Nonlinear Dynamics and Systems Theory, 4(3) (2004) 285–301

Robust $\mathcal{H}_\infty$ Filtering for Discrete Stochastic Time-Delay Systems with Nonlinear Disturbances*

Huijun Gao[1], James Lam[2] and Changhong Wang[1]

[1] *Space Control and Inertial Technology Research Center, P.O.Box 1230, Harbin Institute of Technology, Xidazhi Street 92, Harbin, 150001, P. R. China*
[2] *Department of Mechanical Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong*

Received: September 29, 2004; Revised: November 4, 2004

Abstract: This paper deals with the problem of robust $\mathcal{H}_\infty$ filtering for discrete time-delay systems with stochastic perturbation and nonlinear disturbance. It is assumed that the state-dependent noises and the nonlinearities satisfying global Lipschitz conditions enter into both the state and measurement equations, and the system matrices also contain parameter uncertainties residing in a polytope. Attention is focused on the design of robust full-order and reduced-order filters guaranteeing a prescribed noise attenuation level in an $\mathcal{H}_\infty$ sense with respect to all energy-bounded noise inputs for all admissible uncertainties and time delays. Sufficient conditions for the existence of such filters are formulated in terms of a set of linear matrix inequalities, upon which admissible filters can be obtained from the solution of a convex optimization problem. A numerical example is provided to illustrate the applicability of the developed filter design procedure.

Keywords: *Filter design; linear matrix inequality; robust filtering; state-delay systems; stochastic systems; nonlinearity.*

Mathematics Subject Classification (2000): 93E11, 93C10, 93C23.

1 Introduction

During the past decades, stochastic modelling has played an important role in many branches of science such as biology, economics and engineering applications. Therefore, much attention has been drawn to systems with stochastic perturbations from researchers

*This work was partially supported by RGC HKU Grant 7028/04P.

working in related areas. By stochastic systems, we generally refer to systems whose parameter uncertainties are modelled as white noise processes. The appearance of these parameter uncertainties are usually due to the random changes of the environment under which the systems are operated, and thus it is a natural way to represent them in the model by stochastic parameters fluctuating around some deterministic nominal values. This kind of systems has been called systems with random parametric excitation [1], stochastic bilinear systems [20, 30] and linear stochastic systems with multiplicative noise [15, 17, 31]. Analysis and synthesis of stochastic systems have been investigated extensively and many fundamental results for deterministic systems have been extended to stochastic cases. To mention a few, the analysis of asymptotic behaviour can be found in [21]; the optimal control problems were reported in [17, 31]; and recently with the development of $\mathcal{H}_\infty$ control theory, the robust control and filtering results have also been extended to stochastic systems through Ricatti-like and linear matrix inequality (LMI) approaches [8, 18].

On the other hand, since time delay exists commonly in dynamic systems and is frequently a source of instability and poor performance, much theoretical work has been produced for time-delay systems. The most powerful approach for solving problems arising in time-delay systems so far has been the so-called Lyapunov-Krasovskii approach, in which the asymptotic stability as well as performances can be established by employing appropriate Lyapunov-Krasovskii functionals. Within this framework, a great number of results have been reported, including stability analysis [26], state-feedback control [5, 23, 28], output-feedback control [9, 10], filter design [12, 13] and model reduction [34], etc.

The simultaneous presence of stochastic uncertainty and time delays results in stochastic time-delay systems (STDS) have attracted much attention in recent years, and some useful research results related to STDS have been reported in the literature. Among these results, the exponential stability and asymptotic stability of stochastic differential delay equations are investigated in [22, 24]; the problems of stabilization and $\mathcal{H}_\infty$ control via a memoryless state-feedback are considered in [32]; and the filtering problems have also been addressed in [2, 19] for different classes of STDS. These useful results have greatly advanced the analysis and synthesis of stochastic systems. However, it is worth noting that most of the aforementioned results are developed for continuous-time systems, while few results are available for discrete time-delay systems with stochastic perturbations which are also important in practical applications.

In this paper, we are interested in the problem of robust $\mathcal{H}_\infty$ filtering for discrete stochastic time-delay systems with parameter uncertainties and nonlinear disturbances. The parameter uncertainty is assumed to be of polytopic-type, and the nonlinearity satisfies global Lipschitz conditions, entering into both state and measurement equations. Attention is focused on the design of robust full-order and reduced-order filters guaranteeing a prescribed noise attenuation level in an $\mathcal{H}_\infty$ sense with respect to all energy-bounded noise inputs for all admissible uncertainties and time delays. Sufficient conditions for the existence of such filters are formulated in terms of a set of linear matrix inequalities, upon which admissible filters can be obtained from the solution of a convex optimization problem. A numerical example is provided to illustrate the applicability of the developed filter design procedure.

Notations The notations used throughout the paper are fairly standard. The superscript "T" stands for matrix transposition; R^n denotes the n-dimensional Euclidean space and $R^{m\times n}$ is the set of all real matrices of dimension $m \times n$; the notation $P > 0$ means that P is real symmetric and positive definite; I and 0 represent identity matrix and zero matrices; the notation $|\cdot|$ refers to the Euclidean vector norm; $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalue of the corresponding matrix respectively. In symmetric block matrices or long matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry and $\mathrm{diag}\{\dots\}$ stands for a block-diagonal matrix. In addition, $E\{x\}$ and $E\{x|\,y\}$ will, respectively, mean expectation of x and expectation of x conditional on y. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The space of square summable infinite sequence is denoted by $l_2[0,\infty)$.

2 Problem Formulation

Consider the following discrete stochastic time-delay system with nonlinear disturbance:

$$
\begin{aligned}
\mathcal{S}\colon \quad & x_{t+1} = [Ax_t + A_d x_{t-d} + Ff(x_t, x_{t-d}) + B\omega_t] + [Mx_t + M_d x_{t-d}]v_t,\\
& y_t = [Cx_t + C_d x_{t-d} + Gg(x_t, x_{t-d}) + D\omega_t] + [Nx_t + N_d x_{t-d}]v_t,\\
& z_t = Lx_t,\\
& x_t = \phi_t, \quad t = -d,\, -d+1,\, \dots, 0,
\end{aligned}
\tag{1}
$$

where $x_t \in R^n$ is the state vector; $y_t \in R^m$ is the measured output; $z_t \in R^p$ is the signal to be estimated; $\omega_t \in R^l$ is the disturbance input which belongs to $l_2[0,\infty)$; v_t is a zero mean white noise sequence with covariance I; A, A_d, F, B, M, M_d, C, C_d, G, D, N, N_d, L are system matrices with appropriate dimensions; $d > 0$ is a constant time delay; $\{\phi_t\colon\ t = -d, -d+1, \dots, 0\}$ is a given initial condition sequence; $f(x_t, x_{t-d})$, $g(x_t, x_{t-d})$ are known nonlinear functions. Throughout the paper, we make the following assumptions.

Assumption 1 The nonlinear functions satisfy

(1) $f(0,0) = 0$, $g(0,0) = 0$;

(2) (Lipschitz conditions) there exist known real appropriately dimensioned matrices S_1, S_2, T_1, T_2 such that for all x_1, x_2, y_1, y_2 satisfying

$$
\begin{aligned}
\|f(x_1,x_2) - f(y_1,y_2)\| &\le \|S_1(x_1 - y_1)\| + \|S_2(x_2 - y_2)\|\,,\\
\|g(x_1,x_2) - g(y_1,y_2)\| &\le \|T_1(x_1 - y_1)\| + \|T_2(x_2 - y_2)\|\,.
\end{aligned}
$$

Assumption 2 The system matrices are appropriately dimensioned with partially unknown parameters. We assume that

$$
\Omega \triangleq (A, A_d, F, B, M, M_d, C, C_d, G, D, N, N_d, L) \in \mathcal{R}
$$

where $\mathcal{R}$ is a given convex bounded polyhedral domain described by s vertices

$$
\mathcal{R} \triangleq \left\{ \Omega(\lambda)\colon\ \Omega(\lambda) = \sum_{i=1}^{s} \lambda_i \Omega_i;\ \ \sum_{i=1}^{s} \lambda_i = 1,\ \ \lambda_i \ge 0 \right\}
$$

and $\Omega_i \triangleq (A_i, A_{di}, F_i, B_i, M_i, M_{di}, C_i, C_{di}, G_i, D_i, N_i, N_{di}, L_i)$ denotes the vertices of the polytope $\mathcal{R}$.

Remark 1 The system under investigation in this paper contains both parameter and nonlinear uncertainties. As can be seen in Assumption 2, the parameter uncertainties are assumed to be of polytopic-type, entering into all the matrices of the system model. The polytopic uncertainty has been widely used in the problems of robust control and filtering for uncertain systems, see, e.g., [3, 7, 14] and the references therein and many practical systems possess parameter uncertainties which can be either exactly modeled or over-bounded by the polytope $\mathcal{R}$. In addition, the nonlinear uncertainty in Assumption 1 has also been widely used in the literature, see, e.g., [16, 29, 33].

Remark 2 Although there is only a single delay taken into consideration in system $\mathcal{S}$, the results developed in this paper can be easily extended to systems with multiple state delays. The reason why we consider single delay systems is to make our derivation more lucid and to avoid complicated notations. It is also worth mentioning that the results obtained in this paper can be readily extended to the case where v_t enters system $\mathcal{S}$ in a summation form, that is, the dynamic and measurement equations in system $\mathcal{S}$ have the following form

$$x_{t+1} = [Ax_t + A_d x_{t-d} + Ff(x_t, x_{t-d}) + B\omega_t] + \sum_{i=1}^{r} [M_i x_t + M_{di} x_{t-d}]\, v_{ti},$$

$$y_t = [Cx_t + C_d x_{t-d} + Gg(x_t, x_{t-d}) + D\omega_t] + \sum_{i=1}^{r} [N_i x_t + N_{di} x_{t-d}]\, v_{ti}.$$

Here we are interested in estimating the signal z_t by a linear dynamic filter of general structure described by

$$\begin{aligned} \mathcal{F}: \quad \hat{x}_{t+1} &= A_F \hat{x}_t + B_F y_t, \\ \hat{z}_t &= C_F \hat{x}_t, \\ \hat{x}_t &= \varphi_t, \quad t = -d, -d+1, \ldots, 0, \end{aligned} \tag{2}$$

where $\hat{x}_t \in R^k$ is the filter state vector and (A_F, B_F, C_F) are appropriately dimensioned filter matrices to be determined. It should be pointed out that here we are interested not only in the full-order filtering problem (when $k = n$), but also in the reduced-order filtering problem (when $1 \le k < n$). As can be seen in the following, these two filtering problems are solved in a unified framework.

Augmenting the model of $\mathcal{S}$ to include the states of the filter $\mathcal{F}$, we obtain the filtering error system $\mathcal{E}$:

$$\begin{aligned} \mathcal{E}: \quad \xi_{t+1} &= \left[\bar{A}\xi_t + \bar{A}_d K \xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t\right] + \left[\overline{M}\xi_t + \overline{M}_d K \xi_{t-d}\right] v_t, \\ e_t &= \overline{C}\xi_t, \\ \xi_t &= [\,\phi_t^{\mathrm{T}} \quad \varphi_t^{\mathrm{T}}\,]^{\mathrm{T}}, \quad t \in [-d, 0], \end{aligned} \tag{3}$$

where $\xi_t = [\,x_t^{\mathrm{T}} \quad \hat{x}_t^{\mathrm{T}}\,]^{\mathrm{T}}$, $\eta(x_t, x_{t-d}) = [\,f^{\mathrm{T}}(x_t, x_{t-d}) \quad g^{\mathrm{T}}(x_t, x_{t-d})\,]^{\mathrm{T}}$, $e_t = z_t - \hat{z}_t$ and

$$\begin{aligned} &\bar{A} = \begin{bmatrix} A & 0 \\ B_F C & A_F \end{bmatrix}, \quad \bar{A}_d = \begin{bmatrix} A_d \\ B_F C_d \end{bmatrix}, \quad \overline{F} = \begin{bmatrix} F & 0 \\ 0 & B_F G \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B \\ B_F D \end{bmatrix}, \\ &\overline{M} = \begin{bmatrix} M & 0 \\ B_F N & 0 \end{bmatrix}, \quad \overline{M}_d = \begin{bmatrix} M_d \\ B_F N_d \end{bmatrix}, \quad \overline{C} = [\,L \quad -C_F\,], \quad K = [\,I \quad 0\,]. \end{aligned} \tag{4}$$

We first introduce the following definitions.

Definition 1 The *filtering error system* $\mathcal{E}$ in (3) with $\omega_t = 0$ is said to be *mean-square stable* if for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that $E\{|\xi_t|^2\} < \epsilon$, $t > 0$ when $\sup\limits_{-d \le s \le 0} E\{|\xi_s|^2\} < \delta(\epsilon)$. In addition, if $\lim\limits_{t \to \infty} E\{|\xi_t|^2\} = 0$ for any initial conditions, then it is said to be mean-square asymptotically stable.

Definition 2 The *filtering error system* $\mathcal{E}$ in (3) is said to be *mean-square asymptotically stable with an* $\mathcal{H}_\infty$ *disturbance attenuation level* γ if it is mean-square asymptotically stable and under zero-initial conditions $E\{\|e\|_2\} < \gamma\|\omega\|_2$ for all nonzero disturbances $\omega_t \in l_2[0, \infty)$, where

$$E\{\|e\|_2\} \triangleq E\left\{\left(\sum_{t=0}^{\infty} e_t^{\mathrm{T}} e_t\right)^{1/2}\right\}, \quad \|\omega\|_2 \triangleq \left(\sum_{t=0}^{\infty} \omega_t^{\mathrm{T}} \omega_t\right)^{1/2}.$$

Throughout the paper, we make the following assumption.

Assumption 3 System $\mathcal{S}$ in (2) is mean-square asymptotically stable.

Remark 3 Assumption 3 is made based on the fact that there is no control in the system model $\mathcal{S}$ in (1), therefore the original system $\mathcal{S}$ in (1) to be estimated has to be mean-square asymptotically stable, which is a prerequisite for the filtering error system $\mathcal{E}$ in (3) to be mean-square asymptotically stable.

Then the filtering problem to be addressed in this paper is expressed as follows.

Problem RHF (Robust $\mathcal{H}_\infty$ Filtering): Given system $\mathcal{S}$ in (1), develop full-order and reduced-order robust $\mathcal{H}_\infty$ filters of the form $\mathcal{F}$ in (2) such that for all admissible uncertainties, disturbances and time delays the filtering error system $\mathcal{E}$ in (3) is robustly mean-square asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level γ. Filters satisfying this requirement are called robust $\mathcal{H}_\infty$ filters.

Throughout the paper, $(\bar{A}_i, \bar{A}_{di}, \overline{F}_i, \overline{B}_i, \overline{M}_i, \overline{M}_{di}, \overline{C}_i)$ denotes matrices evaluated at each of the vertices of the polytope $\mathcal{R}$. The following lemma will be useful in our derivation.

Lemma 1 *Let* Φ_1, Φ_2, Φ_3 *and* $\Pi > 0$ *be given constant matrices with appropriate dimensions. Then, for any scalar* $\epsilon > 0$ *satisfying* $\epsilon I - \Phi_2^{\mathrm{T}} \Pi \Phi_2 > 0$ *we have*

$$[\Phi_1 + \Phi_2\Phi_3]^{\mathrm{T}} \Pi [\Phi_1 + \Phi_2\Phi_3] \le \Phi_1^{\mathrm{T}} [\Pi^{-1} - \epsilon^{-1}\Phi_2\Phi_2^{\mathrm{T}}]^{-1} \Phi_1 + \epsilon \Phi_3^{\mathrm{T}} \Phi_3$$

3 Filtering Analysis

This section is concerned with the filtering analysis problem. More specifically, assuming that the matrices (A_F, B_F, C_F) of the filter $\mathcal{F}$ in (2) are already known, we shall study the conditions under which the filtering error system $\mathcal{E}$ in (3) is mean-square asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level γ. To ease the exposition of our results, we first consider the stationary case, i.e. $\Omega \in \mathcal{R}$ is fixed. The following theorem shows that the $\mathcal{H}_\infty$ performance of the filtering error system can be guaranteed if there exist some positive definite matrices satisfying certain LMIs. This theorem will play an instrumental role in the filter design problems.

Theorem 1 *Consider system $\mathcal{S}$ in (1) with $\Omega \in \mathcal{R}$ fixed, and suppose the filter matrices (A_F, B_F, C_F) of $\mathcal{F}$ in (2) are given. Then the filtering error system $\mathcal{E}$ in (3) is mean-square asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level bound γ if there exist matrices $P > 0$, $Q > 0$ and a scalar $\epsilon > 0$ satisfying*

$$\begin{bmatrix} -P & 0 & 0 & P\bar{A} & P\bar{A}_d & P\overline{B} & P\overline{F} \\ * & -P & 0 & P\overline{M} & P\overline{M}_d & 0 & 0 \\ * & * & -I & \overline{C} & 0 & 0 & 0 \\ * & * & * & \Theta_1 & 0 & 0 & 0 \\ * & * & * & * & \Theta_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0, \tag{5}$$

where

$$\Theta_1 \triangleq -P + K^{\mathrm{T}} Q K + 2\epsilon K^{\mathrm{T}} \left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1 \right) K,$$
$$\Theta_2 \triangleq -Q + 2\epsilon \left(S_2^{\mathrm{T}} S_2 + T_2^{\mathrm{T}} T_2 \right).$$

Proof Let $\mathcal{X}_t \triangleq \{\xi_{t-d}, \xi_{t-d+1}, \ldots, \xi_t\}$, choose a Lyapunov functional candidate for the filtering error system $\mathcal{E}$

$$\begin{gathered} W_t(\mathcal{X}_t) \triangleq W_1 + W_2, \\ W_1 = \xi_t^{\mathrm{T}} P \xi_t, \qquad W_2 = \sum_{i=t-d}^{t-1} \xi_i^{\mathrm{T}} K^{\mathrm{T}} Q K \xi_i, \end{gathered} \tag{6}$$

where P, Q are real symmetric positive definite matrices to be determined. Then, along the solution of the filtering error system $\mathcal{E}$ we have

$$\begin{aligned} \mathcal{J} \triangleq E\{W_{t+1}(\mathcal{X}_{t+1}) \mid \mathcal{X}_t\} - W_t(\mathcal{X}_t) &= E\{[W_{t+1}(\mathcal{X}_{t+1}) - W_t(\mathcal{X}_t)] \mid \mathcal{X}_t\} \\ &= E\{\Delta W_1 \mid \mathcal{X}_t\} + E\{\Delta W_2 \mid \mathcal{X}_t\} \end{aligned} \tag{7}$$

where

$$\begin{aligned} E\left\{\Delta W_1 \left| \mathcal{X}_t \right.\right\} &= E\left\{ \left(\xi_{t+1}^{\mathrm{T}} P \xi_{t+1} - \xi_t^{\mathrm{T}} P \xi_t \right) \Big| \mathcal{X}_t \right\} \\ &= E\Big\{ \Big(\left[\bar{A}\xi_t + \bar{A}_d K \xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t \right]^{\mathrm{T}} P \\ &\quad \times \left[\bar{A}\xi_t + \bar{A}_d K \xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t \right] \\ &\quad + 2\left\{ \left[\overline{M}\xi_t + \overline{M}_d K \xi_{t-d} \right] v_t \right\}^{\mathrm{T}} P \left[\bar{A}\xi_t + \bar{A}_d K \xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t \right] \\ &\quad + \left\{ \left[\overline{M}\xi_t + \overline{M}_d K \xi_{t-d} \right] v_t \right\}^{\mathrm{T}} P \left\{ \left[\overline{M}\xi_t + \overline{M}_d K \xi_{t-d} \right] v_t \right\} - \xi_t^{\mathrm{T}} P \xi_t \Big) \Big| \mathcal{X}_t \Big\}, \end{aligned} \tag{8}$$

$$\begin{aligned} E\left\{\Delta W_2 \left| \mathcal{X}_t \right.\right\} &= E\Big\{ \Big(\sum_{i=t+1-d}^{t} \xi_i^{\mathrm{T}} K^{\mathrm{T}} Q K \xi_i - \sum_{i=t-d}^{t-1} \xi_i^{\mathrm{T}} K^{\mathrm{T}} Q K \xi_i \Big) \Big| \mathcal{X}_t \Big\} \\ &= E\left\{ \left(\xi_t^{\mathrm{T}} K^{\mathrm{T}} Q K \xi_t - \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}} Q K \xi_{t-d} \right) \left| \mathcal{X}_t \right.\right\}. \end{aligned} \tag{9}$$

Then from (7)–(9), we obtain

$$\begin{aligned}\mathcal{J} = [\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t]^{\mathrm{T}} P[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t]\\ + [\overline{M}\xi_t + \overline{M}_d K\xi_{t-d}]^{\mathrm{T}} P[\overline{M}\xi_t + \overline{M}_d K\xi_{t-d}] - \xi_t^{\mathrm{T}} P\xi_t\\ + \xi_t^{\mathrm{T}} K^{\mathrm{T}} QK\xi_t - \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}} QK\xi_{t-d}.\end{aligned} \tag{10}$$

In addition, using Assumption 1, we have

$$\begin{aligned}\|f(x_t, x_{t-d})\| \le \|S_1 x_t\| + \|S_2 x_{t-d}\|,\\ \|g(x_t, x_{t-d})\| \le \|T_1 x_t\| + \|T_2 x_{t-d}\|,\end{aligned}$$

which yields

$$\begin{aligned}\|f(x_t, x_{t-d})\|^2 \le 2(\|S_1 x_t\|^2 + \|S_2 x_{t-d}\|^2),\\ \|g(x_t, x_{t-d})\|^2 \le 2(\|T_1 x_t\|^2 + \|T_2 x_{t-d}\|^2).\end{aligned}$$

Then

$$\begin{aligned}\eta^{\mathrm{T}}(x_t, x_{t-d})\eta(x_t, x_{t-d}) &= f^{\mathrm{T}}(x_t, x_{t-d}) f(x_t, x_{t-d}) + g^{\mathrm{T}}(x_t, x_{t-d}) g(x_t, x_{t-d})\\ &\le 2\left(\|S_1 x_t\|^2 + \|S_2 x_{t-d}\|^2 + \|T_1 x_t\|^2 + \|T_2 x_{t-d}\|^2\right)\\ &= 2\xi_t^{\mathrm{T}} K^{\mathrm{T}}\left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1\right) K\xi_t + 2\xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}}\left(S_2^{\mathrm{T}} S_2 + T_2^{\mathrm{T}} T_2\right) K\xi_{t-d}.\end{aligned} \tag{11}$$

Since (5) implies $\epsilon > 0$ and $\epsilon I - \overline{F}^{\mathrm{T}} P\overline{F} > 0$, by identifying $\Phi_1 = \bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{B}\omega_t$, $\Phi_2 = \overline{F}$, $\Phi_3 = \eta(x_t, x_{t-d})$ and $\Pi = P$ in Lemma 1 , we have an upper bound for the first term of $\mathcal{J}$ in (10)

$$\begin{aligned}&[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t]^{\mathrm{T}} P[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{F}\eta(x_t, x_{t-d}) + \overline{B}\omega_t]\\ &\le [\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{B}\omega_t]^{\mathrm{T}}\Psi[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{B}\omega_t] + \epsilon\eta^{\mathrm{T}}(x_t, x_{t-d})\eta(x_t, x_{t-d}),\end{aligned} \tag{12}$$

where $\Psi = \left[P^{-1} - \epsilon^{-1}\overline{F}\overline{F}^{\mathrm{T}}\right]^{-1}$.

Then from (10)–(12) we have

$$\begin{aligned}\mathcal{J} \le\ & \left[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{B}\omega_t\right]^{\mathrm{T}}\Psi\left[\bar{A}\xi_t + \bar{A}_d K\xi_{t-d} + \overline{B}\omega_t\right]\\ & + 2\epsilon\xi_t^{\mathrm{T}} K^{\mathrm{T}}\left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1\right) K\xi_t + 2\epsilon\xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}}\left(S_2^{\mathrm{T}} S_2 + T_2^{\mathrm{T}} T_2\right) K\xi_{t-d}\\ & + \left[\overline{M}\xi_t + \overline{M}_d K\xi_{t-d}\right]^{\mathrm{T}} P\left[\overline{M}\xi_t + \overline{M}_d K\xi_{t-d}\right] - \xi_t^{\mathrm{T}} P\xi_t\\ & + \xi_t^{\mathrm{T}} K^{\mathrm{T}} QK\xi_t - \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}} QK\xi_{t-d}\\ =\ & \sigma_t^{\mathrm{T}}\Xi\sigma_t,\end{aligned} \tag{13}$$

where

$$\sigma_t = [\,\xi_t^{\mathrm{T}} \quad \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}} \quad \omega_t^{\mathrm{T}}\,]^{\mathrm{T}},$$

$$\Xi = \begin{bmatrix} \begin{pmatrix} \bar{A}^{\mathrm{T}}\Psi\bar{A} - P + K^{\mathrm{T}} QK + \overline{M}^{\mathrm{T}} P\overline{M}\\ +2\epsilon K^{\mathrm{T}}\left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1\right) K \end{pmatrix} & \bar{A}^{\mathrm{T}}\Psi\bar{A}_d + \overline{M}^{\mathrm{T}} P\overline{M}_d & \bar{A}^{\mathrm{T}}\Psi\overline{B}\\ * & \begin{pmatrix} -Q + \bar{A}_d^{\mathrm{T}}\Psi\bar{A}_d + \overline{M}_d^{\mathrm{T}} P\overline{M}_d\\ +2\epsilon\left(S_2^{\mathrm{T}} S_2 + T_2^{\mathrm{T}} T_2\right) \end{pmatrix} & \bar{A}_d^{\mathrm{T}}\Psi\overline{B}\\ * & * & \overline{B}^{\mathrm{T}}\Psi\overline{B} \end{bmatrix}.$$

Therefore, when assuming zero disturbance input $\omega_t = 0$, it follows that

$$\mathcal{J} \leq [\,\xi_t^{\mathrm{T}} \quad \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}}\,]\,\bar{\Xi}\,[\,\xi_t^{\mathrm{T}} \quad \xi_{t-d}^{\mathrm{T}} K^{\mathrm{T}}\,]^{\mathrm{T}}$$

where

$$\bar{\Xi} = \begin{bmatrix} \begin{pmatrix} \bar{A}^{\mathrm{T}}\Psi\bar{A} - P + K^{\mathrm{T}}QK + \\ 2\epsilon K^{\mathrm{T}}\left(S_1^{\mathrm{T}}S_1 + T_1^{\mathrm{T}}T_1\right)K + \overline{M}^{\mathrm{T}}P\overline{M} \end{pmatrix} & \bar{A}^{\mathrm{T}}\Psi\bar{A}_d + \overline{M}^{\mathrm{T}}P\overline{M}_d \\ * & \begin{pmatrix} -Q + \bar{A}_d^{\mathrm{T}}\Psi\bar{A}_d + 2\epsilon\left(S_2^{\mathrm{T}}S_2 + T_2^{\mathrm{T}}T_2\right) \\ +\overline{M}_d^{\mathrm{T}}P\overline{M}_d \end{pmatrix} \end{bmatrix}.$$

By Schur complement [4], LMI (5) implies the negative definiteness of $\bar{\Xi}$, therefore, for $\mathcal{X}_t \neq 0$ we have $\mathcal{J} < 0$, that is,

$$E\left\{W_{t+1}(\mathcal{X}_{t+1})\,|\,\mathcal{X}_t\right\} < W_t(\mathcal{X}_t)$$

which means that there exists $0 < \beta_t < 1$ satisfying

$$E\left\{W_{t+1}(\mathcal{X}_{t+1})|\,\mathcal{X}_t\right\} < \beta_t W_t(\mathcal{X}_t).$$

It is easy to obtain by using this relationship recursively that

$$E\left\{W_t(\mathcal{X}_t)|\,\mathcal{X}_0\right\} < \prod_{i=0}^{t-1}\beta_i W_0(\mathcal{X}_0) \leq \alpha^t W_0(\mathcal{X}_0)$$

where $\alpha = \max_t \beta_t$. Thus $0 < \alpha < 1$ and we have

$$E\left\{\sum_{t=0}^{N}\left[W_t(\mathcal{X}_t)|\,\mathcal{X}_0\right]\right\} < (1 + \alpha + \cdots + \alpha^N)\,W_0(\mathcal{X}_0) = \frac{1-\alpha^{N+1}}{1-\alpha}\,W_0(\mathcal{X}_0).$$

Since $Q > 0$, then

$$\lim_{N\to\infty} E\left\{\sum_{t=0}^{N}\left[x_t^{\mathrm{T}}Px_t\,\middle|\,\mathcal{X}_0\right]\right\} < \frac{1}{1-\alpha}\,W_0(\mathcal{X}_0).$$

Using the Rayleigh quotient inequality, we have

$$\lim_{N\to\infty} E\left\{\sum_{t=0}^{N}\left[x_t^{\mathrm{T}}x_t|\,\mathcal{X}_0\right]\right\} < \frac{1}{(1-\alpha)\lambda_{\min}(P)}\,W_0(\mathcal{X}_0)$$

which means $E\{|x_t|^2\} \to 0$ as $t \to \infty$, then from Definition 1, we know that the filtering error system $\mathcal{E}$ in (3) with $\omega_t = 0$ is mean-square asymptotically stable.

To establish the $\mathcal{H}_\infty$ performance, assume zero initial condition, we have $W_0(\mathcal{X}_0) = 0$. Now consider the following index

$$\mathcal{I} \triangleq E\left\{\sum_{t=0}^{\infty}\left(e_t^{\mathrm{T}}e_t - \gamma^2\omega_t^{\mathrm{T}}\omega_t\right)\right\}. \tag{14}$$

Then, with (13) for all nonzero ω_t we have

$$\begin{aligned}\mathcal{I} &= E\Big\{\sum_{t=0}^{\infty}\left(e_t^{\mathrm{T}}e_t-\gamma^2\omega_t^{\mathrm{T}}\omega_t+E\left\{W_{t+1}(\mathcal{X}_{t+1})|\,\mathcal{X}_t\right\}-W_t(\mathcal{X}_t)\right)\Big\}-E\left\{W_\infty(\mathcal{X}_\infty)\right\}\\ &\le E\Big\{\sum_{t=0}^{\infty}\left(e_t^{\mathrm{T}}e_t-\gamma^2\omega_t^{\mathrm{T}}\omega_t+\mathcal{J}\right)\Big\}=E\Big\{\sum_{t=0}^{\infty}\sigma_t^{\mathrm{T}}\tilde{\Xi}\sigma_t\Big\}\end{aligned}$$

where

$$\tilde{\Xi}=\begin{bmatrix}\begin{pmatrix}\bar{A}^{\mathrm{T}}\Psi\bar{A}-P+K^{\mathrm{T}}QK\\ +\overline{M}^{\mathrm{T}}P\overline{M}+\overline{C}^{\mathrm{T}}\overline{C}\\ +2\epsilon K^{\mathrm{T}}\left(S_1^{\mathrm{T}}S_1+T_1^{\mathrm{T}}T_1\right)K\end{pmatrix} & \bar{A}^{\mathrm{T}}\Psi\bar{A}_d+\overline{M}^{\mathrm{T}}P\overline{M}_d & \bar{A}^{\mathrm{T}}\Psi\overline{B}\\ * & \begin{pmatrix}-Q+\bar{A}_d^{\mathrm{T}}\Psi\bar{A}_d+\overline{M}_d^{\mathrm{T}}P\overline{M}_d\\ +2\epsilon\left(S_2^{\mathrm{T}}S_2+T_2^{\mathrm{T}}T_2\right)\end{pmatrix} & \bar{A}_d^{\mathrm{T}}\Psi\overline{B}\\ * & * & -\gamma^2I+\overline{B}^{\mathrm{T}}\Psi\overline{B}\end{bmatrix}.$$

Then, by Schur complement, (5) guarantees $\tilde{\Xi}<0$, which further implies $\mathcal{I}<0$ and $E\{\|e\|_2\}<\gamma\|\omega\|_2$, then the filtering error system $\mathcal{E}$ in (3) is mean-square asymptotically stable with an $\mathcal{H}_\infty$ noise attenuation level bound γ, and the proof is completed.

Remark 4 Theorem 1 presents a sufficient condition for the $\mathcal{H}_\infty$ performance of discrete-time stochastic time-delay systems with nonlinear disturbances. It is worth pointing out that the condition presented in Theorem 1 is an LMI condition and therefore can be easily tested by standard numerical software [11]. In the case when we assume $v_t=0$, that is, no stochastic uncertainty is present in system $\mathcal{S}$, LMI (5) becomes

$$\begin{bmatrix}-P & 0 & P\bar{A} & P\bar{A}_d & P\overline{B} & P\overline{F}\\ * & -I & \overline{C} & 0 & 0 & 0\\ * & * & \Theta_1 & 0 & 0 & 0\\ * & * & * & \Theta_2 & 0 & 0\\ * & * & * & * & -\gamma^2I & 0\\ * & * & * & * & * & -\epsilon I\end{bmatrix}<0. \tag{15}$$

LMI (15) is an $\mathcal{H}_\infty$ performance condition for linear discrete time-delay systems with nonlinear disturbances. In addition, if we further assume $f(x_t,x_{t-d})=0$ and $g(x_t,x_{t-d})=0$, then LMI (5) becomes

$$\begin{bmatrix}-P & 0 & P\bar{A} & P\bar{A}_d & P\overline{B}\\ * & -I & \overline{C} & 0 & 0\\ * & * & -P+K^{\mathrm{T}}QK & 0 & 0\\ * & * & * & -Q & 0\\ * & * & * & * & -\gamma^2I\end{bmatrix}<0. \tag{16}$$

LMI (16) is an $\mathcal{H}_\infty$ performance condition for linear discrete time-delay systems.

Then, the following theorem provides a sufficient condition of robust $\mathcal{H}_\infty$ performance for the filtering error system $\mathcal{E}$ in (3).

Theorem 2 *Consider system $\mathcal{S}$ in (1) with $\Omega \in \mathcal{R}$ representing uncertain matrices, and suppose the filter matrices (A_F, B_F, C_F) of $\mathcal{F}$ in (2) are given. Then the filtering error system $\mathcal{E}$ in (3) is robustly mean-square asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level bound γ if there exist matrices $P_i > 0$, $Q_i > 0$, V and scalars $\epsilon_i > 0$ satisfying*

$$\begin{bmatrix} P_i - V - V^{\mathrm{T}} & 0 & 0 & V^{\mathrm{T}}\bar{A}_i & V^{\mathrm{T}}\bar{A}_{di} & V^{\mathrm{T}}\overline{B}_i & V^{\mathrm{T}}\overline{F}_i \\ * & P_i - V - V^{\mathrm{T}} & 0 & V^{\mathrm{T}}\overline{M}_i & V^{\mathrm{T}}\overline{M}_{di} & 0 & 0 \\ * & * & -I & \overline{C}_i & 0 & 0 & 0 \\ * & * & * & \Pi_1 & 0 & 0 & 0 \\ * & * & * & * & \Pi_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\epsilon_i I \end{bmatrix} < 0 \tag{17}$$

$$\forall\, i = 1, \ldots, s,$$

where

$$\Pi_1 = -P_i + K^{\mathrm{T}} Q_i K + 2\epsilon_i K^{\mathrm{T}}\left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1\right) K,$$
$$\Pi_2 = -Q_i + 2\epsilon_i \left(S_2^{\mathrm{T}} S_2 + T_2^{\mathrm{T}} T_2\right).$$

Proof LMIs (17) guarantee that for any fixed $\Omega \in \mathcal{R}$, there exist matrices $P > 0$, $Q > 0$, V and a scalar $\epsilon > 0$ satisfying

$$\begin{bmatrix} P - V - V^{\mathrm{T}} & 0 & 0 & V^{\mathrm{T}}\bar{A} & V^{\mathrm{T}}\bar{A}_d & V^{\mathrm{T}}\overline{B} & V^{\mathrm{T}}\overline{F} \\ * & P - V - V^{\mathrm{T}} & 0 & V^{\mathrm{T}}\overline{M} & V^{\mathrm{T}}\overline{M}_d & 0 & 0 \\ * & * & -I & \overline{C} & 0 & 0 & 0 \\ * & * & * & \Theta_1 & 0 & 0 & 0 \\ * & * & * & * & \Theta_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0. \tag{18}$$

In the following we will show that (18) is equivalent to (5). On one hand, if (5) holds, (18) is readily established by choosing $V = V^{\mathrm{T}} = P$. On the other hand, if (18) holds, we can explore the fact that V is nonsingular. In addition, we have $(P - V)^{\mathrm{T}} P^{-1} (P - V) \geq 0$, which implies that $-V^{\mathrm{T}} P^{-1} V \leq P - V^{\mathrm{T}} - V$. Therefore we can conclude from (18) that

$$\begin{bmatrix} -V^{\mathrm{T}}P^{-1}V & 0 & 0 & V^{\mathrm{T}}\bar{A} & V^{\mathrm{T}}\bar{A}_d & V^{\mathrm{T}}\overline{B} & V^{\mathrm{T}}\overline{F} \\ * & -V^{\mathrm{T}}P^{-1}V & 0 & V^{\mathrm{T}}\overline{M} & V^{\mathrm{T}}\overline{M}_d & 0 & 0 \\ * & * & -I & \overline{C} & 0 & 0 & 0 \\ * & * & * & \Theta_1 & 0 & 0 & 0 \\ * & * & * & * & \Theta_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\epsilon I \end{bmatrix} < 0. \tag{19}$$

Performing a congruence transformation to (19) by $\mathrm{diag}\left\{I, V^{-1}P, V^{-1}P, I, I, I, I\right\}$ yields (5), then the proof is completed.

Remark 5 Instead of directly extending Theorem 1 to polytopic uncertain systems based on the notion of quadratic stability, here we incorporate a new result of parameter-dependent stability [6] to reduce the conservatism of filter designs in the quadratic framework. Through the introduction of the slack variable V, the sufficient robust $\mathcal{H}_\infty$ performance condition resulting from Theorem 2 entails different positive definite matrices P_i and Q_i for each vertex of the polytope $\mathcal{R}$, thus enabling us to obtain a parameter-dependent performance criteria. To illustrate the benefit of such performance conditions, let $\bar{\Omega}(\lambda)$ denotes any given point of the polytope $\mathcal{R}$. If we can find feasible solutions in the light of (17), then it is not difficult to show that the Lyapunov matrices defined in (6) for any fixed point $\bar{\Omega}(\lambda)$ can be recovered by

$$P(\lambda) = \sum_{i=1}^{s} \lambda_i P_i, \qquad Q(\lambda) = \sum_{i=1}^{s} \lambda_i Q_i,$$

which implies that there are different Lyapunov functionals for different points in the polytope. Then, the Lyapunov functional defined in (6) for the whole uncertainty domain $\mathcal{R}$ can be expressed as

$$W_t(\mathcal{X}_t, \lambda) = \xi_t^{\mathrm{T}} P(\lambda) \xi_t + \sum_{i=t-d}^{t-1} \xi_i^{\mathrm{T}} K^{\mathrm{T}} Q(\lambda) K \xi_i \tag{20}$$

which is dependent of the parameter λ.

4 Filter Design

In this section we will focus on the design of full-order and reduced-order $\mathcal{H}_\infty$ filters of the form $\mathcal{F}$ based on Theorem 2. That is, to determine the filter matrices (A_F, B_F, C_F) which will guarantee the filtering error system $\mathcal{E}$ to be mean-square asymptotically stable with an $\mathcal{H}_\infty$ performance. The following theorem provides sufficient conditions for the existence of such $\mathcal{H}_\infty$ filters for system $\mathcal{S}$.

Theorem 3 *Consider system $\mathcal{S}$ in (1) with $\Omega \in \mathcal{R}$ representing uncertain matrices. Then an admissible robust $\mathcal{H}_\infty$ filter of the form $\mathcal{F}$ in (2) exists if there exist matrices X, Y, Z, $\bar{A}_F$, $\overline{B}_F$, $\overline{C}_F$, P_{1i}, P_{2i}, P_{3i}, Q_i and scalar $\epsilon_i > 0$ for $i = 1, \dots, s$ satisfying*

$$\begin{bmatrix} \Upsilon_2 & 0 & 0 & \Upsilon_4 & \Upsilon_8 & \Upsilon_{10} & \Upsilon_1 \\ * & \Upsilon_2 & 0 & \Upsilon_5 & \Upsilon_9 & 0 & 0 \\ * & * & -I & \Upsilon_6 & 0 & 0 & 0 \\ * & * & * & \Upsilon_7 & 0 & 0 & 0 \\ * & * & * & * & \Pi_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\epsilon_i I \end{bmatrix} < 0, \tag{21}$$

$$\begin{bmatrix} P_{1i} & P_{2i} \\ * & P_{3i} \end{bmatrix} > 0, \tag{22}$$

where

$$\Upsilon_1 = \begin{bmatrix} XF_i & E^{\mathrm{T}} \overline{B}_F G_i \\ Y^{\mathrm{T}} F_i & \overline{B}_F G_i \end{bmatrix},$$

$$\begin{aligned}
\Upsilon_2 &= \begin{bmatrix} P_{1i} - X - X^{\mathrm{T}} & P_{2i} - Y - E^{\mathrm{T}} Z \\ * & P_{3i} - Z^{\mathrm{T}} - Z \end{bmatrix}, \\
\Upsilon_4 &= \begin{bmatrix} X^{\mathrm{T}} A_i + E^{\mathrm{T}} \overline{B}_F C_i & E^{\mathrm{T}} \bar{A}_F \\ Y^{\mathrm{T}} A_i + \overline{B}_F C_i & \bar{A}_F \end{bmatrix}, \\
\Upsilon_5 &= \begin{bmatrix} X^{\mathrm{T}} M_i + E^{\mathrm{T}} \overline{B}_F N_i & 0 \\ Y^{\mathrm{T}} M_i + \overline{B}_F N_i & 0 \end{bmatrix}, \\
\Upsilon_6 &= \begin{bmatrix} L_i & -\overline{C}_F \end{bmatrix}, \\
\Upsilon_7 &= \begin{bmatrix} -P_{1i} + Q_i + 2\epsilon_i \left(S_1^{\mathrm{T}} S_1 + T_1^{\mathrm{T}} T_1 \right) & -P_{2i} \\ -P_{2i}^{\mathrm{T}} & -P_{3i} \end{bmatrix}, \\
\Upsilon_8 &= \begin{bmatrix} X^{\mathrm{T}} A_{di} + E^{\mathrm{T}} \overline{B}_F C_{di} \\ Y^{\mathrm{T}} A_{di} + \overline{B}_F C_{di} \end{bmatrix}, \\
\Upsilon_9 &= \begin{bmatrix} X^{\mathrm{T}} M_{di} + E^{\mathrm{T}} \overline{B}_F N_{di} \\ Y^{\mathrm{T}} M_{di} + \overline{B}_F N_{di} \end{bmatrix}, \\
\Upsilon_{10} &= \begin{bmatrix} X^{\mathrm{T}} B_i + E^{\mathrm{T}} \overline{B}_F D_i \\ Y^{\mathrm{T}} B_i + \overline{B}_F D_i \end{bmatrix}, \\
E &= \begin{bmatrix} I_{k \times k} & 0_{k \times (n-k)} \end{bmatrix}.
\end{aligned}$$

Moreover, if the above condition has a set of feasible solution $(X, Y, Z, \bar{A}_F, \overline{B}_F, \overline{C}_F, P_{1i}, P_{2i}, P_{3i}, Q_i, \epsilon_i)$*, the matrices for an admissible robust* $\mathcal{H}_\infty$ *filter in the form of* $\mathcal{F}$ *in (2) can be calculated by the following steps:*

(1) *find square and nonsingular matrices* $S \in R^{k \times k}$ *and* $T \in R^{k \times k}$ *satisfying* $Z = S^{\mathrm{T}} T^{-1} S$*;*
(2) *calculate the matrices for desired filter matrices by*

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \begin{bmatrix} S^{-T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_F & \overline{B}_F \\ \overline{C}_F & 0 \end{bmatrix} \begin{bmatrix} S^{-1} T & 0 \\ 0 & I \end{bmatrix}. \tag{23}$$

Proof Since LMIs (21) and (22) implies $P_{3i} - Z - Z^{\mathrm{T}} < 0$ and $P_{3i} > 0$, we can infer that $Z + Z^{\mathrm{T}} > 0$, therefore Z is nonsingular. Then we can always find square and nonsingular $k \times k$ matrices S and T satisfying $Z = S^{\mathrm{T}} T^{-1} S$. Therefore, the matrices (A_F, B_F, C_F) are uniquely defined in (23). Now introduce the following matrix variables:

$$J = \begin{bmatrix} I & 0 \\ 0 & T^{-1} S \end{bmatrix}, \quad V = \begin{bmatrix} X & Y S^{-1} T \\ SE & T \end{bmatrix}, \quad P_i = J^{-T} \begin{bmatrix} P_{1i} & P_{2i} \\ P_{2i}^{\mathrm{T}} & P_{3i} \end{bmatrix} J^{-1}. \tag{24}$$

Then, it is easy to see that the matrix J defined above is nonsingular and we have $P_i > 0$. In the following we will prove that the filter $\mathcal{F}$ in (2) with state-space realization (A_F, B_F, C_F) defined in (23) is an admissible robust $\mathcal{H}_\infty$ filter such that the filtering error system $\mathcal{E}$ in (3) is mean-square asymptotically stable with a guaranteed $\mathcal{H}_\infty$ performance.

Now, by some algebraic matrix manipulations, it can be established that (21) is equivalent to

$$\begin{bmatrix} \begin{matrix} J^{\mathrm{T}}(P_i - \\ V - V^{\mathrm{T}})J \end{matrix} & 0 & 0 & J^{\mathrm{T}}V^{\mathrm{T}}\bar{A}_iJ & J^{\mathrm{T}}V^{\mathrm{T}}\bar{A}_{di} & J^{\mathrm{T}}V^{\mathrm{T}}\overline{B}_i & J^{\mathrm{T}}V^{\mathrm{T}}\overline{F}_i \\ * & \begin{matrix} J^{\mathrm{T}}(P_i - \\ V - V^{\mathrm{T}})J \end{matrix} & 0 & J^{\mathrm{T}}V^{\mathrm{T}}\overline{M}_iJ & J^{\mathrm{T}}V^{\mathrm{T}}\overline{M}_{di} & 0 & 0 \\ * & * & -I & \overline{C}_iJ & 0 & 0 & 0 \\ * & * & * & J^{\mathrm{T}}\Pi_1J & 0 & 0 & 0 \\ * & * & * & * & \Pi_2 & 0 & 0 \\ * & * & * & * & * & -\gamma^2I & 0 \\ * & * & * & * & * & * & -\epsilon_iI \end{bmatrix} < 0. \tag{25}$$

The equivalence between (21) and (25) can be verified in a reverse order by the following steps. First, by substituting (A_F, B_F, C_F) defined in (23) into (4), the matrices $\left(\bar{A}, \bar{A}_d, \overline{F}, \overline{B}, \overline{M}, \overline{M}_d, \overline{C}\right)$ of the filtering error system $\mathcal{E}$ in (3) can be obtained as

$$\bar{A} = \begin{bmatrix} A & 0 \\ S^{-T}\overline{B}_FC & S^{-T}\bar{A}_FS^{-1}T \end{bmatrix}, \qquad \bar{A}_d = \begin{bmatrix} A_d \\ S^{-T}\overline{B}_FC_d \end{bmatrix},$$

$$\overline{F} = \begin{bmatrix} F & 0 \\ 0 & S^{-T}\overline{B}_FG \end{bmatrix}, \qquad \overline{B} = \begin{bmatrix} B \\ S^{-T}\overline{B}_FD \end{bmatrix}, \qquad \overline{M} = \begin{bmatrix} M & 0 \\ S^{-T}\overline{B}_FN & 0 \end{bmatrix}, \tag{26}$$

$$\overline{M}_d = \begin{bmatrix} M_d \\ S^{-T}\overline{B}_FN_d \end{bmatrix}, \qquad \overline{C} = [\, L \quad -\overline{C}_FS^{-1}T \,].$$

Then by substituting the matrices J, P_i, V defined in (24) and the matrices $(\bar{A}, \bar{A}_d, \overline{F}, \overline{B}, \overline{M}, \overline{M}_d, \overline{C})$ given by (26) into (25), and by considering the relationship $Z = S^{\mathrm{T}}T^{-1}S$, we obtain inequality (21) after some straightforward matrix manipulations.

Now, performing a congruence transformation to (25) by $\mathrm{diag}\{J^{-1}, J^{-1}, I, J^{-1}, I, I, I\}$ yields (17). Therefore, we conclude from Theorem 2 that the filter $\mathcal{F}$ in (2) with state-space realization (A_F, B_F, C_F) defined in (24) is an admissible robust $\mathcal{H}_\infty$ filter such that the filtering error system $\mathcal{E}$ in (3) is mean-square asymptotically stable with a guaranteed $\mathcal{H}_\infty$ performance, and the proof is completed.

Remark 6 To obtain certain LMI conditions for the existence of desired filters, usually linearization procedures have to be adopted. Since the standard linearization methods adopted in [25, 27] assume the off-diagonal entry of certain matrix (the matrix to be partitioned, in this paper it is V in Theorem 2) to be square and nonsingular, they can only be used to deal with the full-order filtering problem. To keep the reduced-order filter design tractable, here we have sought a different linearization procedure, which solves both the full-order and reduced-order filtering synthesis problems in a unified framework. It is worth noting that the matrix E defined in Theorem 3 plays an instrumental role. For the full-order filtering, the matrix E becomes an identity matrix of dimension n, and for the reduced-order case, we have imposed certain structural restriction on the $(2,1)$ block entry of the matrix V, which introduces some overdesign into the filter design.

Remark 7 Theorem 3 casts the robust $\mathcal{H}_\infty$ filtering problem into an LMI feasibility test, and any feasible solution to the conditions presented in Theorem 3 will yield a suitable filter, which can be obtained by following the two steps presented in Theorem 3. Another formulation of suitable filters upon these feasible solution can be given by

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \begin{bmatrix} Z^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_F & \overline{B}_F \\ \overline{C}_F & 0 \end{bmatrix}. \tag{27}$$

To prove (27), let us denote the filter z transfer function from $y(t)$ to $\hat{z}(t)$ by $T_{\hat{z}y}(z) = C_F(zI - A_F)^{-1}B_F$. By substituting the filter matrices with (23) and by considering the relationship $Z = S^{\mathrm{T}}T^{-1}S$, we have

$$\begin{aligned} T_{\hat{z}y}(s) &= \overline{C}_F S^{-1}T(zI - S^{-T}\bar{A}_F S^{-1}T)^{-1}S^{-T}\overline{B}_F \\ &= \overline{C}_F(zI - Z^{-1}\bar{A}_F)^{-1}Z^{-1}\overline{B}_F. \end{aligned}$$

Therefore, an admissible filter can also be given by (27).

Remark 8 Note that (21) and (22) are LMIs not only over the matrix variables, but also over the scalar γ^2. This implies that the scalar γ^2 can be included as an optimization variable to obtain the minimum noise attenuation level bound. Then the minimum (in terms of the feasibility of (21) and (22)) guaranteed cost of robust $\mathcal{H}_\infty$ filters can be readily found by solving the following convex optimization problems

Problem RHFD (Robust $\mathcal{H}_\infty$ filter design): Minimize γ subject to (21) and (22) over $(X, Y, Z, \bar{A}_F, \overline{B}_F, \overline{C}_F, P_{1i}, P_{2i}, P_{3i}, Q_i, \epsilon_i)$.

Remark 9 Theorem 3 presents a sufficient condition for the existence of robust $\mathcal{H}_\infty$ filters for discrete-time stochastic time-delay systems with nonlinear disturbance. In the case when we assume $v_t = 0$, that is, no stochastic uncertainty is present in system $\mathcal{S}$, LMI (21) becomes

$$\begin{bmatrix} \Upsilon_2 & 0 & \Upsilon_4 & \Upsilon_8 & \Upsilon_{10} & \Upsilon_1 \\ * & -I & \Upsilon_6 & 0 & 0 & 0 \\ * & * & \Upsilon_7 & 0 & 0 & 0 \\ * & * & * & \Theta_2 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -\epsilon_i I \end{bmatrix} < 0.$$

In addition, if we further assume $f(x_t, x_{t-d}) = 0$ and $g(x_t, x_{t-d}) = 0$, then LMI (21) becomes

$$\begin{bmatrix} \Upsilon_2 & 0 & \Upsilon_4 & \Upsilon_8 & \Upsilon_{10} \\ * & -I & \Upsilon_6 & 0 & 0 \\ * & * & \begin{bmatrix} -P_{1i} + Q_i & -P_{2i} \\ -P_{2i}^{\mathrm{T}} & -P_{3i} \end{bmatrix} & 0 & 0 \\ * & * & * & -Q_i & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0.$$

5 Illustrative Example

In this section, we will provide an example to illustrate the applicability of the above filter design method. Consider the following system:

$$\begin{aligned} x_{t+1} &= \begin{bmatrix} 0.9944 & -0.1203 & -0.4302 \\ 0.0017 & 0.9902 & -0.0747 + 0.01\alpha \\ 0 & 0.8187 & 0 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} \omega_t \\ &\quad + \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.02 \end{bmatrix} x_t v_t, \\ y_t &= [\,0.2 \quad 0.1 \quad 0.1 + 0.01\alpha\,]\, x_t + [\,0.1 \quad 0.1 + 0.01\alpha \quad 0\,]\, x_{t-d} \\ &\quad + 0.2\sin([\,0 \quad 0 \quad 0.2\,]\, x_t + [\,0 \quad 0.1 \quad 0\,]\, x_{t-d}) + 0.1\omega_t, \\ z_t &= [\,0 \quad 0.1 \quad 0.2\,]\, x_t, \end{aligned} \tag{28}$$

where α is an unknown parameter satisfying $-1 \le \alpha \le 1$. It is easy to see that system (28) has the structure of system $\mathcal{S}$ in (1) with the following parameters:

$$
\begin{aligned}
A &= \begin{bmatrix} 0.9944 & -0.1203 & -0.4302 \\ 0.0017 & 0.9902 & -0.0747 + 0.01\alpha \\ 0 & 0.8187 & 0 \end{bmatrix}, \\
M &= \begin{bmatrix} 0.1 & 0 & 0.2 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.02 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}, \\
A_d &= 0_{3\times 3}, \quad F = 0_{3\times 1}, \quad M_d = 0_{3\times 3}, \\
C &= [\,0.2 \quad 0.1 \quad 0.1 + 0.01\alpha\,], \\
C_d &= [\,0.1 \quad 0.1 + 0.01\alpha \quad 0\,], \\
G &= 0.2, \quad D = 0.1, \quad N = 0_{1\times 3}, \quad N_d = 0_{1\times 3}, \\
L &= [\,0 \quad 0.1 \quad 0.2\,], \\
f(x_t, x_{t-d}) &= 0 \\
g(x_t, x_{t-d}) &= 0.2 \sin\left([\,0 \quad 0 \quad 0.2\,]\, x_t + [\,0 \quad 0.1 \quad 0\,]\, x_{t-d}\right).
\end{aligned}
$$

In addition, the nonlinear functions $f(x_t, x_{t-d})$ and $g(x_t, x_{t-d})$ satisfy Assumption 1 with

$$S_1 = S_2 = 0_{1\times 3}, \quad T_1 = [\,0 \quad 0 \quad 0.2\,], \quad T_2 = [\,0 \quad 0.1 \quad 0\,].$$

By solving Problem RHFD, the obtained minimum feasible γ^* and the associated matrices for different cases are as follows:

Third-order Filtering: ($\gamma^* = 0.0200$)

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{ccc:c} 0.1864 & 1.3287 & 0.1981 & -3.5599 \\ -0.0268 & 0.9945 & 0.0077 & -0.1232 \\ -0.0132 & 0.2543 & 0.0391 & -0.1472 \\ \hdashline 0.0001 & -0.0999 & -0.2048 & 0 \end{array}\right]. \tag{29}$$

Second-order Filtering: ($\gamma^* = 0.0226$)

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{cc:c} 0.9669 & 0.1947 & -0.0624 \\ 0.0002 & 0.9353 & -0.0084 \\ \hdashline -0.0011 & -0.1400 & 0 \end{array}\right]. \tag{30}$$

First-order Filtering: ($\gamma^* = 0.0228$)

$$\begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix} = \left[\begin{array}{c:c} 0.9589 & -0.1801 \\ \hdashline -0.0031 & 0 \end{array}\right]. \tag{40}$$

6 Concluding Remarks

The problem of robust $\mathcal{H}_\infty$ filtering for a class of stochastic nonlinear time-delay systems in discrete time has been investigated in this paper. Sufficient conditions are obtained in terms of linear matrix inequality for the existence of desired filters which guarantee the filtering error system to be mean-square asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level. A parametrization of the filter matrices can be readily obtained if these conditions have feasible solutions. A numerical example is provided to show the applicability of the developed filter design methods.

References

[1] Arnold, L. *Stochastic Differential Equations: Theory and Applications*. Wiley, New York, 1974.

[2] Basin, M. and Skliar, M. Integral approach to optimal filtering and control of continuous processes with time-varying delays. In: *Proc. 40th Conf. Decision Control*, Orlando, FL, 2001, P.2911–2916.

[3] Boukas, E.K. and Liu, Z.K. Robust H_∞ filtering for polytopic uncertain time-delay systems with Markov jumps. *Computer & Electrical Engineering* **28** (2002) 171–193.

[4] Boyd, S., Ghaoui, L.El, Feron, E. and Balakrishnan, V. *Linear Matrix Inequalities in Systems and Control Theory*. SIAM, Philadelphia, PA, 1994.

[5] Cao, Y.-Y., Sun, Y.-X. and Lam, J. Delay-dependent robust H_∞ control for uncertain systems with time-varying delays. *IEE Proc. Part D: Control Theory Appl.* **145** (1998) 338–344.

[6] De Oliveira, M.C., Bernussou, J. and Geromel, J.C. A new discrete-time robust stability condition. *Systems & Control Letters* **37** (1999) 261–265.

[7] De Souza, C.E., Palhares, R.M. and Peres, P.L.D. Robust H_∞ filter design for uncertain linear systems with multiple time-varying state delays. *IEEE Trans. Signal Processing* **49**(3) (2001) 569–576.

[8] El Bouhtouri, A., Hinrichsen, D. and Pritchard, A.J. H_∞ type control for discrete-time stochastic systems. *Int. J. Robust & Nonlinear Control* **9** (1999) 923–948.

[9] Esfahani, S. and Petersen, I.R. An LMI approach to output-feedback-guaranteed cost control for uncertain time-delay systems. *Int. J. Robust & Nonlinear Control* **10** (2000) 157–174.

[10] Fridman, E. and Shaked, U. A descriptor system approach to H_∞ control of linear time-delay systems. *IEEE Trans. Automat. Control* **47**(2) (2002) 253–270.

[11] Gahinet, P., Nemirovskii, A., Laub, A.J. and Chilali, M. *LMI Control Toolbox User's Guide*. The Math. Works Inc., Natick, MA, 1995.

[12] Gao, H. and Wang, C. Delay-dependent robust H_∞ and L_2-L_∞ filtering for a class of uncertain nonlinear time-delay systems. *IEEE Trans. Automat. Control* **48**(9) (2003) 1661–1666.

[13] Gao, H. and Wang, C. Robust L_2-L_∞ filtering for uncertain systems with multiple time-varying state delays. *IEEE Trans. Circuits and Systems (I)* **50**(4) (2003) 594–599.

[14] Geromel, J.C. and De Oliveira, M.C. H_2 and H_∞ robust filtering for convex bounded uncertain systems. *IEEE Trans. Automat. Control* **46**(1) (2001) 100–107.

[15] Gershon, E., Shaked, U. and Yaesh, I. H_∞ control and filtering of discrete-time stochastic with multiplicative noise. *Automatica* **37** (2001) 409–417.

[16] Guo, L. H_∞ output feedback control for delay systems with nonlinear and parametric uncertainties. *IEE Proc. Part D: Control Theory Appl.* **149** (2002) 226–236.

[17] Hausmann, U.G. Optimal stationary control with state and control dependent noise. *SIAM J. Control Optim.* **9** (1971) 184–198.

[18] Hinrichsen, D. and Pritchard, A.J. Stochastic H_∞. *SIAM J. Control Optim.* **36**(5) (1998) 1504–1538.

[19] Hsiao, F. and Pan, S. Robust Kalman filter synthesis for uncertain multiple time-delay stochastic systems. *J. Dyn. Syst. Meas. Contr.* **118** (1996) 803–808.

[20] Kubrusly, C.S. On discrete stochastic bilinear systems stability. *J. Math. Anal. Appl.* **113** (1986) 36–58.

[21] Kushner, H. *Stochastic stability and control.* Academic Press, New York, 1967.

[22] Liao, X. and Mao, X. Exponential stability of stochastic delay interval systems. *Systems & Control Letters* **40** (2000) 171–181.

[23] Mahmoud, M.S. and Shi, P. Robust stability, stabilization and H_∞ control of time-delay systems with Markovian jump parameters. *Int. J. Robust & Nonlinear Control* **13** (2003) 755–784.

[24] Mao, X., Koroleva, N. and Rodkina, A. Robust stability of uncertain stochastic differential delay equations. *Systems & Control Letters* **35** (1998) 325–336.

[25] Palhares, R.M. and Peres, P.L.D. Robust filtering with guaranteed energy-to-peak performance – an LMI approach. *Automatica* **36** (2000) 851–858.

[26] Park, P. A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Trans. Automat. Control* **44**(4) (1999) 876–877.

[27] Scherer, C., Gahinet, P. and Chilali, M. Multiobjective output-feedback control via LMI optimization. *IEEE Trans. Automat. Control* **42**(7) (1997) 896–911.

[28] Shi, P., Boukas, E.K., Shi, Y. and Agarwal, R.K. Optimal guaranteed cost control of uncertain discrete time-delay systems. *J. of Comput. and Appl. Math.* **157** (2003) 435–451.

[29] Wang, Z., Goodall, D.P. and Burnham, K.J. On designing observers for time-delay systems with nonlinear disturbances. *Int. J. Control* **75** (2002) 803–811.

[30] Wang, Z. and Qiao, H. Robust filtering for bilinear uncertain stochastic discrete-time systems. *IEEE Trans. Signal Processing* **50**(3) (2002) 560–567.

[31] Wonham, W.M. On a matrix Riccati equation of stochastic control. *SIAM J. Control Optim.* **6** (1968) 681–697.

[32] Xie, S. and Xie, L. Stabilization of a class of uncertain large-scale stochastic systems with time delays. *Automatica* **36** (2000) 161–167.

[33] Xu, S. Robust H_∞ filtering for a class of discrete-time uncertain nonlinear systems with state delay. *IEEE Trans. Circuits and Systems (I)* **49** (2002) 1853–1859.

[34] Xu, S., Lam, J., Huang, S. and Yang, C. H_∞ model reduction for linear time-delay systems: continuous-time case. *Int. J. Control* **74**(11) (2001) 1062–1074.