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# Robust Adaptive Control for a Class of Nonlinear Stochastic Time-delay Systems

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**Abstract:** The adaptive control problem of a class of stochastic time-delay systems is investigated. Firstly we consider a simple class of stochastic systems with time-varying delays and design the corresponding adaptive controller based on the solution of linear matrix inequalities (LMIs), which can render the closed-loop asymptotically stable in probability. Then we apply the adaptive idea to the interconnected system case. Under the condition that interconnections satisfy the matching condition, we propose a class of decentralized feedback controllers and the corresponding closed-loop systems are also asymptotically stable in probability. Numerical examples on controlling the two classes of stochastic systems are given to show the validity of obtained theoretical results.

**Keywords:** *Stochastic systems; time-delay systems; interconnected systems; adaptive control.*

**Mathematics Subject Classification (2000):** 93E15, 93C23, 93D09.

## 1 Introduction

Time-delay is often encountered in various engineering systems, such as electrical networks, turbojet engines, microwave oscillators, nuclear reactors, rolling mills, chemical processes, manual control, long transmission lines in pneumatic, and hydraulic systems, etc. Its existence is often a source of instability and poor performance. Therefore, the problem of stability analysis and robust control for dynamic time-delay systems has attracted considerable attention of a number of researchers over the past years, see for example, $[1-4]$ and the references therein.

In this paper we will focus on controlling stochastic time-delay systems. In the existing literature, some work has been done on stability analysis and control for stochastic time-delay systems. The robust stability problem of linear stochastic time-delay systems was studied in [5], while robust stability analysis for stochastic delay interval systems is considered in [6]. In [7], the problem of control for uncertain stochastic time-delay systems was considered, and the results were given in the form of LMIs. Filtering problem for uncertain stochastic systems was considered in [8 − 10]. In the meantime, the problem of control for interconnected stochastic time-delay systems was tackled in [11].

Unlike the existing results in literature, in this paper, we investigate the adaptive control problem of stochastic time-delay systems, whose bounds of uncertainties in matching parts are not required to be known. Firstly we consider a simple class of stochastic systems with time-varying delays. Corresponding adaptive controller is designed based on the solution of LMI. Then we apply the adaptive idea to the interconnected system case. Under the condition that interconnections satisfy the matching condition, we propose a class of decentralized feedback controllers, which can render the closed-loop systems asymptotically stable.

## 2 Problem Formulation

Consider the following time delay system

$$\begin{gathered} dx = (Ax + f(x, x(t - d(t)) + Bu)\,dt + g(x, x(t - h(t)))\,dw, \\ x(t) = \varphi(t), \qquad t \in [-d, 0]. \end{gathered} \tag{1}$$

where $x \in R^n$ and $u \in R^m$ are the state and control input respectively, $d(t)$ and $h(t)$ are time-varying delay parameters, $A$ and $B$ are known constant matrices with appropriate dimensions. $w$ is a zero-mean Wiener process. $f(\cdot)$ and $g(\cdot)$ are uncertain nonlinear function vectors.

For system (1), we introduce the following standard assumptions.

**Assumption 2.1** The time-varying time delays $d(t)$ satisfies

$$\dot{d}(t) \le \tau < 1, \qquad \dot{h}(t) \le k < 1. \tag{2}$$

**Assumption 2.2** The nonlinear function $f(\cdot)$ can be decomposed into the matched form and the unmatched form

$$f(x, x(t - d(t))) = B\xi(x, x(t - d(t))) + \zeta(x, x(t - d(t))), \tag{3}$$

where $\xi(x, x(t - d(t)))$ and $\zeta(x, x(t - d(t)))$ satisfy

$$\|\xi(x, x(t - d(t)))\| \le \beta_1 \|x\| + \beta_2 \|x(t - d(t))\|, \tag{4}$$

$$\|\zeta(x, x(t - d(t)))\| \le \gamma_1 \|x\| + \gamma_2 \|x(t - d(t))\|, \tag{5}$$

where $\gamma_1$ and $\gamma_2$ are known positive scalars, $\beta_1$ and $\beta_2$ are unknown positive scalars.

**Assumption 2.3** There exist matrix $Y$, positive matrix $X$ and positive scalars $\varepsilon_1$ and $\varepsilon_2$ such that the following LMI holds

$$\begin{bmatrix} AX + XA^{\mathrm{T}} + BY + Y^{\mathrm{T}}B^{\mathrm{T}} + \varepsilon_1\gamma_1^2 I + \frac{\varepsilon_2}{1-\tau}\gamma_2^2 I & X & X \\ X & -\varepsilon_1 I & 0 \\ X & 0 & -\varepsilon_2 I \end{bmatrix} < 0. \qquad (6)$$

**Assumption 2.4** The nonlinear function $g$ satisfies

$$\begin{aligned} g^{\mathrm{T}}Pg \le \alpha_2 \left\|B^{\mathrm{T}}Px\right\| \left\|x\left(t-h\left(t\right)\right)\right\| + \alpha_3 \left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\| \left\|x\right\| \\ + \alpha_1 \left\|B^{\mathrm{T}}Px\right\| \left\|x\right\| + \alpha_4 \left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\| \left\|x\left(t-h\left(t\right)\right)\right\|, \end{aligned} \qquad (7)$$

where matrix $P = X^{-1}$, $X$ satisfies LMI (6), $\alpha_i$ $(i = 1, 2, 3, 4)$ are unknown positive scalars.

*Remark 1* Assumption 2.1 is often needed on investigating time-delay systems by employing Lyapunov-Krasovskii method. Different from the existing literatures on control of stochastic time-delay systems, we divide the uncertainties into matched and unmatched parts and the bounds of matched parts are not needed to be known in Assumption 2.2. Assumption 2.3 is to guarantee that the system is asymptotically stable without the matching parts and the stochastic parts. In practical systems we may also not know the function $g$ exactly, so Assumption 2.4 is imposed.

Before giving the problem statement in this paper, we first introduce the following definition of stability in probability.

Consider the nonlinear stochastic system

$$dx = f(x, x(t-d))dt + g(x, x(t-d))dw, \qquad (8)$$

where $x \in R^n$ is the state, $w$ is an $r$-dimensional standard Wiener process, and functions $f$ and $g$ are locally Lipschitz and satisfy $f(0, 0) = 0$ and $g(0, 0) = 0$.

**Definition 2.1** [7] The equilibrium $x = 0$ of the system (8) is said to be globally asymptotically stable in probability for given $x(t)$ if for any $s \ge 0$ and $\varepsilon > 0$

$$\lim_{x\to 0} P\left\{\sup_{s<t} |x_t^{s,x}| > \varepsilon\right\} = 0, \qquad P\left\{\lim_{t\to+\infty} |x_t^{s,x}| = 0\right\} = 1,$$

where $x_t^{s,x}$ denotes the solution at time $t$ of a stochastic differential equation starting from the state $x$ at time $s$ for $s \le t$.

**Lemma 2.1** [12] *Consider system (8) and suppose there exists a positive definite, radially unbounded, twice continuously differentiable function $V(x)$ such that the following inequality holds*

$$LV(x) = \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \operatorname{tr}\left[g(x)^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g(x)\right] < 0,$$

*then system (8) is globally asymptotically stable in probability.*

In this paper, we will firstly consider designing controller to render system (1) globally asymptotically stable in probability under the above assumptions, then further apply the design idea to interconnected stochastic system case and design the corresponding controller.

## 3 Robust Controller Design

In this section we will investigate designing adaptive state feedback controller to stabilize uncertain stochastic system (1).

**Theorem 3.1** *For system (1), the following adaptive state feedback controller*

$$u = Kx - \frac{1}{2}\theta(t)B^{\mathrm{T}}Px, \tag{9}$$

*where $K = YX^{-1}$, and matrices $P$, $Y$ and $X$ satisfy (6), $\theta(t)$ is adaptive parameter whose adaptive law is*

$$\frac{d\theta(t)}{dt} = a\|B^{\mathrm{T}}Px\|^2, \tag{10}$$

*where $a$ is an arbitrary positive scalar, can render the closed-loop system robustly stable in probability.*

*Proof* Substituting (9) into (1), we can obtain

$$dx = \left(Ax + f(x, x(t-d(t)) + BK - \frac{1}{2}B\theta(t)B^{\mathrm{T}}Px)\right)dt + g(x, x(t-h(t)))dw. \tag{11}$$

Choose the following Lyapunov–Krasovskii function

$$\begin{aligned} V = {} & x^{\mathrm{T}}Px + \frac{1}{2}a^{-1}\widetilde{\theta}^{\mathrm{T}}\widetilde{\theta} + \left(\delta_6 + \varepsilon_2^{-1}\right)\int_{t-d(t)}^{t} \|x(\xi)\|^2 d\xi \\ & + \left(\frac{\delta_2}{1-k} + \frac{\delta_4}{1-k}\right)\int_{t-h(t)}^{t} \|x(\xi)\|^2 d\xi \\ & + \left(\frac{\alpha_3^2}{4\delta_3(1-k)} + \frac{\alpha_4^2}{4\delta_4(1-k)}\right)\int_{t-h(t)}^{t} \|B^{\mathrm{T}}Px(\xi)\|^2 d\xi, \end{aligned} \tag{12}$$

where $\delta_i$, $(i = 1, 2, \ldots, 6)$ are positive scalars, $\tilde{\theta} = \theta - \hat{\theta}(t)$, $\hat{\theta}$ is a positive scalar defined in (18).

Taking the time derivative of above Lyapunov function, one can get

$$\begin{aligned} LV \le {} & 2x^{\mathrm{T}}P\left(Ax + f\left(x, x\left(t-d\right)\right) + BK\right) - x^{\mathrm{T}}PB\theta\left(t\right)B^{\mathrm{T}}Px \\ & + g\left(x, x\left(t-h\right)\right)^{\mathrm{T}}Pg\left(x, x\left(t-h\right)\right) + a^{-1}\tilde{\theta}\dot{\tilde{\theta}} \\ & + \left(\delta_6 + \varepsilon_2^{-1}\right)\left[\|x\|^2 - (1-\tau)\left\|x\left(t-d\left(t\right)\right)\right\|^2\right] \\ & + \frac{1}{1-k}\left(\delta_2 + \delta_4\right)\left(\|x\|^2 - (1-k)\left\|x(t-h(t))\right\|^2\right) \\ & + \frac{1}{(1-k)}\left(\frac{\alpha_3^2}{4\delta_3} + \frac{\alpha_4^2}{4\delta_4}\right)\left(\left\|B^{\mathrm{T}}Px\left(t\right)\right\|^2 - (1-k)\left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\|^2\right). \end{aligned} \tag{13}$$

From Assumption 2.4, we obtain that

$$\begin{aligned} g^{\mathrm{T}}Pg \le {} & \alpha_2\left\|B^{\mathrm{T}}Px\right\|\left\|x\left(t-h\left(t\right)\right)\right\| + \alpha_3\left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\|\|x\| \\ & + \alpha_1\left\|B^{\mathrm{T}}Px\right\|\|x\| + \alpha_4\left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\|\left\|x\left(t-h\left(t\right)\right)\right\| \\ \le {} & \frac{\alpha_1^2}{4\delta_1}\left\|B^{\mathrm{T}}Px\right\|^2 + \delta_1\|x\|^2 + \frac{\alpha_2^2}{4\delta_2}\left\|B^{\mathrm{T}}Px\right\|^2 + \delta_2\left\|x\left(t-h\left(t\right)\right)\right\|^2 \\ & + \frac{\alpha_3^2}{4\delta_3}\left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\|^2 + \delta_3\|x\|^2 \\ & + \frac{\alpha_4^2}{4\delta_4}\left\|B^{\mathrm{T}}Px\left(t-h\left(t\right)\right)\right\|^2 + \delta_4\left\|x\left(t-h\left(t\right)\right)\right\|^2. \end{aligned} \tag{14}$$

We know

$$
\begin{aligned}
&2x^{\mathrm{T}}P(A+f\,(x,\,x(t-d))+BK)\\
&= x^{\mathrm{T}}\left(PA+A^{\mathrm{T}}P+PBK+K^{\mathrm{T}}B^{\mathrm{T}}P\right)x+2x^{\mathrm{T}}PB\xi\,(x,x\,(t-d\,(t)))\\
&\qquad +2x^{\mathrm{T}}P\zeta\,(x,x\,(t-d)t))\\
&\le x^{\mathrm{T}}\left(PA+A^{\mathrm{T}}P+PBK+K^{\mathrm{T}}B^{\mathrm{T}}P\right)x+\frac{\beta_1^2}{\delta_5}\,x^{\mathrm{T}}PBB^{\mathrm{T}}Px+\delta_5\,\|x\|^2\\
&\quad +\frac{\beta_2^2}{(1-\tau)\,\delta_6}\,x^{\mathrm{T}}PBB^{\mathrm{T}}Px+(1-\tau)\,\delta_6\,\|x\,(t-d\,(t))\|^2+\varepsilon_1\gamma_1^2x^{\mathrm{T}}PPx\\
&\quad +\varepsilon_1^{-1}\,\|x\|^2+\frac{\varepsilon_2}{(1-\tau)}\gamma_2^2x^{\mathrm{T}}PPx+(1-\tau)\,\varepsilon_2^{-1}\,\|x\,(t-d\,(t))\|^2 .
\end{aligned}
\tag{15}
$$

Substituting (14), (15) into (13), we can further obtain that

$$
LV \le -x^{\mathrm{T}}\Phi x+\left(\widehat{\theta}-\theta\right)\left\|B^{\mathrm{T}}Px\right\|^2+a^{-1}\tilde{\theta}\dot{\tilde{\theta}}, \tag{16}
$$

where

$$
\begin{aligned}
-\Phi &= PA+A^{\mathrm{T}}P+PBK+K^{\mathrm{T}}B^{\mathrm{T}}P+\varepsilon_1\gamma_1^2PP+\varepsilon_1^{-1}I+\frac{\varepsilon_2}{(1-\tau)}\,\gamma_2^2PP\\
&\quad +\varepsilon_2^{-1}I+\delta_1+\frac{1}{1-k}\delta_2+\delta_3+\frac{1}{1-k}\,\delta_4+\delta_5+\delta_6,
\end{aligned}
\tag{17}
$$

$$
\widehat{\theta} = \frac{\beta_1^2}{\delta_5}+\frac{\beta_2^2}{\delta_6\,(1-\tau)}+\frac{\alpha_1^2}{4\delta_1}+\frac{\alpha_2^2}{4\delta_2}+\frac{\alpha_3^2}{4\delta_3\,(1-k)}+\frac{\alpha_4^2}{4\delta_4\,(1-k)}. \tag{18}
$$

As we know if LMI (6) holds, the following inequality stands

$$
AX+XA^{\mathrm{T}}+BY+Y^{\mathrm{T}}B^{\mathrm{T}}+\varepsilon_1\gamma_1^2I+\frac{\varepsilon_2}{1-\tau}\,\gamma_2^2I+\varepsilon_1^{-1}X^{\mathrm{T}}X+\varepsilon_2^{-1}X^{\mathrm{T}}X<0. \tag{19}
$$

Further, the following inequality holds (by multiply $P$ on both sides of (19) with $P=X^{-1}$)

$$
PA+A^{\mathrm{T}}P+PBK+K^{\mathrm{T}}B^{\mathrm{T}}P+\left(\varepsilon_1\gamma_1^2+\frac{\varepsilon_2}{1-\tau}\gamma_2^2\right)PP+\varepsilon_1^{-1}I+\varepsilon_2^{-1}I<0. \tag{20}
$$

Therefore, from (17) and (20) we know there always exist sufficiently small positive scalars $\delta_i$ $(\,i=1,2,\dots,6)$ such that

$$
\Phi > 0. \tag{21}
$$

Substituting (10) into (16), we can obtain

$$
LV \le -x^{\mathrm{T}}\Phi x \tag{22}
$$

which implies that the closed-loop system is robustly stable in probability.

**Corollary 3.1** *If Assumptions 2.1, 2.4 and Assumption 2.2 with $\zeta(\cdot) = 0$ are satisfied, and the pair (A, B) is completely controllable, the following controller*

$$u = -\frac{1}{2}\,\theta(t)B^{\mathrm{T}}Px \tag{23}$$

*with adaptive law*

$$\frac{d\theta(t)}{dt} = a\|B^{\mathrm{T}}Px\|^2, \tag{24}$$

*where $a$ is a positive scalar, will render the closed-loop system (1) robustly stable in probability.*

*Proof* If $(A, B)$ are completely controllable, for a given positive matrix $\Omega$ there always exist positive scalar $\mu$ such that the following Riccati equality

$$PA + A^{\mathrm{T}}P - \mu PBB^{\mathrm{T}}P = -\Omega \tag{25}$$

has positive matrix solution $P$. From the above proof, we can design the following controller

$$u = -\frac{1}{2}\,\mu B^{\mathrm{T}}Px - \frac{1}{2}\,\Theta(t)B^{\mathrm{T}}Px \tag{26}$$

with adaptive law

$$\frac{d\Theta(t)}{dt} = a\|B^{\mathrm{T}}Px\|^2. \tag{27}$$

Further we let $\theta(t) = \Theta(t) + \mu$, where $\mu$ is a positive scalar. Thus the controller (26), (27) will give us the desired result.

**Corollary 3.2** *If $B = I$ ($I$ is an identity matrix) and Assumption 2.1 holds, the following controller*

$$u_i = -\frac{1}{2}\,\Theta(t)x$$

*with adaptive law*

$$\frac{d\Theta(t)}{dt} = a\|x\|^2$$

*will render the closed-loop system (1) robustly stable in probability.*

*Proof* If $B = I$, it is easy to see $(A, B)$ are completely controllable and Assumption 2.4 is satisfied. Therefore, we can design the required adaptive controller to achieve our goal.

*Remark 3.1* In the designed controller, we adopt the adaptive law (10). In fact, we can also use the $\sigma$-modification adaptive law, that is (10) can be changed into

$$\frac{d\theta(t)}{dt} = a\|B^{\mathrm{T}}Px\|^2 - \sigma\theta(t), \tag{28}$$

where $\sigma$ is an adjustable parameter. Compared with the adaptive law (10), the modified adaptive control law (28) can improve the robust performance for the closed-loop systems. Similar to the proof of above, we can also obtain the closed-loop system (1) and (28) is uniformly ultimately bounded stable, and the bounds of the steady-state can be adjusted to be sufficiently small by selecting small parameter $\sigma$ [4].

## 4 Control of Interconnected Time Delay Systems

In this section, we investigate a class of interconnected stochastic time-delay systems. A controller is designed to stabilize the underlying system. Different from the literature, instead of using bounds of uncertainties to design the controller, we assume all the bounds unknown. Therefore, the proposed adaptive decentralized feedback controller can be applied to stabilization of a large class of interconnected time-delay systems.

Consider the following interconnected systems whose $i$-th subsystem is described by

$$\begin{gathered} dx_i = (A_i x_i + B_i u_i)\,dt + f_i(x_i, x_1, x_2, \dots, x_n, x_1(t - d_{i1}(t), \dots, x_n(t - d_{in}(t))))\,dt \\ + g_i(x_i, x_1, x_2, \dots, x_n, x_1(t - h_{i1}(t), \dots, x_n(t - h_{in}(t))))\,dw, \\ i = 1, 2, \dots, N. \end{gathered} \tag{29}$$

We impose the following assumptions on system (29).

**Assumption 4.1** For $i,\, j = 1, 2, ..., N$, the time-varying time delays satisfy

$$\dot{d}_{ij}(t) \le \tau_j < 1, \qquad \dot{h}_{ij}(t) \le k_j < 1. \tag{30}$$

**Assumption 4.2** For $i,\, j = 1, 2, ..., N$ and given $Q_i > 0$, there exist matrix $P_i > 0$ and scalar $\sigma_i > 0$ such that the following equality holds

$$P_i A_i + A_i P_i - \sigma_i P_i B_i B_i^{\mathrm{T}} P_i = -Q_i. \tag{31}$$

**Assumption 4.3** For $i = 1, 2, ..., N$, the nonlinear functions $f_i(\cdot)$ satisfy matching condition

$$f_i\left(\cdot\right) = B_i \xi_i\left(\cdot\right), \tag{32}$$

where $\xi_i\left(\cdot\right)$ satisfies

$$\left\|\xi_i\left(\cdot\right)\right\| \le \sum_{j=1}^{N} \left(\rho_{ij}\left\|x_j\right\| + \varphi_{ij}\left\|x_j\left(t - d_{ij}\left(t\right)\right)\right\|\right). \tag{33}$$

Here $\rho_{ij}$ and $\varphi_{ij}$ are unknown positive scalars, $i,\, j = 1, 2, ..., N$.

**Assumption 4.4** The following inequalities hold

$$\begin{aligned} g_i\left(\cdot\right)^{\mathrm{T}} P_i g_i\left(\cdot\right) \le{}& \sum_{j=1}^{N} \left\|B_i^{\mathrm{T}} P_i x_i\right\| \left(\phi_{ij}\left\|x_j\right\| + \overline{\phi}_{ij}\left\|x_j\left(t - h_{ij}\left(t\right)\right)\right\|\right) \\ &+ \sum_{j=1}^{N} \left\|B_i^{\mathrm{T}} P_i x_i\left(t - h_{ij}\right)\right\| \left(\psi_{ij}\left\|x_j\right\| + \overline{\psi}_{ij}\left\|x_j\left(t - h_{ij}\left(t\right)\right)\right\|\right), \end{aligned} \tag{34}$$

where $\phi_{ij}$, $\overline{\phi}_{ij}$, $\psi_{ij}$ and $\overline{\psi}_{ij}$ are positive scalars, $i, j = 1, 2, ..., N$.

Now we are ready to present our main result in this paper.

**Theorem 4.1** *For interconnected stochastic systems (29) under Assumptions 4.1–4.4, the following decentralized feedback controller, for $i = 1, 2, ..., N$,*

$$u_i = -\frac{1}{2}\,\Theta_i\,(t)\,B_i^{\mathrm{T}} P_i x_i \tag{35}$$

*with adaptive law*

$$\frac{d\Theta_i\,(t)}{dt} = a_i \left\|B_i^{\mathrm{T}} P_i x_i\right\|^2 \tag{36}$$

*will render the closed-loop system robustly stable in probability, where $a_i$ is a positive scalar.*

*Proof* Choose the following Lyapunov function

$$\begin{aligned} V = \sum_{i=1}^{N} V_i + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-\tau_j}\,\delta_{2j} \int_{t-d_{ij}}^{t} \|x_j(\zeta)\|^2 d\zeta + \sum_{i=1}^{N} \frac{1}{2}\,a_i^{-1}\overline{\Theta}_i(t)^2 \\ + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-k_j}\,(\delta_{4j}+\delta_{6j}) \int_{t-h_{ij}}^{t} \|x_j(\zeta)\|^2 d\zeta \\ + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-k_j}\,(\delta_{5j}^{-1}\psi_{ij}^2 + \delta_{6j}^{-1}\overline{\psi}_{ij}^2) \int_{t-h_{ij}}^{t} \|B_i^{\mathrm{T}} P_i x_i(\xi)\|^2 d\xi, \end{aligned} \tag{37}$$

where $\delta_{sj}$ $(s \in [1,6],\ j \in [1,N])$ are positive scalars and

$$\begin{aligned} V_i &= x_i^{\mathrm{T}} P_i x_i, \\ \overline{\Theta}_i\,(t) &= \widehat{\Theta}_i - \Theta_i\,(t)\,, \end{aligned} \tag{38}$$

$\widehat{\Theta}_i$ is defined in (44) (below).

Taking the derivative of $V$ with respect to time $t$, along the closed-loop system, we obtain

$$\begin{aligned} LV = {} & \sum_{i=1}^{N} LV_i + \sum_{i=1}^{N} a_i \overline{\Theta}_i\,(t)\,\dot{\overline{\Theta}}_i\,(t) \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-\tau_j}\delta_{2j} \left(\|x_j\,(t)\|^2 - (1-\tau_j)\,\|x_j\,(t-d_{ij}\,(t))\|^2\right) \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-k_j}\,(\delta_{4j}+\delta_{6j}) \left(\|x_j\,(t)\|^2 - (1-\tau_j)\,\|x_j\,(t-h_{ij}\,(t))\|^2\right) \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{1-k_j} \left(\delta_{5j}^{-1}\psi_{ij}^2 + \delta_{6j}^{-1}\overline{\psi}_{ij}^2\right) \\ & \times \left(\left\|B_i^{\mathrm{T}} P_i x_i\right\|^2 - (1-k_j)\left\|B_i^{\mathrm{T}} P_i x_i\,(t-h_{ij}\,(t))\right\|^2\right). \end{aligned} \tag{39}$$

We know

$$LV_i = 2x_i^{\mathrm{T}} P_i \left( A_i x_i + B_i u_i + f_i \right) + g_i^{\mathrm{T}} P_i g_i \tag{40}$$

and

$$\begin{aligned} 2x_i^{\mathrm{T}} P_i f_i &= 2x_i^{\mathrm{T}} P_i B_i \xi_i (\cdot) \\ &\le \sum_{j=1}^{N} \left[ 2 \left\| x_i^{\mathrm{T}} P_i B_i \right\| \rho_{ij} \left\| x_j \right\| + 2 \left\| x_i^{\mathrm{T}} P_i B_i \right\| \varphi_{ij} \left\| x_j \left( t - d_{ij} (t) \right) \right\| \right] \\ &\le \sum_{j=1}^{N} \left[ \delta_{1j}^{-1} \rho_{ij}^2 \left\| x_i^{\mathrm{T}} P_i B_i \right\|^2 + \delta_{1j} \left\| x_j \right\|^2 \right] \\ &\quad + \sum_{j=1}^{N} \left[ \delta_{2j}^{-1} \varphi_{2j}^2 \left\| x_i^{\mathrm{T}} P_i B_i \right\|^2 + \delta_{2j} \left\| x_j \left( t - d_{ij} (t) \right) \right\|^2 \right]. \end{aligned} \tag{41}$$

From Assumption 4.3 one can get

$$\begin{aligned} g_i^{\mathrm{T}} P_i g_i &\le \sum_{j=1}^{N} \| B_i^{\mathrm{T}} P_i x_i \| ( \phi_{ij} \| x_j \| + \overline{\phi}_{ij} \| x_j (t - h_{ij}(t)) \| ) \\ &\quad + \sum_{j=1}^{N} \| B_i^{\mathrm{T}} P_i x_i (t - h_{ij}(t)) \| ( \psi_{ij} \| x_j \| + \overline{\psi}_{ij} \| x_j (t - h_{ij}(t)) \| ) \\ &\le \sum_{j=1}^{N} \left[ \delta_{3j}^{-1} \phi_{ij}^2 \left\| B_i^{\mathrm{T}} P_i x_i \right\|^2 + \delta_{3j} \left\| x_j \right\|^2 \right] \\ &\quad + \sum_{j=1}^{N} \left[ \delta_{4j}^{-1} \overline{\phi}_{ij}^2 \left\| B_i^{\mathrm{T}} P_i x_i \right\|^2 + \delta_{4j} \left\| x_j \left( t - h_{ij} (t) \right) \right\|^2 \right] \\ &\quad + \sum_{j=1}^{N} \left[ \delta_{5j}^{-1} \psi_{ij}^2 \left\| B_i^{\mathrm{T}} P_i x_i \left( t - h_{ij} (t) \right) \right\|^2 + \delta_{5j} \left\| x_j \right\|^2 \right] \\ &\quad + \sum_{j=1}^{N} \left[ \delta_{6j}^{-1} \overline{\psi}_{ij}^2 \left\| B_i^{\mathrm{T}} P_i x_i \left( t - h_{ij} (t) \right) \right\|^2 + \delta_{6j} \left\| x_j \left( t - h_{ij} (t) \right) \right\|^2 \right]. \end{aligned} \tag{42}$$

Substituting (40)–(42) into (39), we obtain

$$\begin{aligned} LV &\le \sum_{i=1}^{N} \left[ x_i^{\mathrm{T}} \left( P_i A_i + A_i^{\mathrm{T}} P_i - \sigma_i P_i B_i B_i^{\mathrm{T}} P_i \right) x_i + \sigma_i \left\| B_i^{\mathrm{T}} P_i x_i \right\|^2 \right] \\ &\quad + \sum_{i=1}^{N} \left( a_i^{-1} \overline{\Theta}_i (t) \dot{\overline{\Theta}}_i (t) - \Theta_i (t) \left\| x_i^{\mathrm{T}} P_i B_i \right\|^2 \right) \\ &\quad + \sum_{i=1}^{N} \sum_{j=1}^{N} \Bigg( \delta_{1j}^{-1} \rho_{ij}^2 + \delta_{2j}^{-1} \varphi_{ij}^2 + \delta_{3j}^{-1} \phi_{ij}^2 + \delta_{4j}^{-1} \overline{\phi}_{ij}^2 \\ &\quad + \frac{1}{1 - k_j} \left( \delta_{5j}^{-1} \psi_{ij}^2 + \delta_{6j}^{-1} \overline{\psi}_{ij}^2 \right) \Bigg) \left\| B_i^{\mathrm{T}} P_i x_i \right\|^2 \\ &\quad + \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ \delta_{1j} + \frac{1}{1 - \tau_j} \delta_{2j} + \delta_{3j} + \frac{1}{1 - k_j} \left( \delta_{4j} + \delta_{6j} \right) + \delta_{5j} \right] \left\| x_j \right\|^2. \end{aligned} \tag{43}$$

Let

$$\widehat{\Theta}_i = \sum_{j=1}^{N} \left( \delta_{1j}^{-1}\rho_{ij}^2 + \delta_{2j}^{-1}\rho_{ij}^2 + \delta_{3j}^{-1}\phi_{ij}^2 + \delta_{4j}^{-1}\overline{\phi}_{ij}^2 + \frac{1}{1-k_j}\left(\delta_{5j}^{-1}\psi_{ij}^2 + \delta_{6j}^{-1}\overline{\psi}_{ij}^2\right)\right) + \sigma_i,$$

$$\lambda_i = N\left[\delta_{1i} + \frac{1}{1-\tau_i}\delta_{2i} + \delta_{3i} + \frac{1}{1-k_i}\left(\delta_{4i} + \delta_{6i}\right) + \delta_{5i}\right]. \tag{44}$$

Further, we obtain

$$LV \le -\sum_{i=1}^{N}\left[x_i^{\mathrm{T}}(Q_i - \lambda_i I)x_i + (\widehat{\Theta}_i - \Theta_i(t))\|B_i^{\mathrm{T}} P_i x_i\|^2 + \sum_{i=1}^{N} a_i^{-1}\overline{\Theta}_i(t)\dot{\overline{\Theta}}_i(t)\right]. \tag{45}$$

Substituting (36) into (45), we obtain that

$$LV = -\sum_{i=1}^{N} x_i^{\mathrm{T}}(Q_i - \lambda_i I)x_i. \tag{46}$$

From (46), by selecting sufficiently small parameters $\delta_{li}$ $(l \in [1,6])$ we know parameters $\lambda_i$ can be small enough to ensure

$$Q_i - \lambda_i I > 0.$$

It is readily to see that the closed-loop interconnected time-delay systems are robustly asymptotically stable in probability.

## 5 Numerical Examples

In this section, simulation examples on time-delay stochastic systems and interconnected stochastic systems are given to demonstrate the validness and feasibility of the obtained theoretic results in previous sections.

*Example 1* Consider the following stochastic time-delay system

$$\begin{aligned} dx = &\left\{\begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} x_1\left(t - 0.5\left(1+\sin t\right)\right)\sin t \\ \delta_1 x_2\left(t\right)\cos t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right\} dt \\ &+ \begin{bmatrix} \delta_2\left(|x_2|\,|x_1|\right)^{1/2} \\ \delta_3 x_2\left(t - 0.3\left(1+\sin\left(t\right)\right)\right)\cos t \end{bmatrix} dw, \end{aligned} \tag{47}$$

where $\delta_1$, $\delta_2$ and $\delta_3$ are arbitrary scalars. We know the above system satisfying Assumptions 2.1 and 2.2, and when $X = I$, $Y = 0$, $\varepsilon_1 = \varepsilon_2 = 1$, Assumption 2.3 is also satisfied. Further we can verify that Assumption 2.4 also holds.

Therefore, based on Theorem 2.1 we can obtain the following controller

$$u = -\frac{1}{2}\,\theta(t)B^{\mathrm{T}}Px$$

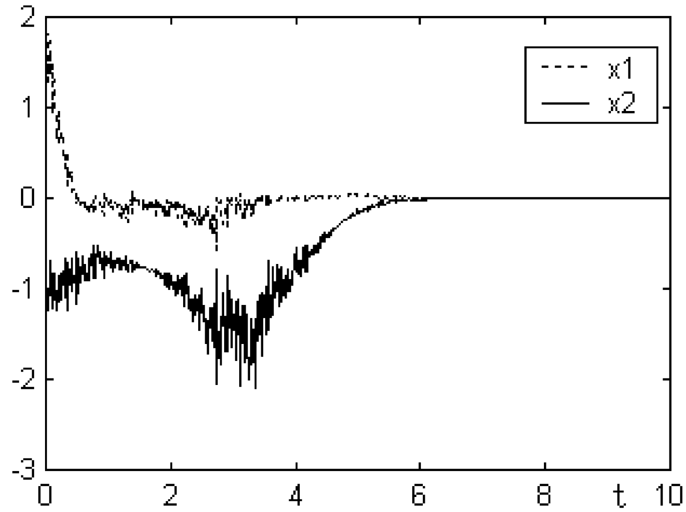


**Figure 5.1**. The states response curves with $\delta_i = 1$.

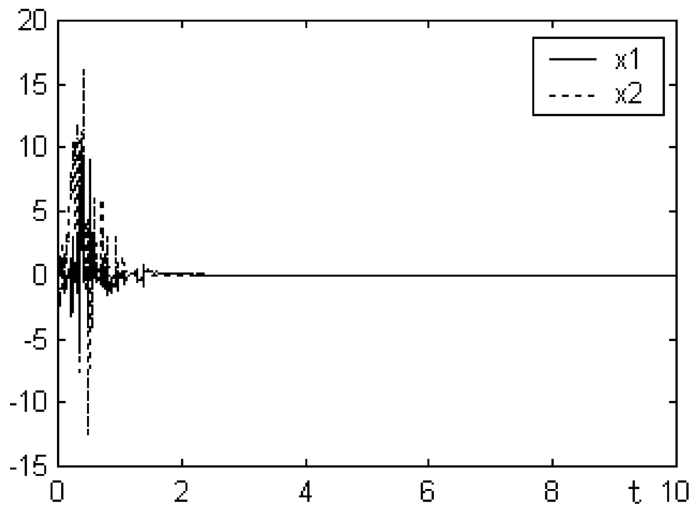


**Figure 5.2**. The states response curves with $\delta_i = 5$.

with adaptive law

$$\frac{d\theta(t)}{dt} = \|x\|^2.$$

The initial values are chosen as

$$x_1(0) = 2, \qquad x_2(0) = -1, \qquad \theta(0) = 2$$

and the sample time is $0.01s$. The simulation results are shown in Figure 5.1 and Figure 5.2. In Figure 5.1, it shows the response curves with above adaptive controller when $\delta_1 = \delta_2 = \delta_3 = 1$. With the same controller, the response curves are shown in Figure 5.2 when $\delta_i = 5$. From the figures, we can see that the designed controller can render the closed-loop system stable.

*Example 2* Consider the following stochastic interconnected time-delay system

$$dx_1 = \left(\begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\right) dt + \begin{bmatrix} \delta_3 \left(|x_{11}x_{21}|\right)^{1/2} \\ \delta_4 x_{12}\left(t - 0.3(1+\sin t)\right)\cos t \end{bmatrix} dw$$

$$+ \begin{bmatrix} 0 \\ \delta_1 x_{21}\left(t - 0.6\left(1+\sin t\right)\right) + \delta_2 x_{11}\left(t - 0.5\left(1 + \cos\left(t\right)\right)\right) \end{bmatrix} dt,$$

$$dx_2 = \left\{\begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} \delta_5 \left(|x_{21}\left(t - 0.6\left(1+\sin\left(t\right)\right)\right) x_{12}|\right)^{1/2} \\ 0 \end{bmatrix}\right\} dt$$

$$+ \begin{bmatrix} \delta_6 x_{21} \\ \delta_7 \left(|x_{12}\left(t - 0.3\left(1 + \cos\left(t\right)\right)\right) x_{21}|\right)^{1/2} \end{bmatrix} dw.$$

We can verify that Assumptions 4.1 − 4.4 hold with $P_i = I$. Therefore the following decentralized feedback controllers can be constructed.

$$u_i = -\frac{1}{2}\widehat{\Theta}_i(t) B_i^{\mathrm{T}} P_i x_i \tag{48}$$

with adaptive law

$$\frac{d\widehat{\Theta}_i(t)}{dt} = \|B_i^{\mathrm{T}} P_i x_i\|^2. \tag{49}$$

The initial values are chosen as

$$x_{11}(0) = 2, \quad x_{12}(0) = 1, \quad x_{21}(0) = -1, \quad x_{22}(0) = -2, \quad \Theta_i(0) = 2.$$

When the parameters $\delta_i = 1$, the states response curves are shown in Figure 5.3, while Figure 5.4 depicts the curves when $\delta_i = 5$. From the two figures, the proposed decentralized feedback controllers guarantee the closed-loop system stable.

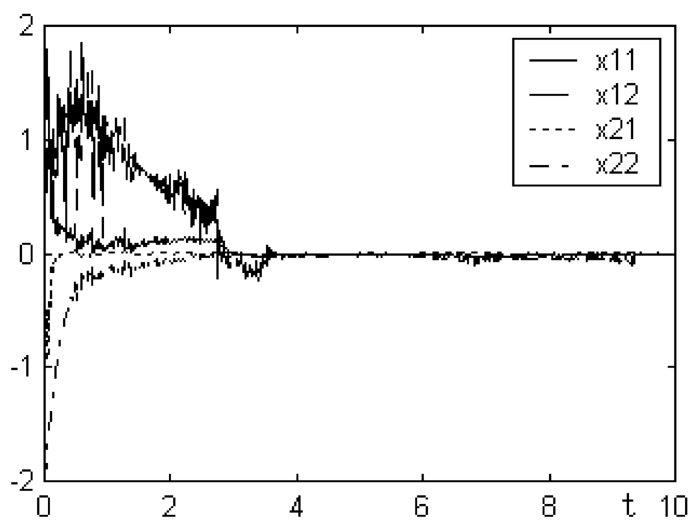


**Figure 5.3**. The states response curves of interconnected systems with $\delta_i = 1$.

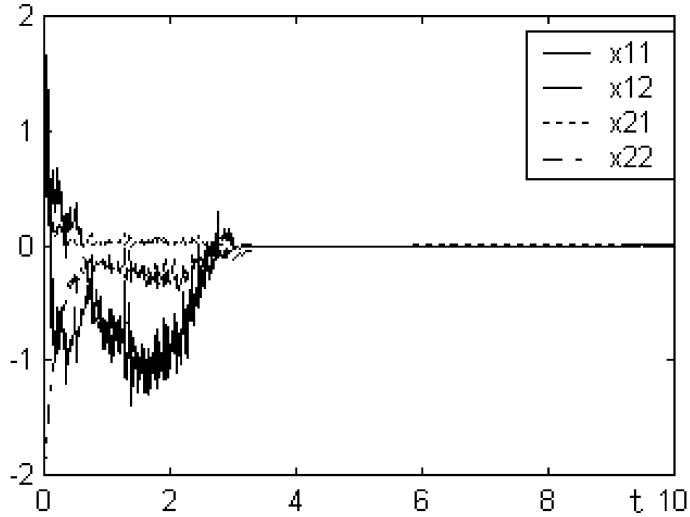


**Figure 5.4.** The states response curves of interconnected systems with $\delta_i = 5$.

## 6 Conclusion

In this paper, the robust control problem for uncertain stochastic time-delay systems is investigated. First we considered a simple class of systems and designed the corresponding adaptive feedback controller. Based on L-K method, we proved that the resulting closed-loop system is asymptotically stable. Next, we studied the problem of adaptive control of a class of time-delay interconnected stochastic systems. Sufficient conditions to construct a desired controller are derived. Simulations on controlling the uncertain systems are conducted and the results showed the potential of the proposed techniques.

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