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Global Stability Properties for a Class of Dissipative Phenomena via One or Several Liapunov Functionals

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Abstract: We find some new results regarding the existence, uniqueness, boundedness, stability and attractivity of the solutions of a class of initial-boundary-value problems characterized by a quasi-linear third order equation which may have non-autonomous forcing terms. The class includes equations arising in superconductor theory, quantum mechanics and in the theory of viscoelastic materials.

Keywords: *Nonlinear higher order PDE; stability; boundedness; boundary value problems.*

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1 Introduction

In this paper we deal with questions regarding the existence, uniqueness, boundedness, stability and attractivity of solutions u of the following class of initial-boundary-value problems:

$$Lu = f(x, t, u, u_x, u_{xx}, u_t), \quad 0 < x < 1, \quad 0 < t < T, \tag{1.1}$$

where $L = -\varepsilon\partial_{xxt} - c^2\partial_{xx} + \partial_{tt}$, f is a continuous function of its arguments, c and ε are positive constants, and

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad 0 < x < 1, \tag{1.2}$$

$$u(0, t) = h_1(t), \quad u(1, t) = h_2(t), \quad 0 < t < T, \tag{1.3}$$

where $T \leq +\infty$, $h_1, h_2 \in C^2([0,T[)$, $u_0, u_1 \in C^2([0,1])$ are assigned and fulfill the consistency condition

$$\begin{aligned} h_1(0) &= u_0(0), \qquad \frac{dh_1(0)}{dt} = u_1(0), \\ h_2(0) &= u_0(1), \qquad \frac{dh_2(0)}{dt} = u_1(1). \end{aligned} \tag{1.4}$$

Solutions u of such problems describe a number of physically remarkable continuous phenomena occurring on a finite space interval. In the operator L the D'Alembertian $-c^2\partial_{xx} + \partial_{tt}$ induces wave propagation, $-\varepsilon\partial_{xxt}$ dissipation. The term on the right-hand side of (1.1) may contain forcing terms, nonlinear (local) couplings of u to itself, further dissipative terms. For instance, when $f = -b\sin u - au_t + F(x,t)$, where a, b are positive constants, we deal with the perturbed Sine-Gordon equation, which can be used e.g. to describe the classical Josephson effect with driving force F in the theory of superconductors [6, 11]. F is a forcing term, $-au_t$ is a dissipative one and $-b\sin u$ is a nonlinear coupling. On the other hand it is well known [12] that equation (1.1) describes the evolution of the displacement $u(x,t)$ of the section of a rod from its rest position x in a Voigt material when an external force f is applied; in this case $c^2 = E/\rho$, $\varepsilon = 1/(\rho\mu)$, where ρ is the (constant) linear density of the rod at rest, and E, μ are respectively the elastic and viscous constants of the rod, which enter the stress-strain relation $\sigma = E\nu + \partial_t\nu/\mu$, where σ is the stress, ν is the strain. As we shall see in the sequel, even considering only one of these examples, e.g. the perturbed Sine-Gordon equation $f = -b\sin u - au_t$, it is important to keep room for a more general f because the latter will naturally appear when asking whether a particular solution u^* of the problem is stable or attractive, or when reducing the original problem to one with trivial boundary conditions.

Several papers [2−5, 7−9, 13] have already been devoted to the analysis of the operator L and more specifically to the investigation of the boundedness, stability and attractivity of the solutions of the above problem. Here we improve previous results, by weakening the assumptions on f, and find some new ones. In Section 2 we improve the existence and uniqueness Theorem 2.1 proved in [2], in that we require f to satisfy only *locally* Lipschitz condition. In Section 3.2 we improve the boundedness and stability Theorem 3.1 of the same reference, in that we require only a suitable *time average* of the quadratic norm of f to be bounded. While doing so we prove two lemmas concerning boundedness and attractivity of the zero solution for a class of first order ordinary differential equations in one unknown; the second lemma is a generalization of a lemma due to Hale [10]. In Sections 4 and 5 we respectively improve the exponential asymptotic stability Theorem 3.3 of [2] and the uniform asymptotic stability Theorem 2 of [5], valid for some special f, by removing the boundedness assumption on the latter. The trick we use is to associate to each neighbourhood of the origin with radius σ (the 'error') a Liapunov functional depending on a parameter γ adapted to σ, instead of fixing γ once and for all.

2 Existence and Uniqueness of the Solution

To discuss the existence and uniqueness of the above problem it is convenient to formulate it as an equivalent integro-differential equation so as to apply the fixed-point theorem.

As in [2], we start from the identity

$$\begin{aligned}\partial_\xi(c^2uw_\xi - c^2u_\xi w + \varepsilon u_\xi w_\tau - \varepsilon u w_{\xi\tau}) + \partial_\tau(u_\tau w - uw_\tau - \varepsilon u_{\xi\xi}w)\\ = fw - u(\varepsilon w_{\xi\xi\tau} - c^2w_{\xi\xi} + w_{\tau\tau}),\end{aligned} \tag{2.1}$$

that follows from (1.1) for any smooth function $w(\xi,\tau)$, assuming $u(\xi,\tau)$ is a smooth solution of (1.1). We choose w as a function depending also on x,t and fulfilling the equation $Lw = 0$, more precisely

$$w(x,\xi,\, t-\tau) = \theta(x-\xi,\, t-\tau) - \theta(x+\xi,\, t-\tau), \tag{2.2}$$

with

$$\theta(x,t) = K(|x|,t) + \sum_{m=1}^{\infty}\left[K(|x+2m|,\, t) + K(|x-2m|,\, t)\right]. \tag{2.3}$$

The function K represents the fundamental solution of the linear equation $LK = 0$. It has been determined and studied in [3], and reads

$$K(|x|,t) = \int_0^t \frac{e^{-c^2\tau/\varepsilon}}{\sqrt{\pi\varepsilon\tau}}\,d\tau \int_0^\infty \frac{x^2(z+1)}{4\varepsilon\tau}\, e^{-x^2(z+1)^2/4\varepsilon\tau} I_0\left(\frac{c}{\varepsilon}\, 2|x|\sqrt{z}\right)dz, \tag{2.4}$$

where I_0 is the modified Bessel function of order zero. Since $\theta(-x,t) = \theta(x,t)$ and $\theta(x+2m,\, t) = \theta(x,t)$, $m \in N$, it is sufficient to restrict our attention to the domain $0 \le x < 2$, and note that θ is continuous together with its partial derivatives and satisfies the equation $L\theta = 0$. Moreover, from the analysis of K developed in [3], we can deduce that θ is a positive function that has properties similar to ones of the analogous function θ used for the heat operator, see [1].

As for the data we shall assume that:

$$\begin{gathered} f(x,t,n,p,q,r) \quad \text{is defined and continuous on the set}\\ \{(x,t,n,p,q,r)\colon\ 0 \le x \le 1,\ \ 0 \le t \le T,\ \ -\infty < n,p,q,r < \infty,\ \ T > 0\},\end{gathered} \tag{2.5}$$

it locally satisfies a Lipschitz condition, namely for any bounded set $\Omega \subset [0,T] \times \mathbb{R}^4$ there exists a constant μ_Ω such that for any $(t,n_1,p_1,q_1,r_1), (t,n_2,p_2,q_2,r_2) \in \Omega$ and $x \in [0,1]$ (2.6)

$$\begin{aligned}&|f(x,t,n_1,p_1,q_1,r_1) - f(x,t,n_2,p_2,q_2,r_2)|\\ &\le \mu_\Omega\{|n_1-n_2| + |p_1-p_2| + |q_1-q_2| + |r_1-r_2|\},\end{aligned}$$

$$u_0,\ u_0',\ u_0'',\ u_1 \quad \text{continuous on} \quad 0 \le x \le 1, \tag{2.7}$$

$$h_i,\ \frac{dh_i}{dt},\ \ i = 1,2, \quad \text{continuous on} \quad 0 \le t \le T, \tag{2.8}$$

$$h_1(0) = u_0(0), \quad h_2(0) = u_0(1), \quad \frac{dh_1(0)}{dt} = u_1(0), \quad \frac{dh_2(0)}{dt} = u_1(1). \tag{2.9}$$

Given a solution u of (1.1) − (1.3), by integrating (2.1) on $\{(\xi,\tau):\ 0<\xi<1,\ \delta<\tau<t-\delta\},\ \delta>0$, and letting $\delta\to 0,$ we find that it satisfies the following integral equation

$$u(x,t)=\int_0^1 w_t(x,\xi,t)u_0(\xi)\,d\xi+\int_0^1 w(x,\xi,t)[u_1(\xi)-\varepsilon u_0''(\xi)]\,d\xi \tag{2.10}$$
$$-2\int_0^t h_1(\tau)(c^2+\varepsilon\partial_t)\theta_x(x,\,t-\tau)\,d\tau+2\int_0^t h_2(\tau)(c^2+\varepsilon\partial_t)\theta_x(1-x,\,t-\tau)\,d\tau$$
$$+\int_0^t d\tau\int_0^1 w(x,\xi,\,t-\tau)f(\xi,\tau,u(\xi,\tau),u_\xi(\xi,\tau),u_\tau(\xi,\tau),u_{\xi\xi}(\xi,\tau))\,d\xi.$$

Conversely, one can immediately verify that under the assumptions (2.5) − (2.9) a solution u of (2.10) satisfies (1.1) using the fact that $L\theta=0$ and $Lw=0$. We refer the reader to [2] for the slightly longer proof that the initial conditions (1.2) and the boundary conditions are satisfied.

If $f=f(x,t)$, (2.10) gives the unique explicit solution of (1.1) − (1.3). On the contrary, if f depends on u (2.10) is an integro-differential equation. We shall now discuss the existence and uniqueness of its solutions.

For any $c,d\in[0,T],\ c\le d$, we shall denote

$$\mathcal{B}_{[c,d]}:=\{u(x,t):\ u,u_x,u_t,u_{xx}\in C([0,1]\times[c,d])\}.$$

For any $a\in[0,T],\ v\in\mathcal{B}_{[0,a]}$ and $t\in[a,T]$ we define a mapping of $\mathcal{B}_{[a,T]}$ into itself by

$$\mathcal{T}_v u(x,t):=\omega_v(x,t)+\int_a^t d\tau\int_0^1 w(x,\xi,\,t-\tau)f(\xi,\tau,u(\xi,\tau),u_\xi(\xi,\tau),u_\tau(\xi,\tau),u_{\xi\xi}(\xi,\tau))\,d\xi \tag{2.11}$$

where

$$\omega_v(x,t)=\int_0^1 w_t(x,\xi,t)u_0(\xi)\,d\xi+\int_0^1 w(x,\xi,t)[u_1(\xi)-\varepsilon u_0^{''}(\xi)]\,d\xi$$
$$-2\int_0^t h_1(\tau)(c^2+\varepsilon\partial_t)\theta_x(x,\,t-\tau)\,d\tau+2\int_0^t h_2(\tau)(c^2+\varepsilon\partial_t)\theta_x(1-x,\,t-\tau)\,d\tau$$
$$+\int_0^a d\tau\int_0^1 w(x,\xi,\,t-\tau)f(\xi,\tau,v(\xi,\tau),v_\xi(\xi,\tau),v_\tau(\xi,\tau),v_{\xi\xi}(\xi,\tau))\,d\xi.$$

We fix a $\rho>0$ and for any $t\in[a,T]$ we consider the domain

$$S_{v,t}:=\{u\in\mathcal{B}_{[a,T]}:\ \forall\,x\in[0,1]\ \ |u(x,t)-\omega_v(x,t)|<\rho,\ \ |u_x(x,t)-\omega_{vx}(x,t)|<\rho,$$
$$|u_{xx}(x,t)-\omega_{vxx}(x,t)|<\rho,\ \ |u_t(x,t)-\omega_{vt}(x,t)|<\rho\}$$

and define

$$M = M(a,T,v,\rho) := \sup_{\substack{\tau\in[a,T]\\ \xi\in[0,1]}} \sup_{u\in S_{v,\tau}} |f(\xi,\tau,u(\xi,\tau),u_\xi(\xi,\tau),u_\tau(\xi,\tau),u_{\xi\xi}(\xi,\tau))|,$$

$$\begin{aligned} b-a &= \min\left\{T-a,\ \frac{\rho}{M},\ \frac{c\rho}{M},\ \frac{\varepsilon\rho}{M},\ \sqrt{\frac{2\rho}{M}}\right\}, \\ R_{a,b,v} &:= \{u\in\mathcal{B}_{[a,b]}\colon\ \forall\,(x,t)\in[0,1]\times[a,b]\quad |u(x,t)-\omega_v(x,t)|\le\rho, \\ &\quad |u_x(x,t)-\omega_{vx}(x,t)|\le\rho,\quad |u_{xx}(x,t)-\omega_{vxx}(x,t)|\le\rho, \\ &\quad |u_t(x,t)-\omega_{vt}(x,t)|\le\rho\}. \end{aligned} \tag{2.12}$$

Note that by its definition M is finite because f is continuous and evaluated on a compact subset of R^6. We denote by $\mu=\mu(a,b,v,\rho)$ the constant μ_Ω of (2.6) corresponding to the choice

$$\Omega = \{(t,n,p,q,r)\colon\ \text{with}\ t\in[a,b],\ \text{and such that}\ \exists\,x\in[0,1],\ \exists\,u\in R_{a,b,v}\ \text{such that}\ n=u(x,t),\ p=u_x(x,t),\ q=u_{xx}(x,t),\ r=u_t(x,t)\},$$

we choose a positive constant λ

$$\lambda=\lambda(a,b,v,\rho)>\max\left\{1,\ \mu\left(2+\frac{1}{c}+\frac{1+2c^2}{\varepsilon}\right)\right\}$$

and we introduce a norm

$$\begin{aligned} \|u\|_{a,b} &:= \sup_{[0,1]\times[a,b]} |\mathrm{e}^{-\lambda t}u(x,t)| + \sup_{[0,1]\times[a,b]} |\mathrm{e}^{-\lambda t}u_x(x,t)| \\ &\quad + \sup_{[0,1]\times[a,b]} |\mathrm{e}^{-\lambda t}u_t(x,t)| + \sup_{[0,1]\times[a,b]} |\mathrm{e}^{-\lambda t}u_{xx}(x,t)|. \end{aligned} \tag{2.14}$$

We now show that $\mathcal{T}_v$ is a map of $R_{a,b,v}$ into itself, more precisely a contraction (w.r.t the above norm). From (2.11) we get for any $(x,t)\in[0,1]\times[a,b]$

$$|\mathcal{T}_v u(x,t)-\omega_v(x,t)| \le M(a,T,v,\rho)\int_a^t d\tau\int_0^1 |w(x,\xi,t-\tau)|\,d\xi,$$

and, because of the inequality [3]

$$\int_0^1 |w(x,\xi,t-\tau)|\,d\xi = \int_0^1 |\theta(x-\xi,\,t-\tau)-\theta(x+\xi,\,t-\tau)|\,d\xi \le t-\tau, \tag{2.15}$$

and the definition of b we find

$$|\mathcal{T}_v u(x,t)-\omega_v(x,t)| \le M(a,T,v,\rho)\,\frac{(b-a)^2}{2} \le \rho. \tag{2.16}$$

Similarly, one can prove that

$$|\mathcal{T}_v u_x(x,t) - \omega_{vx}(x,t)| \le \rho, \tag{2.17}$$
$$|\mathcal{T}_v u_{xx}(x,t) - \omega_{vxx}(x,t)| \le \rho, \tag{2.18}$$
$$|\mathcal{T}_v u_t(x,t) - \omega_{vt}(x,t)| \le \rho, \tag{2.19}$$

making use of the basic properties of K proved in [3], which lead to the following estimates:

$$\begin{aligned} &\int_0^1 |w_x(x,\xi,t-\tau)|\, d\xi \le 1/c, \\ &\int_0^1 |w_t(x,\xi,t-\tau)|\, d\xi \le 1, \\ &\int_0^1 |w_{xx}(x,\xi,t-\tau)|\, d\xi \le \frac{1}{\epsilon}\,[1+2c^2(t-\tau)]. \end{aligned} \tag{2.20}$$

The first two inequalities were already given in [2], together with

$$\int_0^1 |(\partial_t - \partial_x^2) w(x,\xi,t-\tau)|\, d\xi \le 1. \tag{2.21}$$

The third was used but not explicitly written, and easily follows from the latter inequality, the equation $L\theta = 0$, and the relation $\theta(x,0) = 0$. In fact, from $L\theta = 0$ it immediately follows that

$$\theta_t - \theta_{xx} = \partial_t \left[\theta + \frac{\epsilon}{c^2}\theta_{xx} - \frac{1}{c^2}\,\theta_t \right],$$

and therefore

$$w_t(x,\xi,t-\tau) - w_{xx}(x,\xi,t-\tau) = \partial_t \left[w(x,\xi,t-\tau) + \frac{\epsilon}{c^2}\, w_{xx}(x,\xi,t-\tau) - \frac{1}{c^2}\, w_t(x,\xi,t-\tau) \right].$$

Integrating over ξ and using (2.21) we find $|\partial_t A(x,t-\tau)| \le 1$, where

$$A(x,\, t-\tau) := \int_0^1 \left[w(x,\xi,\, t-\tau) + \frac{\epsilon}{c^2}\, w_{xx}(x,\xi,\, t-\tau) - \frac{1}{c^2}\, w_t(x,\xi,\, t-\tau) \right] d\xi.$$

As $\theta(x,0) = 0$, then $A(x,0) = 0$. By the comparison principle we therefore find

$$\tau - t \le A(x,\, t-\tau) = \int_0^1 w\, d\xi + \int_0^1 \frac{\epsilon}{c^2}\, w_{xx}\, d\xi - \int_0^1 \frac{1}{c^2}\, w_t\, d\xi \le t-\tau,$$

implying

$$\left| \int_0^1 \frac{\epsilon}{c^2}\, w_{xx} \right| \le (t-\tau) + \left| \int_0^1 w\, d\xi \right| + \left| \int_0^1 \frac{1}{c^2}\, w_t\, d\xi \right|;$$

using (2.15) and $(2.20)_2$ to bound the integrals on the right hand-side we find $(2.20)_3$.

From the above results we conclude that $\mathcal{T}_v u(x,t) \in R_{a,b,v}$ as claimed.

From (2.11), (2.15)) we get for $t \in [a,b]$

$$
\begin{aligned}
|\mathcal{T}_v u_1(x,t) - \mathcal{T}_v u_2(x,t)| \mathrm{e}^{-\lambda t} &\le \mu \|u_1 - u_2\|_{a,b} \int_a^t \mathrm{e}^{-\lambda(t-\tau)}\, d\tau \int_0^1 |w(x,\xi,t-\tau)|\, d\xi \\
&\le \mu \|u_1 - u_2\|_{a,b} \int_a^t \mathrm{e}^{-\lambda(t-\tau)} (t-\tau)\, d\tau \le \frac{\mu}{\lambda^2} \|u_1 - u_2\|_{a,b}.
\end{aligned}
\tag{2.22}
$$

From (2.11), (2.20) one can get analogous results for the partial derivatives ∂_x, ∂_t, ∂_x^2 of (2.11):

$$
\begin{aligned}
|\mathcal{T}_v u_{1x}(x,t) - \mathcal{T}_v u_{2x}(x,t)|\, \mathrm{e}^{-\lambda t} &\le \frac{\mu}{\lambda c} \|u_1 - u_2\|_{a,b}, \\
|\mathcal{T}_v u_{1t}(x,t) - \mathcal{T}_v u_{2t}(x,t)|\, \mathrm{e}^{-\lambda t} &\le \frac{\mu}{\lambda} \|u_1 - u_2\|_{a,b}, \\
|\mathcal{T}_v u_{1xx}(x,t) - \mathcal{T}_v u_{2xx}(x,t)|\, \mathrm{e}^{-\lambda t} &\le \frac{\mu}{\lambda \varepsilon} \left(1 + \frac{2c^2}{\lambda}\right) \|u_1 - u_2\|_{a,b}.
\end{aligned}
\tag{2.23}
$$

Thus, we obtain

$$
\|\mathcal{T}_v u_1(x,t) - \mathcal{T}_v u_2(x,t)\|_{a,b} \le \frac{\mu}{\lambda} \left[\frac{1}{\lambda} + \frac{1}{c} + 1 + \frac{1}{\varepsilon} + \frac{2c^2}{\varepsilon \lambda} \right] \|u_1 - u_2\|_{a,b}, \tag{2.24}
$$

with $\mu \equiv \mu(a,b,v,\rho)$, $\lambda \equiv \lambda(a,b,v,\rho)$. Under assumption (2.13), inequality (2.24) shows that $\mathcal{T}_v$ is a contraction of $R_{a,b,v}$ into itself. Thus we are in the conditions to apply the fixed point theorem, and we find that there exists a unique solution in $R_{a,b,v}$ of the problem $\mathcal{T}_v u = u$ in the time interval $[a,b]$.

We now apply the above result iteratively. We start by choosing $a = 0 = a_0$, $v = 0$; the last integral disappears from (2.12). From the definition of b we determine the corresponding $b = a_1$ and by the fixed point theorem a unique solution $u^{(1)}(x,t)$ of the problem (1.1)–(1.4) in the time interval $[0,a_1]$. Next we choose $a = a_1$, $v = u^{(1)}$; from (2.12) we determine the corresponding $b = a_2$ and by the fixed point theorem a unique solution of the problem $\mathcal{T}_{u^{(1)}} u = u$ in the time interval $[a_1,a_2]$. This is also a smooth continuation of $u^{(1)}$, therefore we have found a unique solution $u^{(2)}(x,t)$ of the problem (1.1)–(1.4) in the time interval $[0,a_2]$, and so on. We conclude by stating the following

Theorem *Under hypotheses (2.5)–(2.9), the quasilinear problem (1.1)–(1.3) has a unique smooth solution in the time interval* $[0,a_\infty]$*, where*

$$
a_\infty := \lim_{k \to +\infty} a_k \le T.
$$

3 Eventual Boundedness and Asymptotic Stability

3.1 Preliminaries

By the rescaling $t \to t/c$, $\varepsilon \to c\varepsilon$ and of $f \to c^2 f$ we can factor c out of (1.1), so that it completely disappears from the problem, without loosing generality. In the sequel we shall assume we have done this. Moreover, without loss of generality we can also consider $h_1(t) = h_2(t) \equiv 0$ in (1.3), as any problem (1.1)−(1.4) is equivalent to another one of the same kind with trivial boundary conditions and a different f. In fact, setting for any $t \in J = [0, \infty[$

$$v(x,t) := u(x,t) + p(x,t), \qquad p(x,t) := (1-x)h_1(t) + xh_2(t)$$

we immediately find that $v(0,t) = v(1,t) \equiv 0$, that the initial condition for v, v_t are completely determined and that v fulfills the equation

$$-\varepsilon v_{xxt} + v_{tt} - v_{xx} = \tilde{f}(x,t,v,v_x,v_{xx},v_t),$$

where

$$\tilde{f}(x,t,v,v_x,v_{xx},v_t) := f(x,\, t,\, v-p,\, v_x - h_2 + h_1,\, v_{xx},\, v_t - p_t) - p_{tt}.$$

The difference $u = \tilde{u} - u^*$ between a generic solution $\tilde{u}$ and a given one u^* of the problem (1.1)−(1.4) is also a solution of a new problem of the same kind, which we denote by problem $\mathcal{P}$, but with $h_1(t) \equiv h_2(t) \equiv 0$, namely

$$\begin{gathered} -\varepsilon u_{xxt} + u_{tt} - u_{xx} = f(x,t,u,u_x,u_{xx},u_t), \quad x \in]0,1[, \quad t > t_0 \in J, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad t \in J, \end{gathered} \tag{3.1}$$

with the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in]0,1[, \tag{3.2}$$

fulfilling the consistency conditions

$$u_0(0) = u_1(0) = u_0(1) = u_1(1) = 0. \tag{3.3}$$

Here

$$\begin{aligned} f(x,t,u,u_x,u_{xx},u_t) = {} & f(x,t,\, u+u^*,\, u_x + u_x^*,\, u_{xx} + u_{xx}^*,\, u_t + u_{xx}^*) \\ & - f(x,t,u^*,u_x^*,u_{xx}^*,u_t^*) \end{aligned}$$

and $u_0(x) := \tilde{u}_0(x) - u_0^*(x)$, $u_1(x) = \tilde{u}_1(x) - u_1^*(x)$. The two solutions $\tilde{u}$, u^* are 'close' to each other iff u is a 'small' solution of the latter problem, and coincide iff u is the zero solution.

We introduce the distance between the origin O and a nonzero element $\big(u(\cdot,t), u_t(\cdot,t)\big) \in \Gamma := \big(C_0([0,1]) \cap C^2([0,1])\big) \times C_0([0,1])$ as the functional $d(u,u_t)$, where for any $(\varphi,\psi) \in \Gamma$ we define

$$d^2(\varphi,\psi) = \int_0^1 (\varphi^2 + \varphi_x^2 + \varphi_{xx}^2 + \psi^2)\, dx. \tag{3.4}$$

The notions of (eventual) boundedness, stability, attractivity, etc. are formulated using this distance. Imposing the condition that φ, ψ vanish in $0,1$ one easily derives that $|\varphi(x)|, |\varphi_x(x)| \leq d(\varphi,\psi)$ for any x; therefore a convergence in the norm d implies also a uniform pointwise convergence of φ, φ_x.

Definition 3.1 The *solutions* of (3.1) – (3.3) are *eventually uniformly bounded* if for any $\alpha > 0$ there exist an $s(\alpha) \geq 0$ and a $\beta(\alpha) > 0$ such that if $t_0 \geq s(\alpha)$, $d(u_0, u_1) \leq \alpha$, then $d(u(t), u_t(t)) < \beta(\alpha)$ for all $t \geq t_0$. If $s(\alpha) = 0$ the solutions of (3.1) are uniformly bounded.

Definition 3.2 The *origin* O of Γ is *eventually quasi-uniform-asymptotically stable in the large* for the solutions of (3.1) if for any $\rho, \alpha > 0$ there exist $s(\alpha) \geq 0$, and $\widehat{T}(\rho, \alpha) > 0$ such that if $d(u_0, u_1) \leq \alpha$, $t_0 \geq s(\alpha)$ then $d(u, u_t) < \rho$ for any $t \geq t_0 + \widehat{T}$. If $s(\alpha) = 0$, $u(x,t) \equiv 0$ is said to be quasi-uniform-asymptotically stable in the large for the solutions of (3.1).

Suppose now that problem $\mathcal{P}$ admits the solution $u(x,t) \equiv 0$.

Definition 3.3 The *solution* $u(x,t) \equiv 0$ is *uniform-asymptotical stable in the large* if it is uniformly stable and quasi-uniform-asymptotically stable in the large, and the solutions of problem $\mathcal{P}$ are uniformly bounded.

Definition 3.4 The *solution* $u(x,t) \equiv 0$ of the problem $\mathcal{P}$ is *exponential-asymptotically stable in the large* if for any $\alpha > 0$ there are two positive constants $D(\alpha)$, $C(\alpha)$ such that if $d(u_0, u_1) \leq \alpha$, then

$$d(u(t), u_t(t)) \leq D(\alpha) \exp\left[-C(\alpha)(t - t_0)\right] d(u_0, u_1), \quad \forall\, t \geq t_0. \tag{3.5}$$

To prove our theorems we shall use the Liapunov direct method. We introduce the Liapunov functional

$$V(\varphi, \psi) = \frac{1}{2} \int_0^1 \{(\varepsilon\varphi_{xx} - \psi)^2 + \gamma\psi^2 + (1+\gamma)\varphi_x^2\}\, dx, \tag{3.6}$$

where γ is an arbitrary positive constant. Using the inequality $|2\varepsilon\varphi_{xx}\psi| \leq \varepsilon(\varphi_{xx}^2 + \psi^2)$ we find

$$V \leq \frac{1}{2} \int_0^1 \{\varepsilon^2\varphi_{xx}^2 + \psi^2 + \varepsilon\varphi_{xx}^2 + \varepsilon\psi^2 + \gamma\psi^2 + (1+\gamma)\varphi_x^2\}\, dx.$$

Setting

$$c_2^2 = \max\{\varepsilon(1+\varepsilon)/2,\ (1+\varepsilon+\gamma)/2\}, \tag{3.7}$$

we thus derive

$$V(\varphi, \psi) \leq c_2^2 d^2(\varphi, \psi). \tag{3.8}$$

Moreover, it is known that [13]

$$\varphi(0) = 0, \quad \varphi(1) = 0 \quad \Longrightarrow \quad \begin{cases} \int_0^1 \varphi_x^2(x)\, dx \geq \pi^2 \int_0^1 \varphi^2(x)\, dx \\ \int_0^1 \varphi_{xx}^2(x)\, dx \geq \pi^2 \int_0^1 \varphi_x^2(x)\, dx \end{cases} \tag{3.9}$$

(these inequalities can be easily proved by Fourier analysis of φ). In view of the bounds we shall consider below we introduce the notation

$$\omega_1 := \frac{\pi^4}{1+\pi^4} \approx 0.99, \quad \omega_2 := \frac{\pi^4}{1+\pi^2+\pi^4} \approx 0.90, \quad \omega_3 := \frac{\pi^2}{1+\pi^2} \approx 0.91. \tag{3.10}$$

Using (3.9) and an argument employed in [2], we get

$$V(\varphi,\psi) \geq c_1^2 d^2(\varphi,\psi). \tag{3.11}$$

where

$$c_1^2 = \min\left\{\frac{\varepsilon^2}{8}\,\omega_1,\ \frac{1}{2}\left(\gamma-\frac{1}{2}\right)\right\}, \qquad (\gamma > 1/2). \tag{3.12}$$

Therefore, from (3.8) and (3.11) we find

$$\frac{V}{c_2^2} \leq d^2 \leq \frac{V}{c_1^2}. \tag{3.13}$$

On the other hand, choosing $\gamma = 1$ in (3.6) and reasoning as it has been done in [2] it turns out that

$$\begin{aligned}
\frac{dV}{dt} &= \int_0^1 \left\{ -\frac{\varepsilon}{2}\,u_{xx}^2 - \varepsilon u_{xt}^2 + \frac{\varepsilon}{2}\,u_t^2 - \frac{\varepsilon}{2}(u_{xx}+f)^2 - \frac{\varepsilon}{2}(u_t - 2f/\varepsilon)^2 + Af^2 \right\} dx \\
&\leq -\int_0^1 \left\{ \frac{\varepsilon}{2}\omega_2(u^2+u_x^2+u_{xx}^2) + \varepsilon\left(\pi^2-\frac{1}{2}\right)u_t^2 + Af^2 \right\} dx \\
&\leq -c_3^2 d^2(u,u_t) + \int_0^1 Af^2 dx
\end{aligned} \tag{3.14}$$

where we have set

$$A := \frac{\varepsilon}{2} + \frac{2}{\varepsilon}, \qquad c_3^2 := \frac{\omega_2}{2}\,\varepsilon, \tag{3.15}$$

and we have used (3.9). In the sequel we shall set also $p := c_3^2/c_2^2$.

3.2 Eventual boundedness and asymptotic stability

We assume that

$$A\int_0^1 f^2\,dx \leq g(t)c_1^2 d^2 + \tilde{g}_1(t,d^2) + \tilde{g}_2(t,d^2), \tag{3.16}$$

where f is the function appearing in (3.1), and $g(t)$, $\tilde{g}_i(t,\eta)$ $(i = 1,2$ and $t \in J,\ \eta > 0)$ denote suitable nonnegative continuous functions fulfilling the following conditions:

(1) there exists a constant $\sigma > 0$ such that for any $t \geq t_0 \geq 0$

$$\int_{t_0}^t g(\tau)\,d\tau - p(t-t_0) \leq \sigma; \tag{3.17}$$

(2) there exist constants $\chi \in [0,1]$, $\kappa \in [0,1]$ and $q \geq 0$ (with $q < p$ if $\chi = 1$) and $M > 0$ such that

$$\left| \frac{\int_0^t g(\tau)d\tau}{1+t^\chi} - q \right| < \frac{M}{1+t^\kappa}; \tag{3.18}$$

(3) for any $\eta > 0$

$$\lim_{t\to+\infty} \tilde{g}_1(t,\eta)e^{\xi(t^\chi - t^\kappa)} = 0,$$
$$\int_0^\infty \tilde{g}_2(\tau,\eta)e^{\xi(\tau^\chi-\tau^\kappa)}\,d\tau = \sigma_2(\eta) < +\infty, \tag{3.19}$$

where ξ is some positive constant if $\chi > \kappa$, while $\xi = 0$ if $\chi \le \kappa$.

Without loss of generality we can assume that $\tilde{g}_i(t,\eta)$ are non-decreasing in η; if originally this is not the case, we just need to replace $\tilde{g}_i(t,\eta)$ by $\max\limits_{0\le u\le\eta} \tilde{g}_i(t,u)$ to fulfill this condition.

From (3.14), using (3.4), (3.16), (3.13) we now find

$$\begin{aligned} \frac{dV(u,u_t)}{dt} &\le -c_3 d^2(u,u_t) + g(t)c_1^2 d^2 + \tilde{g}_1(t,d^2) + \tilde{g}_2(t,d^2) \\ &\le -p\,V + g(t)V + g_1(t,V) + g_2(t,V), \end{aligned} \tag{3.20}$$

where we have set

$$g_i(t,\eta) = \tilde{g}_i\left(t, \frac{\eta}{c_1^2}\right). \tag{3.21}$$

By the "Comparison Principle" (see e.g. [14]) V is bounded from above

$$V(t) \le y(t), \tag{3.22}$$

by the solution $y(t)$ of the Cauchy problem

$$\frac{dy}{dt} = -p\,y + g(t)y + g_1(t,y) + g_2(t,y), \qquad y(t_0) = y_0 \equiv V(t_0) \ge 0. \tag{3.23}$$

We therefore study the latter, proving first a theorem of eventual uniform boundedness.

Lemma 1 *Assume $g(t)$, $\tilde{g}_i(t,\eta)$ $(i = 1,2$ and $t \in J$, $\eta > 0)$ are continuous nonnegative functions fulfilling the conditions (3.17) – (3.19). Then $\forall\,\tilde{\alpha} > 0$ there exist $\tilde{s}(\tilde{\alpha}) \ge 0$, $\tilde{\beta}(\tilde{\alpha}) > 0$ such that if $|y_0| \le \tilde{\alpha}$, $t_0 \ge \tilde{s}(\tilde{\alpha})$, the modulus of the solution $y(t;t_0,y_0)$ of (3.23) is bounded by $\tilde{\beta}$:*

$$|y(t;t_0,y_0)| < \tilde{\beta}, \quad t \ge t_0 \ge \tilde{s}(\tilde{\alpha}); \tag{3.24}$$

if in particular $y_0 \in [0,\tilde{\alpha}]$, then

$$0 \le y(t;t_0,y_0) < \tilde{\beta}, \quad t \ge t_0 \ge \tilde{s}(\tilde{\alpha}). \tag{3.25}$$

Proof Problem (3.23) is equivalent to the integral equation

$$\begin{aligned} y(t) &= y_0\, e^{-p(t-t_0)+\int_{t_0}^t g(\tau)d\tau} \\ &\quad + e^{-pt+\int_0^t g(\tau)d\tau} \int_{t_0}^t \left[g_1\left(\tau, y(\tau)\right) + g_2\left(\tau, y(\tau)\right)\right] e^{p\tau - \int_0^\tau g(z)dz}\, d\tau. \end{aligned} \tag{3.26}$$

Take $\tilde{\beta}(\tilde{\alpha}) := \tilde{\alpha}(e^{\sigma} + \frac{e^{2M}}{m} + e^{2M})$, where

$$m = \begin{cases} \dfrac{p}{2} & \text{if } \chi < 1 \\[2mm] \dfrac{p-q}{2} & \text{if } \chi = 1. \end{cases} \tag{3.27}$$

Because of (3.17), if $|y_0| \leq \tilde{\alpha}$, for any $t \geq t_0$ one finds

$$|y_0| e^{-p(t-t_0) + \int\limits_{t_0}^{t} g(\tau) d\tau} \leq \tilde{\alpha} e^{\sigma}. \tag{3.28}$$

Moreover, because of (3.18), we obtain

$$q(1+t^{\chi}) - M\, \frac{1+t^{\chi}}{1+t^{\kappa}} < \int\limits_0^t g(z)\, dz < q(1+t^{\chi}) + M\, \frac{1+t^{\chi}}{1+t^{\kappa}}. \tag{3.29}$$

Let

$$\vartheta := \begin{cases} 0 & \text{if } \chi \leq \kappa, \\ \min\left\{1,\ \dfrac{\xi}{2M}\right\} & \text{if } 1 > \chi > \kappa \\ \min\left\{1,\ \dfrac{p-q}{2M},\ \dfrac{\xi}{2M}\right\} & \text{if } 1 = \chi > \kappa, \end{cases}$$

$$t_{\vartheta} := \begin{cases} 0 & \text{if } \vartheta = 0, \\ \left(\dfrac{1-\vartheta}{\vartheta}\right)^{1/\kappa} & \text{if } \vartheta > 0; \end{cases}$$

considering separately the cases $\chi \leq \kappa$, $\chi > \kappa$ and recalling the definition of ξ, we find

$$\frac{1+t^{\chi}}{1+t^{\kappa}} = 1 + \frac{t^{\chi} - t^{\kappa}}{1+t^{\kappa}} \leq 1 + \vartheta(t^{\chi} - t^{\kappa})$$

for any $t \geq t_{\vartheta}$. Then from (3.29)

$$q(1+t^{\chi}) - M[1+\vartheta(t^{\chi} - t^{\kappa})] < \int\limits_0^t g(z)\, dz < q(1+t^{\chi}) + M[1+\vartheta(t^{\chi} - t^{\kappa})] \tag{3.30}$$

for any $t \geq t_{\vartheta}$. Consequently, for $i = 1, 2$ and $|y| \leq \tilde{\beta}$

$$\begin{aligned} & e^{-pt + \int\limits_0^t g(\tau) d\tau} \int\limits_{t_0}^{t} g_i(\tau, y) e^{p\tau - \int\limits_0^{\tau} g(z) dz} d\tau \\ & < e^{-pt + q(1+t^{\chi}) + M[1+\vartheta(t^{\chi} - t^{\kappa})]} \int\limits_{t_0}^{t} g_i(t, \tilde{\beta}) e^{p\tau - q(1+\tau^{\chi}) + M[1+\vartheta(\tau^{\chi} - \tau^{\kappa})]}\, d\tau \\ & = e^{qt^{\chi} + M\vartheta(t^{\chi} - t^{\kappa}) - pt} e^{2M} \int\limits_{t_0}^{t} g_i(t, \tilde{\beta}) e^{p\tau - q\tau^{\chi} + M\vartheta(\tau^{\chi} - \tau^{\kappa})} d\tau, \end{aligned} \tag{3.31}$$

where we have used also the fact that $g_i(t,\eta)$ are non-decreasing functions of η.

Now consider the function

$$h(\tau) := p\tau - q\tau^{\chi} - M\vartheta(\tau^{\chi} - \tau^{\kappa}) \tag{3.32}$$

and its derivative $h'(\tau) = p - q\chi\tau^{\chi-1} - M\vartheta(\chi\tau^{\chi-1} - \kappa\tau^{\kappa-1})$. We now show that, for any $\chi \in [0,1]$,

$$h'(\tau) \geq h'(\tilde{t}) = m \quad \text{if} \quad \tau \geq \tilde{t} := \left[\frac{\chi(2q+\xi)}{p}\right]^{\frac{1}{1-\chi}} \tag{3.33}$$

with the m defined in (3.27) (this implies that for $\tau \geq \tilde{t}$ the function $h(\tau)$ is increasing). In fact, if $\vartheta > 0$, then it is either $0 \leq \kappa < \chi < 1$, implying

$$h'(\tau) > p - (q + M\vartheta)\chi\tau^{\chi-1} \geq \frac{p}{2} = m$$

for any $\tau \geq \tilde{t}$, or $0 \leq \kappa < \chi = 1$, implying (because of the inequality $p - q > 0$ and the definition of ϑ)

$$h'(\tau) = p - q - M\vartheta + M\vartheta\kappa\tau^{\kappa-1} > p - q - M\vartheta \geq \frac{p-q}{2} = m$$

for any $\tau > 0$, in particular for $\tau \geq \tilde{t}$. If $\vartheta = 0$, then it is either $0 \leq \chi \leq \kappa \leq 1$ with $\chi < 1$, implying

$$h'(\tau) > p - q\chi\tau^{\chi-1} \geq \frac{p}{2} = m$$

for any $\tau \geq \left[\dfrac{2q\chi}{p}\right]^{\frac{1}{1-\chi}} \equiv \tilde{t}$, or $\chi = \kappa = 1$, implying also $h'(\tau) = p - q > m$ (for any τ), as claimed.

On the other hand, because of (3.19) there exist $s_1(\tilde{\alpha}), s_2(\tilde{\alpha}) \geq 0$ (recall that $\tilde{\beta} = \tilde{\beta}(\tilde{\alpha})$) such that

$$\begin{aligned} &g_1(\tau,\tilde{\beta})e^{\xi(\tau^{\chi}-\tau^{\kappa})} \leq \tilde{\alpha} \quad \text{if} \quad \tau \geq t_0 \geq s_1(\tilde{\alpha}), \\ &\int_{t_0}^{\infty} g_2(\tau,\tilde{\beta})e^{\xi(\tau^{\chi}-\tau^{\kappa})}\,d\tau \leq \tilde{\alpha} \quad \text{if} \quad t_0 \geq s_2(\tilde{\alpha}). \end{aligned} \tag{3.34}$$

Hence, for $t \geq t_0 \geq \max\{\tilde{t}, t_\vartheta, s_1(\tilde{\alpha})\}$ we find that if $|\,y(\tau)| \leq \tilde{\beta}$ for any $\tau \in [t_0, t[$ then

$$\begin{aligned} &e^{-pt+\int_0^t g(\tau)d\tau} \int_{t_0}^{t} g_1\big(\tau, y(\tau)\big) e^{p\tau - \int_0^\tau g(z)dz} d\tau \\ &< e^{-h(t)+2M} \int_{t_0}^{t} g_1\big(\tau,\tilde{\beta}\big) e^{h(\tau)+\xi(\tau^{\chi}-\tau^{\kappa})} d\tau \\ &\leq e^{-h(t)+2M}\tilde{\alpha} \int_{t_0}^{t} \frac{h'(\tau)}{m}\, e^{h(\tau)} d\tau = \tilde{\alpha}\, \frac{e^{-h(t)+2M}}{m}\, (e^{h(t)} - e^{h(t_0)}) < \frac{e^{2M}}{m}\,\tilde{\alpha} \end{aligned} \tag{3.35}$$

where in the first line we have used (3.31) and the definition of ϑ, in the second (3.33) and $(3.34)_1$. Similarly, for $t \geq t_0 \geq \max\{s_2(\tilde{\alpha}), t_\vartheta, \tilde{t}\}$ we find that if $|y(\tau)| \leq \tilde{\beta}$ for any $\tau \in [t_0, t[$ then

$$\begin{aligned} & e^{-pt+\int\limits_0^t g(\tau)d\tau} \int\limits_{t_0}^{t} g_2\big(\tau, y(\tau)\big) e^{p\tau - \int\limits_0^\tau g(z)dz} d\tau \\ & < e^{-h(t)+2M} \int\limits_{t_0}^{t} g_2\big(\tau, \tilde{\beta}\big) e^{h(\tau)+\xi(\tau^\chi - \tau^\kappa)} d\tau \\ & < e^{-h(t)+h(t)+2M} \int\limits_{t_0}^{\infty} g_2(\tau, \tilde{\beta}) e^{\xi(\tau^\chi - \tau^\kappa)} d\tau < \tilde{\alpha} e^{+2M}, \end{aligned} \tag{3.36}$$

(in the first inequality we have used (3.31) and again the definition of ϑ, in the second the monotonicities of h and g_2, in the third $(3.34)_2$). Summarizing, the inequalities (3.28), (3.35), (3.36) are fulfilled for $t \geq t_0 \geq \tilde{s}(\tilde{\alpha}) = \max\{\tilde{t}, t_\vartheta, s_1(\tilde{\alpha}), s_2(\tilde{\alpha}), \}$.

Now let us suppose *per absurdum* that there exists $t_1 > t_0 \geq \tilde{s}(\tilde{\alpha})$ such that

$$|y(t; t_0, y_0)| < \tilde{\beta} \quad \text{for} \quad t_0 \leq t < t_1, \tag{3.37}$$

$$|y(t_1; t_0, y_0)| = \tilde{\beta}. \tag{3.38}$$

Because of (3.37) the inequalities (3.35), (3.36) are fulfilled; together with equations (3.26), (3.28) for $t = t_1$ they imply

$$|y(t_1; t_0, y_0)| < \tilde{\beta},$$

against the assumption (3.38). Finally, from (3.26) and the nonnegativity of the functions g_i we find that $0 \leq y_0 < \tilde{\alpha}$ implies $y(t) > 0$ for any t, whence (3.25).

As a result of the previous lemma, for any $\tilde{\alpha} > 0$ the solution $y(t)$ of the Cauchy problem (3.23) remains eventually uniformly bounded by $\tilde{\beta}(\tilde{\alpha})$ if $0 \leq y_0 \leq \tilde{\alpha}$. By (3.22) and (3.13), the same applies with $V(t)$ and $d^2(u, u_t)$.

By the monotonicity of $g_i(t, \eta)$ in η and the comparison principle we find that $y(t)$ is also bounded

$$y(t) \leq z(t), \quad t \geq t_0 \tag{3.39}$$

by the solution $z(t)$ of the Cauchy problem

$$\frac{dz}{dt} = -p\,z + g(t)z + g_1(t, \tilde{\beta}) + g_2(t, \tilde{\beta}), \qquad z(t_0) = z_0 \tag{3.40}$$

(which differs from (3.23) in that the second argument of g_i is now a fixed constant $\tilde{\beta} > 0$), provided that $z_0 = y_0$, and $t_0 \geq \tilde{s}(\tilde{\alpha})$.

We therefore study the Cauchy problem (3.40), keeping in mind that for our final purposes we will choose $\tilde{\beta} = \tilde{\beta}(\tilde{\alpha})$, $t_0 = t_0(\tilde{\alpha}) \geq \tilde{s}(\tilde{\alpha})$.

Lemma 2 *Assume $g(t)$, $\tilde g_i(t,\eta)$ $(i=1,2$ and $t\in J,\ \eta>0)$ are continuous functions fulfilling the conditions (3.17)–(3.19). Then for any $\tilde\rho>0$, $t_0>0$, $\tilde\alpha>0$ there exists $\widehat T(\tilde\rho,\tilde\alpha,\tilde\beta,t_0)>0$ such that for $|z_0|\le\tilde\alpha\in[0,\tilde\alpha]$ the solution $z(t;t_0,z_0)$ of (3.40) is bounded as follows:*

$$|z(t;t_0,z_0)|<\tilde\rho, \quad \text{if} \quad t\ge t_0+\widehat T. \tag{3.41}$$

If in particular $z_0\in[0,\tilde\alpha[$, then

$$0\le z(t;t_0,z_0)<\tilde\rho, \quad \text{if} \quad t\ge t_0+\widehat T. \tag{3.42}$$

Proof The solution $z(t)=z(t;t_0,z_0)$ is of the form

$$\begin{aligned} z(t) &= z_0\, e^{-p(t-t_0)+\int_{t_0}^{t} g(\tau)d\tau} \\ &\quad + e^{-pt+\int_0^t g(\tau)d\tau}\int_{t_0}^{t}\Big[g_1(\tau,\tilde\beta)+g_2(\tau,\tilde\beta)\Big]\, e^{p\tau-\int_0^\tau g(\lambda)d\lambda}\,d\tau. \end{aligned} \tag{3.43}$$

We now consider each of the three terms on the right-hand side of (3.43) separately.

By equation (3.30) for $t\ge t_\vartheta$

$$-p(t-t_0)+\int_{t_0}^{t} g(\tau)\,d\tau \le -(t-t_0)\left[p-q\frac{1+t^\chi}{t-t_0}-M\frac{1+\vartheta(t^\chi-t^\kappa)}{t-t_0}\right];$$

the right-hand side is negatively divergent for $t-t_0\to+\infty$, and so will be the left-hand side; this implies that there exists a $T_0(\tilde\rho,\tilde\alpha,t_0)\ge 0$ such that

$$|z_0|\, e^{-p(t-t_0)+\int_{t_0}^{t} g(\tau)d\tau} < \frac{\tilde\rho}{3}, \quad t\ge t_0+T_0, \quad z_0\in[-\tilde\alpha,\tilde\alpha]. \tag{3.44}$$

As for the second term, given $\tilde\beta>0$, $\tilde\rho>0$, because of $(3.19)_1$ there exist $T_1(\tilde\beta,\tilde\rho)\ge \max\{\tilde t,t_\vartheta\}$ and $\sigma_1(\tilde\beta)$ such that

$$\begin{aligned} g_1(\tau,\tilde\beta) &\le \sigma_1 && \text{if} \quad \tau\ge 0, \\ g_1(\tau,\tilde\beta)e^{\xi(\tau^\chi-\tau^\kappa)} &\le \frac{1}{6}\, m\tilde\rho\, e^{-2M} && \text{if} \quad \tau\ge T_1 \end{aligned} \tag{3.45}$$

($\tilde t, m$ have been defined respectively in (3.33), (3.27)). Since the function $h(t)$ defined in (3.32) is increasing as the first power of t for $t\ge\tilde t$, there exists $T_2(\tilde\beta,\tilde\rho)\ge T_1$ such that for $t\ge T_2$

$$\frac{\sigma_1}{p}\, e^{-h(t)+M+q+pT_1} < \frac{\tilde\rho}{6}. \tag{3.46}$$

Therefore, for $t \geq T_2$,

$$
\begin{aligned}
& e^{-pt+\int_0^t g(\tau)d\tau} \int_{t_0}^{t} g_1(\tau,\tilde{\beta}) e^{p\tau - \int_0^\tau g(\lambda)d\lambda} d\tau \\
& < e^{-pt+q(1+t^\chi)+M[1+\vartheta(t^\chi - t^\kappa)]} \int_0^t g_1(\tau,\tilde{\beta}) e^{p\tau-\int_0^\tau g(\lambda)d\lambda} d\tau \\
& < e^{-h(t)+M+q} \int_0^{T_1} g_1(\tau,\tilde{\beta}) e^{p\tau} d\tau + e^{-h(t)+2M} \int_{T_1}^{t} g_1(\tau,\tilde{\beta}) e^{h(\tau)+\xi(\tau^\chi-\tau^\kappa)} d\tau \\
& < e^{-h(t)+M+q}\sigma_1 \int_0^{T_1} e^{p\tau} d\tau + e^{-h(t)+2M} e^{-2M} \frac{m\tilde{\rho}}{6} \int_{T_1}^{t} \frac{h'(\tau)}{m} e^{h(\tau)} d\tau \\
& < e^{-h(t)+M+q}\sigma_1 \frac{e^{pT_1}}{p} + \frac{\tilde{\rho}}{6} e^{-h(t)} (e^{h(t)} - e^{h(T_1)}) \\
& < \frac{\tilde{\rho}}{6}(1+1) = \frac{\tilde{\rho}}{3},
\end{aligned}
\tag{3.47}
$$

where in the first and in the second inequality we have used (3.30), the nonnegativity of g_1, the fact that $\xi(\tau^\chi - \tau^\kappa) \geq 0$ and the definition of $h(t)$, in the third (3.45) and (3.33), in the fourth we have performed integration over τ, and in the last we have used (3.46).

As for the third term on the right-hand side of (3.43), from $(3.19)_2$ it follows that there exists $T_3(\tilde{\beta},\tilde{\rho}) \geq \max\{\tilde{t}, t_\vartheta\}$ such that for $t \geq T_3$

$$
e^{2M} \int_{T_3}^{t} g_2(\tau,\tilde{\beta}) e^{\xi(\tau^\chi-\tau^\kappa)} d\tau < \frac{\tilde{\rho}}{6}
\tag{3.48}
$$

and on the other hand that

$$
\int_0^{T_3} g_2(\tau,\tilde{\beta}) e^{p\tau} d\tau < \sigma_2,
\tag{3.49}
$$

where σ_2 has been defined in (3.19). Moreover, there exists $T_4(\tilde{\beta},\tilde{\rho}) \geq T_3$ such that for $t \geq \tilde{t} + T_4$

$$
\sigma_2 e^{-h(t)+q+M} < \frac{\tilde{\rho}}{6}.
\tag{3.50}
$$

Therefore for $t \geq T_4$

$$
\begin{aligned}
& e^{-pt+\int\limits_0^t g(\tau)d\tau} \int\limits_{t_0}^{t} g_2(\tau,\tilde\beta) e^{p\tau - \int\limits_0^\tau g(z)dz} d\tau \\
& < e^{-pt+q(1+t^\chi)+M[\vartheta(t^\chi - t^\kappa)+1]} \int\limits_0^t g_2(\tau,\tilde\beta) e^{p\tau - \int\limits_0^\tau g(z)dz} d\tau \\
& < e^{-h(t)+q+M} \int\limits_0^{T_3} g_2(\tau,\tilde\beta) e^{p\tau} d\tau + e^{-h(t)+2M} \int\limits_{T_3}^{t} g_2(\tau,\tilde\beta) e^{\xi(\tau^\chi - \tau^\kappa)} e^{h(\tau)} d\tau \\
& < e^{-h(t)+q+M}\sigma_2 + e^{-h(t)+2M+h(t)} \int\limits_{T_3}^{t} g_2(\tau,\tilde\beta) e^{\xi(\tau^\chi - \tau^\kappa)} d\tau \\
& < \frac{\tilde\rho}{6} + \frac{\tilde\rho}{6} = \frac{\tilde\rho}{3},
\end{aligned} \tag{3.51}
$$

where we have used the nonnegativity of g_2 and (3.30) in the first inequality, again (3.30), the fact that $\xi(\tau^\chi - \tau^\kappa) \geq 0$ and the nonnegativity of g in the second, (3.49) and the monotonicity of $h(\tau)$ for $\tau \geq T_3$ in the third, (3.48) and (3.50) in the last one.

Let $\widehat{T}(\tilde\rho,\tilde\beta,t_0) := \max\{T_0, T_2, T_4\}$. Collecting the results (3.44), (3.47), (3.51) we find that the solution $z(t)$ of (3.40) fulfills the condition

$$
z(t,t_0,z_0) < \frac{\tilde\rho}{3}[1+1+1] = \tilde\rho, \quad t \geq t_0 + \widehat{T}.
$$

Remark 1 This lemma is a generalization of Lemma 24.3 in [14], based in turn on an argument due to Hale [10].

Remark 2 If $\chi \leq \kappa$ then in the previous proof T_0, and therefore $\widehat{T}$, becomes independent of t_0. In fact, $\vartheta = 0$ and from (3.30) we find

$$
\int\limits_{t_0}^{t} g(z)\,dz = \int\limits_0^t g(z)\,dz - \int\limits_0^{t_0} g(z)\,dz < q(t^\chi - t_0^\chi) + 2M.
$$

By Lagrange's theorem there exists a $\tau \in]t_0, t[$ such that $t^\chi - t_0^\chi = \dfrac{\chi}{\tau^{1-\chi}}(t-t_0)$. Since $t_0 \geq \tilde t \equiv (2q\chi/p)^{1/(1-\chi)}$ we find $t^\chi - t_0^\chi < \dfrac{p}{2q}(t-t_0)$

$$
-p(t-t_0) + \int\limits_{t_0}^{t} g(z)dz < -p(t-t_0)\left[1-\frac{1}{2}\right] + 2M = -\frac{p}{2}(t-t_0) + 2M.
$$

This implies that the left-hand side is negatively divergent for $t - t_0 \to +\infty$ uniformly in t_0, as anticipated. The argument is not applicable in the case $\chi > \kappa$.

We are now in the conditions to prove the following

Theorem 1 *Assume that the function f of (3.1) is bounded as in (3.16), where $g(t)$, $\tilde{g}_i(t,\eta)$ $(i=1,2$ and $t\in J,\ \eta>0)$ are continuous functions fulfilling the conditions (3.17) – (3.19). Then the solutions of the problem (3.1), (3.2) are eventually uniformly bounded. Moreover, the origin O is eventually quasi-uniform-asymptotically stable in the large with respect to the metric d.*

Proof Set $\tilde{\alpha} := \alpha^2 c_2^2$, and apply Lemma 1. Under the assumption $d(u_0,u_1)\le\alpha$, by (3.13) we find $y_0 = V(t_0)\le\tilde{\alpha}$, by (3.22) and the application of the lemma we find that $y(t)$ (and therefore $V(t)$) is bounded by $\tilde{\beta}(\alpha^2c_2^2)$, and again by (3.13) we find $d(t)\le\beta(\alpha) := \sqrt{\tilde{\beta}(\alpha^2c_2^2)/c_1^2}$ for $t\ge s(\alpha) := \tilde{s}(\alpha^2c_2^2)$, as claimed. Moreover, we can now apply the comparison principle (3.39) – (3.40) and Lemma 2: chosen $\rho>0$, we set $\tilde{\rho} := c_1^2\rho^2$. As a consequence of (3.39), (3.42), (3.13) we thus find that for $t_0\ge s(\alpha)$ and $t\ge\widehat{T}(c_1^2\rho, t_0(\alpha), c_2^2\alpha^2)\equiv T(\rho,\alpha)$

$$d^2(t)\le\frac{V(t)}{c_1^2}\le\frac{y(t)}{c_1^2}\le\frac{z(t,\tilde{\beta}(\alpha))}{c_1^2}<\frac{\tilde{\rho}}{c_1^2}=\rho^2.$$

Remark 3 This theorem is a generalization of Theorem 3.1 in reference [2]: the claims are the same, but the hypotheses on the function f are weakened. First, (3.16) is an upper bound condition only on the mean square value of f^2, rather than on its supremum (as in [2]). Second, this upper bound may depend on t in a more general way than in that reference. The hypotheses (3.17), (3.18), (3.19) considered here are fulfilled by the ones considered there with $g(t)\equiv\text{const}$ and $\chi=\kappa=1$. The former, but not the latter, are satisfied e.g. by the following family of examples.

Examples Let $f=b(t)\sin\varphi$, with a function $b(t)$ such that the integral $\int\limits_0^t b^2(\tau)\,d\tau$ grows as some power t^χ, where $\chi\le1$, and in the case $\chi=1$ is smaller than pt for sufficiently large t; then we can set $\hat{g}(t,\eta)\equiv b^2(t)$. For instance we could take b^2 a continuous function that vanishes everywhere except in intervals centered, say, at equally spaced points, where it takes maxima increasing with some power law $\sim t^\beta$, but keeps the integral bounded, e.g.

$$b^2(t)=b_0^2\begin{cases}4n^{\alpha+\beta}(t-n+\dfrac{1}{2n^\alpha}) & \text{if}\quad t\in\left[n-\dfrac{1}{2n^\alpha},\,n\right],\\[2ex] 4n^\beta-4n^{\alpha+\beta}(t-n) & \text{if}\quad t\in\left]n,\,n+\dfrac{1}{2n^\alpha}\right],\\[2ex] 0 & \text{otherwise},\end{cases}\tag{3.52}$$

with $b_0^2<p$, $\alpha\ge1$, $\beta\in\,]\,\alpha-1,\,\alpha]$ and $n\in N$. (The case $\alpha=\beta=1$ has already been considered in [5]).

The graph of $(b(t)/b_0)^2$ consists of a sequence of isosceles triangles enumerated by n, having bases of length $1/n^\alpha$ and upper vertices with coordinates $(x,y)=(n,2n^\beta)$ (see the Figure 3.1). Their areas are $A_n=1/n^\gamma$, where $\gamma:=\alpha-\beta\in[0,1[$.

If $0\le t-t_0<2$ then we immediately find

$$\int\limits_{t_0}^t g(\tau)\,d\tau\le b_0^2\,2.\tag{3.53}$$

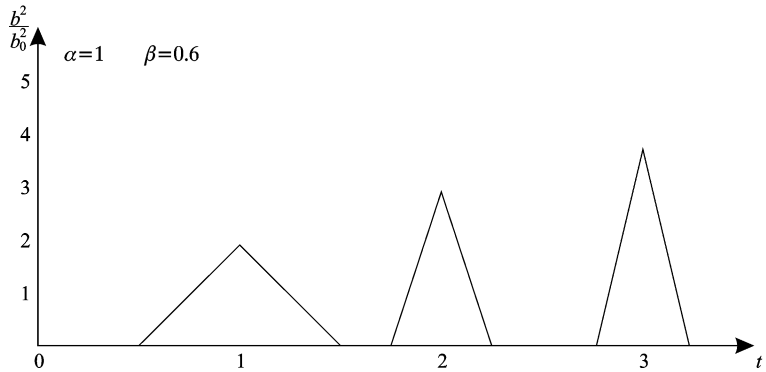

Figure 3.1

If on the contrary $t - t_0 \geq 2$, then there exist integers m, n with $0 \leq m \leq n-2$ and $t > t_0 \geq 0$ such that $t \in]n-1/2,\, n+1/2]$ and $t_0 \in]m-1/2,\, m+1/2]$. Then we find

$$\int\limits_{m+1/2}^{n-1/2} g(\tau)\, d\tau \leq \int\limits_{t_0}^{t} g(\tau)\, d\tau \leq \int\limits_{m-1/2}^{n+1/2} g(\tau)\, d\tau,$$

namely

$$\sum_{k=m+1}^{n-1} \frac{b_0^2}{k^\gamma} = b_0^2 \sum_{k=m+1}^{n-1} A_k \leq \int\limits_{t_0}^{t} g(\tau)\, d\tau \leq b_0^2 \sum_{\substack{k=m \\ k\geq 1}}^{n} A_k = \sum_{\substack{k=m \\ k\geq 1}}^{n} \frac{b_0^2}{k^\gamma}. \tag{3.54}$$

Consider the function $e(y) := y^{1-\gamma}$, $\gamma \in [0,1[$. Applying Lagrange's theorem we find that for any $h \in N$ there exists a $\xi_h \in]h,\, h+1[$ such that

$$(h+1)^{1-\gamma} - h^{1-\gamma} = (1-\gamma)\frac{1}{\xi_h^\gamma},$$

whence, taking $h = k$ and $h = k-1$ respectively,

$$(k+1)^{1-\gamma} - k^{1-\gamma} < (1-\gamma)\frac{1}{k^\gamma},$$
$$k^{1-\gamma} - (k-1)^{1-\gamma} > (1-\gamma)\frac{1}{k^\gamma};$$

therefore

$$\frac{1}{1-\gamma}\,[(k+1)^{1-\gamma} - k^{1-\gamma}] < \frac{1}{k^\gamma} < \frac{1}{1-\gamma}\,[k^{1-\gamma} - (k-1)^{1-\gamma}]. \tag{3.55}$$

From (3.54), (3.55) we find

$$\frac{b_0^2[n^{1-\gamma} - (m+1)^{1-\gamma}]}{1-\gamma} < \int\limits_{t_0}^{t} g(\tau)\, d\tau < \frac{b_0^2[n^{1-\gamma} - (m-1)^{1-\gamma}(1-\delta_0^m)]}{1-\gamma}. \tag{3.56}$$

where δ_0^m denotes a Kronecker δ. Hence,

$$\int_{t_0}^{t} g(\tau)\,d\tau = \frac{b_0^2}{1-\gamma}\,[n^{1-\gamma}-(m+1)^{1-\gamma}] + L_{m,n}(t) \tag{3.57}$$

where the remainder $L_{m,n}(t)$ is bounded by the difference d_m on the right-hand side and left-hand side of (3.56),

$$0 < L_{m,n}(t) < d_m := \frac{b_0^2}{1-\gamma}\,[(m+1)^{1-\gamma}-(m-1)^{1-\gamma}(1-\delta_0^m)].$$

The expression in square bracket equals 1 for $m=0$ and $2^{1-\gamma}$ for $m=1$. It is immediate to check that the function $\tilde{e}(y) := (y+1)^{1-\gamma}-(y-1)^{1-\gamma}$ is decreasing for $y \geq 1$ and therefore takes its maximum in $y=1$. We therefore derive the bound

$$0 < L_{m,n}(t) < d_m \leq \frac{b_0^2\,\tilde{e}(1)}{1-\gamma} = \frac{b_0^2\,2^{1-\gamma}}{1-\gamma}. \tag{3.58}$$

Moreover, since $t > n-1$, $t_0 < m+1$ and g is nonnegative, from (3.57) we find

$$\int_{t_0}^{t} g(\tau)\,d\tau < \frac{b_0^2}{1-\gamma}\,[(t+1)^{1-\gamma}-t_0^{1-\gamma}] + L_{m,n}(t).$$

If $t_0 \geq 1$, applying again Lagrange's theorem to the function $e(t) = t^{1-\gamma}$ we find

$$\frac{b_0^2}{1-\gamma}[(t+1)^{1-\gamma}-t_0^{1-\gamma}] = b_0^2\frac{t-t_0+1}{\bar{t}^{\gamma}} < b_0^2(t-t_0+1)$$

with a suitable $\bar{t} \in\,]t_0,\, t+1[$, and therefore

$$\int_{t_0}^{t} g(\tau)\,d\tau - b_0^2(t-t_0) < b_0^2\left(1+\frac{2^{1-\gamma}}{1-\gamma}\right). \tag{3.59}$$

If $0 \leq t_0 < 1$,

$$\int_{t_0}^{t} g(\tau)\,d\tau - b_0^2(t-t_0) \leq \int_{0}^{1} g(\tau)\,d\tau - b_0^2(1-t_0) + \int_{1}^{t} g(\tau)\,d\tau - b_0^2(t-1) < b_0^2\left(2+\frac{2^{1-\gamma}}{1-\gamma}\right) =: \sigma,$$

where we have used (3.59) with $t_0 = 1$ and $\int_0^1 g(\tau)d\tau \leq b_0^2$, showing (together with (3.59) itself and (3.53)) that g fulfills condition (3.17) in any case.

On the other hand, choosing $t_0 = 0$ (and therefore $m = 0$) in (3.57), dividing by $1+t^{1-\gamma}$ and subtracting $b_0^2/(1-\gamma)$ we find

$$\frac{\int_0^t g(\tau)\,d\tau}{1+t^{1-\gamma}} - \frac{b_0^2}{1-\gamma} = \frac{b_0^2}{1-\gamma}\left[\frac{n^{1-\gamma}-(1+t^{1-\gamma})-1}{1+t^{1-\gamma}}\right] + \frac{L_{0,n}(t)}{1+t^{1-\gamma}}.$$

But it is $n-1<t<n+1$, what implies

$$1-2^{1-\gamma} \le n^{1-\gamma}-(n+1)^{1-\gamma} < n^{1-\gamma}-t^{1-\gamma} < (t+1)^{1-\gamma}-t^{1-\gamma} < 1$$

(in fact the function $\hat{e}(y) := (y+1)^{1-\gamma}-y^{1-\gamma}$ is decreasing and therefore has maximum at the lower extremum of the interval in which we define it); hence, using also (3.58), we find

$$-\frac{b_0^2}{1-\gamma}\left[\frac{2^{1-\gamma}+1}{1+t^{1-\gamma}}\right] < \frac{\int\limits_0^t g(\tau)d\tau}{1+t^{1-\gamma}} - \frac{b_0^2}{1-\gamma} < \frac{b_0^2}{1-\gamma}\left[\frac{2^{1-\gamma}-1}{1+t^{1-\gamma}}\right] < \frac{b_0^2}{1-\gamma}\left[\frac{2^{1-\gamma}+1}{1+t^{1-\gamma}}\right].$$

We have proved these inequalities under the current assumption $t \ge 2$, showing that in this domain also condition (3.18), with $q = b_0^2/(1-\gamma)$, $\chi = \kappa = 1-\gamma$ and $M = b_0^2(2^{1-\gamma}+1)/(1-\gamma)$, is satisfied. For $0 \le t \le 2$ the left-hand side of (3.18) is certainly bounded by $b_0^2\, 3/[2(1-\gamma)]$, therefore it is sufficient to choose e.g. $M = b_0^2\, 9/[2(1-\gamma)]$ to fulfill (3.18) for any $t \ge 0$.

4 Exponential-Asymptotic Stability for Special f's via a Family of Liapunov Functionals

In this section we specialize the function f of (3.1) as $f = F(u) - a(x,t,u,u_x,u_t,u_{xx})u_t$, where $F \in C(R)$ and $a \in C(\,]0,1[\, \times J \times R^4)$, and examine the particular problem

$$\begin{gathered} Lu = F(u) - a(x,t,u,u_x,u_t,u_{xx})u_t, \quad x \in]0,1[, \quad t > t_0, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad t > t_0, \end{gathered} \tag{4.1}$$

with initial and consistency conditions (3.2)–(3.3). We shall use the one-parameter family of modified Liapunov functionals

$$\begin{aligned} W_\gamma(\varphi,\psi) = \frac{1}{2}\int\limits_0^1 &\left\{(\varepsilon\varphi_{xx}-\psi)^2 + \gamma\psi^2 + (1+\gamma)\varphi_x^2\right\} dx \\ &- (1+\gamma)\int\limits_0^1 \left(\int\limits_0^{\varphi(x)} F(z)dz\right) dx \end{aligned} \tag{4.2}$$

where $\gamma > 1/2$ is for the moment an unspecified parameter.

Theorem 2 *Under the following assumptions*

(1) $F(u) \in C^1(R)$, $F(0)=0$, *and moreover there exists a positive constant* K *such that*

$$F_u \le K < 3\pi^2/4; \tag{4.3}$$

(2) *the function* a *satisfies*

$$\nu := \varepsilon\pi^2 + \inf a > 0; \tag{4.4}$$

(3) *there exist* $\tau \in [0,2[$ *and constants* $A > 0$, $A' \ge 0$ *such that*

$$a(x,t,\varphi,\varphi_x,\varphi_{xx},\psi) \le A[d(\varphi,\psi)]^\tau + A', \tag{4.5}$$

the zero solution of the problem (4.1) is exponential-asymptotically stable in the large.

As anticipated in the introduction, this should be compared with Theorem 3.3. in the main reference, [2]: by replacing the requirement that $\sup a < \infty$ and adding the assumption (4.5) we are still able to prove the exponential-asymptotic stability in the large of the zero solution. The trick is to associate to each neighbourhood of the origin with radius σ (the 'error') a Liapunov functional (4.2) with parameter γ adapted to σ, instead of fixing γ once and for all.

Proof We start by improving or recalling some inequalities proved in [2]. From (4.3) we find

$$\int_0^{\varphi} F(z)\,dz = \int_0^{\varphi} dz \int_0^{z} F_s(s)\,ds \leq K \int_0^{\varphi} dz \int_0^{z} ds = K\varphi^2/2. \tag{4.6}$$

Employing this inequality and the estimate (3.9) we find

$$\begin{aligned} W_\gamma(\varphi,\psi) = \frac{1}{2}\int_0^1 \Big\{ &(\varepsilon\varphi_{xx} - 2\psi)^2/4 + (\varepsilon\varphi_{xx} - \psi)^2/2 + (\gamma - 1/2)\psi^2 \\ &+ (1+\gamma)\varphi_x^2 + \varepsilon^2\varphi_{xx}^2/4 - 2(1+\gamma)\int_0^{\varphi} F(z)\,dz \Big\}\,dx. \end{aligned} \tag{4.7}$$

It easy to see that

$$\begin{aligned} W_\gamma(\varphi,\psi) &\geq \frac{1}{2}\int_0^1 \left[\left(\gamma - \frac{1}{2}\right)\psi^2 + (1+\gamma)\varphi_x^2 + \frac{\varepsilon^2}{4}\varphi_{xx}^2 - 2(1+\gamma)\int_0^{\varphi} F(z)\,dz \right] dx \\ &\geq \frac{1}{2}\int_0^1 \left[\left(\gamma - \frac{1}{2}\right)\psi^2 + (1+\gamma)\pi^2\varphi^2 + \frac{\varepsilon^2}{4}\omega_3(\varphi_{xx}^2 + \varphi_x^2) - (1+\gamma)K\varphi^2 \right] dx \\ &\geq k_1^2 d^2(\varphi,\psi), \end{aligned} \tag{4.8}$$

where we have used again (4.3) and we have introduced the constant k_1^2

$$k_1^2 = \min\{\varepsilon^2\omega_3/8,\, (2\gamma-1)/4\}, \quad \gamma > 1/2. \tag{4.9}$$

Another inequality of [2] reads

$$W_\gamma(\varphi,\psi) \leq c_2^2\,[1 + m(d(\varphi,\psi))]\, d^2(\varphi,\psi), \tag{4.10}$$

where

$$m(|\varphi|) = \max\{|F_\zeta(\zeta)|:\ |\zeta| \leq |\varphi|\}. \tag{4.11}$$

The map $B(d) := [1 + m(d)]^{1/2} d$ is increasing and continuous, therefore invertible. Finally,

$$
\begin{aligned}
\frac{dW_\gamma(u,u_t)}{dt} &= -\int_0^1 \{\varepsilon u_{xx}^2 + \varepsilon\gamma u_{xt}^2 + a(1+\gamma)u_t^2 + \varepsilon F(u)u_{xx} - \varepsilon a u_{xx}u_t\}\,dx \\
= -\int_0^1 &\left\{\frac{3}{4}\varepsilon u_{xx}^2 + \varepsilon\left[\frac{c}{2}u_{xx} - \frac{a}{c}u_t\right]^2 + \varepsilon\gamma u_{tx}^2 + a\left[1+\gamma-\varepsilon a\right]u_t^2 - \varepsilon F_u u_x^2\right\}dx \\
&\le -\int_0^1 \{3\varepsilon(1-\lambda)u_{xx}^2/4 + \varepsilon(3\lambda\pi^2/4 - K)u_x^2 \\
&\quad + [(\varepsilon\pi^2 + a)\gamma + a(1-\varepsilon a)]u_t^2\}dx,
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
\frac{dW_\gamma(u,u_t)}{dt} \le &-\int_0^1 \{3\varepsilon(1-\lambda)\omega_1(u_{xx}^2 + u^2)/4 + \varepsilon(3\lambda\pi^2/4 - K)u_x^2 \\
&+ [(\varepsilon\pi^2 + a)\gamma + a(1-\varepsilon a)]u_t^2\}\,dx,
\end{aligned}
\tag{4.13}
$$

where $\lambda \in\,]0,1[$ is a constant chosen in such a way that $3\lambda\pi^2/4 - K > 0$, and we have used (3.9), (4.3).

Now we are going to show that for any "error" $\sigma > 0$ there exists a $\delta \in\,]0,\sigma[$ such that $d(t_0) \equiv d(u_0,u_1) < \delta$ implies

$$
d(t) \equiv d(u(x,t), u_t(x,t)) < \sigma \quad \forall\, t \ge t_0. \tag{4.14}
$$

To this end we associate to the neighbourhood with radius σ of the zero solution the Liapunov functional (4.2) choosing the parameter γ and δ as the following functions of σ:

$$
\gamma(\sigma) = (A\sigma^\tau + A')\varepsilon + M, \qquad M := \frac{1+\varepsilon\pi^2+\varepsilon^3\pi^4}{\nu} + \frac{1}{\varepsilon\pi^2} + \frac{1}{2}, \tag{4.15}
$$

$$
\delta(\sigma) = B^{-1}\left(\frac{\sigma k_1(\gamma(\sigma))}{c_2(\gamma(\sigma))}\right); \tag{4.16}
$$

we shall call the corresponding Liapunov functional W_σ. *Per absurdum*, assume that there exist a $t_1 > t_0$ such that (4.14) is fulfilled for any $t \in [t_0, t_1[$, whereas

$$
d(t_1) = \sigma. \tag{4.17}
$$

Consider the term in the square bracket on the right-hand side of (4.13). From (4.15), (4.4), (4.5) considering separately the cases $a > 0$, $-\varepsilon\pi^2 < a \le 0$, we find

$$
-\left[(\varepsilon\pi^2 + a)\gamma + a(1-\varepsilon a)\right] \le -1, \tag{4.18}
$$

whence

$$
\frac{dW_\sigma(u(t),u_t(t))}{dt} \le -k_3^2 d^2(u(t),u_t(t)) < 0, \tag{4.19}
$$

where

$$k_3^2 = \min\{3\varepsilon(1-\lambda)\omega_1/4,\ \varepsilon(3\lambda\pi^2/4 - K),\ 1\}. \tag{4.20}$$

From (4.8), (4.19), (4.10), (4.16), it follows

$$\begin{aligned} k_1^2 d^2(t_1) &\le W_\sigma(u(t_1), u_t(t_1)) < W_\sigma(u(t_0), u_t(t_0)) \le c_2^2\,[1 + m(d(t_0))]\, d^2(t_0) \\ &< c_2^2\,[1 + m(\delta)]\,\delta^2 = c_2^2\,[B(\delta)]^2 = c_2^2\left[B\left(B^{-1}\left(\frac{\sigma k_1}{c_2}\right)\right)\right]^2 = k_1^2\sigma^2, \end{aligned}$$

against (4.17).

Having proved (4.14), it follows $m(d(t)) < m(\sigma)$, which replaced in (4.10) gives

$$W_\sigma \le c_2^2(\sigma)\,[1 + m(\sigma)]\, d^2(t);$$

together with (4.19) this in turn implies

$$\frac{dW_\sigma(u(t), u_t(t))}{dt} \le -C(\sigma) W_\sigma(u(t), u_t(t)),$$

with $C(\sigma) := k_3^2/[c_2^2(\sigma)(1 + m(\sigma))]$. Using the comparison principle we find that $d(t_0) \equiv d(u_0, u_1) < \delta$ implies

$$d(u(t), u_t(t)) \le D(\sigma) e^{-\frac{C(\sigma)}{2}(t - t_0)} d(u_0, u_1), \tag{4.21}$$

with $D(\sigma) := \dfrac{c_2}{k_1}\sqrt{1 + m(\delta(\sigma))}$.

Last, we show that under the present assumptions the function (4.16) can be inverted. It is evident from (4.9) that $k_1(\sigma)$ is non-decreasing, from (3.7) and (4.5) that $\sigma/c_2(\gamma(\sigma))$ is strictly increasing, therefore that $\sigma k_1(\sigma)/c_2(\gamma(\sigma))$ is strictly increasing too, hence invertible. Since B^{-1} is invertible, $\delta(\sigma)$ is invertible and its range is J.

Thus we can express $D(\sigma)$, $C(\sigma)$ as functions of δ, proving the exponential asymptotic stability of the zero solution.

Remark 4 The theorem holds also if we replace the right-hand side of (4.5) with $A(d)$, where $A\colon\ [0, +\infty[\ \to R^+$ is any nondecreasing function such that $A(\sigma)/\sigma^2 \xrightarrow{\sigma\to+\infty} 0$.

Remark 5 If (4.5) holds with $\tau = 2$ the function $\dfrac{\sigma}{c_2(\gamma(\sigma))}$ is still increasing but its range is $[0,\, 2/\varepsilon A]$, implying that the function $\dfrac{\sigma k_1(\gamma(\sigma))}{c_2(\gamma(\sigma))}$ is still increasing but its range is $[0,\, \sqrt{\omega_3}/\sqrt{2}A]$. Therefore the condition (3.5) of Definition 3.4 is fulfilled only for $\alpha \in\ ]0,\, B^{-1}(\sqrt{\omega_3}/\sqrt{2}A)[$, and the attraction region includes the set $d(u_0, u_1) < B^{-1}(\sqrt{\omega_3}/\sqrt{2}A)$.

We now give a variant of the preceding theorem, based on a hypothesis slightly different from (4.5). Beside the distance (3.4), we need also a "weaker" distance $d_1(u, u_t)$ between the zero and a nonzero solution $u(x,t)$ of the problem (3.1)–(3.2): for any $(\varphi, \psi) \in C_0^2([0,1]) \times C_0([0,1])$ we define

$$d_1^2(\varphi, \psi) = \int_0^1 (\varphi^2 + \varphi_x^2 + \psi^2)\, dx. \tag{4.22}$$

Clearly,

$$d_1(\varphi,\psi) \le d(\varphi,\psi). \tag{4.23}$$

The "Hamiltonian" Liapunov functional $v(u,u_t)$, with

$$v(\varphi,\psi) := \frac{1}{2}\int_0^1 \left\{\psi^2 + \varphi_x^2 - 2\left(\int_0^{\varphi(x)} F(z)\,dz\right)\right\} dx, \tag{4.24}$$

will play w.r.t. the distance d_1 a role similar to the one played by the Liapunov functionals V or W_γ w.r.t. the distance d.

Theorem 3 *Under the following assumptions*

(1) $F(u) \in C^1(R)$, $F(0) = 0$, *and there exists a positive constant* K *such that*

$$F_u \le K < 3\pi^2/4; \tag{4.25}$$

(2) *the function* a *satisfies*

$$\inf a > -\varepsilon\pi^2; \tag{4.26}$$

(3) *there exists a nondecreasing map* $A\colon J \to J$ *such that*

$$|a(x,t,\varphi,\varphi_x,\varphi_{xx},\psi)| \le A\left[d_1(\varphi,\psi)\right], \tag{4.27}$$

the zero solution of the problem (4.1) is exponential-asymptotically stable in the large.

Proof Some steps of the proof are exactly as in the previous theorem. Employing inequality (4.6) and the estimate (3.9) we find

$$v \ge \frac{1}{2}\int_0^1 \left\{\left(\frac{1}{8}\,u_x^2 + \frac{7}{8}\,u^2\pi^2\right) + u_t^2 - \frac{3}{4}\,\pi^2u^2\right\}dx \ge \frac{1}{16}\,d_1^2. \tag{4.28}$$

Setting $v(t) \equiv v(u,u_t)$, integrating by parts and using (4.1), (4.26), (3.9) we also find

$$\begin{aligned} \frac{dv}{dt} &= \int_0^1 \{u_t[-u_{xx} + u_{tt} - F(u)]\}\,dx = -\int_0^1 \left\{\varepsilon u_{xt}^2 + au_t^2\right\} dx \\ &\le -\int_0^1 (\varepsilon\pi^2 + a)u_t^2\,dx < 0 \end{aligned} \tag{4.29}$$

Now we are going to prove the uniform boundedness of the solutions of the problem (4.1). To this end first note that from the definition (4.11) it follows

$$\left|\int_0^{\varphi} F(z)dz\right| \le m(|\varphi|)\frac{\varphi^2}{2};$$

employing this inequality and the one $\varphi^2 \le d_1^2(\varphi,\psi)$ we find

$$v \le \frac{1}{2}\left[1 + m\left(d_1(u,u_t)\right)\right] d_1^2(u,u_t). \tag{4.30}$$

From (4.29) we derive the inequality $v(t) < v(t_0)$ for any $t > t_0$, whence

$$\frac{1}{16}\, d_1^2(t) \le v(t) < v(t_0) \le \frac{1}{2}\left[1 + m\left(d_1(t_0)\right)\right] d_1^2(t_0).$$

Therefore, for any $t > t_0$

$$d(t_0) \le \alpha \quad \Longrightarrow \quad d_1(t_0) \le \alpha \quad \Longrightarrow \quad d_1(t) < \beta_1(\alpha) := 2\sqrt{2}\left[1 + m(\alpha)\right]^{1/2} \alpha,$$

so that, in view of the assumption (4.27),

$$d(t_0) \le \alpha \quad \Longrightarrow \quad |a(x,t,u,u_x,u_t,u_{xx})| \le A\left[\beta_1(\alpha)\right] \equiv A(\alpha). \tag{4.31}$$

Now we associate to any $\alpha > 0$ the Liapunov functional (4.2) with the parameter γ chosen as the following function of α:

$$\gamma(\alpha) = A(\alpha)\varepsilon + M, \qquad M := \frac{1 + \varepsilon\pi^2 + \varepsilon^3\pi^4}{\nu} + \frac{1}{\varepsilon\pi^2} + \frac{1}{2}; \tag{4.32}$$

we shall call the corresponding Liapunov functional W_α. Consider the term in the square bracket on the right-hand side of (4.13). From (4.31), (4.32), we find again (4.18), whence

$$\frac{dW_\alpha(u(t),u_t(t))}{dt} \le -k_3^2 d^2(u(t),u_t(t)) < 0, \tag{4.33}$$

with the same k_3^2 of (4.20). From (4.8), (4.33), (4.10), it follows for any $t > t_0$

$$\begin{aligned} k_1^2 d^2(t) &\le W_\alpha(u(t),u_t(t)) < W_\alpha(u(t_0),u_t(t_0)) \le c_2^2\left[1 + m(d(t_0))\right] d^2(t_0) \\ &< c_2^2(\gamma(\alpha))\left[1 + m(\alpha)\right]\alpha^2 = c_2^2(\gamma(\alpha))B^2(\alpha), \end{aligned}$$

proving the uniform boundedness of u:

$$d(u(t),u_t(t)) < \frac{c_2(\gamma(\alpha))}{k_1(\gamma(\alpha))}\, B(\alpha) \equiv \beta(\alpha). \tag{4.34}$$

Having proved this, it follows $m(d(t)) < m(\beta(\alpha))$, which replaced in (4.10) gives

$$W_\alpha \le c_2^2(\gamma(\alpha))[1 + m(\beta(\alpha))]d^2(t);$$

together with (4.33) this in turn implies

$$\frac{dW_\alpha(u(t),u_t(t))}{dt} \le -C(\alpha)W_\alpha(u(t),u_t(t)),$$

with $C(\alpha) := k_3^2(\gamma(\alpha))/\{c_2^2(\gamma(\alpha))[1+m(\beta(\alpha))]\}$. Using the comparison principle we find that $d(t_0) \equiv d(u_0,u_1) \le \alpha$ implies

$$d(u(t),u_t(t)) \le D(\alpha)e^{-C(\alpha)(t-t_0)}d(u_0,u_1), \tag{4.35}$$

with $D(\alpha) := \dfrac{c_2(\gamma(\alpha))}{k_1(\gamma(\alpha))}\sqrt{1 + m(\beta(\alpha))}$, namely the exponential-asymptotical stability.

5 Uniform Asymptotic Stability in the Large for a Class of Non-Analytic f's

Here we give a generalization of Theorem 2 in [5]. As in the preceding sections, using the trick of the one-parameter family of Liapunov functionals we are able to replace the boundedness assumption for the function a by a weaker one.

Theorem 4 *Under the following assumptions*

$$F(\varphi) \in C(R) \ \textit{ such that } \ F(0) = 0, \tag{5.1}$$

$$\textit{there exist } \ \tau \in [0,1[\ \textit{ and } \ D > 0 \ \textit{ such that, for any } \varphi, \psi \tag{5.2}$$

$$0 \le -\int_0^1 \left(\int_0^{\varphi(x)} F(z)dz \right) dx \le D d^{\tau+1}(\varphi, \psi),$$

$$\int_0^1 F(\varphi(x))\varphi_{xx}(x)\, dx \ge 0 \ \ \textit{for any} \ \ \varphi \in C_0^2([0,1]), \tag{5.3}$$

$$\textit{the function } a \textit{ satisfies } \ \inf a > -\varepsilon\pi^2, \tag{5.4}$$

$$\textit{there exists a nondecreasing map } \ A\colon \ [0,\infty[\to R^+ \ \textit{ such that} \tag{5.5}$$

$$|a(x,t,\varphi,\varphi_x,\varphi_{xx})| \le A\left(d(\varphi,\psi)\right),$$

the zero solution of the problem (4.1) is uniformly asymptotically stable in the large.

Proof From (4.7), (5.2)

$$\begin{aligned} W_\gamma(\varphi,\psi) &\ge \frac{1}{2}\int_0^1 \{(\gamma - 1/2)\psi^2 + (1+\gamma)\varphi_x^2 + \varepsilon^2\varphi_{xx}^2/4\}\, dx \\ &\ge \frac{1}{2}\int_0^1 \{(\gamma - 1/2)\psi^2 + (1+\gamma)\omega_3(\varphi^2 + \varphi_x^2) + \varepsilon^2\varphi_{xx}^2/4\}\, dx \ge {k_1'}^2 d^2(\varphi,\psi), \end{aligned} \tag{5.6}$$

where

$$ {k_1'}^2 := \frac{1}{2}\min\left\{\gamma - \frac{1}{2},\, \frac{\varepsilon^2}{4},\, (1+\gamma)\omega_3\right\}, \quad \gamma > \frac{1}{2}. \tag{5.7}$$

Moreover, taking into account (4.2), assumption (5.2), noting that $(\varepsilon\varphi_{xx} - \psi)^2 \le \varepsilon^2\varphi_{xx}^2 + \psi^2 + \varepsilon(\varphi_{xx}^2 + \psi^2)$, and considering (3.7) it follows

$$W_\gamma(\varphi,\psi) \le G_\gamma(d(\varphi,\psi)), \tag{5.8}$$

where

$$G_\gamma(d) := c_2^2(\gamma)d^2 + D(\gamma+1)d^{\tau+1}. \tag{5.9}$$

For any choice of $\gamma > \dfrac{1}{2}$ the map $G_\gamma(d)$ is increasing and continuous in d, therefore invertible. Finally, with the help of (3.9) we obtain from (4.12)

$$\begin{aligned} \frac{dW_\gamma(u,u_t)}{dt} &\le -\int_0^1 \{(3/4)\varepsilon u_{xx}^2 + [\varepsilon\gamma + a(1+\gamma-\varepsilon a)]u_t^2\}\, dx \\ &\le -\int_0^1 \{\varepsilon\omega_2(u_{xx}^2 + u_x^2 + u^2)/4 + [(\varepsilon + a)\gamma + a(1-\varepsilon a)]u_t^2\}\, dx. \end{aligned} \tag{5.10}$$

Now we are going to show that for any "error" $\sigma > 0$ there exists a $\delta \in \,]0, \sigma[$ such that $d(t_0) \equiv d(u_0, u_1) < \delta$ implies

$$d(t) \equiv d\left(u(x,t), u_t(x,t)\right) < \sigma \quad \forall\, t \geq t_0. \tag{5.11}$$

To this end we choose the parameter γ in the Liapunov functional (4.2) as in (4.32) and δ as the following function of the error σ:

$$\delta(\sigma) = G^{-1}_{\gamma(\sigma)}\left(\sigma^2 {k'_1}^2\left(\gamma(\sigma)\right)\right); \tag{5.12}$$

we shall indicate the corresponding Liapunov functional $W_{\gamma(\sigma)}$ simply by W_σ. *Per absurdum*, assume that there exist a $t_1 > t_0$ such that (4.14) is fulfilled for any $t \in [t_0, t_1[$, whereas (4.17) holds for $t = t_1$. Consider the term in the square bracket on the right-hand side of (5.10). From (4.32), (4.4), (5.5) we get again (4.18), whence

$$\frac{dW_\sigma(u(t), u_t(t))}{dt} \leq -{k'_3}^2 d^2(u(t), u_t(t)) < 0, \tag{5.13}$$

where now ${k'_3}^2 := \min\{\varepsilon\omega_2/4,\, 1\}$. From (5.6), (5.8), (5.13),(5.12), it follows

$$\begin{aligned} {k'_1}^2 d^2(t_1) &\leq W_\sigma(u(t_1), u_t(t_1)) < W_\sigma(u(t_0), u_t(t_0)) \\ &\leq G_{\gamma(\sigma)}(d(t_0)) < G_{\gamma(\sigma)}(\delta(\sigma)) = {k'_1}^2\sigma^2, \end{aligned}$$

against (4.17). So we have proved the uniform stability of the zero solution.

Note now that the function $\delta(\sigma)$ is invertible, since it is the composition of two increasing functions. Therefore W_σ can be expressed as a function W_δ of the parameter δ. By (5.13) it is $W_\delta(t) \leq W_\delta(t_0)$ so by (5.6), (5.8) we find that for $d(t_0) \equiv d(u_0, u_1) \leq \delta$

$$d^2(t) \leq \frac{W_\delta(t)}{{k'_1}^2} \leq \frac{W_\delta(t_0)}{{k'_1}^2} \leq \frac{G_{\gamma(d(t_0))}}{{k'_1}^2} \leq \frac{G_{\gamma(\delta)}}{{k'_1}^2\big(\gamma(\delta)\big)} =: \beta^2(\delta),$$

proving the uniform boundedness of u.

Employing an argument of [5] one can now show that for any choice of the initial condition $d(t_0) < \delta$ the functional W_δ decreases to zero (at least) as a negative power of $(t - t_0)$ as $(t - t_0) \to \infty$. From (5.8) we find

$$d^2 \geq \min\left\{\frac{W_\delta}{2c_2^2(\gamma(\sigma))}, \left(\frac{W_\delta}{2D(\gamma+1)}\right)^{\frac{2}{\tau+1}}\right\},$$

which considered in (5.13) gives

$$\frac{dW_\delta(u, u_t)}{dt} \leq -k'_3 \min\left\{\frac{W_\delta}{2c_2^2}, \left(\frac{W_\delta}{2D(\gamma+1)}\right)^{\frac{2}{\tau+1}}\right\} \leq 0. \tag{5.14}$$

If at $t = t_0$

$$\frac{W_\delta}{2c_2^2} \geq \left(\frac{W_\delta}{2D(\gamma+1)}\right)^{\frac{2}{\tau+1}}, \tag{5.15}$$

then setting

$$E(\delta) := \frac{k'_3}{\left[2D\big(\gamma(\delta)+1\big)\right]^{\frac{2}{\tau+1}}} \frac{1-\tau}{1+\tau} > 0$$

one finds

$$d^2(t) \leq \frac{W_\delta(t)}{{k'_1}^2} \leq \frac{1}{{k'_1}^2[W_\delta(t_0) + E(t-t_0)]^{\frac{1+\tau}{1-\tau}}} \leq \frac{1}{{k'_1}^2[E(t-t_0)]^{\frac{1+\tau}{1-\tau}}} \tag{5.16}$$

for $t \geq t_0$. If on the contrary

$$\frac{W_\delta(t_0)}{2c_2^2} < \left(\frac{W_\delta(t_0)}{2D(\gamma+1)}\right)^{\frac{2}{\tau+1}},$$

(5.14) will imply for some time

$$\frac{dW_\delta(u,u_t)}{dt} \leq -k_3' W_\delta$$

and by the comparison principle an (at least) exponential decrease of W_δ. Hence there will exist a $\widetilde{T}(\delta) > 0$ such that

$$\frac{W_\delta(t_0+\tilde{T})}{2c_2^2} = \left(\frac{W_\delta(t_0+\widetilde{T})}{2D(\gamma+1)}\right)^{\frac{2}{\tau+1}},$$

after which (5.14) will take again the form considered in the previous case and thus imply

$$\begin{aligned} d^2(t) \leq \frac{W_\delta(t)}{k_1'} &\leq \frac{1}{{k_1'}^2[W_\delta(t_0+\widetilde{T}) + E(t-t_0-\widetilde{T})]^{\frac{1+\tau}{1-\tau}}} \\ &\leq \frac{1}{{k_1'}^2[E(t-t_0-\widetilde{T})]^{\frac{1+\tau}{1-\tau}}} \end{aligned} \tag{5.17}$$

for $t \geq t_0 + \widetilde{T}$. Formula (5.17) will be valid also if δ is so small that inequality (5.15) occurs, provided we correspondingly define $\widetilde{T} := 0$, so that it reduces to (5.16). Formula (5.17) evidently implies the quasi-uniform asymptotic stability in the large of the zero solution.

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