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Robust Observers for a Class of Uncertain Nonlinear Stochastic Systems with State Delays*

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Abstract: This paper investigates the problem of robust observer design for a class of nonlinear stochastic systems with state delays and time-varying norm-bounded parameter uncertainties. The nonlinearities are assumed to satisfy the global Lipschitz conditions and appear in both the state and measured output equations. The purpose is to design a nonlinear observer ensuring mean square asymptotic stability for the error system, irrespective of the uncertainties and the time delays. A sufficient condition for the solvability of this problem is derived in terms of a linear matrix inequality and the explicit formula of a desired robust observer is also given. An example is given to illustrate the proposed approach.

Keywords: *Linear matrix inequality; nonlinear systems; robust observers; stochastic systems; time-delay systems; uncertain systems.*

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1 Introduction

Observer design for linear as well as nonlinear systems has been an active research area in the past years. Various approaches, such as transfer-function, geometric, algebraic, singular value decomposition and so on, have been successfully proposed and many results on the observer design have been reported in the literature. For some representative work on this general topic, to name a few, we refer readers to [6, 7, 9, 10, 12] and the

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references therein. However, one of the limitations of classical observer theory is that it cannot guarantee the observer performance when parameter uncertainty appears in a system model. This has motivated the study of robust observer design problem; see, e.g. [1, 3, 15], and the references cited therein. It is worth noting that in the context of stochastic nonlinear systems, the robust observer design problem has been investigated in [20], in which a method for the design of time-invariant observers with guaranteed exponential convergence has been proposed.

On the other hand, it is well known that time delays are inherent in many physical and engineering systems due to transportation lags, and conduction or computation times [4, 8]. It has been shown that time delay is often a main cause of instability of a dynamic system. A number of estimation and control problems related to time-delay systems have been addressed by many researchers [5, 11, 13, 16 − 18]. Recently, a great deal of interest has been devoted to the observer design for time-delay systems. A general form of linear observers for time-delay systems by using the factorization approach was proposed in [19], where a necessary and sufficient condition for the existence of the state functional observers was presented. For discrete-time delay systems, a memoryless state observer was designed by the state augmentation approach in [13]. However, it should be pointed out that disturbances as well as nonlinearities may be present in time-delay systems. Therefore, the observer design problem for nonlinear time-delay stochastic systems is important in both theory and practice and challenging, thus should be considered. To date, to the authors' best knowledge, little work has been done for such stochastic systems.

In this paper, we are concerned with the problem of robust observer design for a class of nonlinear stochastic systems with state delay and parameter uncertainties. The class of systems under consideration is described by a linear stochastic differential delay equation with the addition of known nonlinearities which depend not only on the state but also on the delayed state and are assumed to satisfy the global Lipschitz conditions. The nonlinearities appear in both the state and measured output equations. The parameter uncertainties are real time-varying norm-bounded and appear in both the state and output matrices of the linear part of the system model. The problem under study is the design of a nonlinear observer that guarantees mean square asymptotic stability of the error dynamics for the whole set of admissible systems. A linear matrix inequality (LMI) approach is proposed to solve this problem and a solution is given in terms of an LMI, which defines a convex set of solutions and can be easily computed by the available LMI algorithms ([2]).

Notation Throughout this paper, for symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite); I is the identity matrix with appropriate dimension. The notation M^{T} represents the transpose of the matrix M. While, $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. The notation $\mathcal{E}\{\cdot\}$ stands for the expectation operator; $\|x\|$ stands for the Euclidean norm of the vector x. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2 Problem Formulation

Consider the following class of nonlinear stochastic systems with state-delay and parameter uncertainties:

$$
\begin{aligned}
(\Sigma):\quad dx(t) &= [(A+\Delta A(t))\,x(t) + (A_d+\Delta A_d(t))\,x(t-\tau) + Gg(x(t),x(t-\tau))]\,dt \\
&\quad + [(B+\Delta B(t))\,x(t) + (B_d+\Delta B_d(t))\,x(t-\tau)]\,d\omega(t), && (1)\\
dy(t) &= [(C+\Delta C(t))\,x(t) + (C_d+\Delta C_d(t))\,x(t-\tau) + Hh(x(t),x(t-\tau))]\,dt \\
&\quad + [(D+\Delta D(t))\,x(t) + (D_d+\Delta D_d(t))\,x(t-\tau)]\,d\omega(t), && (2)\\
x(t) &= \phi(t), \quad \forall\, t\in[-\tau,0], && (3)
\end{aligned}
$$

where $x(t)\in R^n$ is the system state, $y(t)\in R^m$ is the measurement; $\omega(t)$ is a zero-mean real scalar Wiener process on $(\Omega,\ \mathcal{F},\ \mathcal{P})$ relative to an increasing family $(\mathcal{F}_t)_{t>0}$ of σ-algebras $\mathcal{F}_t\subset\mathcal{F}$. We assume

$$\mathcal{E}\left\{d\omega(t)\right\}=0, \qquad \mathcal{E}\left\{d\omega(t)^2\right\}=dt. \tag{4}$$

In system (Σ), $\phi(t)$ is a real-valued continuous initial function on $[-\tau,0]$, $\tau>0$ is a known time delay of the system, $g(\cdot,\cdot)\colon R^n\times R^n\to R^{n_g}$ and $h(\cdot,\cdot)\colon R^n\times R^n\to R^{n_h}$ are known nonlinear functions, A, A_d, B, B_d, C, C_d, D, D_d, G and H are known real constant matrices, $\Delta A(t)$, $\Delta A_d(t)$, $\Delta B(t)$, $\Delta B_d(t)$, $\Delta C(t)$, $\Delta C_d(t)$, $\Delta D(t)$ and $\Delta D_d(t)$ are unknown matrices representing time-varying parameter uncertainties, and are assumed to be of the form

$$\begin{bmatrix}\Delta A(t) & \Delta A_d(t) & \Delta B(t) & \Delta B_d(t)\\ \Delta C(t) & \Delta C_d(t) & \Delta D(t) & \Delta D_d(t)\end{bmatrix} = \begin{bmatrix}M_1\\ M_2\end{bmatrix} F(t)\,[\,N_1 \quad N_2 \quad N_3 \quad N_4\,], \tag{5}$$

where M_1, M_2, N_1, N_2, N_3 and N_4 are known real constant matrices and $F(\cdot)\colon R\to R^{k\times l}$ is a unknown real-valued time-varying matrix satisfying

$$F(t)^{\mathrm{T}}F(t)\le I, \quad \forall\, t. \tag{6}$$

It is assumed that all the elements of $F(t)$ are Lebesgue measurable. $\Delta A(t)$, $\Delta A_d(t)$, $\Delta B(t)$, $\Delta B_d(t)$, $\Delta C(t)$, $\Delta C_d(t)$, $\Delta D(t)$ and $\Delta D_d(t)$ are said to be admissible if both (5) and (6) hold.

Remark 1 The parameter uncertainty structure as in (5) and (6) has been widely used in the problems of robust control and robust filtering of uncertain systems, see, for example, [11, 12, 15] and the references therein and many practical systems possess parameter uncertainties which can be either exactly modeled, or overbounded by (6). Observe that the unknown matrix $F(t)$ in (5) can even be allowed to be state-dependent, i.e. $F(t)=F(t,x(t))$, as long as (6) is satisfied.

Throughout the paper, we make the following assumption on the nonlinear functions in system (Σ).

Assumption 1

(I) $g(0,0)=0$;

(II) $\|g(x_1,x_2)-g(y_1,y_2)\|\le\|S_{1g}(x_1-y_1)\|+\|S_{2g}(x_2-y_2)\|$,
$\|h(x_1,x_2)-h(y_1,y_2)\|\le\|S_{1h}(x_1-y_1)\|+\|S_{2h}(x_2-y_2)\|$,
for all x_1, x_2, y_1, $y_2\in R^n$, where S_{1g}, S_{2g}, S_{1h} and S_{2h} are known real constant matrices.

Before formulating the problem to be addressed in this paper, we first introduce the following concept of stochastic stability.

Definition 1 The *equilibrium* $x = 0$ of the system (1) is said to be *mean square stable* if for any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that

$$\mathcal{E}\left|x(t)\right|^2 < \varepsilon, \ t > 0$$

when $\sup\limits_{-\tau \le s \le 0} \mathcal{E}\left|\phi(s)\right|^2 < \delta(\varepsilon)$. If, in addition,

$$\lim_{t\to\infty} x(t) = 0$$

for any initial conditions, then the equilibrium $x = 0$ of the system (1) is said to be mean square asymptotically stable.

Now, the observer design problem we address in this paper can be formulated as follows: given the uncertain nonlinear stochastic time-delay system (Σ), we are concerned with obtaining an estimate $\hat{x}(t)$ of the state $x(t)$ by using the measurement $y(t)$, such that the error dynamics remain mean square asymptotically stable for all admissible uncertainties satisfying (5) and (6) and the nonlinearities satisfying Assumption 1.

3 Main Results

In this section, an LMI approach is proposed to solve the robust observe design problems formulated in the previous section. Before presenting the main results, we give the following lemmas which will be used in the proof of our main results.

Lemma 1 [14] *Let $\mathcal{A}$, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{W}$ and F be real matrices of appropriate dimensions such that $\mathcal{W} > 0$ and $F^{\mathrm{T}}F \le I$. Then we have the following:*

(1) *for scalar $\epsilon > 0$ and vectors $x, y \in R^n$,*

$$2x^{\mathrm{T}}\mathcal{D}F\mathcal{S}y \le \epsilon^{-1}x^{\mathrm{T}}\mathcal{D}\mathcal{D}^{\mathrm{T}}x + \epsilon y^{\mathrm{T}}\mathcal{S}^{\mathrm{T}}\mathcal{S}y;$$

(2) *for any scalar $\epsilon > 0$ such that $\mathcal{W} - \epsilon\mathcal{D}\mathcal{D}^{\mathrm{T}} > 0$,*

$$(\mathcal{A} + \mathcal{D}F\mathcal{S})^{\mathrm{T}}\mathcal{W}^{-1}(\mathcal{A} + \mathcal{D}F\mathcal{S}) \le \mathcal{A}^{\mathrm{T}}(\mathcal{W} - \epsilon\mathcal{D}\mathcal{D}^{\mathrm{T}})^{-1}\mathcal{A} + \epsilon^{-1}\mathcal{S}^{\mathrm{T}}\mathcal{S}.$$

Theorem 1 *Consider the uncertain nonlinear stochastic time-delay system (1) and (3), that is,*

$$\begin{aligned}(\Sigma_1):\quad dx(t) &= [(A + \Delta A(t))\,x(t) + (A_d + \Delta A_d(t))\,x(t-\tau) + Gg(x(t), x(t-\tau))]\,dt \\ &\quad + [(B + \Delta B(t))\,x(t) + (B_d + \Delta B_d(t))\,x(t-\tau)]\,d\omega(t), \qquad (7)\\ x(t) &= \phi(t), \quad \forall\, t \in [-\tau, 0]. \qquad (8)\end{aligned}$$

Then system (Σ_1) is mean square asymptotically stable if there exist matrices $P > 0$, $Q > 0$ and scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$, such that the following LMI holds:

$$\begin{bmatrix} \Omega_1 & PA_d + \epsilon_2 N_1^{\mathrm{T}}N_2 + \epsilon_3 N_3^{\mathrm{T}}N_4 & PG & PM_1 & 0 & B^{\mathrm{T}}P \\ A_d^{\mathrm{T}}P + \epsilon_2 N_2^{\mathrm{T}}N_1 + \epsilon_3 N_4^{\mathrm{T}}N_3 & \Omega_2 & 0 & 0 & 0 & B_d^{\mathrm{T}}P \\ G^{\mathrm{T}}P & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ M_1^{\mathrm{T}}P & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon_3 I & M_1^{\mathrm{T}}P \\ PB & PB_d & 0 & 0 & PM_1 & -P \end{bmatrix} < 0 \qquad (9)$$

where

$$\Omega_1 = A^{\mathrm{T}}P + PA + Q + 2\epsilon_1 S_{1g}^{\mathrm{T}} S_{1g} + \epsilon_2 N_1^{\mathrm{T}} N_1 + \epsilon_3 N_3^{\mathrm{T}} N_3 \tag{10}$$

$$\Omega_2 = 2\epsilon_1 S_{2g}^{\mathrm{T}} S_{2g} + \epsilon_2 N_2^{\mathrm{T}} N_2 + \epsilon_3 N_4^{\mathrm{T}} N_4 - Q. \tag{11}$$

Proof Define the following Lyapunov function candidate:

$$V(x_t, t) = x(t)^{\mathrm{T}} P x(t) + \int_{t-\tau}^{t} x(s)^{\mathrm{T}} Q x(s)\, ds \tag{12}$$

where

$$x_t = x(t+\beta), \quad \beta \in [-\tau, 0] .$$

By Itô's formula, we obtain the stochastic differential as

$$dV(x_t) = LV(x_t,t)dt + 2x(t)^{\mathrm{T}} P[(B+\Delta B(t))\, x(t) + (B_d + \Delta B_d(t))\, x(t-\tau)]\, d\omega(t), \tag{13}$$

where

$$\begin{aligned} LV(x_t,t) = 2x(t)^{\mathrm{T}} P\, &[(A+\Delta A(t))\, x(t) + (A_d + \Delta A_d(t))\, x(t-\tau) + Gg(x(t), x(t-\tau))] \\ &+ [(B+\Delta B(t))\, x(t) + (B_d + \Delta B_d(t))\, x(t-\tau)]^{\mathrm{T}} \\ &\times P\, [(B+\Delta B(t))\, x(t) + (B_d + \Delta B_d(t))\, x(t-\tau)] \\ &+ x(t)^{\mathrm{T}} Q x(t) - x(t-\tau)^{\mathrm{T}} Q x(t-\tau). \end{aligned} \tag{14}$$

From Assumption 1, it follows that

$$\|g(x(t), x(t-\tau))\| \le \|S_{1g} x(t)\| + \|S_{2g} x(t-\tau)\| .$$

Therefore

$$\|g(x(t), x(t-\tau))\|^2 \le 2\, \|S_{1g} x(t)\|^2 + 2\, \|S_{2g} x(t-\tau)\|^2 . \tag{15}$$

Considering this and (5) and using Lemma 1, we have that for any scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$,

$$\begin{aligned} &2x(t)^{\mathrm{T}} PGg(x(t), x(t-\tau)) \\ &\le \epsilon_1^{-1} x(t)^{\mathrm{T}} PGG^{\mathrm{T}} P x(t) + \epsilon_1 g(x(t), x(t-\tau))^{\mathrm{T}} g(x(t), x(t-\tau)) \\ &\le \epsilon_1^{-1} x(t)^{\mathrm{T}} PGG^{\mathrm{T}} P x(t) + 2\epsilon_1 \left[x(t)^{\mathrm{T}} S_{1g}^{\mathrm{T}} S_{1g} x(t) + x(t-\tau)^{\mathrm{T}} S_{2g}^{\mathrm{T}} S_{2g} x(t-\tau) \right] \end{aligned} \tag{16}$$

and

$$\begin{aligned} &2x(t)^{\mathrm{T}} P\, [\Delta A(t) x(t) + \Delta A_d(t) x(t-\tau)] = 2x(t)^{\mathrm{T}} P M_1 F(t)\, [N_1 x(t) + N_2 x(t-\tau)] \\ &\le \epsilon_2^{-1} x(t)^{\mathrm{T}} P M_1 M_1^{\mathrm{T}} P x(t) + \epsilon_2\, [N_1 x(t) + N_2 x(t-\tau)]^{\mathrm{T}}\, [N_1 x(t) + N_2 x(t-\tau)] . \end{aligned} \tag{17}$$

Furthermore, from (9) it is easy to see that

$$\epsilon_3 I - M_1^{\mathrm{T}} P M_1 > 0$$

which implies

$$P - \epsilon_3^{-1} P M_1 M_1^{\mathrm{T}} P > 0.$$

Therefore, by using Lemma 1 again, we have

$$\left[\overline{B} + M_1 F(t) \overline{N}\right]^{\mathrm{T}} P \left[\overline{B} + M_1 F(t) \overline{N}\right] \leq \overline{B}^{\mathrm{T}} P \left(P - \epsilon_3^{-1} P M_1 M_1^{\mathrm{T}} P\right)^{-1} P \overline{B} + \epsilon_3 \overline{N}^{\mathrm{T}} \overline{N} \tag{18}$$

where

$$\overline{B} = [\, B \quad B_d \,], \quad \overline{N} = [\, N_3 \quad N_4 \,].$$

Noting

$$\begin{aligned} & [(B + \Delta B(t)) x(t) + (B_d + \Delta B_d(t)) x(t-\tau)]^{\mathrm{T}} P \\ & \times [(B + \Delta B(t))\, x(t) + (B_d + \Delta B_d(t))\, x(t-\tau)] \\ = {} & [\, x(t)^{\mathrm{T}} \quad x(t-\tau)^{\mathrm{T}} \,] \left[\overline{B} + M_1 F(t) \overline{N}\right]^{\mathrm{T}} P \left[\overline{B} + M_1 F(t) \overline{N}\right] \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \end{aligned}$$

and using (16) – (18) we obtain

$$LV(x_t, t) \leq [\, x(t)^{\mathrm{T}} \quad x(t-\tau)^{\mathrm{T}} \,]\, W \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \tag{19}$$

where

$$\begin{aligned} W = {} & \begin{bmatrix} \Omega_1 + \epsilon_1^{-1} P G G^{\mathrm{T}} P + \epsilon_2^{-1} P M_1 M_1^{\mathrm{T}} P & P A_d + \epsilon_2 N_1^{\mathrm{T}} N_2 + \epsilon_3 N_3^{\mathrm{T}} N_4 \\ A_d^{\mathrm{T}} P + \epsilon_2 N_2^{\mathrm{T}} N_1 + \epsilon_3 N_4^{\mathrm{T}} N_3 & \Omega_2 \end{bmatrix} \\ & + \overline{B}^{\mathrm{T}} P \left(P - \epsilon_3^{-1} P M_1 M_1^{\mathrm{T}} P\right)^{-1} P \overline{B}. \end{aligned}$$

On the other hand, pre and post-multiplying (9) by

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}$$

and using Schur complement, we have $W < 0$, this together with (19) implies

$$LV(x_t, t) < 0$$

for

$$\begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \neq 0,$$

which, by the result in [8], guarantees the mean square asymptotic stability of system (Σ_1).

Now, we are in a position to give a solution to the robust observer design problem formulated in the previous section.

Theorem 2 *Consider the uncertain nonlinear stochastic time-delay system* (Σ) *under Assumption 1. If there exist matrices* $P_1 > 0,\ P_2 > 0,\ Q_1 > 0,\ Q_2 > 0$ *and* Z *and scalars* $\epsilon_1 > 0,\ \epsilon_2 > 0$ *and* $\epsilon_3 > 0$*, such that the following LMI holds:*

$$\begin{bmatrix} \Xi_1 & \Lambda_1 & \Lambda_2 & 0 & \Lambda_3 \\ \Lambda_1^{\mathrm{T}} & \Xi_2 & 0 & 0 & \Pi_1 \\ \Lambda_2^{\mathrm{T}} & 0 & -\Upsilon_1 & 0 & 0 \\ 0 & 0 & 0 & -\Upsilon_2 & \Pi_2 \\ \Lambda_3^{\mathrm{T}} & \Pi_1^{\mathrm{T}} & 0 & \Pi_2^{\mathrm{T}} & -\Upsilon_3 \end{bmatrix} < 0 \tag{20}$$

where

$$\begin{aligned}
\Xi_1 &= \mathrm{diag}\,(\Xi_{11}, \Xi_{12})\,, \\
\Xi_2 &= \mathrm{diag}\,(\Xi_{21}, \Xi_{22})\,, \\
\Xi_{11} &= A^{\mathrm{T}}P_1 + P_1A + Q_1 + 2\epsilon_1 S_{1g}^{\mathrm{T}}S_{1g} + \epsilon_2 N_1^{\mathrm{T}}N_1 + \epsilon_3 N_3^{\mathrm{T}}N_3, \\
\Xi_{12} &= A^{\mathrm{T}}P_2 + P_2A - ZC - C^{\mathrm{T}}Z^{\mathrm{T}} + Q_2 + 2\epsilon_1 S_1^{\mathrm{T}}S_1, \\
\Xi_{21} &= 2\epsilon_1 S_{2g}^{\mathrm{T}}S_{2g} + \epsilon_2 N_2^{\mathrm{T}}N_2 + \epsilon_3 N_4^{\mathrm{T}}N_4 - Q_1, \\
\Xi_{22} &= 2\epsilon_1 S_2^{\mathrm{T}}S_2 - Q_2, \\
\Lambda_1 &= \begin{bmatrix} P_1A_d + \epsilon_2 N_1^{\mathrm{T}}N_2 + \epsilon_3 N_3^{\mathrm{T}}N_4 & 0 \\ 0 & P_2A_d - ZC_d \end{bmatrix}, \\
\Lambda_2 &= \begin{bmatrix} P_1G & 0 & 0 & P_1M_1 \\ 0 & P_2G & -ZH & P_2M_1 - ZM_2 \end{bmatrix}, \\
\Lambda_3 &= \begin{bmatrix} B^{\mathrm{T}}P_1 & B^{\mathrm{T}}P_2 - D^{\mathrm{T}}Z^{\mathrm{T}} \\ 0 & 0 \end{bmatrix}, \\
\Pi_1 &= \begin{bmatrix} B_d^{\mathrm{T}}P_1 & B_d^{\mathrm{T}}P_2 - D_d^{\mathrm{T}}Z^{\mathrm{T}} \\ 0 & 0 \end{bmatrix}, \\
\Pi_2 &= \begin{bmatrix} M_1^{\mathrm{T}}P_1 & M_1^{\mathrm{T}}P_2 - M_2^{\mathrm{T}}Z^{\mathrm{T}} \end{bmatrix}, \\
\Upsilon_1 &= \mathrm{diag}\,(\epsilon_1 I, \epsilon_1 I, \epsilon_1 I, \epsilon_2 I)\,, \\
\Upsilon_2 &= \epsilon_3 I, \\
\Upsilon_3 &= \mathrm{diag}(P_1, P_2).
\end{aligned}$$

Then the robust observer design problem is solvable, where

$$S_1 = \begin{bmatrix} S_{1g} \\ S_{1h} \end{bmatrix}, \qquad S_2 = \begin{bmatrix} S_{2g} \\ S_{2h} \end{bmatrix}. \tag{21}$$

Furthermore, when LMI (20) is satisfied, a suitable nonlinear observer is given as follows:

$$\begin{aligned} d\hat{x}(t) &= [A\hat{x}(t) + A_d\hat{x}(t-\tau) + Gg(\hat{x}(t), \hat{x}(t-\tau))]\,dt \\ &+ L\,[dy(t) - (C\hat{x}(t) + C_d\hat{x}(t-\tau) + Hh(\hat{x}(t), \hat{x}(t-\tau)))\,dt]\,, \end{aligned} \tag{22}$$

where $L = P_2^{-1}Z$*.*

Proof Let

$$\tilde{x}(t) = x(t) - \hat{x}(t)$$

then from (1) – (3) and (22), we obtain

$$\begin{aligned} d\tilde{x}(t) = {} & [(A - LC)\tilde{x}(t) + (A_d - LC_d)\tilde{x}(t-\tau) + (\Delta A(t) - L\Delta C(t))\, x(t) \\ & + (\Delta A_d(t) - L\Delta C_d(t))\, x(t-\tau) + \bar{G}\xi(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau))]\, dt \\ & + [((B - LD) + (\Delta B(t) - L\Delta D(t)))\, x(t) \\ & + ((B_d - LD_d) + (\Delta B_d(t) - L\Delta D_d(t)))\, x(t-\tau)]\, d\omega(t), \end{aligned} \tag{23}$$

where $\bar{G} = [\, G \quad -LH \,]$ and

$$\xi(x(t),\, x(t-\tau),\, \hat{x}(t),\, \hat{x}(t-\tau)) = \begin{bmatrix} g(x(t),\, x(t-\tau)) - g(\hat{x}(t),\, \hat{x}(t-\tau)) \\ h(x(t),\, x(t-\tau)) - h(\hat{x}(t),\, \hat{x}(t-\tau)) \end{bmatrix}.$$

Setting

$$\eta(t)^{\mathrm{T}} = [\, x(t)^{\mathrm{T}} \quad \tilde{x}(t)^{\mathrm{T}} \,]^{\mathrm{T}}$$

and considering (1) – (3) and (18), we have

$$\begin{aligned} d\eta(t) = {} & [(A_c + \Delta A_c(t))\, \eta(t) + (A_{cd} + \Delta A_{cd}(t))\, \eta(t-\tau) \\ & + G_c \xi_c(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau))]dt \\ & + [(B_c + \Delta B_c(t))\, \eta(t) + (B_{cd} + \Delta B_{cd}(t))\, \eta(t-\tau)]\, d\omega(t), \end{aligned} \tag{24}$$

where

$$A_c = \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix}, \qquad \Delta A_c(t) = \begin{bmatrix} \Delta A(t) & 0 \\ \Delta A(t) - L\Delta C(t) & 0 \end{bmatrix},$$
$$A_{cd} = \begin{bmatrix} A_d & 0 \\ 0 & A_d - LC_d \end{bmatrix}, \qquad \Delta A_{cd}(t) = \begin{bmatrix} \Delta A_d(t) & 0 \\ \Delta A_d(t) - L\Delta C_d(t) & 0 \end{bmatrix},$$
$$B_c = \begin{bmatrix} B & 0 \\ B - LD & 0 \end{bmatrix}, \qquad \Delta B_c(t) = \begin{bmatrix} \Delta B(t) & 0 \\ \Delta B(t) - L\Delta D(t) & 0 \end{bmatrix},$$
$$B_{cd} = \begin{bmatrix} B_d & 0 \\ B_d - LD_d & 0 \end{bmatrix}, \qquad \Delta B_{cd}(t) = \begin{bmatrix} \Delta B_d(t) & 0 \\ \Delta B_d(t) - L\Delta D_d(t) & 0 \end{bmatrix},$$
$$G_c = \begin{bmatrix} G & 0 \\ 0 & \bar{G} \end{bmatrix}$$

and

$$\xi_c(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau)) = [\, g(x(t), x(t-\tau))^{\mathrm{T}} \quad \xi(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau))^{\mathrm{T}} \,]^{\mathrm{T}}.$$

Using Assumption 1 yields

$$\|\xi_c(x(t), x(t-\tau), \hat{x}(t), \hat{x}(t-\tau))\|^2 \le 2 \left\| \tilde{S}_1 \eta(t) \right\|^2 + 2 \left\| \tilde{S}_2 \eta(t-\tau) \right\|^2, \tag{25}$$

where

$$\tilde{S}_1 = \begin{bmatrix} S_{1g} & 0 \\ 0 & S_1 \end{bmatrix}, \qquad \tilde{S}_2 = \begin{bmatrix} S_{2g} & 0 \\ 0 & S_2 \end{bmatrix}. \tag{26}$$

Noting (5), it can be easily seen that

$$[\,\Delta A_c(t) \quad \Delta A_{cd}(t) \quad \Delta B_c(t) \quad \Delta B_{cd}(t)\,] = M_{1c}F(t)\,[\,N_{1c} \quad N_{2c} \quad N_{3c} \quad N_{4c}\,]\,,$$

where

$$M_{1c} = \begin{bmatrix} M_1 \\ M_1 - LM_2 \end{bmatrix}, \qquad N_{1c} = [\,N_1 \quad 0\,]\,, \qquad N_{2c} = [\,N_2 \quad 0\,]\,,$$
$$N_{3c} = [\,N_3 \quad 0\,]\,, \qquad N_{4c} = [\,N_4 \quad 0\,]\,.$$

Define

$$\begin{aligned}
P_c &= \operatorname{diag}(P_1, P_2)\,,\\
Q_c &= \operatorname{diag}(Q_1, Q_2)\,,\\
\Omega_{1c} &= A_c^{\mathrm{T}} P_c + P_c A_c + Q_c + 2\epsilon_1 \tilde{S}_1^{\mathrm{T}} \tilde{S}_1 + \epsilon_2 N_{1c}^{\mathrm{T}} N_{1c} + \epsilon_3 N_{3c}^{\mathrm{T}} N_{3c},\\
\Omega_{2c} &= 2\epsilon_1 \tilde{S}_2^{\mathrm{T}} \tilde{S}_2 + \epsilon_2 N_{2c}^{\mathrm{T}} N_{2c} + \epsilon_3 N_{4c}^{\mathrm{T}} N_{4c} - Q_c,
\end{aligned}$$

then by some algebraic manipulations and noting (20), it follows that

$$\begin{bmatrix}
\Omega_{1c} & \begin{matrix} P_c A_{cd} + \epsilon_2 N_{1c}^{\mathrm{T}} N_{2c} \\ + \epsilon_3 N_{3c}^{\mathrm{T}} N_{4c} \end{matrix} & P_c G_c & P_c M_{1c} & 0 & B_c^{\mathrm{T}} P_c \\
\begin{matrix} A_{cd}^{\mathrm{T}} P_c + \epsilon_2 N_{2c}^{\mathrm{T}} N_{1c} \\ + \epsilon_3 N_{4c}^{\mathrm{T}} N_{3c} \end{matrix} & \Omega_{2c} & 0 & 0 & 0 & B_{cd}^{\mathrm{T}} P_c \\
G_c^{\mathrm{T}} P_c & 0 & -\epsilon_1 I & 0 & 0 & 0 \\
M_{1c}^{\mathrm{T}} P_c & 0 & 0 & -\epsilon_2 I & 0 & 0 \\
0 & 0 & 0 & 0 & -\epsilon_3 I & M_{1c}^{\mathrm{T}} P_c \\
P_c B_c & P_c B_{cd} & 0 & 0 & P_c M_{1c} & -P_c
\end{bmatrix}$$
$$= \begin{bmatrix}
\Xi_1 & \Lambda_1 & \Lambda_2 & 0 & \Lambda_3 \\
\Lambda_1^{\mathrm{T}} & \Xi_2 & 0 & 0 & \Pi_1 \\
\Lambda_2^{\mathrm{T}} & 0 & -\Upsilon_1 & 0 & 0 \\
0 & 0 & 0 & -\Upsilon_2 & \Pi_2 \\
\Lambda_3^{\mathrm{T}} & \Pi_1^{\mathrm{T}} & 0 & \Pi_2^{\mathrm{T}} & -\Upsilon_3
\end{bmatrix} < 0.$$

Finally, using this inequality and Theorem 1, the desired result follows immediately.

Remark 2 Theorem 2 provides an LMI method for designing robust observers for system (Σ). It is worth pointing out that the LMI in (20) can be solved by means of numerically efficient convex programming algorithms, and no tuning of parameters is required [2, though there are several parameters and matrices to be determined.

4 Numerical Example

In this section, we provide an example to demonstrate the effectiveness of the proposed method.

Consider the following class of nonlinear stochastic systems with state-delay and parameter uncertainties:

$$
\begin{aligned}
dx_1(t) = {} & [-1.8x_1(t) + (0.2 - 0.4f(t))x_2(t) - (0.1 + 0.2f(t))x_1(t-1.5) + 0.2x_2(t-1.5) \\
& + 0.3\sin(-0.2x_1(t) + 0.1x_2(t) + 0.1x_1(t-0.5) + 0.2x_2(t-1.5))]\,dt \\
& + [(0.1 + 0.2f(t))x_1(t) + (0.3 + 0.2f(t))x_2(t) \\
& + 0.4f(t)x_1(t-1.5) - 0.2x_2(t-1.5)]\,d\omega(t), \\
dx_2(t) = {} & [-0.4x_1(t) - (2.5 + 0.2f(t))x_2(t) - 0.1f(t)x_1(t-1.5) - 0.1x_2(t-1.5) \\
& + 0.2\sin(-0.2x_1(t) + 0.1x_2(t) + 0.1x_1(t-1.5) + 0.2x_2(t-1.5))]\,dt \\
& + [(0.1f(t) - 0.4)x_1(t) + (1 + 0.1f(t))x_2(t) \\
& + (0.6 + 0.2f(t))x_1(t-1.5) + 0.1x_2(t-1.5)]\,d\omega(t), \\
dy(t) = {} & [0.1x_1(t) - (0.4 + 0.2f(t))x_2(t) + (0.4 - 0.1f(t))x_1(t-1.5) + 0.6x_2(t-1.5) \\
& + 0.5\sin(0.2x_1(t) - 0.1x_2(t) + 0.2x_1(t-1.5))]\,dt \\
& + [0.1f(t)x_1(t) + (0.1f(t) - 0.2)x_2(t) \\
& + (0.2f(t) - 0.5)x_1(t-1.5) + 0.2x_2(t-1.5)]\,d\omega(t),
\end{aligned}
$$

where $f(t)$ is unknown but satisfies $|f(t)| \le 1$. It is easy to see that the above system has the form (1) and (2) with parameters as follows

$$
\begin{aligned}
A &= \begin{bmatrix} -1.8 & 0.2 \\ -0.4 & -2.5 \end{bmatrix}, & A_d &= \begin{bmatrix} -0.1 & 0.2 \\ 0 & -0.1 \end{bmatrix}, \\
B &= \begin{bmatrix} 0.1 & 0.3 \\ -0.4 & 1 \end{bmatrix}, & B_d &= \begin{bmatrix} 0 & -0.2 \\ 0.6 & 0.1 \end{bmatrix}, \\
C &= \begin{bmatrix} 0.1 & -0.4 \end{bmatrix}, & C_d &= \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}, \\
D &= \begin{bmatrix} 0 & -0.2 \end{bmatrix}, & D_d &= \begin{bmatrix} -0.5 & 0.2 \end{bmatrix}, \\
G &= \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, & H &= 0.5, \\
M_1 &= \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}, & M_2 &= 0.2, \\
N_1 &= \begin{bmatrix} 0 & -1 \end{bmatrix}, & N_2 &= \begin{bmatrix} -0.5 & 0 \end{bmatrix}, \\
N_3 &= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, & N_4 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
S_{1g} &= \begin{bmatrix} -0.2 & 0.1 \end{bmatrix}, & S_{2g} &= \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \\
S_{1h} &= \begin{bmatrix} 0.2 & -0.1 \end{bmatrix}, & S_{2h} &= \begin{bmatrix} 0.2 & 0 \end{bmatrix}.
\end{aligned}
$$

Now, using the Matlab LMI Control Toolbox, we obtain the solution to the LMI (20) as follows:

$$
\begin{aligned}
P_1 &= \begin{bmatrix} 5.0934 & -0.7812 \\ -0.7812 & 4.3022 \end{bmatrix}, & P_2 &= \begin{bmatrix} 2.8203 & -0.5012 \\ -0.5012 & 1.6465 \end{bmatrix}, \\
Q_1 &= \begin{bmatrix} 10.9532 & -0.5227 \\ -0.5227 & 3.6335 \end{bmatrix}, & Q_2 &= \begin{bmatrix} 4.3914 & -0.7795 \\ -0.7795 & 4.7745 \end{bmatrix},
\end{aligned}
$$

$$
Z = \begin{bmatrix} 0.1271 \\ -2.1537 \end{bmatrix},
$$

$$
\epsilon_1 = 4.8400, \qquad \epsilon_2 = 2.6078, \qquad \epsilon_3 = 2.7588.
$$

Therefore, by Theorem 2, it follows that the robust observer design problem is solvable, and the desired nonlinear observer can be chosen by

$$\begin{aligned} d\hat{x}(t) = \bigg(& \begin{bmatrix} -1.8 & 0.2 \\ -0.4 & -2.5 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} -0.1 & 0.2 \\ 0 & -0.1 \end{bmatrix} \hat{x}(t-1.5) \\ & + \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix} \sin([\,-0.2 \quad 0.1\,]\,\hat{x}(t) + [\,0.1 \quad 0.2\,]\,\hat{x}(t-1.5)) \bigg) dt \\ & + \begin{bmatrix} -0.1981 \\ -1.3684 \end{bmatrix} (dy(t) - ([\,0.1 \quad -0.4\,]\,\hat{x}(t) + [\,0.4 \quad 0.6\,]\,\hat{x}(t-1.5) \\ & + 0.5 \sin([\,-0.2 \quad 0.1\,]\,\hat{x}(t) + [\,0.1 \quad 0.2\,]\,\hat{x}(t-1.5)))\,dt)\,. \end{aligned}$$

5 Conclusions

In this paper, we have studied the robust observer design problem for a class of nonlinear stochastic systems with state delays and time-varying norm-bounded parameter uncertainties. In terms of an LMI, a nonlinear observer has been developed to guarantee mean square asymptotic stability of the error dynamics for all admissible uncertainties. A numerical example has been provided to show the effectiveness of the proposed methods.

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