode shapes, beginning from the fifth one.

Keywords: *Composite delaminated beam; piezoelectric actuator; contact of crack faces; Lagrange multipliers; penalty function method; shear deformation theory; nonlinear partial differential equations; nonlinear finite element analysis.*

Mathematics Subject Classification (2000): 74A40, 74F15, 74S99.

1 Introduction

A model of a composite beam with the delamination and with a piezoelectric actuator, with account of contact interaction of the delamination crack faces, based on the classical beam theory, was presented in Reference [1]. This model did not take the shear strain energy into account, and, therefore, produced sufficiently accurate results only for thin beams. To model thicker beams with delamination, one needs to use a beam theory, based on simplifying assumptions, which do not lead to vanishing of the shear strains. The first order shear deformation theory [2], based on assumed linear variation of a longitudinal displacement in the thickness direction, is the simplest approach that satisfies the requirement of a non-zero shear strain. This approach is used in the present paper for modeling a composite delaminated beam with a piezoelectric actuator. In this model, the interpenetration of the crack faces is prevented by imposing a constraint, written with the use of the Heaviside function in one of its analytical forms, leading to taking account of a force of contact interaction of the crack faces and to nonlinearity of the formulated boundary value problem.

2 Variational Formulation of The Problem

Total Potential Energy for Zone 0 (Part 0), i.e. for $0 \le x \le a$ (Figure 2.1).

Assumptions of the first-order shear deformation beam theory:

$$u_0(x,z,t) = z\phi_0(x,t), \quad w_0(x,z,t) = w_0(x,t), \tag{1}$$

where $u_0(x,z,t)$ and $w_0(x,z,t)$ are longitudinal and transverse displacements of Zone 0 (Part 0). The subscript 0 in the notations $u_0(x,z,t)$ and $w_0(x,z,t)$ indicates that the quantities u_0 and w_0 are associated with the Zone 0 (Part 0). The notation $u_0 = u_0(x,z,t)$ is not a notation for the axial longitudinal displacement (at $z=0$). The axial longitudinal displacement is considered to be negligibly small here, because this model is developed for the beam to which an external longitudinal force is not applied.

Strain-displacement relations:

$$\varepsilon_{xx}^{(0)} = \frac{\partial u_0}{\partial x}, \quad \varepsilon_{xz}^{(0)} = \frac{1}{2}\left(\frac{\partial u_0}{\partial z} + \frac{\partial w_0}{\partial x}\right). \tag{2a}$$

In this text, ε_{xz} is a notation for a component of the strain tensor, not an engineering strain. With account of Equation (1), Equations (2a) take the form

$$\varepsilon_{xx}^{(0)} = z\phi_0', \quad \varepsilon_{xz}^{(0)} = \frac{1}{2}(\phi_0 + w_0'), \tag{2b}$$

where prime denotes differentiation with respect to x.

Stress-strain relations for an orthotropic piezoelectric layer of a composite beam (plane stress with respect to the y-direction), Appendix A:

$$\sigma_{xx}^{(p)} = \frac{1}{\overline{S}_{11}}\varepsilon_{xx}^{(p)} - \frac{\overline{d}_{31}}{\overline{S}_{11}}\frac{V}{\tau}, \quad \sigma_{xz}^{(p)} = \frac{1}{\overline{S}_{55}}2\varepsilon_{xz}^{(p)}, \tag{3}$$

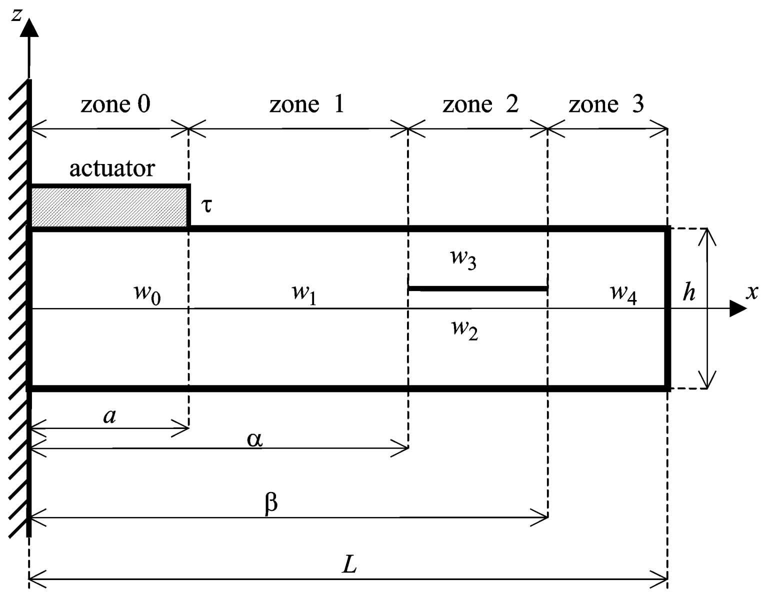

Figure 2.1. Cantilever beam with delamination and piezoelectric actuator.

a is length of the actuator; α is x-coordinate of the left crack tip; β is x-coordinate of the right crack tip; γ is z-coordinate of the crack (distance from x-axis to crack); τ is thickness of the actuator; w_0 is transverse displacement of zone 0; w_1 is transverse displacement of zone 1; w_2 is transverse displacement of lower part of zone 2 (under the crack); w_3 is transverse displacement of upper part of zone 2 (above the crack); w_4 is transverse displacement of zone 3.

where τ is thickness of the actuator, and $V = V(x,t)$ is a voltage, applied to the actuator. It is implied that this voltage creates an electric field in the z-direction.

Stress-strain relations for an orthotropic layer that does not have piezoelectric properties, Appendix A:

$$\sigma_{xx}^{(0)} = \frac{1}{\overline{S}_{11}}\varepsilon_{xx}^{(0)}, \quad \sigma_{xz}^{(0)} = \frac{1}{\overline{S}_{55}}2\varepsilon_{xz}^{(0)}. \tag{4}$$

Total potential energy where K is a shear correction factor and b is the beam's width

$$\begin{aligned} U_0 = {} & \frac{1}{2}\,b\int_0^a\int_{-h/2}^{h/2}\frac{1}{\overline{S}_{11}^{(0)}(z)}\left(\varepsilon_{xx}^{(0)}\right)^2 dz\,dx \\ & + \frac{1}{2}\,b\int_0^a\int_{h/2}^{h/2+\tau}\left(\frac{1}{\overline{S}_{11}^{(p)}(z)}\left(\varepsilon_{xx}^{(p)}\right)^2 - \frac{2\overline{d}_{31}(z)}{\overline{S}_{11}(z)}\frac{V}{\tau}\,\varepsilon_{xx}^{(p)}\right)dz\,dx \\ & + 2Kb\int_0^a\int_{-h/2}^{h/2}\frac{1}{\overline{S}_{55}^{(0)}(z)}\left(\varepsilon_{xz}^{(0)}\right)^2 dz\,dx + 2Kb\int_0^a\int_{-h/2}^{h/2+\tau}\frac{1}{\overline{S}_{55}^{(p)}(z)}\left(\varepsilon_{xz}^{(p)}\right)^2 dz\,dx, \end{aligned} \tag{5}$$

where K is a shear correction factor and b is the beam's width. Substitution of equation (2b) into Equation (5) yields

$$U_0 = \int\limits_0^a \left(\frac{A_0}{2} (\phi_0')^2 + K \frac{G_0}{2} (\phi_0 + w_0')^2 - I_p V \phi_0' \right) dx, \tag{6}$$

where the constants A_0, I_p and G_0 are defined as

$$\begin{aligned} A_0 &= b \int\limits_{-h/2}^{h/2} \frac{z^2}{\overline{S}_{11}^{(0)}(z)}\, dz + b \int\limits_{h/2}^{h/2+\tau} \frac{z^2}{\overline{S}_{11}^{(p)}(z)}\, dz, \\ I_p &= \left(\frac{b}{\tau} \int\limits_{h/2}^{h/2+\tau} \frac{\overline{d}_{31}(z)}{\overline{S}_{11}^{(p)}(z)}\, z\, dz \right), \quad G_0 = b \int\limits_{-h/2}^{h/2} \frac{1}{\overline{S}_{55}^{(0)}(z)}\, dz + b \int\limits_{h/2}^{h/2+\tau} \frac{1}{\overline{S}_{55}^{(p)}(z)}\, dz. \end{aligned} \tag{7}$$

Kinetic Energy for Zone 0 (Part 0), i.e. for $0 \le x \le a$:

$$T_0 = \frac{1}{2}\, b \int\limits_0^a \int\limits_{-h/2}^{h/2} \rho^{(0)}(z)(\dot{w}_0^2 + \dot{u}_0^2)\, dz\, dx + \frac{1}{2}\, b \int\limits_0^a \int\limits_{h/2}^{h/2+\tau} \rho^{(p)}(z)(\dot{w}_0^2 + \dot{u}_0^2)\, dz\, dx, \tag{8}$$

where $\rho^{(0)}(z)$ is a mass density of composite layers of Zone 0 without piezoelectric properties and $\rho^{(p)}(z)$ is the mass density of the piezoelectric actuator ($\rho^{(p)}$ may depend on the z-coordinate if the actuator has plies with different densities).

Substitution of Equations (1) into Equation (8) produces the result

$$T_0 = \int\limits_0^a \left(\frac{1}{2} B_0 \dot{w}_0^2 + \frac{1}{2} C_0 \dot{\phi}_0^2 \right) dx, \tag{9}$$

where the constants B_0 and C_0 are defined as follows:

$$\begin{aligned} B_0 &= b \left(\int\limits_{-h/2}^{h/2} \rho^{(0)}(z)\, dz + \int\limits_{h/2}^{h/2+\tau} \rho^{(p)}(z)\, dz \right), \\ C_0 &= b \left(\int\limits_{-h/2}^{h/2} \rho^{(0)}(z)\, z^2\, dz + \int\limits_{h/2}^{h/2+\tau} \rho^{(p)}(z)\, z^2\, dz \right). \end{aligned} \tag{10}$$

In a similar manner we obtain the strain and kinetic energy for Zone 1 (Part 1) and Zone 3 (Part 4).

Strain Energy for Zone 1 (Part 1), i.e. for $a \le x \le \alpha$:

$$U_1 = \int\limits_a^\alpha \left(\frac{A_1}{2} (\phi_1')^2 + K \frac{G_1}{2} (\phi_1 + w_1')^2 \right) dx, \tag{11}$$

where the constants A_1 and G_1 are defined as follows:

$$A_1 \equiv b \int\limits_{-h/2}^{h/2} \frac{z^2}{\overline{S}_{11}^{(1)}(z)}\, dz \qquad G_1 \equiv b \int\limits_{-h/2}^{h/2} \frac{1}{\overline{S}_{55}^{(1)}(z)}\, dz. \tag{12}$$

Kinetic energy for Zone 1 (Part 1), i.e. for $a \le x \le \alpha$:

$$T_1 = \int\limits_{a}^{\alpha} \left(\frac{1}{2} B_1 \dot{w}_1^2 + \frac{1}{2} C_1 \dot{\phi}_1^2 \right) dx, \tag{13}$$

where the constants B_1 and C_1 are defined as follows:

$$B_1 = b \int\limits_{-h/2}^{h/2} \rho^{(1)}(z)\, dz, \qquad C_1 = b \int\limits_{-h/2}^{h/2} \rho^{(1)}(z)\, z^2\, dz. \tag{14}$$

Strain Energy for Zone 3 (Part 4), i.e. for $\beta \le x \le L$:

$$U_4 = \int\limits_{\beta}^{L} \left(\frac{A_4}{2} (\phi_4')^2 + K \frac{G_4}{2} (\phi_4 + w_4')^2 \right) dx, \tag{15}$$

where the constants A_4 and G_4 are defined as follows:

$$A_4 = b \int\limits_{-h/2}^{h/2} \frac{z^2}{\overline{S}_{11}^{(4)}(z)}\, dz \qquad G_4 = b \int\limits_{-h/2}^{h/2} \frac{1}{\overline{S}_{55}^{(4)}(z)}\, dz. \tag{16}$$

Kinetic Energy for Zone 3 (Part 4), i.e. for $\beta \le x \le L$:

$$T_4 = \int\limits_{\beta}^{L} \left(\frac{1}{2} B_4 \dot{w}_4^2 + \frac{1}{2}\, C_4 \dot{\phi}_4^2 \right) dx, \tag{17}$$

where the constants B_4 and C_4 are defined as follows:

$$B_4 = b \int\limits_{-h/2}^{h/2} \rho^{(4)}(z)\ dz, \quad C_4 = b \int\limits_{-h/2}^{h/2} \rho^{(4)}(z)\ z^2\ dz. \tag{18}$$

Strain Energy for Zone 2 (Part 2 and Part 3), i.e. for $\alpha \leq x \leq \beta$.

In the Zone 2, which contains the delamination crack, the longitudinal force resultants in the delaminated lower and upper parts (Part 2 and Part 3),

$$N_x^{(2)} = b \int_{-h/2}^{\gamma} \sigma_{xx}^{(2)}\, dz, \qquad N_x^{(3)} = b \int_{\gamma}^{h/2} \sigma_{xx}^{(3)}\, dz,$$

can be not negligibly small, even if external longitudinal forces are not applied to the beam. In order for these force resultants to be taken into account, a nonlinear term $\frac{1}{2}(w')^2$ in the Green-Lagrange strain-displacement relation for the strain component ε_{xx} must be taken into account. So, for the Part 2 (lower part of Zone 2) the following relations are used:

strain-displacement relations:

$$\varepsilon_{xx}^{(2)} = \frac{\partial u_2}{\partial x} + \frac{1}{2}\left(\frac{\partial w_2}{\partial x}\right)^2, \tag{19a}$$

$$\varepsilon_{xz}^{(2)} = \frac{1}{2}\left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x}\right); \tag{19b}$$

simplifying assumptions:

$$u_2(x,z,t) = z\phi_2(x,t) \quad w_2(x,z,t) = w_2(x,t); \tag{20}$$

stress-strain relations:

$$\sigma_{xx}^{(2)} = \frac{1}{\overline{S}_{11}^{(2)}}\varepsilon_{xx}^{(2)}, \quad \sigma_{xz}^{(2)} = \frac{1}{\overline{S}_{55}^{(2)}} 2\varepsilon_{xz}^{(2)}; \tag{21}$$

strain energy:

$$U_2 = \frac{1}{2}\, b \int_{\alpha}^{\beta} \int_{-h/2}^{\gamma} \sigma_{xx}^{(2)} \varepsilon_{xx}^{(2)}\, dz\, dx + Kb \int_{\alpha}^{\beta} \int_{-h/2}^{\gamma} \sigma_{xz}^{(2)} \varepsilon_{xz}^{(2)}\, dz\, dx. \tag{22}$$

From Equations (18) – (22) we obtain the following expression for the strain energy:

$$U_2 = \int_{\alpha}^{\beta} \left[\frac{1}{2} A_2 (\phi_2')^2 + \frac{1}{2} K G_2 (\phi_2 + w_2')^2 + \frac{1}{4} H_2 (w_2')^2 \phi_2' + \frac{1}{4} N_x^{(2)} (w_2')^2\right] dx, \tag{23}$$

where A_2, G_2, H_2 are constants, defined as

$$A_2 = b \int_{-h/2}^{\gamma} \frac{1}{\overline{S}_{11}^{(2)}(z)} z^2\, dz, \quad G_2 = b \int_{-h/2}^{\gamma} \frac{1}{\overline{S}_{55}^{(2)}(z)}\, dz, \quad H_2 = b \int_{-h/2}^{\gamma} \frac{1}{\overline{S}_{11}^{(2)}(z)}\, z\, dz, \tag{24}$$

and $N_x^{(2)}$ is a longitudinal force resultant in the lower delaminated part (Part 2):

$$N_x^{(2)} = b \int_{-h/2}^{\gamma} \sigma_{xx}^{(2)}\, dz = H_2\phi_2' + \frac{1}{2}\, Q_2(w_2')^2, \tag{25}$$

where Q_2 is a constant, defined as

$$Q_2 = b \int_{-h/2}^{\gamma} \frac{1}{\overline{S}_{11}^{(2)}(z)}\, dz. \tag{26}$$

Similarly, for the Part 3 (upper part of Zone 2) the expression for the strain energy has the form

$$U_3 = \int_{\alpha}^{\beta} \left[\frac{1}{2}A_3(\phi_3')^2 + \frac{1}{2}KG_3(\phi_3 + w_3')^2 + \frac{1}{4}H_3(w_3')^2\phi_3' + \frac{1}{4}N_x^{(3)}(w_3')^2\right] dx, \tag{27}$$

where

$$A_3 = b \int_{\gamma}^{h/2} \frac{1}{\overline{S}_{11}^{(3)}(z)}\, z^2\, dz, \qquad G_3 = b \int_{\gamma}^{h/2} \frac{1}{\overline{S}_{55}^{(3)}(z)}\, dz, \qquad H_3 = b \int_{\gamma}^{h/2} \frac{1}{\overline{S}_{11}^{(3)}(z)}\, z\, dz, \tag{28}$$

$$N_x^{(3)} = b \int_{\gamma}^{h/2} \sigma_{xx}^{(3)}\, dz = H_3\phi_3' + \frac{1}{2}\, Q_3(w_3')^2, \tag{29}$$

where

$$Q_3 = b \int_{\gamma}^{h/2} \frac{1}{\overline{S}_{11}^{(3)}(z)}\, dz. \tag{30}$$

Kinetic Energy for Zone 2 (Part 2 and Part 3), i.e. for $\alpha \le x \le \beta$.

Expressions for kinetic energy of Part 2 and Part 3 are obtained similarly to the expressions for the kinetic energies of all other parts, and they have the form:

$$\begin{aligned} T_2 &= \int_{\alpha}^{\beta} \left(\frac{1}{2}B_2\dot{w}_2^2 + \frac{1}{2}C_2\dot{\phi}_2^2\right) dx, \\ T_3 &= \int_{\alpha}^{\beta} \left(\frac{1}{2}B_3\dot{w}_3^2\ dx + \frac{1}{2}C_3\dot{\phi}_3^2\right) dx, \end{aligned} \tag{31}$$

where

$$\begin{aligned} B_2 &= b \int_{-h/2}^{\gamma} \rho^{(2)}(z)\, dz, \qquad C_2 = b \int_{-h/2}^{\gamma} \rho^{(2)}(z) z^2\, dz, \\ B_3 &= b \int_{\gamma}^{h/2} \rho^{(3)}(z)\, dz, \qquad C_3 = b \int_{\gamma}^{h/2} \rho^{(3)}(z) z^2\, dz. \end{aligned} \tag{32}$$

In view of the expressions for strain and kinetic energies, derived above, the Lagrangian function density (potential energy minus kinetic energy per unit length) for the delaminated composite beam with the piezoelectric actuator (Figure 2.1) can be written as follows

$$\widetilde{\mathfrak{L}} = \begin{cases} \widetilde{\mathfrak{L}}_0(\dot{w}_0,\, w_0',\, \phi_0,\, \dot{\phi}_0,\, \phi_0') & \text{in Zone 0}\ \ (0 \le x \le a) \\ \widetilde{\mathfrak{L}}_1(\dot{w}_1,\, w_1',\, \phi_1,\, \dot{\phi}_1,\, \phi_1') & \text{in Zone 1}\ \ (a \le x \le \alpha) \\ \widetilde{\mathfrak{L}}_2(\dot{w}_2,\, w_2',\, \phi_2,\, \dot{\phi}_2,\, \phi_2',\, \dot{w}_3,\, w_3',\, \phi_3,\, \dot{\phi}_3,\, \phi_3') & \text{in Zone 2}\ \ (\alpha \le x \le \beta) \\ \widetilde{\mathfrak{L}}_3(\dot{w}_4,\, w_4',\, \phi_4,\, \dot{\phi}_4,\, \phi_4') & \text{in Zone 3}\ \ (\beta \le x \le L), \end{cases} \tag{33}$$

where

$$\widetilde{\mathfrak{L}}_0 = \frac{A_0}{2}(\phi_0')^2 + K\frac{G_0}{2}(\phi_0 + w_0')^2 - I_pV\phi_0' - \frac{B_0}{2}\dot{w}_0^2 - \frac{C_0}{2}\dot{\phi}_0^2, \tag{34a}$$

$$\widetilde{\mathfrak{L}}_1 = \frac{A_1}{2}(\phi_1')^2 + K\frac{G_1}{2}(\phi_1 + w_1')^2 - \frac{B_1}{2}\dot{w}_1^2 - \frac{C_1}{2}\dot{\phi}_1^2, \tag{34b}$$

$$\begin{aligned} \widetilde{\mathfrak{L}}_2 = {} & \frac{1}{2}A_2(\phi_2')^2 + \frac{1}{2}KG_2(\phi_2 + w_2')^2 + \frac{1}{2}H_2(w_2')^2\phi_2' + \frac{1}{8}Q_2(w_2')^4 \\ & - \frac{1}{2}B_2\dot{w}_2^2 - \frac{1}{2}C_2\dot{\phi}_2^2 + \frac{1}{2}A_3(\phi_3')^2 + \frac{1}{2}KG_3(\phi_3 + w_3')^2 \\ & + \frac{1}{2}H_3(w_3')^2\phi_3' + \frac{1}{8}Q_3(w_3')^4 - \frac{1}{2}B_3\dot{w}_2^2 - \frac{1}{2}C_3\dot{\phi}_3^2, \end{aligned} \tag{34c}$$

$$\widetilde{\mathfrak{L}}_3 = \frac{A_4}{2}(\phi_4')^2 + K\frac{G_4}{2}(\phi_4 + w_4')^2 - \frac{B_4}{2}\dot{w}_4^2 - \frac{C_4}{2}\dot{\phi}_4^2. \tag{34d}$$

A variational formulation of the problem includes essential boundary conditions at the ends of each zone, which will be treated as point-wise constraints, and a nonpenetration condition for the delamination crack faces (subdomain constraints for Zone 2), Reference [1]. For a clamped-free beam, the *point-wise constraints* have the form

$$R_i(t) = 0 \quad (i = 1,\, 2, \ldots,\, 12), \tag{35a}$$

where

$$\begin{aligned} R_1 &\equiv w_0(0,t), & R_2 &\equiv \phi_0(0,t), \\ R_3 &\equiv w_0(a,t) - w_1(a,t) & R_4 &\equiv \phi_0(a,t) - \phi_1(a,t), \\ R_5 &\equiv w_1(\alpha,t) - w_2(\alpha,t), & R_6 &\equiv \phi_1(\alpha,t) - \phi_2(\alpha,t), \\ R_7 &\equiv w_1(\alpha,t) - w_3(\alpha,t), & R_8 &\equiv \phi_1(\alpha,t) - \phi_3(\alpha,t), \\ R_9 &\equiv w_2(\beta,t) - w_4(\beta,t), & R_{10} &\equiv \phi_2(\beta,t) - \phi_4(\beta,t), \\ R_{11} &\equiv w_3(\beta,t) - w_4(\beta,t), & R_{12} &\equiv \phi_3(\beta,t) - \phi_4(\beta,t). \end{aligned} \tag{35b}$$

In case of other kinds of fixation of the beam's ends, the first two point-wise constraints will be different, of course, but the other point-wise constraints will be the same.

During the vibration of the delaminated beam, the upper and lower delaminated parts touch each other, and the force of their interaction needs to be taken into account. This force enters into the differential equations of motion as a reaction of constraint, which prevents overlapping of the upper and lower delaminated parts. Such a constraint can be expressed by a relationship between w_2 and w_3 (i.e. displacements of the lower and upper delaminated parts) that prevents the difference $w_3 - w_2$ to take on negative values:

$$f(w_2(x,t),\, w_3(x,t)) = f(x,t) \equiv (w_3 - w_2)\,[1 - \mathfrak{H}_0(w_3 - w_2)] = 0, \tag{36a}$$

where $\mathfrak{H}_0$ is a Heaviside function, defined in Appendix B. If delaminated sublaminates "attempt" to overlap during the vibration (if $w_3 - w_2 < 0$), or if the crack is closed $(w_3 - w_2 = 0)$, then $\mathfrak{H}_0(w_3 - w_2) = 0$, and, therefore, due to equation (10), the difference $w_3 - w_2$ is set equal to zero. If the crack is open $(w_3 - w_2 > 0)$, then $\mathfrak{H}_0(w_3 - w_2) = 1$, and no constraints are imposed on the difference $w_3 - w_2$. With the use of the analytical representation of the Heaviside function (equation B-5), the nonpenetration constraint, expressed by equation (36a), can be written as follows:

$$f(x,t) \equiv (w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}\right) = 0, \tag{36b}$$

where ϵ is some small number. The nonpenetration constraint (36) is a *subdomain constraint* for the Zone 2 $(\alpha \le x \le \beta)$.

Now, the problem can be formulated as a problem of finding a constrained (conditional) extremum of the functional

$$J = \int_{t_1}^{t_2}\int_0^L \widetilde{\mathfrak{L}}\, dx\, dt \tag{37}$$

with constraints expressed by Equations (35) and (36). The constraints (35) and (36) can be included into the functional by the method of Lagrange multipliers. This will produce a modified functional $\overline{J}$:

$$\overline{J} = J + \int_{t_1}^{t_2}\sum_{i=1}^{12}\lambda_i(t)R_i(t) + \int_{t_1}^{t_2}\int_\alpha^\beta \mu(x,t)f(x,t)\,dx\,dt, \tag{39}$$

where $\lambda_i(t)$ and $\mu(x,t)$ are the Lagrange multipliers. Now we have a problem of an unconstrained (unconditional) extremum of the modified functional $\overline{J}$. Derivation of the partial differential equations of motion and natural boundary conditions from the condition of extremum of the functional (39) can be performed using standard methods of calculus of variations. In the following text, partial differential equations of motion with boundary conditions, a weak form of the partial differential equations and a finite element formulation on the basis of the weak form will be obtained.

3 Partial Differential Equations with Boundary Conditions

To derive the partial differential equations of motion with boundary conditions, the condition of unconstrained extremum of the functional $\overline{J}$ (Equation (39)) will be used. The condition $\delta\overline{J} = 0$ leads to the following partial differential equations and natural boundary conditions.

Partial differential equations:

$$-\frac{\partial}{\partial t}\frac{\partial\widetilde{\mathfrak{L}}_0}{\partial\dot{w}_0}-\frac{\partial}{\partial x}\frac{\partial\widetilde{\mathfrak{L}}_0}{\partial w_0'}=0 \quad \text{in} \quad x\in[0,\,a]\,, \tag{41}$$

$$\frac{\partial\widetilde{\mathfrak{L}}_0}{\partial\phi_0}-\frac{\partial}{\partial t}\frac{\partial\widetilde{\mathfrak{L}}_0}{\partial\dot{\phi}_0}-\frac{\partial}{\partial x}\frac{\partial\widetilde{\mathfrak{L}}_0}{\partial \phi_0'}=0 \quad \text{in} \quad x\in[0,\,a]\,, \tag{42}$$

$$-\frac{\partial}{\partial t}\frac{\partial\widetilde{\mathfrak{L}}_1}{\partial\dot{w}_1}-\frac{\partial}{\partial x}\frac{\partial\widetilde{\mathfrak{L}}_1}{\partial w_1'}=0 \quad \text{in} \quad x\in[a,\,\alpha]\,, \tag{43}$$

$$\frac{\partial\widetilde{\mathfrak{L}}_1}{\partial\phi_1}-\frac{\partial}{\partial t}\frac{\partial\widetilde{\mathfrak{L}}_1}{\partial\dot{\phi}_1}-\frac{\partial}{\partial x}\frac{\partial\widetilde{\mathfrak{L}}_1}{\partial \phi_1'}=0 \quad \text{in} \quad x\in[a,\,\alpha], \tag{44}$$

$$\mu\frac{\partial f}{\partial w_2}-\frac{\partial}{\partial t}\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial\dot{w}_2}-\frac{\partial}{\partial x}\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial w_2'}=0 \quad \text{in} \quad x\in[\alpha,\,\beta]\,, \tag{45}$$

$$\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial\phi_2}-\frac{\partial}{\partial t}\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial\dot{\phi}_2}-\frac{\partial}{\partial x}\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial \phi_2'}=0 \quad \text{in} \quad x\in[\alpha,\,\beta]\,, \tag{46}$$

$$\mu\frac{\partial f}{\partial w_3}-\frac{\partial}{\partial t}\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial\dot{w}_3}-\frac{\partial}{\partial x}\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial w_3'}=0 \quad \text{in} \quad x\in[\alpha,\,\beta]\,, \tag{47}$$

$$\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial\phi_3}-\frac{\partial}{\partial t}\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial\dot{\phi}_3}-\frac{\partial}{\partial x}\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial \phi_3'}=0 \quad \text{in} \quad x\in[\alpha,\,\beta]\,, \tag{48}$$

$$-\frac{\partial}{\partial t}\frac{\partial\widetilde{\mathfrak{L}}_3}{\partial\dot{w}_4}-\frac{\partial}{\partial x}\frac{\partial\widetilde{\mathfrak{L}}_3}{\partial w_4'}=0 \quad \text{in} \quad x\in[\beta,\,L]\,, \tag{49}$$

$$\frac{\partial\widetilde{\mathfrak{L}}_3}{\partial\phi_4}-\frac{\partial}{\partial t}\frac{\partial\widetilde{\mathfrak{L}}_3}{\partial\dot{\phi}_4}-\frac{\partial}{\partial x}\frac{\partial\widetilde{\mathfrak{L}}_3}{\partial \phi_4'}=0 \quad \text{in} \quad x\in[\beta,\,L]\,. \tag{50}$$

Natural boundary conditions:

$$\frac{\partial\widetilde{\mathfrak{L}}_0}{\partial w_0'}+\lambda_3=0 \quad \text{at} \quad x=a, \qquad \frac{\partial\widetilde{\mathfrak{L}}_0}{\partial \phi_0'}+\lambda_4=0 \quad \text{at} \quad x=a, \tag{51}$$

$$-\frac{\partial\widetilde{\mathfrak{L}}_1}{\partial w_1'}-\lambda_3=0 \quad \text{at} \quad x=a, \qquad \frac{\partial\widetilde{\mathfrak{L}}_1}{\partial w_1'}+\lambda_5+\lambda_7=0 \quad \text{at} \quad x=\alpha, \tag{52}$$

$$-\frac{\partial\widetilde{\mathfrak{L}}_1}{\partial \phi_1'}-\lambda_4=0 \quad \text{at} \quad x=a, \qquad \frac{\partial\widetilde{\mathfrak{L}}_1}{\partial \phi_1'}+\lambda_6+\lambda_8=0 \quad \text{at} \quad x=\alpha \tag{53}$$

$$-\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial w_2'}-\lambda_5=0 \quad \text{at} \quad x=\alpha, \qquad \frac{\partial\widetilde{\mathfrak{L}}_2}{\partial w_2'}+\lambda_9=0 \quad \text{at} \quad x=\beta, \tag{54}$$

$$-\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial \phi_2'}-\lambda_6=0 \quad \text{at} \quad x=\alpha, \qquad \frac{\partial\widetilde{\mathfrak{L}}_2}{\partial \phi_2'}+\lambda_{10}=0 \quad \text{at} \quad x=\beta, \tag{55}$$

$$-\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial w_3'}-\lambda_7=0 \quad \text{at} \quad x=\alpha, \qquad \frac{\partial\widetilde{\mathfrak{L}}_2}{\partial w_3'}+\lambda_{11}=0 \quad \text{at} \quad x=\beta, \tag{56}$$

$$-\frac{\partial\widetilde{\mathfrak{L}}_2}{\partial \phi_3'}-\lambda_8=0 \quad \text{at} \quad x=\alpha, \qquad \frac{\partial\widetilde{\mathfrak{L}}_2}{\partial \phi_3'}+\lambda_{12}=0 \quad \text{at} \quad x=\beta, \tag{57}$$

$$-\frac{\partial \widetilde{\mathfrak{L}}_3}{\partial w_4'} - \lambda_9 - \lambda_{11} = 0 \quad \text{at} \quad x = \beta, \qquad \frac{\partial \widetilde{\mathfrak{L}}_3}{\partial w_4'} = 0 \quad \text{at} \quad x = L, \tag{58}$$

$$-\frac{\partial \widetilde{\mathfrak{L}}_3}{\partial \phi_4'} - \lambda_{10} - \lambda_{12} = 0 \quad \text{at} \quad x = \beta, \qquad \frac{\partial \widetilde{\mathfrak{L}}_3}{\partial \phi_4'} = 0 \quad \text{at} \quad x = L. \tag{59}$$

Elimination of the Lagrange multipliers from Equations (51) – (59) leads to the following eight natural boundary conditions:

$$\frac{\partial \widetilde{\mathfrak{L}}_0}{\partial w_0'} - \frac{\partial \widetilde{\mathfrak{L}}_1}{\partial w_1'} = 0 \quad \text{at} \quad x = a, \tag{60}$$

$$\frac{\partial \widetilde{\mathfrak{L}}_0}{\partial \phi_0'} - \frac{\partial \widetilde{\mathfrak{L}}_1}{\partial \phi_1'} = 0 \quad \text{at} \quad x = a, \tag{61}$$

$$\frac{\partial \widetilde{\mathfrak{L}}_1}{\partial w_1'} - \frac{\partial \widetilde{\mathfrak{L}}_2}{\partial w_2'} - \frac{\partial \widetilde{\mathfrak{L}}_2}{\partial w_3'} = 0 \quad \text{at} \quad x = \alpha, \tag{62}$$

$$\frac{\partial \widetilde{\mathfrak{L}}_1}{\partial \phi_1'} - \frac{\partial \widetilde{\mathfrak{L}}_2}{\partial \phi_2'} - \frac{\partial \widetilde{\mathfrak{L}}_2}{\partial \phi_3'} = 0 \quad \text{at} \quad x = \alpha, \tag{63}$$

$$\frac{\partial \widetilde{\mathfrak{L}}_2}{\partial w_2'} + \frac{\partial \widetilde{\mathfrak{L}}_2}{\partial w_3'} - \frac{\partial \widetilde{\mathfrak{L}}_3}{\partial w_4'} = 0 \quad \text{at} \quad x = \beta, \tag{64}$$

$$\frac{\partial \widetilde{\mathfrak{L}}_2}{\partial \phi_2'} + \frac{\partial \widetilde{\mathfrak{L}}_2}{\partial \phi_3'} - \frac{\partial \widetilde{\mathfrak{L}}_3}{\partial \phi_4'} = 0 \quad \text{at} \quad x = \beta, \tag{65}$$

$$\frac{\partial \widetilde{\mathfrak{L}}_3}{\partial w_4'} = 0 \quad \text{at} \quad x = L, \tag{66}$$

$$\frac{\partial \widetilde{\mathfrak{L}}_3}{\partial \phi_4'} = 0 \quad \text{at} \quad x = L. \tag{67}$$

Substitution of Equations (34) into Equations (41) – (50) and into Equations (60) – (67) produces the following result.

Partial differential equations:

$$KG_0(w_0'' + \phi_0') - B_0\ddot{w}_0 = 0 \quad \text{in} \quad x \in [0,\, a]\,, \tag{68}$$

$$A_0\phi_0'' - KG_0(w_0' + \phi_0) - C_0\ddot{\phi}_0 = I_pV' \quad \text{in} \quad x \in [0,\, a]\,, \tag{69}$$

$$KG_1(w_1'' + \phi_1') - B_1\ddot{w}_1 = 0 \quad \text{in} \quad x \in [a,\, \alpha]\,, \tag{70}$$

$$A_1\phi_1'' - KG_1(w_1' + \phi_1) - C_1\ddot{\phi}_1 = 0 \quad \text{in} \quad x \in [a,\, \alpha]\,, \tag{71}$$

$$\begin{aligned} KG_2(w_2'' + \phi_2') - B_2\ddot{w}_2 - \mu\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}\right) \\ - H_2\phi_2'w_2'' - \frac{3}{2}\,Q_2(w_2')^2w_2'' = 0 \quad \text{in} \quad x \in [\alpha,\, \beta]\,, \end{aligned} \tag{72}$$

$$A_2\phi_2'' - KG_2(w_2' + \phi_2) - C_2\ddot{\phi}_2 - H_2w_2'w_2'' = 0 \quad \text{in} \quad x \in [\alpha,\, \beta]\,, \tag{73}$$

$$\begin{aligned} KG_3(w_3'' + \phi_3') - B_3\ddot{w}_3 + \mu\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}\right) \\ - H_3\phi_3'w_3'' - \frac{3}{2}\,Q_3(w_3')^2w_3'' = 0 \quad \text{in} \quad x \in [\alpha,\, \beta]\,, \end{aligned} \tag{74}$$

$$A_3\phi_3'' - KG_3(w_3' + \phi_3) - C_3\ddot{\phi}_3 - H_3 w_3' w_3'' = 0 \quad \text{in} \quad x \in [\alpha,\, \beta]\,, \tag{75}$$

$$KG_4(w_4'' + \phi_4') - B_4\ddot{w}_4 = 0 \quad \text{in} \quad x \in [\beta,\, L]\,, \tag{76}$$

$$A_4\phi_4'' - KG_4(w_4' + \phi_4) - C_4\ddot{\phi}_4 = 0 \quad \text{in} \quad x \in [\beta,\, L]\,, \tag{77}$$

Natural boundary conditions:

$$G_0(\phi_0 + w_0') - G_1(\phi_1 + w_1') = 0 \quad \text{at} \quad x = a, \tag{78}$$

$$A_0\phi_0' - A_1\phi_1' = I_p V \quad \text{at} \quad x = a, \tag{79}$$

$$\begin{aligned} &KG_1(\phi_1 + w_1') - KG_2(\phi_2 + w_2') - H_2 w_2'\phi_2' \\ &- \frac{1}{2}\, Q_2(w_2')^3 - KG_3(\phi_3 + w_3') - H_3 w_3'\phi_3' - \frac{1}{2}\, Q_3(w_3')^3 = 0 \quad \text{at} \quad x = \alpha, \end{aligned} \tag{80}$$

$$A_1\phi_1' - A_2\phi_2' - \frac{1}{2}\, H_2(w_2')^2 - A_3\phi_3' - \frac{1}{2}\, H_3(w_3')^2 = 0 \quad \text{at} \quad x = \alpha, \tag{81}$$

$$\begin{aligned} &KG_2(\phi_2 + w_2') + H_2 w_2'\phi_2' + \frac{1}{2}\, Q_2(w_2')^3 + KG_3(\phi_3 + w_3') \\ &+ H_3 w_3'\phi_3' + \frac{1}{2}\, Q_3(w_3')^3 - KG_4(\phi_4 + w_4') = 0 \quad \text{at} \quad x = \beta, \end{aligned} \tag{82}$$

$$A_2\phi_2' + A_3\phi_3' - A_4\phi_4' = 0 \quad \text{at} \quad x = \beta, \tag{83}$$

$$\phi_4 + w_4' = 0 \quad \text{at} \quad x = L, \tag{84}$$

$$\phi_4' = 0 \quad \text{at} \quad x = L. \tag{85}$$

In computation of derivatives $\dfrac{\partial f}{\partial w_2}$ and $\dfrac{\partial f}{\partial w_3}$ in Equations (45) and (47), which led to Equations (72) and (74), the following chain of transformations was used:

$$\begin{aligned} \frac{\partial f}{\partial w_2} &= \lim_{\epsilon \to 0} \frac{\partial}{\partial w_2}\left((w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{w_3 - w_2}{\epsilon}\right)\right) \\ &= \lim_{\epsilon \to 0}\left(-\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{w_3 - w_2}{\epsilon}\right) + \frac{1}{\epsilon\pi} \frac{\epsilon^2 (w_3 - w_2)}{\epsilon^2 + (w_3 - w_2)^2}\right) \\ &= -\lim_{\epsilon \to 0}\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{w_3 - w_2}{\epsilon}\right). \end{aligned}$$

So,

$$\frac{\partial f}{\partial w_2} \approx -\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{w_3 - w_2}{\epsilon}\right), \tag{86}$$

where ϵ is some small number. Similarly

$$\frac{\partial f}{\partial w_3} \approx \left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{w_3 - w_2}{\epsilon}\right). \tag{87}$$

So, the formulation of the problem includes eleven equations for subdomains: ten partial differential equations (68) – (77) and one algebraic equation of constraint (36b) for Zone 2. The number of unknown functions is also eleven. The unknown functions are: $\mu(x,t)$, $w_k(x,t)$, $\phi_k(x,t)$ $(k = 0, 1, 2, 3, 4)$. The total order of spatial derivatives of the partial differential equations is twenty, and the number of boundary conditions is also twenty: twelve essential boundary conditions (Equations (35)) and eight natural boundary conditions (Equations (78)–(85)).

The formulation of the problem in terms of partial differential equations can be simplified, if the *penalty function method* [2] is used for the nonpenetration constraint, i.e. if the Lagrange multiplier $\mu(x,t)$, associated with the nonpenetration constraint (36), is written as

$$\mu(x,t) = \chi f(x,t), \tag{88}$$

where the function $f(x,t)$ is defined by Equation (36b), and χ is some large number, which has to be chosen by an analyst. Then, Equation (72) takes the form

$$KG_2(w_2'' + \phi_2') - B_2\ddot{w}_2 - \chi(w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}\right) = 0 \quad \text{in} \quad x \in [\alpha,\, \beta]\,, \tag{89}$$

and Equation (74) takes the form

$$KG_3(w_3'' + \phi_3') - B_3\ddot{w}_3 + \chi(w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}\right) = 0 \quad \text{in} \quad x \in [\alpha,\, \beta]\,. \tag{90}$$

In transition from Equations (72) and (74) to Equations (89) and (90) respectively, the following transformation was used

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}\right)^2 &= (1 - \mathfrak{H}_0(w_3 - w_2))^2 \\ &= 1 - \mathfrak{H}_0(w_3 - w_2) = \frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}.\end{aligned} \tag{91}$$

Now, the formulation of the problem contains ten partial differential equations (68)–(71), (89), (73), (90), (75)–(77) with ten unknown functions $w_k(x,t)$, $\phi_k(x,t)$, $(k = 0,1,2,3,4)$.

The natural boundary condition (79) is nonhomogeneous, because the externally applied voltage $V(a,t)$ enters into it. To avoid having the nonhomogeneous boundary condition, one can consider that the voltage, applied to the actuator, is distributed not over the subdomain $x \in [0,a]$, but over the subdomain $x \in [0,\, a-\varepsilon]$, where ε is some very small positive number. Then the physics of the problem is not changed, and the voltage $V(a,t)$ does not enter into the boundary condition (79), i.e. this boundary condition takes a simpler homogeneous form

$$A_0\phi_0' - A_1\phi_1' = 0 \quad \text{at} \quad x = a. \tag{92}$$

Let us consider, for example, the *voltage distributed uniformly over the length of the actuator*, i.e.

$$V(x,t) = V(t) \quad \text{in} \quad x \in [0,a]. \tag{93a}$$

Then, without altering the physical formulation of the problem, we can write

$$V(x,t) = V(t) \quad \text{in} \quad x \in [0,\, a-\varepsilon]. \tag{93b}$$

Then, the voltage $V(x,t)$ can be presented in the form

$$V(x,t) = [1 - \mathfrak{H}_{a-\varepsilon}(x)]\; V(t) \quad \text{in} \quad x \in [0,a] \tag{94}$$

where $\mathfrak{H}_{a-\varepsilon}(x)$ is the Heaviside function (Appendix B). Then, the right side of the differential equation (69) takes the form

$$I_p V' = -I_p V(t)\ \mathfrak{H}'_{a-\varepsilon}(x) = -I_p V(t)\ \delta_{a-\varepsilon}(x), \tag{95}$$

where $\delta_{a-\varepsilon}(x)$ is the delta-function (Appendix B).

In computation of the example problems for the clamped-free beams, presented below, the formulation based on the penalty function method will be used, and the voltage will be distributed uniformly along the length of the actuator. For convenience, this formulation is summarized below.

Partial differential equations:

$$KG_0(w_0'' + \phi_0') - B_0\ddot{w}_0 = 0 \quad \text{in} \quad x \in [0, a], \tag{96}$$

$$A_0\phi_0'' - KG_0(w_0' + \phi_0) - C_0\ddot{\phi}_0 = -I_p V(t)\delta_{a-\varepsilon}(x) \quad \text{in} \quad x \in [0, a], \tag{97}$$

$$KG_1(w_1'' + \phi_1') - B_1\ddot{w}_1 = 0 \quad \text{in} \quad x \in [a, \alpha], \tag{98}$$

$$A_1\phi_1'' - KG_1(w_1' + \phi_1) - C_1\ddot{\phi}_1 = 0 \quad \text{in} \quad x \in [a, \alpha], \tag{99}$$

$$\begin{aligned} &KG_2(w_2'' + \phi_2') - B_2\ddot{w}_2 - \chi(w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}\right) \\ &\quad - H_2\phi_2' w_2'' - \frac{3}{2}Q_2(w_2')^2 w_2'' = 0 \quad \text{in} \quad x \in [\alpha, \beta], \end{aligned} \tag{100}$$

$$A_2\phi_2'' - KG_2(w_2' + \phi_2) - C_2\ddot{\phi}_2 - H_2 w_2' w_2'' = 0 \quad \text{in} \quad x \in [\alpha,\ \beta], \tag{101}$$

$$\begin{aligned} &KG_3(w_3'' + \phi_3') - B_3\ddot{w}_3 + \chi(w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}\right) \\ &\quad - H_3\phi_3' w_3'' - \frac{3}{2}Q_3(w_3')^2 w_3'' = 0 \quad \text{in} \quad x \in [\alpha,\ \beta]. \end{aligned} \tag{102}$$

$$A_3\phi_3'' - KG_3(w_3' + \phi_3) - C_3\ddot{\phi}_3 - H_3 w_3' w_3'' = 0 \quad \text{in} \quad x \in [\alpha,\ \beta], \tag{103}$$

$$KG_4(w_4'' + \phi_4') - B_4\ddot{w}_4 = 0 \quad \text{in} \quad x \in [\beta, L], \tag{104}$$

$$A_4\phi_4'' - KG_4(w_4' + \phi_4) - C_4\ddot{\phi}_4 = 0 \quad \text{in} \quad x \in [\beta, L]. \tag{105}$$

Essential boundary conditions:

$$R_i = 0 \quad (i = 1, 2, \ldots, 12), \tag{106a}$$

where

$$\begin{aligned} R_1 &\equiv w_0(0,t), & R_2 &\equiv \phi_0(0,t), \\ R_3 &\equiv w_0(a,t) - w_1(a,t), & R_4 &\equiv \phi_0(a,t) - \phi_1(a,t), \\ R_5 &\equiv w_1(\alpha,t) - w_2(\alpha,t), & R_6 &\equiv \phi_1(\alpha,t) - \phi_2(\alpha,t), \\ R_7 &\equiv w_1(\alpha,t) - w_3(\alpha,t), & R_8 &\equiv \phi_1(\alpha,t) - \phi_3(\alpha,t), \\ R_9 &\equiv w_2(\beta,t) - w_4(\beta,t), & R_{10} &\equiv \phi_2(\beta,t) - \phi_4(\beta,t), \\ R_{11} &\equiv w_3(\beta,t) - w_4(\beta,t), & R_{12} &\equiv \phi_3(\beta,t) - \phi_4(\beta,t). \end{aligned} \tag{106b}$$

Natural boundary conditions:

$$G_0(\phi_0 + w_0') - G_1(\phi_1 + w_1') = 0 \quad \text{at} \quad x = a, \tag{107}$$

$$A_0\phi_0' - A_1\phi_1' = 0 \quad \text{at} \quad x = a, \tag{108}$$

$$\begin{aligned} &KG_1(\phi_1 + w_1') - KG_2(\phi_2 + w_2') - H_2 w_2'\phi_2' - \frac{1}{2}\,Q_2(w_2')^3 \\ &- KG_3(\phi_3 + w_3') - H_3 w_3'\phi_3' - \frac{1}{2}\,Q_3(w_3')^3 = 0 \quad \text{at} \quad x = \alpha, \end{aligned} \tag{109}$$

$$A_1\phi_1' - A_2\phi_2' - \frac{1}{2}\,H_2(w_2')^2 - A_3\phi_3' - \frac{1}{2}\,H_3(w_3')^2 = 0 \quad \text{at} \quad x = \alpha, \tag{110}$$

$$\begin{aligned} &KG_2(\phi_2 + w_2') + H_2 w_2'\phi_2' + \frac{1}{2}\,Q_2(w_2')^3 + KG_3(\phi_3 + w_3') \\ &+ H_3 w_3'\phi_3' + \frac{1}{2}\,Q_3(w_3')^3 - KG_4(\phi_4 + w_4') = 0 \quad \text{at} \quad x = \beta, \end{aligned} \tag{111}$$

$$A_2\phi_2' + A_3\phi_3' - A_4\phi_4' = 0 \quad \text{at} \quad x = \beta, \tag{112}$$

$$\phi_4 + w_4' = 0 \quad \text{at} \quad x = L, \tag{113}$$

$$\phi_4' = 0 \quad \text{at} \quad x = L. \tag{114}$$

4 Finite Element Formulation

The finite element formulation is made on the basis of weak forms for the derived partial differential equations (96) – (105).

Finite element within Zone 0 (Part 0), i.e. within a subdomain $x \in [0, a]$.

The weak form for a finite element within Zone 0 is obtained by multiplying equations (96) and (97) by weight functions (variations) δw_0 and $\delta\phi_0$ respectively, integrating them over an element's length, performing integration by parts and adding the resulting equations. The weak form thus obtained is

$$\begin{aligned} 0 = \int_{X_A}^{X_B} \Big[&A_0\phi_0'\,\delta\phi_0' + KG_0(w_0' + \phi_0)\,\delta w_0' + KG_0(w_0' + \phi_0)\,\delta\phi_0 \\ &+ B_0\ddot{w}_0\,\delta w_0 + C_0\ddot{\phi}_0\,\delta\phi_0 - I_p V(t)\,\delta_{a-\varepsilon}(x)\,\delta\phi_0 \Big] dx \\ &+ \Big[KG_0(w_0' + \phi_0)\,\delta w_0 \Big]_{x=X_A} - \Big[KG_0(w_0' + \phi_0)\,\delta w_0 \Big]_{x=X_B} \\ &+ \Big(A_0\phi_0'\,\delta\phi_0 \Big)_{x=X_A} - \Big(A_0\phi_0'\,\delta\phi_0 \Big)_{x=X_B}, \end{aligned} \tag{115}$$

where X_A and X_B are coordinates of the element's left and right boundary points. In the boundary terms of the weak form, variations of the unknown functions w_0 and ϕ_0 themselves (not their derivatives) are present, therefore, the Lagrange interpolation polynomials are appropriate for approximation of the unknown functions within a finite element [3]. In this analysis, the author chose to approximate both unknown functions $w_0(x,t)$ and $\phi_0(x,t)$, within a finite element, by the Lagrange interpolation polynomials of a fifth degree:

$$w_0(x,t) \approx \sum_{i=1}^{6} N_i(x) w_{0i}(t), \qquad \phi_0(x,t) \approx \sum_{i=1}^{6} N_i(x)\phi_{0i}(t), \tag{116}$$

where

$$w_{0i}(t) \equiv w_0(x_i,\, t), \quad \phi_{0i}(t) \equiv \phi_0(x_i,\, t), \tag{117}$$

$$N_i(x) = \prod_{\substack{j=1\\ j\neq i}}^{6} \frac{x - x_j}{x_i - x_j}, \tag{118}$$

$$x_1 \equiv X_A, \quad x_6 \equiv X_B. \tag{119}$$

So, the finite element has six nodes, and two unknown nodal parameters w_{0i} and ϕ_{0i} are associated with each i-th node. The nodes are chosen to be equidistant. Denoting the element's length as l, the nodal coordinates, in the local element coordinate system (the origin of which coincides with the left boundary point of the element), can be written as

$$x_i = \frac{(i-1)l}{5} \quad (i = 1, \ldots, 6). \tag{120}$$

Explicit expressions for the shape functions are written below

$$\begin{aligned}
N_1(x) &= -\frac{625}{24l^5}x^5 + \frac{625}{8l^4}x^4 - \frac{2125}{24l^3}x^3 + \frac{375}{8l^2}x^2 - \frac{137}{12l}x + 1,\\
N_2(x) &= \frac{3125}{12}\frac{x^5}{l^5} - 625\frac{x^4}{l^4} + \frac{6625}{12}\frac{x^3}{l^3} - \frac{425}{2}\frac{x^2}{l^2} + 30\frac{x}{l},\\
N_3(x) &= -\frac{3125}{12}\frac{x^5}{l^5} + \frac{8125}{12}\frac{x^4}{l^4} - \frac{7375}{12}\frac{x^3}{l^3} + \frac{2675}{12}\frac{x^2}{l^2} - 25\frac{x}{l},\\
N_4(x) &= \frac{3125}{12}\frac{x^5}{l^5} - 625\frac{x^4}{l^4} + \frac{6125}{12}\frac{x^3}{l^3} - \frac{325}{2}\frac{x^2}{l^2} + \frac{50}{3}\frac{x}{l},\\
N_5(x) &= -\frac{3125}{24}\frac{x^5}{l^5} + \frac{6875}{24}\frac{x^4}{l^4} - \frac{5125}{24}\frac{x^3}{l^3} + \frac{1525}{24}\frac{x^2}{l^2} - \frac{25}{4}\frac{x}{l},\\
N_6(x) &= \frac{625}{24}\frac{x^5}{l^5} - \frac{625}{12}\frac{x^4}{l^4} + \frac{875}{24}\frac{x^3}{l^3} - \frac{125}{12}\frac{x^2}{l^2} + \frac{x}{l}.
\end{aligned} \tag{121}$$

The column-matrix of element nodal parameters is introduced as follows

$$\underset{(12\times1)}{\{\theta\}} \equiv \lfloor\, w_{01} \quad \phi_{01} \quad w_{02} \quad \phi_{02} \quad w_{03} \quad \phi_{03} \quad w_{04} \quad \phi_{04} \quad w_{05} \quad \phi_{05} \quad w_{06} \quad \phi_{06} \,\rfloor^{\mathrm{T}}. \tag{122}$$

Then, in view of formulas (116), the unknown functions $w_0(x,t)$ and $\phi_0(x,t)$ can be expressed in terms of the column-matrix of nodal parameters $\{\theta\}$ by the formulas

$$w_0 = \underset{(1\times12)}{\lfloor\Phi\rfloor}\,\{\theta\}, \qquad \phi_0 = \underset{(1\times12)}{\lfloor\Psi\rfloor}\,\{\theta\}, \tag{123}$$

where

$$\underset{(1\times12)}{\lfloor\Phi\rfloor} \equiv \lfloor\, N_1 \quad 0 \quad N_2 \quad 0 \quad N_3 \quad 0 \quad N_4 \quad 0 \quad N_5 \quad 0 \quad N_6 \quad 0 \,\rfloor, \tag{124a}$$

$$\underset{(1\times12)}{\lfloor\Psi\rfloor} \equiv \lfloor\, 0 \quad N_1 \quad 0 \quad N_2 \quad 0 \quad N_3 \quad 0 \quad N_4 \quad 0 \quad N_5 \quad 0 \quad N_6 \,\rfloor. \tag{124b}$$

Substitution of equations (124) into the integral part of the weak form (115) produces the result

$$\underset{(1\times12)}{\{\delta\theta\}}^{\mathrm{T}}\Big(\underset{(12\times12)}{[m]}\underset{(12\times1)}{\{\ddot{\theta}\}}+\underset{(12\times12)}{[k]}\underset{(12\times1)}{\{\theta\}}-\underset{(12\times1)}{\{f\}}\Big)=0, \tag{125}$$

where $\{\delta\theta\}$ is a column-matrix of variations of the nodal parameters, and the other matrices are defined as follows:

element mass matrix:

$$\underset{(12\times12)}{[m]}=B_0\int_0^l\underset{(12\times1)}{\lfloor\Phi\rfloor}^{\mathrm{T}}\underset{(1\times12)}{\lfloor\Phi\rfloor}\,dx+C_0\int_0^l\underset{(12\times1)}{\lfloor\Psi\rfloor}^{\mathrm{T}}\underset{(1\times12)}{\lfloor\Psi\rfloor}\,dx, \tag{126}$$

element stiffness matrix:

$$\begin{aligned}\underset{(12\times12)}{[k]}&=A_0\int_0^l\Big(\frac{d}{dx}\underset{(12\times1)}{\lfloor\Psi\rfloor}^{T}\Big)\Big(\frac{d}{dx}\underset{(1\times12)}{\lfloor\Psi\rfloor}\Big)dx\\&+KG_0\int_0^l\underset{(12\times1)}{\Big(\frac{d}{dx}\lfloor\Phi\rfloor+\lfloor\Psi\rfloor\Big)^{\mathrm{T}}}\underset{(1\times12)}{\Big(\frac{d}{dx}\lfloor\Phi\rfloor+\lfloor\Psi\rfloor\Big)}\,dx,\end{aligned} \tag{127}$$

element force vector for the element adjacent to the right boundary of Zone 0:

$$\underset{(12\times1)}{\{f\}}=\left\{\begin{matrix}\underset{(11\times1)}{\{0\}}\\ I_pV(t)\end{matrix}\right\}, \tag{128a}$$

element force vector for all other elements of Zone 0:

$$\underset{(12\times1)}{\{f\}}=\underset{(12\times1)}{\{0\}}\,. \tag{128b}$$

Similar derivations can be used for deriving equations of motion of a finite element within Zone 1 (Part 1) and Zone 3 (Part 4).

Finite element within Zone 2 (Part 2 and Part 3), i.e. within a subdomain $x\in[\alpha,\,\beta]$ and $z\in[-h/2,\,\gamma]$.

The weak form for a finite element within Zone 2 is obtained by multiplying equations (100) and (101) by weight functions (variations) δw_2 and $\delta\phi_2$ respectively, multiplying equations (102) and (103) by δw_3 and $\delta\phi_3$ respectively, integrating them over an element's length, performing integration by parts and adding the resulting equations. The integral part of the weak form thus obtained is

$$
\begin{aligned}
0 = &\int_0^l [A_2\phi_2'\,\delta\phi_2' + KG_2(w_2'+\phi_2)\,\delta w_2' + KG_2(w_2'+\phi_2)\,\delta\phi_2 \\
&\qquad + B_2\ddot{w}_2\,\delta w_2 + C_2\ddot{\phi}_2\,\delta\phi_2]\,dx \\
&+ \int_0^l [A_3\phi_3'\,\delta\phi_3' + KG_3(w_3'+\phi_3)\,\delta w_3' + KG_3(w_3'+\phi_3)\,\delta\phi_3 \\
&\qquad + B_3\ddot{w}_3\,\delta w_3 + C_3\ddot{\phi}_3\,\delta\phi_3]\,dx \\
&- \int_0^l \left[\left(H_2\phi_2' + \frac{3}{2}\,Q_2(w_2')^2\right)w_2''\,\delta w_2 + H_2 w_2' w_2''\,\delta\phi_2\right]dx \\
&- \int_0^l \left[\left(H_3\phi_3' + \frac{3}{2}\,Q_3(w_3')^2\right)w_3''\,\delta w_3 + H_3 w_3' w_3'',\ \delta\phi_3\right]dx \\
&- \int_0^l \chi(w_3-w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3-w_2}{\epsilon}\right)\delta w_2\,dx \\
&+ \int_0^l \chi(w_3-w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3-w_2}{\epsilon}\right)\delta w_3\,dx.
\end{aligned}
\tag{129}
$$

The same interpolation polynomials are used for the Zone 2 as for the Zone 0, i.e.

$$
\begin{aligned}
w_2(x,t) &\approx \sum_{i=1}^{6} N_i(x) w_{2i}(t), \qquad w_3(x,t) \approx \sum_{i=1}^{6} N_i(x) w_{3i}(t), \\
\phi_2(x,t) &\approx \sum_{i=1}^{6} N_i(x) \phi_{2i}(t), \qquad \phi_3(x,t) \approx \sum_{i=1}^{6} N_i(x) \phi_{3i}(t),
\end{aligned}
\tag{130}
$$

where

$$
w_{2i}(t) \equiv w_2(x_i,t), \quad w_{3i}(t) \equiv w_3(x_i,t), \quad \phi_{2i}(t) \equiv \phi_2(x_i,t), \quad \phi_{3i}(t) \equiv \phi_3(x_i,t), \tag{131}
$$

and shape functions $N_i(x)$ are defined by equations (121).

The column-matrix of the element nodal parameters for Zone 2 are introduced as follows:

$$
\underset{(24\times 1)}{\{\theta\}} = \left\{ \begin{matrix} \underset{(12\times 1)}{\{\theta\}^{(2)}} \\ \underset{(12\times 1)}{\{\theta\}^{(3)}} \end{matrix} \right\}, \tag{132}
$$

where

$$
\{\theta\}^{(2)} \equiv \lfloor\, w_{21} \quad \phi_{21} \quad w_{22} \quad \phi_{22} \quad w_{23} \quad \phi_{23} \quad w_{24} \quad \phi_{24} \quad w_{25} \quad \phi_{25} \quad w_{26} \quad \phi_{26} \,\rfloor^{\mathrm{T}}, \tag{133}
$$

$$
\{\theta\}^{(3)} \equiv \lfloor\, w_{31} \quad \phi_{31} \quad w_{32} \quad \phi_{32} \quad w_{33} \quad \phi_{33} \quad w_{34} \quad \phi_{34} \quad w_{35} \quad \phi_{35} \quad w_{36} \quad \phi_{36} \,\rfloor^{\mathrm{T}} \tag{134}
$$

are column-matrices of nodal parameters of Part 2 and Part 3 respectively (of lower and upper delaminated parts of Zone 2). Then, using the weak form (equation (129)) and following the same procedures as for an element in Zone 0, the following expressions for the element mass and stiffness matrices of Zone 2 are obtained.

Element mass matrix for Zone 2:

$$\underset{(24\times 24)}{[m]} = \begin{bmatrix} \underset{(12\times 12)}{[m]^{(2)}} & \underset{(12\times 12)}{[0]} \\ \underset{(12\times 12)}{[0]} & \underset{(12\times 12)}{[m]^{(3)}} \end{bmatrix}, \tag{135}$$

where

$$\underset{(12\times 12)}{[m]^{(i)}} = B_i \int_0^l \underset{(12\times 1)}{\lfloor \Phi \rfloor^{\mathrm{T}}} \underset{(1\times 12)}{\lfloor \Phi \rfloor} \, dx + C_i \int_0^l \underset{(12\times 1)}{\lfloor \Psi \rfloor^{\mathrm{T}}} \underset{(1\times 12)}{\lfloor \Psi \rfloor} \, dx \quad (i = 2, 3), \tag{136}$$

and row-matrices $\lfloor \Phi \rfloor$ and $\lfloor \Psi \rfloor$ are defined by equations (124).

Element stiffness matrix for Zone 2:

$$\underset{(24\times 24)}{[k]} = \begin{bmatrix} \underset{(12\times 12)}{[k]^{(2)}} & \underset{(12\times 12)}{[0]} \\ \underset{(12\times 12)}{[0]} & \underset{(12\times 12)}{[k]^{(3)}} \end{bmatrix}, \tag{137}$$

where

$$\begin{aligned} \underset{(12\times 12)}{[k]^{(i)}} &= A_i \int_0^l \left(\frac{d}{dx} \underset{(12\times 1)}{\lfloor \Psi \rfloor^{\mathrm{T}}} \right) \left(\frac{d}{dx} \underset{(1\times 12)}{\lfloor \Psi \rfloor} \right) dx \\ &+ KG_i \int_0^l \underset{(12\times 1)}{\left(\frac{d}{dx} \lfloor \Phi \rfloor + \lfloor \Psi \rfloor \right)^{\mathrm{T}}} \underset{(1\times 12)}{\left(\frac{d}{dx} \lfloor \Phi \rfloor + \lfloor \Psi \rfloor \right)} dx \quad (i = 2, 3). \end{aligned} \tag{138}$$

The last two integrals in the weak form (129) represent virtual works of forces of mutual impact of the crack's faces, acting, correspondingly, on the lower and upper crack's face. The computation of contribution of these integrals to the discretized equations of motion of a finite element within Zone 2 is presented below. Let us consider one of these integrals

$$I_2 \equiv \int_0^l \chi (w_3 - w_2) \left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{w_3 - w_2}{\epsilon} \right) (\delta w_2) \, dx, \tag{140}$$

which represent virtual work of force of impact acting on the lower face of the crack. Substitution of functions $w_2(x,t)$ and $w_3(x,t)$ by their polynomial approximation (equations (130)) yields

$$\begin{aligned} I_2 = \int_0^l \chi &\left(\sum_{i=1}^{6} (\delta w_{2i}) N_i \right) \left(\sum_{j=1}^{6} (w_{3j} - w_{2j}) N_j \right) \\ &\times \left(\frac{1}{2} - \frac{1}{\pi} \arctan \left(\epsilon^{-1} \sum_{m=1}^{6} (w_{3m} - w_{2m}) N_m \right) \right) dx. \end{aligned} \tag{141}$$

Let the function under the integral sign in the last integral be denoted as $g(x)$. Then, using the trapezoidal rule of numerical integration, with evaluation of the function $g(x)$ at the nodal points $x_1 = 0,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 = l$ of the finite element,

$$\int_0^l g(x)\,dx \approx \frac{l}{10}\left[g(0) + g(l) + 2\sum_{k=2}^{5} g(x_k)\right], \tag{142}$$

and using the property $N_i(x_j) = \delta_{ij}$ of the shape functions, defined by equation (118), one can obtain

$$I_2 = \chi \underset{(1\times 12)}{\left\{\delta\theta^{(2)}\right\}^{\mathrm{T}}} \underset{(12\times 1)}{\{f\}}, \tag{143}$$

where

$$\begin{gathered} f_i = \frac{l}{10}(w_{3i} - w_{2i})\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_{3i} - w_{2i}}{\epsilon}\right) \quad \text{for} \quad i = 1, 11, \\ f_i = \frac{l}{5}(w_{3i} - w_{2i})\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_{3i} - w_{2i}}{\epsilon}\right) \quad \text{for} \quad i = 3,\ 5,\ 7,\ 9, \\ f_i = 0 \quad \text{for} \quad i = 2,\ 4,\ 6,\ 8,\ 10,\ 12. \end{gathered} \tag{144}$$

Similarly, the last integral in equation (129) can be written as

$$I_3 \equiv \int_0^l \chi(w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}\right)(\delta w_3)\,dx = \chi \underset{(1\times 12)}{\left\{\delta\theta^{(3)}\right\}^{\mathrm{T}}} \underset{(12\times 1)}{\{f\}}. \tag{145}$$

The nonlinear terms

$$\begin{aligned} &-\int_0^l \left[\left(H_2\phi_2' + \frac{3}{2}\,Q_2(w_2')^2\right)w_2''\,\delta w_2 + H_2 w_2' w_2''\,\delta\phi_2\right]dx \\ &-\int_0^l \left[\left(H_3\phi_3' + \frac{3}{2}\,Q_3(w_3')^2\right)w_3''\,\delta w_3 + H_3 w_3' w_3''\,\delta\phi_3\right]dx \end{aligned}$$

in the weak form (129), which are due to taking account of longitudinal force resultants in the delaminated parts (i.e. due to the von Karman nonlinearity of the strain-displacement relations), lead to the presence of a column-matrix in the equations of motion of a finite element, the components of which depend nonlinearly on the nodal parameters w_{2i} and w_{3i} $(i = 1, 2, \ldots, 6)$. This column-matrix will be denoted as

$$\underset{(24\times 1)}{\{g\}} \equiv \left\{\begin{matrix} \underset{(12\times 1)}{\{g\}^{(2)}} \\ \underset{(12\times 1)}{\{g\}^{(3)}} \end{matrix}\right\}, \tag{146}$$

where $\{g\}^{(2)}$ is a column-matrix the components of which depend nonlinearly on nodal parameters w_{2i} (associated with the lower delaminated part), and $\{g\}^{(3)}$ is a column-matrix the components of which depend nonlinearly on nodal parameters w_{3i} (associated with the upper delaminated part). Components of $\{g\}^{(2)}$ and $\{g\}^{(3)}$ are not written here explicitly, because of their large size.

So, equations of motion of a finite element in the delaminated zone of the beam (Zone 2) have the form

$$\begin{bmatrix} \underset{(12\times12)}{[m]^{(2)}} & \underset{(12\times12)}{[0]} \\ \underset{(12\times12)}{[0]} & \underset{(12\times12)}{[m]^{(3)}} \end{bmatrix} \begin{Bmatrix} \underset{(12\times1)}{\{\ddot{\theta}\}^{(2)}} \\ \underset{(12\times1)}{\{\ddot{\theta}\}^{(3)}} \end{Bmatrix} + \begin{bmatrix} \underset{0(12\times12)}{[k]^{(2)}} & \underset{(12\times12)}{[0]} \\ \underset{(12\times12)}{[0]} & \underset{(12\times12)}{[k]^{(3)}} \end{bmatrix} \begin{Bmatrix} \underset{(12\times1)}{\{\theta\}^{(2)}} \\ \underset{(12\times1)}{\{\theta\}^{(3)}} \end{Bmatrix}$$
$$+\chi \begin{Bmatrix} -\underset{(12\times1)}{\{f\}} \\ \underset{(12\times1)}{\{f\}} \end{Bmatrix} + \begin{Bmatrix} \underset{(12\times1)}{\{g\}^{(2)}} \\ \underset{(12\times1)}{\{g\}^{(3)}} \end{Bmatrix} = \underset{(24\times1)}{\{0\}}. \tag{147}$$

In Equation (147), the nonlinear internal force vector $\chi \lfloor -\{f\}^{\mathrm{T}} \quad \{f\}^{\mathrm{T}} \rfloor^{\mathrm{T}}$ depends on nodal parameters, associated with both lower and upper delaminated parts (Part 2 and Part 3). Therefore, in the system of equations (147), the nodal parameters $\{\theta\}^{(2)}$, associated with the lower delaminated part (Part 2) are coupled to the nodal parameters $\{\theta\}^{(3)}$, associated with the upper delaminated part.

So, the derivation of the finite element matrices is completed, and an example problem will be considered next.

5 Solution of Example Problems

As an example problem, a clamped-free wooden beam with the following characteristics (Figure 2.1) is considered: length $L = 20 \times 10^{-2} m$, width $b = 2.76 \times 10^{-2} m$, thickness $h = 0.99 \times 10^{-2} m$, wood density $\rho^{(0)} = 418.02 \frac{kg}{m^3}$, Young's modulus of the wood in the direction of fibers $E_1^{(0)} = 1.0897 \times 10^{10} \frac{N}{m^2}$. The piezoelectric actuator is QP10W (Active Control Experts). Thickness of the actuator is $\tau = 3.81 \times 10^{-4} m$, its length is $a = 5.08 \times 10^{-2} m$, the piezoelectric constant in the range of applied voltage (from 0 to $200V$) is $\overline{d}_{31} \approx -1.05 \times 10^{-9} \frac{m}{V}$, the Young's modulus of the actuator with its packaging is $E_1^{(p)} = 2.57 \times 10^{10} \frac{N}{m^2}$, mass density of the actuator with its packaging is $\rho^{(p)} = 6151.1 \frac{kg}{m^3}$. The voltage $V(t)$, applied to the piezoelectric actuator, is distributed uniformly along the length of the actuator and varies with time as

$$V(t) = V_a \sin(\Omega t),$$

where $V_a = 200\,V,\ \Omega = 600 \frac{1}{s}$. The wooden beam is cut along its fibers, so that the angle θ in the formula (6) is equal to zero, and, therefore, the elastic compliance coefficient $\overline{S}_{11}$ for the wood is equal to $\overline{S}_{11}^{(0)} = \frac{1}{E_1^{(0)}} = 9.1768 \times 10^{-11} \frac{m^2}{N}$. For the piezoelectric actuator, the material coordinate system coincides with the problem coordinate system, so that the elastic compliance coefficient $\overline{S}_{11}$ for the material of the piezo-actuator is $\overline{S}_{11}^{(p)} = \frac{1}{E_1^{(p)}} = 3.8911 \times 10^{-11} \frac{m^2}{N}$. Coordinates of the crack tips are: $\alpha = 10 \times 10^{-2} m$, $\beta = 15 \times 10^{-2} m,\ \gamma = 0.66 \times 10^{-2} - \frac{h}{2} = 1.65 \times 10^{-3} m$. Then the constants, entering into the variational formulation and the differential equations of the problem, have the following values in SI units: $A_0 = 31.463$, $B_0 = 0.178\,9$, $C_0 = 2.642\,9 \times 10^{-6}$, $G_0 = 1.299\,10 \times 10^6$, $A_1 = 24.319$, $B_1 = 0.114\,22$, $C_1 = 9.328\,9 \times 10^{-7}$, $G_1 = 1.190999 \times 10^6$, $A_2 = 12.61$, $B_2 = 7.614\,7 \times 10^{-2}$, $C_2 = 4.837\,2 \times 10^{-7}$, $G_2 = 7.93999 \times 10^5$, $A_3 = 11.709$, $B_3 = 3.807\,3 \times 10^{-2}$, $C_3 = 4.491\,7 \times 10^{-7}$, $G_3 = 3.969995 \times 10^5$, $A_4 = 24.319$, $B_4 = 0.114\,22$, $C_4 = 9.328\,9 \times 10^{-7}$, $G_4 = 1.190999 \times 10^6$, $I_p = -3.828\,5 \times 10^{-3}$, $a = 5.08 \times 10^{-2}$, $V_a = 200$, $\Omega = 600$, $\alpha = 10 \times 10^{-2}$, $\beta = 15 \times 10^{-2}$, $\gamma = 1.65 \times 10^{-3}$, $b = 2.76 \times 10^{-2}$, $h = 0.99 \times 10^{-2}$, $Q_2 = 1.985005666 \times 10^6$, $Q_3 = 9.925028332 \times 10^5$, $H_2 = -3275.25935$, $H_3 = 3275.25935$. The small constant ϵ and the large constant χ are chosen to be $\epsilon = 1 \times 10^{-3}$ and $\chi = 1 \times 10^6$.

5.1 Time-domain response to dynamic excitation

Time integration of a system of ordinary differential equations of a global (assembled) semi-discrete finite element model

$$[M]\{\ddot{\Theta}\} + [K]\{\Theta\} + \{R\}_{\text{nonlin}} = \{F\}$$

was performed with the use of the backward-difference method [4]. In the last equation, $\{R\}_{\text{nonlin}}$ is a column-matrix, which contains components that depend nonlinearly on the unknown nodal parameters Θ_i. Transverse displacements as functions of time at free ends of delaminated and undelaminated beams are shown in graphs of Figure 5.1. These graphs are noticeably different. Numerical experiments show that neglecting nonlinear terms in the strain-displacement relations (19a), and, therefore, neglecting the longitudinal force resultants $N_x^{(2)}$ and $N_x^{(3)}$ in the delaminated parts of the beam (equations (25) and (29)), produces much smaller effect on the transverse displacement of the delaminated beam than neglecting the force of contact interaction of the crack faces.

5.2 Eigenvalue analysis

For the same beam, natural frequencies and mode shapes were computed from a linear eigenvalue analysis. Results of calculation of frequencies for beams with different crack lengths are presented in tables below. For some crack lengths, comparison is made between frequencies computed on the basis of the first order shear deformation theory, presented in this paper, and the frequencies computed on the basis of the Euler-Bernoulli beam theory, presented in Reference [1]. Rotary inertia terms are taken into account in both types of solutions.

Let us consider, at first, the first seven circular frequencies of a clamped-free beam **without the delamination** and with the actuator, obtained by setting equal the x-coordinates of the crack tips. The frequencies for this case are presented below. Notation FOSDT stands for the First Order Shear Deformation Theory of the beam, notation E-B stands for the Euler-Bernoulli beam theory.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
FOSDT	1395.535	8130.531	21436.8	40361.9	64915.31	9.36739×10^4	1.25461×10^5
E-B	1397.435	8217.911	21986.6	42205.0	69331.23	1.02371×10^5	1.40641×10^5

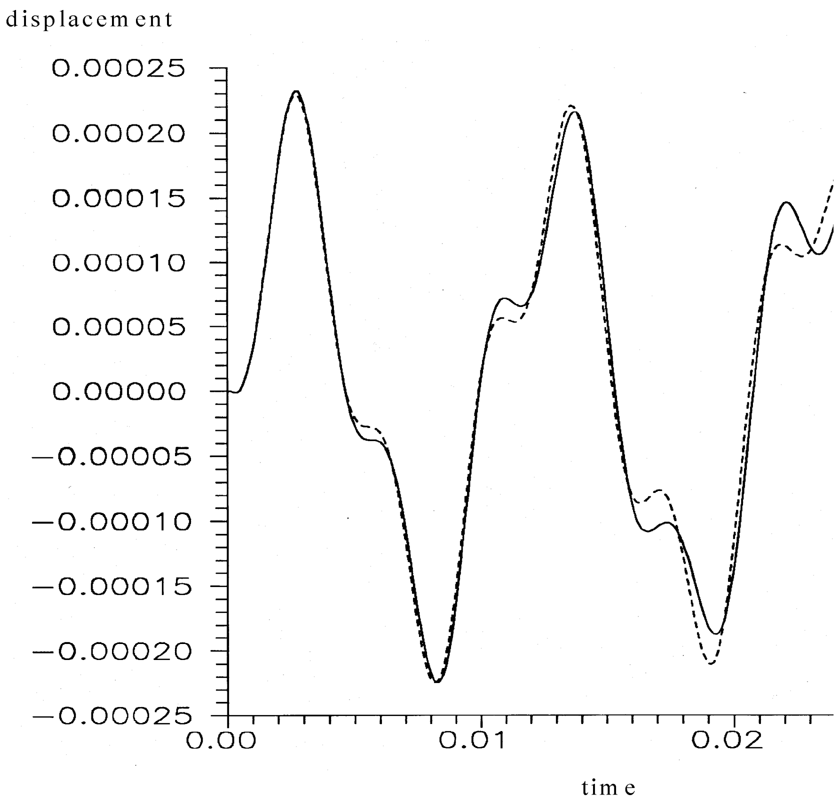

Figure 5.1. Transverse displacement of free end of delaminated beam (solid line) and undelaminated beam (dashed line). Coordinates of the crack tips of the delaminated beam are $\alpha = 0,1m$, $\beta = 0,15m$, $\gamma = 1,65 \times 10^{-3}m$.

In the next table, results are presented for a beam **with the delamination** and with the actuator, with the following coordinates of the crack tips: $\alpha = 0.1m$, $\beta = 0.11m$, $\gamma = 1.65 \times 10^{-3}m$.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
FOSDT	1395.5	8130.5	21436.0	40361.5	64909.9	9.3669×10^4	1.2545×10^5
E-B	1397.435	8217.909	21986.1	42204.9	69331.2	1.02371×10^5	1.40641×10^5

In the next table, results are presented for a beam with the delamination and with the actuator, with the following coordinates of the crack tips: $\alpha = 0.1m$, $\beta = 0.12m$, $\gamma = 1.65 \times 10^{-3}m$.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
FOSDT	1395.5	8130.5	21433.6	40356.8	64900.7	9.3629×10^4	1.2544×10^5
E-B	1397.433	8217.9	21986.0	42200.0	69330.0	1.02368×10^5	1.40625×10^5

In the next table, results are presented for a beam with the delamination and with the actuator, with the following coordinates of the crack tips: $\alpha = 0.1m$, $\beta = 0.13m$, $\gamma = 1.65 \times 10^{-3}m$.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
FOSDT	1395.49	8130.46	21431.655	40345.03	64894.09	9.3576×10^4	1.2535×10^5

In the next table, results are presented for a beam with the delamination and with the actuator, with the following coordinates of the crack tips: $\alpha = 0.1m$, $\beta = 0.14m$, $\gamma = 1.65 \times 10^{-3}m$.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
FOSDT	1395.47	8130.28	21430.371	40330.58	64850.16	9.3541×10^4	1.2504×10^5

In the next table, results are presented for a beam with the delamination and with the actuator, with the following coordinates of the crack tips: $\alpha = 0.1m$, $\beta = 0.15m$, $\gamma = 1.65 \times 10^{-3}m$.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
FOSDT	1395.456	8129.95	21427.848	40320.39	64719.1	9.31501×10^4	1.10100×10^5
E-B	1397.432	8217.62	21980.0	42201.0	69094.0	1.01932×10^5	1.33019×10^5

So, the frequencies decrease as the crack length increases. This phenomenon is more pronounced for higher frequencies.

The first four mode shapes of delaminated beams are nearly the same as the corresponding mode shapes of the undelaminated beams, so that the difference is not noticeable on graphs. But the higher mode shapes of the delaminated beams, beginning from the fifth mode shape, show the crack opening and closure during the vibration, as can be seen in Figures 5.2, 5.3 and 5.4.

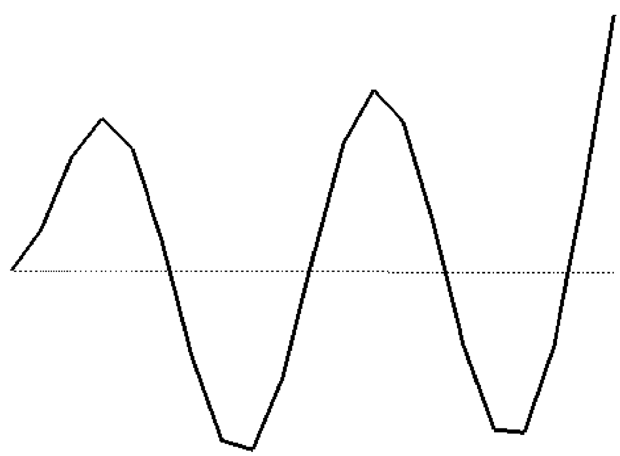

Figure 5.2a. Fifth mode shape of clamped-free beam without delamination.

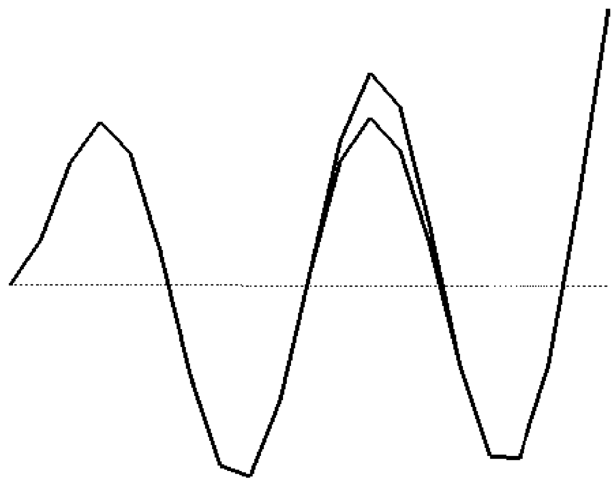

Figure 5.2b. Fifth mode shape of clamped-free beam with delamination.

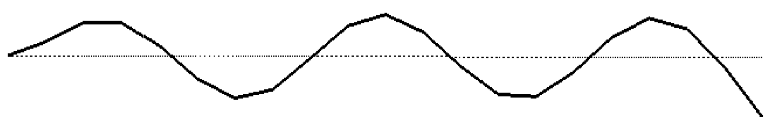

Figure 5.3a. Sixth mode shape of clamped-free beam without delamination.

Figure 5.3b. Sixth mode shape of clamped-free beam with delamination.

Figure 5.4a. Seventh mode shape of clamped-free beam without delamination.

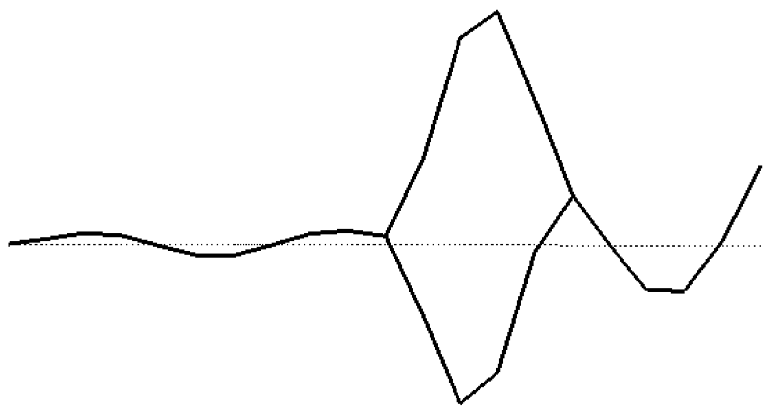

Figure 5.4b. Seventh mode shape of clamped-free beam with delamination.

Experimental verification of the developed theory and the finite element program will be presented in a subsequent publication. The theory, presented in this work, and the finite element program, based on this theory, are developed for the purpose of their subsequent use in nondestructive detection of delamination cracks in composite structures.

Appendix A
Constitutive Equations for a Piezoelectric Orthotropic Layer of a Thin Composite Beam

The constitutive equations of a generally anisotropic piezoelectric material can be written in a matrix form as follows (in these equations, the bars over characters are put to emphasize that the quantities are presented in a problem coordinate system, the coordinate planes of which do not necessarily coincide with the planes of elastic or dielectric symmetry)

$$\underset{(6\times1)}{\{\overline{\varepsilon}\}} = \underset{(6\times6)}{[\overline{S}]}\,\underset{(6\times1)}{\{\overline{\sigma}\}} + \underset{(6\times3)}{[\overline{d}]}^{\mathrm{T}}\,\underset{(3\times1)}{\{\overline{\mathcal{E}}\}}, \tag{A-1}$$

$$\underset{(3\times1)}{\{\overline{D}\}} = \underset{(3\times6)}{[\overline{d}]}\,\underset{(6\times1)}{\{\overline{\sigma}\}} + \underset{(3\times3)}{[\overline{\zeta}]}\,\underset{(3\times1)}{\{\overline{\mathcal{E}}\}}, \tag{A-2}$$

where

$$\underset{(6\times1)}{\{\overline{\varepsilon}\}} = \lfloor\, \varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad 2\varepsilon_{yz} \quad 2\varepsilon_{xz} \quad 2\varepsilon_{xy} \,\rfloor^{\mathrm{T}} \tag{A-3}$$

is a column-matrix of components of the strain tensor,

$$\underset{(6\times1)}{\{\overline{\sigma}\}} = \lfloor\, \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{yz} \quad \sigma_{xz} \quad \sigma_{xy} \,\rfloor^{\mathrm{T}} \tag{A-4}$$

is a column-matrix of components of the stress tensor,

$$\underset{(3\times1)}{\{\overline{\mathcal{E}}\}} = \lfloor\, \mathcal{E}_x \quad \mathcal{E}_y \quad \mathcal{E}_z \,\rfloor^{\mathrm{T}} \tag{A-5}$$

is a column-matrix of components of the electric field intensity vector, $\underset{(6\times6)}{[\bar{S}]}$ is a matrix of elastic coefficients (compliance coefficients) and $\underset{(3\times6)}{[\bar{d}]}$ and $\underset{(3\times3)}{[\bar{\zeta}]}$ are matrices of material constants that characterize electrical properties.

For an orthotropic material, in the principal material coordinate system (whose coordinate planes coincide with the planes of elastic symmetry), the matrix of compliance coefficients is denoted as $[S]$ (without a bar) and has the form

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \tag{A-6}$$

where the compliance coefficients S_{ij} are expressed in terms of engineering constants by the formulas

$$\begin{gathered} S_{11} = \frac{1}{E_1}, \quad S_{12} = -\frac{\nu_{12}}{E_1}, \quad S_{13} = -\frac{\nu_{13}}{E_1}, \quad S_{22} = \frac{1}{E_2}, \quad S_{23} = -\frac{\nu_{23}}{E_2}, \\ S_{33} = \frac{1}{E_3}, \quad S_{44} = \frac{1}{G_{23}}, \quad S_{55} = \frac{1}{G_{13}}, \quad S_{66} = \frac{1}{G_{12}}. \end{gathered} \tag{A-7}$$

The matrices, characterizing electric properties of the material, in the principle material coordinate system, will be denoted without the bar also, i.e. as $\underset{(3\times6)}{[d]}$ and $\underset{(3\times3)}{[\zeta]}$.

In the laminate (problem) coordinate system, rotated clockwise by an angle θ with respect to the principle material coordinate system, the matrix of compliance coefficients and the matrices, characterizing electric properties of the material, $\underset{(3\times6)}{[\bar{d}]}$ and $\underset{(3\times3)}{[\bar{\zeta}]}$, have the form

$$\underset{(6\times6)}{[\bar{S}]} = \underset{(6\times6)}{[T]}^{\mathrm{T}} \underset{(6\times6)}{[S]} \underset{(6\times6)}{[T]} , \tag{A-8}$$

$$\underset{(3\times3)}{[\bar{\zeta}]} = \underset{(3\times3)}{[R]}^{\mathrm{T}} \underset{(3\times3)}{[\zeta]} \underset{(3\times3)}{[R]} , \tag{A-9}$$

$$\underset{(3\times6)}{[\bar{d}]} = \underset{(3\times3)}{[R]}^{\mathrm{T}} \underset{(3\times6)}{[d]} \underset{(6\times6)}{[T]} , \tag{A-10}$$

where the transformation matrices $[T]$ and $[R]$ are defined as follows (with the use of notation $c = \cos\theta$, $s = \sin\theta$):

$$\underset{(6\times6)}{[T]} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & 2sc \\ s^2 & c^2 & 0 & 0 & 0 & -2sc \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -sc & sc & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}, \tag{A-11}$$

$$\underset{(3\times3)}{[R]} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{A-12}$$

For the composite piezoelectric layer of a *thin and narrow composite beam, which bends in the x-z plane*, the following assumptions can be adopted

$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = \sigma_{yy} = 0. \tag{A-13}$$

Besides, in the problem under consideration, the electrical field is applied to the actuator only in the thickness direction (in the direction of the z-axis), i.e.

$$\mathcal{E}_x = \mathcal{E}_y = 0. \tag{A-14}$$

If equations (A-13) and (A-14) are substituted into the constitutive equations (A-1) and (A-2) with account of transformation relations (A-8), (A-9) and (A-10) and with account of equations (A-6) and (A-7) for compliance matrix in the principle material coordinate system, then the constitutive equations take the form

$$\begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & 0 \\ 0 & \overline{S}_{55} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} + \begin{bmatrix} \overline{d}_{31} \\ \overline{d}_{35} \end{bmatrix} \{ \mathcal{E}_z \}, \tag{A-15}$$

$$\{ D_z \} = [\overline{d}_{31} \quad \overline{d}_{35}] \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} + [\overline{\zeta}_{33}] \{ \mathcal{E}_z \}. \tag{A-16}$$

From the constitutive equations (A-15) and (A-16), one can obtain the constitutive equations in a different form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \\ D_z \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\overline{S}_{11}} & 0 & -\dfrac{\overline{d}_{31}}{\overline{S}_{11}} \\ 0 & \dfrac{1}{\overline{S}_{55}} & -\dfrac{\overline{d}_{35}}{\overline{S}_{55}} \\ \dfrac{\overline{d}_{31}}{\overline{S}_{11}} & \dfrac{\overline{d}_{35}}{\overline{S}_{55}} & \left(\overline{\zeta}_{33} - \dfrac{\overline{d}_{31}^2}{\overline{S}_{11}} - \dfrac{\overline{d}_{35}^2}{\overline{S}_{55}}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \mathcal{E}_z \end{Bmatrix}, \tag{A-17a}$$

or, in view of the relationship $\mathcal{E}_z = -\frac{\partial \varphi}{\partial z}$, where φ is the electric potential,

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \\ D_z \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\overline{S}_{11}} & 0 & \dfrac{\overline{d}_{31}}{\overline{S}_{11}} \\ 0 & \dfrac{1}{\overline{S}_{55}} & \dfrac{\overline{d}_{35}}{\overline{S}_{55}} \\ \dfrac{\overline{d}_{31}}{\overline{S}_{11}} & \dfrac{\overline{d}_{35}}{\overline{S}_{55}} & \left(-\overline{\zeta}_{33} + \dfrac{\overline{d}_{31}^2}{\overline{S}_{11}} + \dfrac{\overline{d}_{35}^2}{\overline{S}_{55}}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \frac{\partial \varphi}{\partial z} \end{Bmatrix}. \tag{A-17b}$$

According to equations (A-7) and (A-8), the compliance coefficients $\overline{S}_{11}$ and $\overline{S}_{55}$ in the problem coordinate system that enter into equations (A-17), are expressed in terms of the engineering constants by the formulas

$$\begin{aligned} \overline{S}_{55} &= \frac{1}{G_{23}} s^2 + \frac{1}{G_{13}} c^2, \\ \overline{S}_{11} &= \frac{1}{E_1} c^4 + \frac{1}{E_2} s^4 + \left(\frac{1}{G_{12}} - 2\frac{\nu_{12}}{E_1}\right) s^2 c^2. \end{aligned} \tag{A-18}$$

The material constants $\overline{d}_{31}$ and $\overline{d}_{35}$, which characterize the piezoelectric properties in the problem coordinate system, are expressed in terms of the piezoelectric constants d_{ij} of the material coordinate system by the formulas (derived from matrix transformation equations A-10)

$$\overline{d}_{31} = d_{31}c^2 + d_{32}s^2 - d_{36}sc, \tag{A-19a}$$

$$\overline{d}_{35} = -d_{34}s + d_{35}c, \tag{A-19b}$$

and, according to the transformation equation (A-9),

$$\overline{\zeta}_{33} = \zeta_{33}. \tag{A-20}$$

We consider a piezoelectric material with orthorhombic $mm2$ symmetry, such as polyvinylidene (PVDF) or lead zirconate-titanate (PZT), in which the planes of elastic symmetry are made, in the manufacturing process, the same as the planes of piezoelectric symmetry. In this case, the piezoelectric constants d_{34} and d_{35} are equal to zero (see [5] and [6]). Then, according to equation (A-19b), $\overline{d}_{35} = 0$, and equation (A-17b) takes the form

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \\ D_z \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\overline{S}_{11}} & 0 & \dfrac{\overline{d}_{31}}{\overline{S}_{11}} \\ 0 & \dfrac{1}{\overline{S}_{55}} & 0 \\ \dfrac{\overline{d}_{31}}{\overline{S}_{11}} & 0 & \left(-\overline{\zeta}_{33} + \dfrac{\overline{d}_{31}^2}{\overline{S}_{11}} \right) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \frac{\partial\varphi}{\partial z} \end{Bmatrix}. \tag{A-21}$$

These are the constitutive equations for a layer of orthotropic piezoelectric material with orthorhombic $mm2$ symmetry, in which the planes of elastic symmetry are the same as the planes of piezoelectric symmetry, in a narrow and thin composite beam. Obviously, for a layer of orthotropic material, in a thin narrow beam, which does not have piezoelectric properties, the constitutive equations have the form

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\overline{S}_{11}} & 0 \\ 0 & \dfrac{1}{\overline{S}_{55}} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \end{Bmatrix}. \tag{A-22}$$

Appendix B
Properties of the Heaviside Function

It can be shown [7] that the Heaviside function (unit step-function) $\mathfrak{H}_\alpha(x)$, defined by formula

$$\mathfrak{H}_\alpha(x) = \begin{cases} 0 & \text{for } \ x < \alpha, \\ 1 & \text{for } \ x > \alpha, \end{cases} \tag{B-1}$$

has the following property

$$\frac{d\mathfrak{H}_\alpha(x)}{dx} = \delta_\alpha(x), \tag{B-2}$$

where $\delta_\alpha(x)$ is the Dirac's delta-function, defined as a function that has the following properties:

$$\delta_\alpha(x) = \begin{cases} 0 & \text{for } \; x \neq \alpha, \\ \infty & \text{for } \; x = \alpha \end{cases} \tag{B-3}$$

and

$$\int\limits_{x_1}^{x_2} f(x)\delta_\alpha(x)\,dx = \begin{cases} f(\alpha) & \text{for } \; x_1 < \alpha < x_2, \\ 0 & \text{for } \; \alpha < x_1 \text{ and for } \alpha > x_2. \end{cases} \tag{B-4}$$

The delta-function has several analytical representations, one of which has the form [8]:

$$\delta_\alpha(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + (x-\alpha)^2}. \tag{B-5}$$

According to formula (B-2), the analytical representation of the Heaviside function, corresponding to the analytical representation of the delta-function (B-5) is

$$\mathfrak{H}_\alpha(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \arctan \frac{x-\alpha}{\epsilon} + \frac{1}{2} = \begin{cases} 0 & \text{for } \; x < \alpha, \\ \frac{1}{2} & \text{for } \; x = \alpha, \\ 1 & \text{for } \; x > \alpha. \end{cases} \tag{B-6}$$

Carrying out the Heaviside function $\mathfrak{H}_\alpha(x)$ beyond the integral sign in an indefinite integral is done with the use of the formula

$$\int \mathfrak{H}_\alpha(x)\, f(x)\, dx = \mathfrak{H}_\alpha(x) \int\limits_{\alpha}^{x} f(\eta)\, d\eta. \tag{B-7}$$

With the use of properties (B-2) and (B-4), it can be shown that

$$\int\limits_{x_1}^{x_2} f(x)\, \frac{d^2\mathfrak{H}_\alpha(x)}{dx^2}\, dx = \begin{cases} -\dfrac{df}{dx}(\alpha) & \text{for } \; x_1 < \alpha < x_2, \\ 0 & \text{for } \; \alpha < x_1 \; \text{ and for } \; \alpha > x_2. \end{cases} \tag{B-8}$$

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