Nonlinear Dynamics and Systems Theory, 4(2) (2004) 195–216

# Robust Active Control for Structural Systems with Structured Uncertainties

Sheng-Guo Wang[1], H.Y. Yeh[2] and P.N. Roschke[3]

[1] *Department of Engineering Technology and Department of ECE, University of North Carolina at Charlotte, Charlotte, NC 28223-0001, USA*
[2] *Department of Civil Engineering, Prairie View A&M University, Prairie View, TX 77446-0397, USA*
[3] *Department of Civil Engineering, Texas A&M University, College Station, TX 77843-3136, USA*

Received: April 23, 2003; Revised: March 4, 2004

**Abstract:** Although control theory has been widely applied to constrain motion response of tall, slender structures and long bridges undergoing large forces from natural hazards such as earthquakes and strong wind, numerous uncertainties in these structures such as model errors, stress calculations, material properties, and load environments need to be included in design of the control algorithm. This paper develops a robust active control approach to treat structured uncertainties in the system, control input, and especially, disturbance input matrices that have not been treated previously. Special SVD decomposition is applied to all forms of the structured uncertainties. Robust active control provides multi-objectives, including robust $\alpha$-degree relative stability, robust $H_\infty$ disturbance attenuation and robust $H_2$ optimality. The $H_\infty$ norm of the transfer function from the external disturbance forces (e.g., earthquake, wind, and etc.) to the observed system states is restricted by a prescribed attenuation index $\delta$. Settling time of the controlled structural system is robustly less than $4/\alpha$. Preservation of robust $H_2$ optimality of uncertain structural systems is also discussed. Numerical simulations of a four-story building under robust control are carried out for motion induced by the 1940 El Centro earthquake. Evaluation of controller performance is measured by application of six indices, including a comparison with an LQR controller. Results of the proposed approach may be applied to robust control design of structural systems.

**Keywords:** *Robust active control; structural systems; structured uncertainties; multi-objective; $H_\infty/H_2$ optimality.*

**Mathematics Subject Classification (2000):** 93B36, 93D21, 93C05.

 

## 1 Introduction

Since the advance of new technologies and the advent of high strength materials, civil engineering structures are becoming taller, longer, and more flexible. To warrant safety and comfort of inhabitants, it is deemed necessary to limit the motion of these structures. Application of modern control theory to restrain the structural motion was first proposed by Yao [18]. Since then, considerable progress has been made to reduce effects of undesirable external forces such as earthquakes and strong winds. Among noteworthy contributions to this field of research are those by Soong [9], Spencer, *et al.* [11], Fujino, *et al.* [2], Yang, *et al.* [17], and many others. Housner, *et al.* [5] detailed recent developments in active control strategies for civil engineering structures. In 1997, Housner, *et al.* provided a summary and general overview of structural control: past and present [4]. A survey paper by Spencer and Sain [10] extensively summarizes recent research progress and describes new efforts in feedback control of buildings.

Most control strategies of structural systems focus on application of linear models and control laws. However, structural uncertainties occur from modeling errors, linearization approximations, stress calculations, material properties, and external disturbances. Effects of these uncertainties on stability and robustness of structural control have been previously examined [3, 11]. Consequently, one primary research issue is robustness of control systems. In particular, numerous studies of this kind have focused on control of buildings. In this regard the $H_\infty$ approach is advantageous in that it may consider both attenuation of disturbance effects and perturbation of unstructured parameters. $H_\infty$ design methods may be found in many references such as [7, 19].

It is well known that dynamics of a civil engineering structure can be described by a Lagrangian system of equations. Many physical problems, such as aeronautical systems, mechanical systems, structural systems, and flexible structures can be described via Lagrange's equation using a state-space model [14]. Since there are numerous uncertainties in stresses, material properties, and loadings that pertain to descriptive numerical models, unanticipated variations of these design parameters may cause instability or degradation of a structural system. In such cases robustness of a control system for stability and its performance toward attenuating disturbance from external hazards is important. Wang, *et al.* [14] have discussed robust optimal pole clustering in a vertical strip and $H_\infty$-norm disturbance rejection for uncertain Lagrangian systems. Considered uncertainties are in both the system matrix and the control input matrix. Wang, *et al.* [12, 14] have also discussed a state-feedback controller and an observer-based output-feedback controller for robust pole clustering in a vertical strip and disturbance attenuation in general uncertain systems with structured and unstructured uncertainties, respectively. They [12] also show that this new method is more flexible and less conservative than the traditional approaches. However, no uncertainties are considered in the disturbance input matrix.

Furthermore, there have been no recent treatments of uncertainties with regard to the disturbance input matrix in the literature [7, 12, 14, 15, 19]. However, the disturbance input matrix has uncertainties, e.g., in view of the uncertainties existing in mass, as well as in the inverse mass matrix, and so on. Recently, Wang, *et al.* [13] discussed robust control for structural systems via Lagrange's model with unstructured uncertainties, including those in the disturbance input matrix. In [16] they further discussed parametric uncertainties in system and control input matrices, as well as unstructured uncertainties in the disturbance input matrix. However, some uncertainties may be structured uncertainties, such as from mass, spring constants, and damping ratios. Thus, it is meaningful to investigate robust control for structural systems with structured uncertainties in the disturbance input matrix. Herein lies the motivation for research reported in this paper.

Therefore, the objective of this paper is to develop an approach for active control of structural systems that includes robust stability and performance control with $H_\infty$-norm disturbance attenuation that takes into account structured uncertainties in the structural systems, including those in the disturbance input matrices, to reject/attenuate disturbances such as earthquake and wind forces for a family of structural systems with these uncertainties. Applicable uncertain structural systems include uncertainties among system, control input, and disturbance input matrices. Robust state feedback control is considered here, while robust output feedback control is considered in a future paper. The proposed control algorithm provides a robust $\alpha$-degree relative stability, i.e., the closed-loop system poles robustly stay in the left-half plane with the real part less than $-\alpha$. It also guarantees a prescribed $H_\infty$-norm disturbance attenuation constraint $\delta$ from the external hazard forces to the observed states of the structure. The approach is based on the algebraic Riccati equation (ARE). A group of several flexible scalars is introduced to enable solution of the ARE. Then, $H_2$ optimality of the design controller is also proved.

It is noted that there are many publications concerning robust $H_\infty$ control and multi-objective control in the literature [19]. For structured uncertainties, the $\mu$-theory [19] makes a breakthrough. However, calculation and design based on $\mu$ theory is an NP-hard problem. Therefore, this paper uses a new method to deal with structured uncertainties, extended from Wang, *et al.* [12] to include structured uncertainties in the disturbance input matrix. It uses special SVD-type decomposition and introduces a group of adjustable design parameters to control design to enable control with robust performance, including robust relative stability, robust $H_\infty$ disturbance attenuation and robust $H_2$ optimal control for the whole uncertain system family. It is noticed that the treatment may be taken into some conventional framework from $H_\infty$-control view point. However, Wang, *et al.* [12] have shown that the conventional framework will not be as flexible and is more-conservative than the proposed method that renders conventional treatment of this problem as a special case of their approach as shown by theoretical proof and an example. Therefore, this paper develops an approach that extends work presented by Wang, *et al.* [12, 14].

Salient contributions of this paper are as follows:

1) an uncertain Lagrangian system with uncertainties not only in system and control input matrices but also in disturbance input matrix is treated;
2) structured uncertainties in the disturbance input matrix are taken into account;
3) a special weighted SVD-type decomposition for all structured uncertainties is described;
4) a group of tuning scalars is used;
5) discussion of robust $H_2$ optimality together with robust $H_\infty$ disturbance attenuation and robust relative stability is included;
6) numerical simulation of control for an uncertain building model, including a nominal model and a worst case model, excited by the 1940 El Centro, California, earthquake data is demonstrated; and
7) finally, six performance indices are used for evaluation and comparison with the traditional LQR control.

The paper is organized as follows. Section 2 formulates an analytical approach to control of uncertain structural systems with structured uncertainties. Section 3 provides robust control algorithms with robust relative stability and $H_\infty$-norm disturbance attenuation for uncertain structural systems. Furthermore, in Section 4 preservation of $H_2$ optimality of the design controller with respect to a special performance index is derived. In Section 5, a numerical example of robust control design is presented that illustrates robust controller design. Section 6 provides six indices for performance evaluation and Section 7 demonstrates simulations excited by the 1940 El Centro earthquake data and compares results from three robust controllers and an LQR controller. Finally, Section 8 concludes the paper.

## 2 Control System Formulation

It is well known that motion of a structural system can be described by Lagrange's equations in state-space as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D_f}{\partial \dot{q}_i} = Q_i, \tag{1}$$

where $L = T - V$, $T$ is the system kinetic energy, $V$ is the system potential energy, $D_f$ is the system dissipation function, $Q_i$ represents the generalized force, and $q_i$ is the partial state. For example, dynamic motion of a structural system may be described by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}_d\dot{\mathbf{q}} + \mathbf{K}_s\mathbf{q} = \mathbf{f} \tag{2}$$

where $\mathbf{q}$ is a displacement vector, $\mathbf{M}$ is a mass matrix, $\mathbf{C}_d$ is a damping coefficient matrix, $\mathbf{K}_s$ is a stiffness coefficient matrix, and $\mathbf{f}$ is an external force vector that includes both undesired forces from an external hazard and desired control forces. Mass matrix $\mathbf{M}$ is a full rank matrix, i.e., its inverse exists. Sometimes, it is simply considered to be a diagonal matrix. The dynamic system (2) may be rewritten as

$$\ddot{\mathbf{q}} + \mathbf{M}^{-1}\mathbf{C}_d\dot{\mathbf{q}} + \mathbf{M}^{-1}\mathbf{K}_s\mathbf{q} = \mathbf{M}^{-1}\mathbf{f}. \tag{3}$$

However, uncertainties in structural parameters that are derived from modeling errors, linearized approximation, stress calculations, variation in materials properties, and external disturbances are inevitable. If uncertainties, perturbations, and disturbances are taken into account, equations (1) − (3) can be reformulated as a monic vector differential equation with parametric perturbations and external disturbances as follows:

$$\ddot{\mathbf{q}} + (\mathbf{D}_c + \Delta\mathbf{D}_c)\dot{\mathbf{q}} + (\mathbf{D}_k + \Delta\mathbf{D}_k)\mathbf{q} = (\mathbf{B}_u + \Delta\mathbf{B}_u)\mathbf{u} + (\mathbf{F}_w + \Delta\mathbf{F}_w)\mathbf{w}, \tag{4a}$$

$$\mathbf{z} = \mathbf{C}_1\mathbf{q} + \mathbf{C}_2\dot{\mathbf{q}}, \tag{4b}$$

where $\mathbf{q} \in R^n$, $\mathbf{u} \in R^m$, $\mathbf{w} \in R^\omega$, and $\mathbf{z} \in R^p$ are the partial state, input, disturbance, and output (vibration specification signals), respectively; $\mathbf{D}_c$, $\mathbf{D}_k$, $\mathbf{B}_u$, $\mathbf{F}_w$, $\mathbf{C}_1$, and $\mathbf{C}_2$ are nominal structural system parameter matrices with appropriate dimensions; $\Delta\mathbf{D}_c$, $\Delta\mathbf{D}_k$, $\Delta\mathbf{B}_u$ and $\Delta\mathbf{F}_w$ are perturbation matrices that can be time-varying with appropriate dimensions. The considered disturbance vector $\mathbf{w}(t)$ may include an earthquake force vector $\mathbf{w}_e(\mathrm{t})$ and/or a wind force vector $\mathbf{w}_w(\mathrm{t})$. Thus, the uncertain structural system can be described by the following specific state-space block companion form:

$$\dot{\mathbf{x}}(\mathbf{t}) = (\mathbf{A} + \Delta\mathbf{A})\mathbf{x}(t) + (\mathbf{B} + \Delta\mathbf{B})\mathbf{u}(t) + (\mathbf{F} + \Delta\mathbf{F})\mathbf{w}(t), \tag{5a}$$

$$\mathbf{z}(t) = \mathbf{C}\mathbf{x}(t) \tag{5b}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{D}_k & -\mathbf{D}_c \end{bmatrix}, \quad \Delta\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\Delta\mathbf{D}_k & -\Delta\mathbf{D}_c \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}u \end{bmatrix},$$
$$\Delta\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \Delta\mathbf{B}u \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_w \end{bmatrix}, \quad \Delta\mathbf{F} = \begin{bmatrix} \mathbf{0} \\ \Delta\mathbf{F}_w \end{bmatrix},$$
$$\text{and} \quad \mathbf{C} = [\,\mathbf{C}_1 \quad \mathbf{C}_2\,], \tag{5c}$$

where the state $\mathbf{x} = \begin{bmatrix} \mathbf{q}^{\mathrm{T}} & \dot{\mathbf{q}}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in R^{2n}$, all matrices have appropriate dimensions, and $(\mathbf{A}, \mathbf{B})$ is assumed to be controllable.

Based on the form of block matrices given in equation (5), results are directly derived with respect to the low dimensional uncertainties $\Delta\mathbf{D}_c$, $\Delta\mathbf{D}_k$, $\Delta\mathbf{B}_u$, and $\Delta\mathbf{F}_w$ for simple and less conservative constraints to robust active structural control problems. In light of perturbations of physical parameters, structured or unstructured uncertainties, especially structured ones, usually exist in $\mathbf{A}(\mathbf{D}_k, \mathbf{D}_c)$, $\mathbf{B}_u$, and $\mathbf{F}_w$, and are described as $\Delta\mathbf{A}(\Delta\mathbf{D}_k, \Delta\mathbf{D}_c)$, $\Delta\mathbf{B}_u$, and $\Delta\mathbf{F}_w$, respectively. The case of unstructured uncertainties is considered in [13]. Here, structured uncertainties are treated.

Structured uncertainties can be described as:

$$\begin{aligned} \Delta\mathbf{D}_k &= \sum_{j=1}^{l_k} a_{kj}\mathbf{A}_{kj}, & \Delta\mathbf{D}_c &= \sum_{j=1}^{l_c} a_{cj}\mathbf{A}_{cj}, \\ \Delta\mathbf{B}_{\mathbf{u}} &= \sum_{j=1}^{l_b} b_j\mathbf{B}_j, & \Delta\mathbf{F}_w &= \sum_{j=1}^{l_f} f_j\mathbf{F}_j, \end{aligned} \tag{6}$$

where $\Delta\mathbf{D}_c$, $\Delta\mathbf{D}_k$, $\Delta\mathbf{B}_u$, and $\Delta\mathbf{F}_w$ are the uncertain stiffness matrix, uncertain damping matrix, uncertain control input matrix, and uncertain disturbance input matrix, respectively. They are described as structured uncertainties, i.e., matrices $\mathbf{A}_{kj}$ , $\mathbf{A}_{cj}$ , $\mathbf{B}_j$, and $\mathbf{F}_j$ represent the structures of uncertainties, while scalars $a_{kj}$, $a_{cj}$, $b_j$, and $f_j$ represent the uncertain values of uncertainties on their corresponding structures, respectively, and are bounded by $\pm 1$ without loss of generality.

Here, weighted SVD (singular value decomposition) is applied to the uncertainty structure matrices $\mathbf{A}_{kj}$, $\mathbf{A}_{cj}$, $\mathbf{B}_j$, and $\mathbf{F}_j$. Then, it follows that

$$\mathbf{A}_{kj} = \mathbf{T}_{kj}\mathbf{U}_{kj}^{\mathrm{T}}, \quad \mathbf{A}_{cj} = \mathbf{T}_{cj}\mathbf{U}_{cj}^{\mathrm{T}}, \quad \mathbf{B}_j = \mathbf{T}_{bj}\mathbf{U}_{bj}^{\mathrm{T}}, \quad \text{and} \quad \mathbf{F}_j = \mathbf{T}_{fj}\mathbf{U}_{fj}^{\mathrm{T}}, \tag{7}$$

respectively. Next, the following definitions are made:

$$\begin{aligned} \mathbf{T}_k &= \sum_{j=1}^{l_k} \mathbf{T}_{kj}\mathbf{T}_{kj}^{\mathrm{T}}, & \mathbf{U}_k &= \sum_{j=1}^{l_k} \mathbf{U}_{kj}\mathbf{U}_{kj}^{\mathrm{T}}, & \mathbf{T}_c &= \sum_{j=1}^{l_c} \mathbf{T}_{cj}\mathbf{T}_{cj}^{\mathrm{T}}, & \mathbf{U}_c &= \sum_{j=1}^{l_c} \mathbf{U}_{cj}\mathbf{U}_{cj}^{\mathrm{T}}, \\ \mathbf{T}_b &= \sum_{j=1}^{l_b} \mathbf{T}_{bj}\mathbf{T}_{bj}^{\mathrm{T}}, & \mathbf{U}_b &= \sum_{j=1}^{l_b} \mathbf{U}_{bj}\mathbf{U}_{bj}^{\mathrm{T}}, & \mathbf{T}_f &= \sum_{j=1}^{l_f} \mathbf{T}_{fj}\mathbf{T}_{fj}^{\mathrm{T}}, & \mathbf{U}_f &= \sum_{j=1}^{l_f} \mathbf{U}_{fj}\mathbf{U}_{fj}^{\mathrm{T}} \end{aligned} \tag{8}$$

Furthermore, it is defined that

$$\mathbf{T}_A = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{T}_k + \mathbf{T}_c \end{bmatrix}, \quad \mathbf{U}_A = \begin{bmatrix} \mathbf{U}_k & 0 \\ 0 & \mathbf{U}_c \end{bmatrix}, \quad \mathbf{T}_B = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{T}_b \end{bmatrix},$$
$$\mathbf{U}_B = \mathbf{U}_b, \quad \mathbf{T}_F = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{T}_f \end{bmatrix}, \quad \mathbf{U}_F = \mathbf{U}_f, \quad \mathbf{F}_\Delta = \sum_{j=1}^{l_f} \mathbf{F}_j \mathbf{F}_j^{\mathrm{T}}. \tag{9}$$

Notice that some Lagrangian representations of structures with the block companion form in (5) may be formulated as matched uncertain systems (extended from [8]). That is, the matched uncertainties are within the range of the nominal control-input matrix $\mathbf{B}$. This implies that all uncertainties can be reached by suitable control signals through the control-input matrix $\mathbf{B}$. Thus, a system with matched uncertainties can be compensated if a suitable designed robust controller is applied. In other words, a robust controller is guaranteed to exist and there exists a robust controller that can overcome all these matched uncertainties. In this case these structured matched uncertainties can be described as follows:

$$\Delta\mathbf{A} = \mathbf{B}\cdot\Delta\mathbf{A}_B, \quad \Delta\mathbf{B} = \mathbf{B}\cdot\Delta\mathbf{B}_B, \quad \mathbf{F} = \mathbf{B}\cdot\mathbf{F}_B, \quad \Delta\mathbf{F} = \mathbf{B}\cdot\Delta\mathbf{F}_B, \quad \text{i.e.,} \tag{10a}$$

$$\Delta\mathbf{A}_B = [-\Delta\mathbf{D}_{Bk} \ \ -\Delta\mathbf{D}_{Bc}], \quad \Delta\mathbf{D}_{Bk} = \sum_{j=1}^{l_k} a_{bkj}\mathbf{A}_{bkj}, \quad \Delta\mathbf{D}_{Bc} = \sum_{j=1}^{l_c} a_{bcj}\mathbf{A}_{bcj};$$
$$\Delta\mathbf{B}_B = \Delta\mathbf{B}_{Bu} = \sum_{j=1}^{l_b} b_{bj}\mathbf{B}_{bj}, \quad \Delta\mathbf{F}_w = \mathbf{B}_u\Delta\mathbf{F}_{Bw}, \quad \Delta\mathbf{F}_B = \Delta\mathbf{F}_{Bw} = \sum_{j=1}^{l_f} f_{bj}\mathbf{F}_{bj}; \tag{10b}$$

$$\text{with} \quad \Delta\mathbf{B}_{Bu} + \Delta\mathbf{B}_{Bu}^{\mathrm{T}} + 2\mathbf{I} > \mathbf{0}. \tag{10c}$$

This uncertain system can be called a matched uncertain system, i.e., with matched uncertainties. Applying weighted SVD for all of the above uncertainty structures similar to the above (7) – (9) leads to the following:

$$\mathbf{A}_{bkj} = \mathbf{T}_{bkj}\mathbf{U}_{bkj}^{\mathrm{T}}, \quad \mathbf{A}_{bcj} = \mathbf{T}_{bcj}\mathbf{U}_{bcj}^{\mathrm{T}}, \quad \mathbf{B}_{bj} = \mathbf{T}_{bbj}\mathbf{U}_{bbj}^{\mathrm{T}}, \quad \mathbf{F}_{bj} = \mathbf{T}_{bfj}\mathbf{U}_{bfj}^{\mathrm{T}}, \tag{11}$$

Finally, it is defined that

$$\mathbf{T}_{bk} = \sum_{j=1}^{l_k} \mathbf{T}_{bkj}\mathbf{T}_{bkj}^{\mathrm{T}}, \quad \mathbf{U}_{bk} = \sum_{j=1}^{l_k} \mathbf{U}_{bkj}\mathbf{U}_{bkj}^{\mathrm{T}}, \quad \mathbf{T}_{bc} = \sum_{j=1}^{l_c} \mathbf{T}_{bcj}\mathbf{T}_{bcj}^{\mathrm{T}}, \quad \mathbf{U}_{bc} = \sum_{j=1}^{l_c} \mathbf{U}_{bcj}\mathbf{U}_{bcj}^{\mathrm{T}}, \tag{12a}$$

$$\mathbf{T}_{bb} = \sum_{j=1}^{l_b} \mathbf{T}_{bbj}\mathbf{T}_{bbj}^{\mathrm{T}}, \quad \mathbf{U}_{bb} = \sum_{j=1}^{l_b} \mathbf{U}_{bbj}\mathbf{U}_{bbj}^{\mathrm{T}}, \quad \mathbf{T}_{bf} = \sum_{j=1}^{l_f} \mathbf{T}_{bfj}\mathbf{T}_{bfj}^{\mathrm{T}}, \quad \mathbf{U}_{bf} = \sum_{j=1}^{l_f} \mathbf{U}_{bfj}\mathbf{U}_{bfj}^{\mathrm{T}}, \tag{12b}$$

$$\mathbf{T}_{bA} = \mathbf{T}_{bk} + \mathbf{T}_{bc}, \quad \mathbf{U}_{bA} = \begin{bmatrix} \mathbf{U}_{bk} & 0 \\ 0 & \mathbf{U}_{bc} \end{bmatrix}, \quad \mathbf{T}_{bB} = \mathbf{T}_{bb}, \quad \mathbf{U}_{bB} = \mathbf{U}_{bb}, \quad \mathbf{T}_{bF} = \mathbf{T}_{bf},$$
$$\mathbf{U}_{bF} = \mathbf{U}_{bf}, \quad \mathbf{F}_{b\Delta} = \sum_{j=1}^{l_f} \mathbf{F}_{bj}\mathbf{F}_{bj}^{\mathrm{T}}. \tag{12c}$$

The objective is to find a linear state-feedback control law such that it can accomplish the above-mentioned robust active control that is valid for the whole family of uncertain structural systems in (5) in face of disturbances and perturbations in (6) – (12). Thus, the goal is to design a state feedback controller

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) \tag{13}$$

such that the closed loop uncertain linear system

$$\dot{\mathbf{x}}(\mathbf{t}) = (\mathbf{A} + \Delta\mathbf{A} - \mathbf{B}\mathbf{K} - \Delta\mathbf{B}\mathbf{K})\mathbf{x}(t) + (\mathbf{F} + \Delta\mathbf{F})\mathbf{w}(t), \tag{14a}$$

$$\mathbf{z}(t) = \mathbf{C}\mathbf{x}(t), \tag{14b}$$

has a robust disturbance attenuation with a prescribed $H_\infty$-norm constraint $\delta$ (a specified disturbance attenuation index) that satisfies the following:

$$\|\mathbf{T}_{zw}(s)\|_\infty = \left\|\mathbf{C}(s\mathbf{I} - \mathbf{A}_c)^{-1}(\mathbf{F} + \Delta\mathbf{F})\right\|_\infty \le \delta \tag{15}$$

and a robust $\alpha$-degree relative stability, i.e.,

$$\mathrm{Re}\,\{\lambda(\mathbf{A}_c)\} < -\alpha, \tag{16}$$

where $\mathbf{A}_c = \mathbf{A} + \Delta\mathbf{A} - \mathbf{B}\mathbf{K} - \Delta\mathbf{B}\mathbf{K}$, and $\mathbf{T}_{zw}(s)$ is a transfer function matrix from the disturbance vector $\mathbf{w}$ to the observation vector $\mathbf{z}$ of the structural system. The disturbance vector $\mathbf{w}$ may include a wind and/or earthquake disturbance. The observation vector $\mathbf{z}$ may include a vibration vector, i.e., displacement vector, velocity vector, and other salient observation states. This indicates that the gain of the structural system from the disturbance energy $\|\mathbf{w}\|^2$ to the structural vibration energy $\|\mathbf{z}\|^2$ is bounded by $\delta$ even in the worst case in view of the $H_\infty$-norm property. The control law also provides robust relative stability with an index $\alpha$ to the structural system. In the case of matched uncertainties in (10) – (14), the existence of this desired controller is guaranteed. Also, the optimality of the controller is proved in an $H_2$ sense.

## 3 Robust Feedback Control

In this section, a state feedback controller is developed in (13) that provides robust $\alpha$-degree relative stability in (16) and an $H_\infty$ disturbance attenuation with a prescribed index $\delta$ in (15) for the uncertain structural system given in (5) and (14). The controller (13) is obtained by solving a Riccati equation as derived in this section. A set of tuning parameters is introduced to enhance flexibility in defining the controller.

Before deriving the main result, the following lemmas are cited to provide a basis for the derivation. As a preliminary statement, a matrix $\mathbf{Q}$ that is $> 0$, $\ge 0$, and $< 0$ is said to be positive definite, positive semi-definite, and negative definite, respectively.

**Lemma 1** ([1]) *Matrix $\mathbf{A}$ is robust $\alpha$-degree relatively stable if and only if there exists a unique positive matrix $\mathbf{P}$ for any positive definite matrix $\mathbf{Q}$ such that*

$$(\mathbf{A} + \alpha\mathbf{I})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A} + \alpha\mathbf{I}) = -\mathbf{Q}, \tag{17}$$

*i.e., all eigenvalues of matrix $\mathbf{A}$ lie in the left plane of the line $-\alpha$, $\mathrm{Re}\,\{\lambda(\mathbf{A})\} < -\alpha$.*

**Lemma 2** ([14]) *For any $n \times m$ matrices $\mathbf{X}$ and $\mathbf{Y}$, and any scalar $\xi > 0$,*

$$\xi \mathbf{X}\mathbf{X}^* + \frac{1}{\xi}\mathbf{Y}\mathbf{Y}^* \pm (\mathbf{X}\mathbf{Y}^* + \mathbf{Y}\mathbf{X}^*) \geq 0. \tag{18}$$

**Lemma 3** *For given scalars $\alpha \geq 0$ and $\delta > 0$, if there exist a positive definite matrix $\mathbf{P}$ and positive adjustable scalars $\varepsilon$ and $\varepsilon_3$ such that*

$$(\mathbf{A}_c + \alpha\mathbf{I})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A}_c + \alpha\mathbf{I}) + \frac{\varepsilon}{\delta}\mathbf{P}[\mathbf{F}(\mathbf{I} + \varepsilon_3\mathbf{U}_F)\mathbf{F}^{\mathrm{T}} + \frac{1}{\varepsilon_3}\mathbf{T}_F + l_f\mathbf{F}_\Delta]\mathbf{P} + \frac{1}{\varepsilon\delta}\mathbf{C}^{\mathrm{T}}\mathbf{C} < 0, \tag{19}$$

*then the closed-loop system (14) with structured uncertainties in (6)–(9) is of the $\alpha$-degree relatively stable as (16) and $\delta$-degree disturbance attenuated as (15).*

*Proof* By Lemma 1, it is obvious that system $A_c$ is of $\alpha$-degree relatively stable. By extension of Lemma 2 in [14], it is known that the closed-loop system (14) with structured uncertainties in (6)–(9) is of $\alpha$-degree relatively stable as (16) and there is $\delta$-degree disturbance attenuation in (15) if

$$(\mathbf{A}_c + \alpha\mathbf{I})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A}_c + \alpha\mathbf{I}) + \frac{\xi}{\delta}\mathbf{P}(\mathbf{F} + \Delta\mathbf{F})(\mathbf{F} + \Delta\mathbf{F})^{\mathrm{T}}\mathbf{P} + \frac{1}{\xi\delta}\mathbf{C}^{\mathbf{T}}\mathbf{C} < 0. \tag{20a}$$

In view of Lemma 2, it follows that

$$\mathbf{P}(\mathbf{F} + \Delta\mathbf{F})(\mathbf{F} + \Delta\mathbf{F})^{\mathrm{T}}\mathbf{P} \leq \mathbf{P}\Big[\mathbf{F}(\mathbf{I} + \varepsilon_3\mathbf{U}_F)\mathbf{F}^{\mathrm{T}} + \frac{1}{\varepsilon_3}\mathbf{T}_F + l_f\mathbf{F}_\Delta\Big]\mathbf{P}. \tag{20b}$$

Then, it is obvious that this Lemma holds.

Now, a primary concept for this paper is described as follows.

**Theorem 3.1** *Let the disturbance attenuation index $\delta > 0$ and the robust relative stability index $\alpha > 0$, where $\delta$ and $\alpha$ are prescribed scalars that are determined according to performance requirements of the structure. Consider a given uncertain structural system (5) with structured uncertainties in (6)–(9). Then, if there exist positive adjustable scalars $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, and $\varepsilon$, an adjustable matrix $\mathbf{Q} > 0$, and a solution matrix $\mathbf{P} > 0$ satisfying the following Riccati equation:*

$$\begin{aligned} &(\mathbf{A} + \alpha\mathbf{I})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A} + \alpha\mathbf{I}) - \mathbf{P}\Big\{\mathbf{B}(\mathbf{I} - \frac{\varepsilon_2}{2}\mathbf{U}_B)\mathbf{B}^{\mathrm{T}} \\ &-\varepsilon_1\mathbf{T}_A - \frac{1}{2\varepsilon_2}\mathbf{T}_B - \frac{\varepsilon}{\delta}\Big[\mathbf{F}(\mathbf{I} + \varepsilon_3\mathbf{U}_F)\mathbf{F}^{\mathrm{T}} + \frac{1}{\varepsilon_3}\mathbf{T}_F + l_f\mathbf{F}_\Delta\Big]\Big\}\mathbf{P} \\ &+\frac{1}{\varepsilon_1}\mathbf{U}_A + \frac{1}{\varepsilon\delta}\mathbf{C}^{\mathrm{T}}\mathbf{C} + \mathbf{Q} = \mathbf{0}, \qquad 0 < \varepsilon_2 < \frac{2}{\bar{\sigma}(\mathbf{U}_B)} \end{aligned} \tag{21}$$

*where $\mathbf{T}_A$, $\mathbf{U}_A$, $\mathbf{T}_B$, $\mathbf{U}_B$, $\mathbf{T}_F$, $\mathbf{U}_F$ and $\mathbf{F}_\Delta$ are as in (9), then the state-feedback controller*

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) = -r\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{x}(t), \tag{22}$$

$$\frac{1}{\varepsilon_2\bar{\sigma}(\mathbf{U}_B)} - 0.5 \geq r \geq 0.5 \qquad \text{or} \qquad 0.5 \geq r \geq \frac{1}{\varepsilon_2\underline{\sigma}(\mathbf{U}_B)} - 0.5, \tag{23}$$

*guarantees a robust $\alpha$-degree relative stability (16) and a $\delta$-degree $H_\infty$ disturbance attenuation (15) for the uncertain structural system (5) with all admissible structured uncertainties as shown in (6) – (9).*

*Proof* To prove this theorem, equation (19) is investigated for the uncertain system (5) with uncertainties in (6) – (9). Control vector $\mathbf{u}(t)$ is given by equation (22), and

$$\mathbf{A}_c = \mathbf{A} + \Delta\mathbf{A} - \mathbf{BK} - \Delta\mathbf{BK} = \mathbf{A} + \Delta\mathbf{A} - r\mathbf{BB}^{\mathrm{T}}\mathbf{P} - r\Delta\mathbf{BB}^{\mathrm{T}}\mathbf{P}.$$

Thus, by using the Riccati equation (21), Lemmas 2 and 3, and conditions in (21) and (23), we have

$$\begin{aligned}
(\mathbf{A}_c{+}\alpha\mathbf{I})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A}_c + \alpha\mathbf{I}) + \frac{\varepsilon}{\delta}\mathbf{P}\Big[\mathbf{F}(\mathbf{I} + \varepsilon_3\mathbf{U}_F)\mathbf{F}^{\mathrm{T}} + \frac{1}{\varepsilon_3}\mathbf{T}_F + l_f\mathbf{F}_\Delta\Big]\mathbf{P} + \frac{1}{\varepsilon\delta}\mathbf{C}^{\mathrm{T}}\mathbf{C}& \\
= (\mathbf{A} + \alpha\mathbf{I})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A} + \alpha\mathbf{I}) + \frac{\varepsilon}{\delta}\mathbf{P}\Big[\mathbf{F}(\mathbf{I} + \varepsilon_3\mathbf{U}_F)\mathbf{F}^{\mathrm{T}} + \frac{1}{\varepsilon_3}\mathbf{T}_F + l_f\mathbf{F}_\Delta\Big]\mathbf{P}& \\
+ \frac{1}{\varepsilon\delta}\mathbf{C}^{\mathrm{T}}\mathbf{C} + (\Delta\mathbf{A} - r\mathbf{BB}^{\mathrm{T}}P - r\Delta\mathbf{BB}^{\mathrm{T}}\mathbf{P})^{\mathrm{T}}\mathbf{P}& \\
+ \mathbf{P}(\Delta\mathbf{A} - r\mathbf{BB}^{\mathrm{T}}\mathbf{P} - r\Delta\mathbf{BB}^{\mathrm{T}}\mathbf{P}) = \mathbf{P}\Big[\mathbf{B}(\mathbf{I} - \frac{\varepsilon_2}{2}\mathbf{U}_B)\mathbf{B}^{\mathrm{T}} - \varepsilon_1\mathbf{T}_A - \frac{1}{2\varepsilon_2}\mathbf{T}_B\Big]\mathbf{P}& \\
- \frac{1}{\varepsilon_1}\mathbf{U}_A - \mathbf{Q} + \Delta\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\Delta\mathbf{A} - r\mathbf{P}(2\mathbf{BB}^{\mathrm{T}} + \Delta\mathbf{BB}^{\mathrm{T}} + \mathbf{B}\Delta\mathbf{B}^{\mathrm{T}})\mathbf{P}& \\
\le \mathbf{P}\Big[ - 2r^2\varepsilon_2\mathbf{BU}_B\mathbf{B}^{\mathrm{T}} - \frac{1}{2\varepsilon_2}\mathbf{T}_B - r(\Delta\mathbf{BB}^{\mathrm{T}} + \mathbf{B}\Delta\mathbf{B}^{\mathrm{T}})\Big]\mathbf{P} - \varepsilon_1\mathbf{PT}_A\mathbf{P}& \\
- \frac{1}{\varepsilon_1}\mathbf{U}_A + \mathbf{P}(\Delta\mathbf{A}^{\mathrm{T}} + \Delta\mathbf{A})\mathbf{P} - \mathbf{Q} \le -\mathbf{Q} < \mathbf{0}.&
\end{aligned}$$

Thus, controller (22) makes inequality (19) hold. Then, by Lemma 3, Theorem 3.1 is proved.

The proposed controller (22) in Theorem 3.1 is not only a robust controller with $H_\infty$ disturbance attenuation and robust relative stability, but also an optimal controller in the $H_2$ optimal sense under a certain meaning as discussed in the next section.

Now, consider matched uncertain systems with matched uncertainties in (10) – (12).

**Theorem 3.2** *Consider a matched uncertain system (5) with the matched structured uncertainties in (10) – (12), a specified relative stability degree $\alpha$, and a disturbance attenuation index $\delta$. Select an assigned matrix $\mathbf{Q} > 0$, and positive adjustable scalars $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, $\varepsilon$, and $r$ within the following regions*

$$\begin{gathered}
\frac{\underline{\sigma}\Big(\mathbf{I} - 0.5\varepsilon_2\mathbf{U}_{bB} - 0.5\dfrac{1}{\varepsilon_2}\mathbf{T}_{bB}\Big)}{\bar{\sigma}(\mathbf{T}_{bA})} > \varepsilon_1 > 0, \\
\frac{\underline{\sigma}\Big[\Big(\mathbf{I} - 0.5\varepsilon_2\mathbf{U}_{bB} - 0.5\dfrac{1}{\varepsilon_2}\mathbf{T}_{bB}\Big) - \varepsilon_1\mathbf{T}_{bA}\Big]\delta}{\bar{\sigma}\Big[\dfrac{1}{\varepsilon_3}\mathbf{T}_{bF} + \mathbf{F}_B(\mathbf{I} + \varepsilon_3\mathbf{U}_{bF})\mathbf{F}_B^{\mathrm{T}} + l_f\mathbf{F}_{b\Delta}\Big]} > \varepsilon > 0, \qquad r \ge 0.5,
\end{gathered} \tag{24}$$

*where $\bar{\sigma}$ and $\underline{\sigma}$ denote the maximum and minimum singular values of a matrix, respectively. Then, there always exists a solution matrix $\mathbf{P} > 0$ that satisfies the following Riccati equation*

$$\begin{aligned}(\mathbf{A}+\alpha\mathbf{I})^{\mathrm{T}}\mathbf{P}+\mathbf{P}(\mathbf{A}+\alpha\mathbf{I})-\mathbf{P}\mathbf{B}\Big\{\mathbf{I}-\frac{\varepsilon_2}{2}\mathbf{U}_{bB}-\frac{1}{2\varepsilon_2}\mathbf{T}_{bB}-\varepsilon_1\mathbf{T}_{bA} \\ -\frac{\varepsilon}{\delta}\Big[\mathbf{F}_B(I+\varepsilon_3\mathbf{U}_{bF})\mathbf{F}_B^{\mathrm{T}}+\frac{1}{\varepsilon_3}\mathbf{T}_{bF}+l_f\mathbf{F}_{b\Delta}\Big]\Big\}\mathbf{B}^{\mathrm{T}}\mathbf{P}+\frac{1}{\varepsilon_1}\mathbf{U}_{bA}+\frac{1}{\varepsilon\delta}\mathbf{C}^{\mathrm{T}}\mathbf{C}+\mathbf{Q}=\mathbf{0}\end{aligned} \tag{25}$$

*The robust active state-feedback controller in (22) guarantees a robust $\alpha$-degree relative stability (16) and a $\delta$-degree disturbance attenuation (15) for the uncertain structural system (5) with all admissible matched structured uncertainties as shown in (10) – (12).*

*Proof* Because of matched uncertainty conditions, $\mathbf{I}-0.5\varepsilon_2\mathbf{U}_{bB}-0.5\,\frac{1}{\varepsilon_2}\,\mathbf{T}_{bB}>0$ for some $\varepsilon_2$. Based on optimal control theory [1] it is obvious that selection of $\varepsilon_1$ and $\varepsilon$ in (24) guarantees that the Riccati equation (25) has a solution matrix $\mathbf{P}>0$ for any selected positive semi-definite matrix $\mathbf{Q}$. Following a line of proof similar to that used in Theorem 3.1 and using Lemma 2 lead to the following:

$$\begin{aligned}&(\mathbf{A}_c+\alpha\mathbf{I})^{\mathrm{T}}\mathbf{P}+\mathbf{P}(\mathbf{A}_c+\alpha\mathbf{I})+\frac{\varepsilon}{\delta}\mathbf{P}\Big[\mathbf{F}(\mathbf{I}+\varepsilon_3\mathbf{U}_F)\mathbf{F}^{\mathrm{T}}+\frac{1}{\varepsilon_3}\mathbf{T}_F+l_f\mathbf{F}_\Delta\Big]\mathbf{P}+\frac{1}{\varepsilon\delta}\mathbf{C}^{\mathrm{T}}\mathbf{C} \\ &\quad=\mathbf{P}\mathbf{B}\Big[\mathbf{I}-\frac{\varepsilon_2}{2}\mathbf{U}_{bB}-\frac{1}{2\varepsilon_2}\mathbf{T}_{bB}-\varepsilon_1\mathbf{T}_{bA}\Big]\mathbf{B}^{\mathrm{T}}\mathbf{P}-\frac{1}{\varepsilon_1}\mathbf{U}_{bA}+\Delta\mathbf{A}_B^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{B}\Delta\mathbf{A}_B \\ &\quad-r\mathbf{P}\mathbf{B}(2\mathbf{I}+\Delta\mathbf{B}_B+\Delta\mathbf{B}_B^{\mathrm{T}})\mathbf{B}^{\mathrm{T}}\mathbf{P}-\mathbf{Q}\le-\mathbf{Q}<0\end{aligned}$$

Thus, by Lemma 3, the proof is complete.

*Remark 3.1* The disturbance attenuation index $\delta>0$ and the robust relative stability index $\alpha>0$ are prescribed based on engineering requirements. Riccati equations (21) or (25) are solved for matrix $\mathbf{P}$ after selection of a set of adjustable parameters. $\mathbf{Q}$ is a small positive definite matrix. Then, the robust active control law in equation (22) is used with $\mathbf{P}$ from Riccati equation (21) or (25).

*Remark 3.2* For tuning the adjustable scalars in Theorem 3.2, $\varepsilon_2$ is usually selected such that $\underline{\sigma}(\mathbf{I}-0.5\varepsilon_2\mathbf{U}_{bB}-0.5\frac{1}{\varepsilon_2}\mathbf{T}_{bB})$ is large, and $\varepsilon_3$ is selected such that $\bar{\sigma}(\varepsilon_3\mathbf{F}_B\mathbf{U}_{bF}\mathbf{F}_B^{\mathrm{T}} + \frac{1}{\varepsilon_3}\mathbf{T}_{bF})$ is small.

It is noticed that for uncertain structural systems $\Delta\mathbf{F}=\mathbf{0}$ is a special case of what was discussed above. The following remark addresses this case.

*Remark 3.3* Theorems 3.1 – 3.2 are valid for the case in which disturbance input uncertainties are not considered, i.e., $\Delta\mathbf{F}=\mathbf{0}$. For this special case, we simply let $\mathbf{T}_F=\mathbf{0}$, $\mathbf{U}_F=\mathbf{0}$, and $\mathbf{F}_\Delta=\mathbf{0}$ for Theorem 3.1 and $\mathbf{T}_{bF}=\mathbf{0}$, $\mathbf{U}_{bF}=\mathbf{0}$, and $\mathbf{F}_{b\Delta}=\mathbf{0}$ for Theorem 3.2. Therefore, for the case of $\Delta\mathbf{F}=\mathbf{0}$, Riccati equations (21) and (25) in Theorems 3.1 and 3.2 are reduced to

$$\begin{aligned}(\mathbf{A}+\alpha\mathbf{I})^{\mathrm{T}}\mathbf{P}+\mathbf{P}(\mathbf{A}+\alpha\mathbf{I})-\mathbf{P}\Big[\mathbf{B}(\mathbf{I}-\frac{\varepsilon_2}{2}\mathbf{U}_B)\mathbf{B}^{\mathrm{T}}-\varepsilon_1\mathbf{T}_A-\frac{1}{2\varepsilon_2}\mathbf{T}_B-\frac{\varepsilon}{\delta}\mathbf{F}\mathbf{F}^{\mathrm{T}}\Big]\mathbf{P} \\ +\frac{1}{\varepsilon_1}\mathbf{U}_A+\frac{1}{\varepsilon\delta}\mathbf{C}^{\mathrm{T}}\mathbf{C}+\mathbf{Q}=\mathbf{0},\end{aligned} \tag{26}$$

$$(\mathbf{A}+\alpha\mathbf{I})^{\mathrm{T}}\mathbf{P}+\mathbf{P}(\mathbf{A}+\alpha\mathbf{I})-\mathbf{PB}\Big\{\mathbf{I}-\frac{\varepsilon_2}{2}\,\mathbf{U}_{bB}-\frac{1}{2\varepsilon_2}\,\mathbf{T}_{bB}-\varepsilon_1\mathbf{T}_{bA}-\frac{\varepsilon}{\delta}\,\mathbf{F}_B\mathbf{F}_B^{\mathrm{T}}\Big\}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \frac{1}{\varepsilon_1}\,\mathbf{U}_{bA}+\frac{1}{\varepsilon\delta}\,\mathbf{C}^{\mathrm{T}}\mathbf{C}+\mathbf{Q}=\mathbf{0}, \tag{27}$$

respectively. These equations coincide with the results in [14], in which no uncertainty is considered for the disturbance input matrix, i.e., $\Delta\mathbf{F}=\mathbf{0}$ and also $\varepsilon_2=1$.

Selection of the set of adjustable scalars $\varepsilon_i$ $(i=1,2,3)$, $\varepsilon$, gain parameter $r$, and adjustable positive definite matrix $\mathbf{Q}$ requires some experience. However, these adjustable scalars, parameter, and matrix provide flexibility for obtaining a desired robust active controller for an uncertain structural system. Some general guidance for selection of this adjustable set is summarized in the following remarks.

*Remark 3.4* The set of adjustable scalars $\varepsilon_i$ $(i=1,2,3)$, and $\varepsilon$ is usually chosen in (21) or (25) of Theorems 3.1 − 3.2, such that

$$\mathbf{B}\Big(\mathbf{I}-\frac{\varepsilon_2}{2}\mathbf{U}_B\Big)\mathbf{B}^{\mathrm{T}}-\varepsilon_1\mathbf{T}_A-\frac{1}{2\varepsilon_2}\mathbf{T}_B-\frac{\varepsilon}{\delta}\Big[\mathbf{F}(\mathbf{I}+\varepsilon_3\mathbf{U}_F)\mathbf{F}^{\mathrm{T}}+\frac{1}{\varepsilon_3}\mathbf{T}_F+l_f\mathbf{F}_\Delta\Big] \tag{28a}$$

$$\text{or}\quad \mathbf{I}-\frac{\varepsilon_2}{2}\mathbf{U}_{bB}-\frac{1}{2\varepsilon_2}\mathbf{T}_{bB}-\varepsilon_1\mathbf{T}_{bA}-\frac{\varepsilon}{\delta}\Big[\frac{1}{\varepsilon_3}\mathbf{T}_{bF}+\mathbf{F}_B(\mathbf{I}+\varepsilon_3\mathbf{U}_{bF})\mathbf{F}_B^{\mathrm{T}}+l_f\mathbf{F}_{b\Delta}\Big] \tag{28b}$$

is semi-positive definite if possible. Positive definite matrix $\mathbf{Q}$ is usually assigned as a small matrix. Then, matrix $\mathbf{P}$ is solved from Riccati equations (21) and (25), respectively. Gain parameter $r$ is selected to satisfy Riccati equations. A small $r$ means a small energy requirement for the controller. However, a large $r$ provides a fast decay response to disturbances (earthquake and wind disturbances, etc.). Also, another consideration for selection of gain $r$ is to let conditions in Section 4 hold for $H_2$ optimality in Theorems 4.1 − 4.2. Therefore, selection of gain parameter $r$ depends on physical conditions and requirements. Due to the special block companion form of structural systems, and even the special block diagonal structure, selection of appropriate adjustable scalars is accomplished easily.

*Remark 3.5* For a matched uncertain structural system, selection of adjustable scalars $\varepsilon$ and $\varepsilon_i$ $(i=1,2,3)$, is very easy from (24) since solution of the Riccati equation (25) always exists from (24).

## 4 Preservation of $H_2$ Optimality

The proposed controllers (22) in Theorems 3.1 − 3.2 are not only robust with $H_\infty$ disturbance attenuation and robust relatively stability, but also optimal in the $H_2$ optimal sense as discussed in this section. Thus, many $H_2$ optimal properties [1] hold for these robust controlled uncertain structural systems via the designed controller. The following theorems provide these results with $H_2$ optimality.

**Theorem 4.1** *Under conditions in Theorem 3.1, if*

$$2\alpha\mathbf{P}+\mathbf{P}\Big\{r\Big[\mathbf{B}(\mathbf{I}-\varepsilon_2\mathbf{U}_B)\mathbf{B}^{\mathrm{T}}-\frac{1}{\varepsilon_2}\mathbf{T}_B\Big]-\Big[\mathbf{B}(\mathbf{I}-\frac{\varepsilon_2}{2}\mathbf{U}_B)\mathbf{B}^{\mathrm{T}}-\frac{1}{2\varepsilon_2}\mathbf{T}_B\Big] + \frac{\varepsilon}{\delta}\Big[\mathbf{F}(\mathbf{I}+\varepsilon_3\mathbf{U}_F)\mathbf{F}^{\mathrm{T}}+\frac{1}{\varepsilon_3}\mathbf{T}_F+l_f\mathbf{F}_\Delta\Big]\Big\}\mathbf{P}+\frac{1}{\varepsilon\delta}\mathbf{C}^{\mathrm{T}}\mathbf{C}+\mathbf{Q}\geq\mathbf{0}, \tag{29}$$

*then the robust active controller (24) is also $H_2$ optimal for the uncertain structural system (5) regarding a specific performance index*

$$J = \int [\mathbf{x}^{\mathrm{T}}(t)\tilde{\mathbf{Q}}\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\tilde{\mathbf{R}}\mathbf{u}(t)]\, dt \tag{30}$$

*with*

$$\tilde{\mathbf{Q}} = -\tilde{\mathbf{A}}^{\mathrm{T}}\mathbf{P} - \mathbf{P}\tilde{\mathbf{A}} + \mathbf{P}\mathbf{B}\tilde{\mathbf{R}}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} \geq 0, \qquad \tilde{\mathbf{R}} = \frac{1}{r}\mathbf{I}, \tag{31}$$

*where*

$$\tilde{\mathbf{A}} = \mathbf{A} + \Delta\mathbf{A} - r\Delta\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{P}. \tag{32}$$

*Proof* To show that the designed controller (22) is optimal, matrix $\tilde{\mathbf{Q}}$ is expanded as follows:

$$\begin{aligned}\tilde{\mathbf{Q}} &= -\tilde{\mathbf{A}}^{\mathrm{T}}\mathbf{P} - \mathbf{P}\tilde{\mathbf{A}} + \mathbf{P}\mathbf{B}\tilde{\mathbf{R}}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\\ &= -(\mathbf{A} + \Delta\mathbf{A} - r\Delta\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{P})^{\mathrm{T}}\mathbf{P} - \mathbf{P}(\mathbf{A} + \Delta\mathbf{A} - r\Delta\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{P}) + r\mathbf{P}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{P}.\end{aligned}$$

It follows from the proof of Theorem 3.1 and equation (21) that

$$\begin{aligned}\tilde{\mathbf{Q}} \geq 2\alpha\mathbf{P} + \mathbf{P}\Bigg\{ r\left[\mathbf{B}(\mathbf{I} - \varepsilon_2\mathbf{U}_B)\mathbf{B}^{\mathrm{T}} - \frac{1}{\varepsilon_2}\mathbf{T}_B\right] - \left[\mathbf{B}\left(\mathbf{I} - \frac{\varepsilon_2}{2}\mathbf{U}_B\right)\mathbf{B}^{\mathrm{T}} - \frac{1}{2\varepsilon_2}\mathbf{T}_B\right]\\ + \frac{\varepsilon}{\delta}\left[\mathbf{F}(\mathbf{I} + \varepsilon_3\mathbf{U}_F)\mathbf{F}^{\mathrm{T}} + \frac{1}{\varepsilon_3}\mathbf{T}_F + l_f\mathbf{F}_\Delta\right]\Bigg\}\mathbf{P} + \frac{1}{\varepsilon\delta}\mathbf{C}^{\mathrm{T}}\mathbf{C} + \mathbf{Q}.\end{aligned}$$

Therefore, if (29) holds, $\tilde{\mathbf{Q}} \geq 0$, and equation (31) is true. Since $\tilde{\mathbf{R}} > 0$ and $\tilde{\mathbf{Q}} \geq 0$, the perturbed uncertain system (14) is optimal with respect to the specific performance index (30) by the active robust controller (22) based on well-known $H_2$ optimal control theory. Thus, this theorem is proved.

For other cases, the following theorem for $H_2$ optimality is listed. Due to the similarity of proofs, details are omitted here.

*Theorem 4.2 Under conditions in Theorem 3.2, if*

$$\begin{aligned}2\alpha\mathbf{P} + P\mathbf{B}\Bigg\{ r\left(\mathbf{I} - \varepsilon_2\mathbf{U}_{bB} - \frac{1}{\varepsilon_2}\mathbf{T}_{bB}\right) - \left(I - \frac{\varepsilon_2}{2}\mathbf{U}_{bB} - \frac{1}{2\varepsilon_2}\mathbf{T}_{bB}\right)\\ + \frac{\varepsilon}{\delta}\left[\mathbf{F}_B(I + \varepsilon_3\mathbf{U}_{bF})\mathbf{F}_B^{\mathrm{T}} + \frac{1}{\varepsilon_3}\mathbf{T}_{bF} + l_f\mathbf{F}_{b\Delta}\right]\Bigg\}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \frac{1}{\varepsilon\delta}\mathbf{C}^{\mathrm{T}}\mathbf{C} + \mathbf{Q} \geq \mathbf{0},\end{aligned} \tag{33}$$

*then the robust active controller (22) is also $H_2$ optimal for the uncertain structural system (5) regarding a specific performance index $J$ in (30) with $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{R}}$ in (31).*

*Remark 4.1* Notice that $H_2$ optimality is for the uncertain structural system (5) regarding a specific performance index $J$ in (30) with $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{R}}$ in (31), where $\tilde{\mathbf{Q}}$ is uncertain. Also, it is noticed that uncertainties in the uncertain system are unknown but only their bound and structures are known. The importance of the above theorems is that when the respective condition of (29) or (33) holds, the robust controller (22) provides $H_2$ optimality in face of any admissible uncertainties as described in the respective theorems

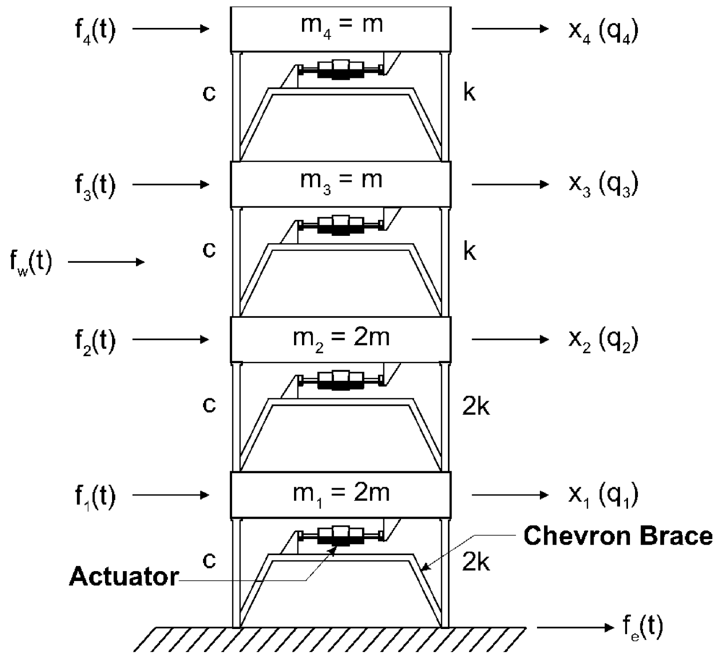


**Figure 5.1**. Building model for numerical example.

in Section 3 even though their exact values are not known. This means that the robust controller (22) provides a gain margin of infinity and at least a 60°-phase margin for whole uncertain structural systems with all admissible uncertainties even though the exact performance due to unknown uncertainties is not known.

## 5 Numerical Example

In order to illustrate effectiveness of the proposed approach for robust control, a numerical example of a four-degree-of-freedom system is taken and extended from [6] (see Figure 5.1). For this model of a tall building, stiffness, mass, and damping values of $k = 350 \times 10^6$N/m, $m = 1.05 \times 10^6$kg, and $c = 1.575 \times 10^6$N-s/m, respectively, are assumed. Total weight of the building is 61.74MN. In order to design a robust controller that is valid for both earthquake and wind disturbances, the considered external disturbance force applied to each floor level is $f_{di}(t) = f_{wi}(t) + f_{ei}(t)$, $i = 1, \dots, 4$, where $f_{wi}(t)$ is from a strong wind event and $f_{ei}(t)$ is from an earthquake event. The total external force for each floor level is $f_i(t) = f_{ui}(t) + f_{di}(t)$, where $f_{ui}(t)$ is the control force. The system is described as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}_d\dot{\mathbf{q}} + \mathbf{K}_s\mathbf{q} = \mathbf{f} \quad \text{or} \quad \ddot{\mathbf{q}} + \mathbf{M}^{-1}\mathbf{C}_d\dot{\mathbf{q}} + \mathbf{M}^{-1}\mathbf{K}_s\mathbf{q} = \mathbf{M}^{-1}\mathbf{f}, \tag{34}$$

where $\mathbf{M}$ only has elements on the diagonal, $\mathbf{q}$ is a relative displacement vector to the ground,

$$\mathbf{K}_s = \begin{bmatrix} 4k & -2k & 0 & 0 \\ -2k & 3k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & k \end{bmatrix} = 175 \cdot 10^6 \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 6 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix},$$

$$\mathbf{C}_d = \begin{bmatrix} 2c & -c & 0 & 0 \\ -c & 2c & -c & 0 \\ 0 & -c & 2c & -c \\ 0 & 0 & -c & c \end{bmatrix} = 1.575 \cdot 10^6 \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \tag{35}$$

$$\mathbf{M} = 1.05 \cdot 10^6 \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix}.$$

From equations (34) and (35), we have

$$\mathbf{D}_{k0} = \begin{bmatrix} 2000/3 & -1000/3 & 0 & 0 \\ -1000/3 & 500 & -500/3 & 0 \\ 0 & -1000/3 & 2000/3 & -1000/3 \\ 0 & 0 & -1000/3 & 1000/3 \end{bmatrix},$$

$$\mathbf{D}_{c0} = \begin{bmatrix} 1.5 & -0.75 & 0 & 0 \\ -0.75 & 1.5 & -0.75 & 0 \\ 0 & -1.5 & 3.0 & -1.5 \\ 0 & 0 & -1.5 & 1.5 \end{bmatrix}, \qquad \mathbf{A}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{D}_{k0} & -\mathbf{D}_{c0} \end{bmatrix} \tag{36}$$

If it is assumed that each story has a controller and is connected to a Chevron brace, then

$$\mathbf{B}_{ch} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B}_0 = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{B}_{ch} \end{bmatrix}, \quad \text{and} \quad \mathbf{B}_{u0} = \mathbf{M}^{-1}\mathbf{B}_{ch}. \tag{37}$$

State variables are chosen to be the displacement and velocity of each level (relative to the ground), $\mathbf{z}(t) = \mathbf{x}(t)$, i.e., $\mathbf{C} = \mathbf{I}$. Consider $\mathbf{w}(t) = [\, w_w^{\mathrm{T}}(t) \quad w_e(t)\,]^{\mathrm{T}}$, where $w_w(t)$ and $w_e(t)$ are wind and earthquake forces, respectively. Then,

$$\mathbf{F}_{w0} = \mathbf{M}^{-1}\mathbf{F}_0,$$

where

$$\mathbf{F}_0 = \begin{bmatrix} & 2 \\ \mathbf{I}_4 & 2 \\ & 1 \\ & 1 \end{bmatrix}, \qquad w_w(t) = \begin{bmatrix} f_{w1}(t) \\ f_{w2}(t) \\ f_{w3}(t) \\ f_{w4}(t) \end{bmatrix}$$

and $w_e(t)$ is an earthquake force for a mass $m$. For simplicity, only an earthquake loading is considered here and it follows that $\mathbf{F}_{w0}$ and $\mathbf{w}(t)$ reduce to

$$\mathbf{F}_{w0} = \mathbf{M}^{-1} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{m} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{m}\mathbf{F}_{wI},$$

$$\mathbf{F}_{wI} = [\,1, 1, 1, 1\,]^{\mathrm{T}}, \quad \text{and} \quad \mathbf{w}(t) = w_e(t). \tag{38}$$

Uncertainties are taken to be as follows: $\Delta m = \pm 10\% \cdot m$, $\Delta k = \pm 10\% \cdot k$, and $\Delta c = \pm 10\% \cdot c$. Thus, $\Delta\mathbf{K}_s = a_k \cdot 0.1\mathbf{K}_s$, $\Delta\mathbf{C}_d = a_c \cdot 0.1\mathbf{C}_d$, and $\Delta\mathbf{M} = a_m \cdot 0.1\mathbf{M}$, where $|a_k| \le 1$, $|a_c| \le 1$, and $|a_m| \le 1$. Then, $(\mathbf{M}+\Delta\mathbf{M})^{-1} = (0.90909 \sim 1.11111)\mathbf{M}^{-1}$. These are parametric perturbations, i.e., structured uncertainties. Also, it is obvious that the disturbance input matrix $\mathbf{F}_w$ has uncertainties when parameter $m$ changes with uncertainties. $\mathbf{D}_k$ is perturbed by a factor $(0.818181 \sim 1.22222)$, as is $\mathbf{D}_c$. The central matrix $1.0101\mathbf{M}^{-1}$ is taken as a nominal $\mathbf{M}_0^{-1}$ and $\Delta\mathbf{M}_0^{-1} = a_{m1} \cdot 0.10101\mathbf{M}^{-1} = a_{m1} \cdot 0.1\mathbf{M}_0^{-1}$, where $|a_{m1}| \le 1$. Further, take central matrices as nominal models for new $\mathbf{D}_k$ and $\mathbf{D}_c$ for design, i.e.,

$$\mathbf{D}_k = 1.0202\mathbf{D}_{k0}, \quad \mathbf{D}_c = 1.0202\mathbf{D}_{c0}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{D}_k & -\mathbf{D}_c \end{bmatrix}, \quad \mathbf{B}_u = \mathbf{M}_0^{-1}\mathbf{B}_{ch},$$

then

$$\begin{gathered} \Delta\mathbf{D}_k = a_{k1} 0.198\mathbf{D}_k, \quad \Delta\mathbf{D}_c = a_{c1} 0.198\mathbf{D}_c, \quad \Delta\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\Delta\mathbf{D}_k & -\Delta\mathbf{D}_c \end{bmatrix}, \\ \text{and} \quad \Delta\mathbf{B}_u = 0.1 a_{m1}\mathbf{M}_0^{-1}\mathbf{B}_{ch}, \end{gathered} \tag{39}$$

where $|a_{k1}| \le 1$, and $|a_{c1}| \le 1$. These matrices are actually structured uncertainties. Similarly, a new central disturbance input matrix is taken as follows: $\mathbf{F}_w = \mathbf{M}_0^{-1}[2,2,1,1]^{\mathrm{T}}$, $\Delta\mathbf{F}_w = 0.1 f_1 \mathbf{F}_w$ and $|f_1| \le 1$. This case is obviously a matched uncertainty model, so that $\Delta\mathbf{B}_{Bu}$, $\Delta\mathbf{D}_{Bk}$, $\Delta\mathbf{D}_{Bc}$, $\mathbf{F}_{Bw}$ and $\Delta\mathbf{F}_{Bw}$ are available and can be obtained by a left multiplication of $\mathbf{B}_w$ with the respective uncertainties and matrices.

Next, the SVD decomposition is applied to all of the above uncertainty structures to obtain $\mathbf{T}_{bk}$, $\mathbf{U}_{bk}$, $\mathbf{T}_{bc}$, $\mathbf{U}_{bc}$, $\mathbf{T}_{bb}$, $\mathbf{U}_{bb}$, $\mathbf{T}_{bf}$, $\mathbf{U}_{bf}$, and $\mathbf{F}_{b\Delta}$. The final step is to design a robust controller for this uncertain structure system with all above structured uncertainties in $\Delta\mathbf{A}$, $\Delta\mathbf{B}$, and $\Delta\mathbf{F}$. A relative degree of stability and a disturbance attenuation index are taken to be $\alpha = 1.5$ and $\delta = 0.01$, respectively. Based on Theorem 3.2 and Remarks 3.2, 3.4, and 3.5, $\mathbf{Q} = 0.05\mathbf{I}$, $\varepsilon_1 = 3.9 \cdot 10^{-9}$, $\varepsilon_2 = 1$, $\varepsilon_3 = 1$, and $\varepsilon = 10^{-7}$. From Theorem 3.2, Riccati equation (25) has the solution matrix $\mathbf{P}$. For optimality, we choose $r = 1.04$. Also, equation (33) satisfies Theorem 4.2. Then, the robust control law (22) is $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) = -r\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{x}(t)$ with

$$\mathbf{K} = r\mathbf{B}^{\mathrm{T}}\mathbf{P} = r \cdot 10^8 \times \begin{bmatrix} 0.1611 & 1.1376 & -0.7897 & 0.7668 & 0.4093 & 0.0358 & 0.0693 & 0.0949 \\ -0.8795 & -0.7291 & 2.1767 & -0.4508 & -0.3735 & 0.4524 & 0.1012 & 0.0631 \\ 3.2609 & -2.8101 & -0.0195 & 0.7873 & 0.1028 & -0.1471 & 0.2127 & 0.1394 \\ -2.4035 & 1.7595 & -0.7564 & 0.4764 & 0.0511 & -0.0252 & -0.0859 & 0.1903 \end{bmatrix}. \tag{40}$$

## 6 Evaluation Indices

In order to evaluate the controller, special consideration is given to absolute accelerations $\mathbf{a}_a(t)$, interstory drifts $\mathbf{d}_x(t)$, and control forces $\mathbf{u}(t)$. Maximum peak values and maximum RMS values for all four floors and over the entire simulation period are monitored and recorded. Elements of the relative acceleration vector $\mathbf{a}(t)$ are determined by numerical differentiation of the output velocities. Then, the absolute accelerations $\mathbf{a}_a(t)$ are computed as follows:

$$\begin{aligned}\mathbf{a}_a(t) &= \begin{bmatrix} a_{a1}(t) \\ a_{a2}(t) \\ a_{a3}(t) \\ a_{a4}(t) \end{bmatrix} = \mathbf{a}(t) + (1+\Delta f)a_e(t)\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \end{bmatrix} + (1+\Delta f)a_e(t)\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \\ &= \begin{bmatrix} \dot{x}_5(t) \\ \dot{x}_6(t) \\ \dot{x}_7(t) \\ \dot{x}_8(t) \end{bmatrix} + (1+\Delta f)a_e(t)\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\end{aligned} \tag{41}$$

where $a_e(t)$ is the acceleration time-history of the earthquake, and $\Delta f$ is the enlarged ratio of earthquake acceleration (for the nominal model $\Delta f = 0$). The interstory drift vector is

$$\mathbf{d}_x = [d_{x1},\, d_{x2},\, d_{x3},\, d_{x4}]^{\mathrm{T}} = [x_1,\, x_2 - x_1,\, x_3 - x_2,\, x_4 - x_3]^{\mathrm{T}}. \tag{42}$$

In order to facilitate an evaluation of the merits of the proposed approach for control, six performance indices are defined as listed in Tables 7.1 and 7.2. The maximum peak value of absolute acceleration is defined as follows:

$$J_1 = \max_{i,t}\{|a_{ai}(t)|\}. \tag{43}$$

The second evaluation criterion, $J_2$, is the maximum RMS value of absolute acceleration and is given by:

$$J_2 = \max_i \left[\frac{1}{T_f}\int_0^{T_f} a_{ai}^2(t)\,dt\right]^{1/2}. \tag{44}$$

The third and fourth indices, $J_3$ and $J_4$, are the maximum peak value and the maximum RMS value of the interstory drifts, respectively:

$$J_3 = \max_{i,t}\{|d_{xi}(t)|\}, \qquad J_4 = \max_i \left[\frac{1}{T_f}\int_0^{T_f} d_{xi}^2(t)\,dt\right]^{1/2}. \tag{45}$$

Finally, $J_5$ and $J_6$ are the maximum peak value and the maximum RMS value of the control forces, respectively,

$$J_5 = \max_{i,t}\{|u_i(t)|\}, \qquad J_6 = \max_i \left[\frac{1}{T_f}\int_0^{T_f} u_i^2(t)\,dt\right]^{1/2}.$$

## 7 Simulations

Numerical simulations are carried out for both a nominal case without perturbations and a worst case where $\Delta\mathbf{M} = 0.1\mathbf{M}$, $\Delta\mathbf{K}_s = 0.1\mathbf{K}_s$, and $\Delta\mathbf{C}_d = -0.1\mathbf{C}_d$ (i.e., $\Delta m = 0.1m$,

$\Delta k = 0.1k$, $\Delta c = -0.1c$). The basic concept is to take the worst case for $\mathbf{D}_c + \Delta\mathbf{D}_c$, i.e., the smallest one from $\Delta c = -0.1c$ and $\Delta m = 0.1m$, and the largest $\mathbf{K}_s + \Delta\mathbf{K}_s$ from $\Delta k = 0.1k$ when $\Delta m = 0.1m$. It follows that $\Delta\mathbf{D}_k = 0$, $\Delta\mathbf{D}_c = -0.1818\mathbf{D}_{c0}$, $\Delta\mathbf{B}_B = [\,\mathbf{0}\;\; -0.1\mathbf{I}\,]^{\mathrm{T}}$, and $\Delta\mathbf{F}_w = 0.1\mathbf{F}_{w0}$. Thus, one simulated uncertain system model in the worst case is taken as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{D}_{k0} & -0.81818\mathbf{D}_{c0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ 0.9\mathbf{B}_{u0} \end{bmatrix} \mathbf{K} \right\} \mathbf{x}(t) + 1.1 \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_{w0} \end{bmatrix} \mathbf{w}(t) \\ &= \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{D}_{k0} & -0.81818\mathbf{D}_{c0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ 0.9\mathbf{B}_{u0} \end{bmatrix} \mathbf{K} \right\} \mathbf{x}(t) - 1.1 a_e(t) \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_{wI} \end{bmatrix}, \\ \mathbf{z}(t) &= \mathbf{x}(t), \end{aligned} \tag{47}$$

where $\mathbf{D}_{k0}$ and $\mathbf{D}_{c0}$ are given by (36), $\mathbf{K}$ is from (40), $\mathbf{w}(t) = w_e(t)$, and $a_e(t)$ is the earthquake acceleration time-history. The simulated nominal system model is

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{D}_{k0} & -\mathbf{D}_{c0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_{u0} \end{bmatrix} \mathbf{K} \right\} \mathbf{x}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_{w0} \end{bmatrix} \mathbf{w}(t) \\ &= \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{D}_{k0} & -\mathbf{D}_{c0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_{u0} \end{bmatrix} \mathbf{K} \right\} \mathbf{x}(t) - a_e(t) \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_{wI} \end{bmatrix}, \\ \mathbf{z}(t) &= \mathbf{x}(t). \end{aligned} \tag{48}$$

A time history of acceleration $a_e(t)$ from the 1940 El Centro, California, earthquake is applied to the base of the structure.

It is noted that numerical simulations for the perturbed building apply the disturbance earthquake forces and corresponding accelerations enlarged by 10%.

For comparison, the numerical simulations are also conducted on the same structure using an LQR controller. Weighting matrices for the LQR design, $\mathbf{Q} = 10^{12} \times \mathbf{I}$ and $\mathbf{R} = \mathbf{I}$, are selected by a trial and error procedure in order to produce an allowable maximum peak control force that is physically realizable. Under these conditions, the maximum control force for the LQR controller is 811 kN. Likewise, the robust control force is limited to 810 kN for comparison. Then, a small gain robust controller is included with an adjustable gain of $r = 1.637 \times 10^{-2}$ which requires a maximum force 810 kN. Finally, a clipped robust controller with an 810 kN force limit is simulated as well, which is also physically realizable. However, by contrast, the robust controller provides information about how much force is required for a very high level of performance, without a trial and error procedure.

Output of numerical simulations for the uncontrolled, LQR controlled, and clipped robust controlled cases is shown in Figures 7.1 – 7.4. These graphs show fourth floor interstory drift and absolute acceleration for 30-sec of motion. Figures 7.1 and 7.2 illustrate results for the nominal model. Figures 7.3 and 7.4 show the corresponding information for the perturbed model. Results indicate that reduction in response of the structure is very good for both interstory drift and absolute acceleration. Note that the robust controller requires a much larger maximum control force if it is not clipped.

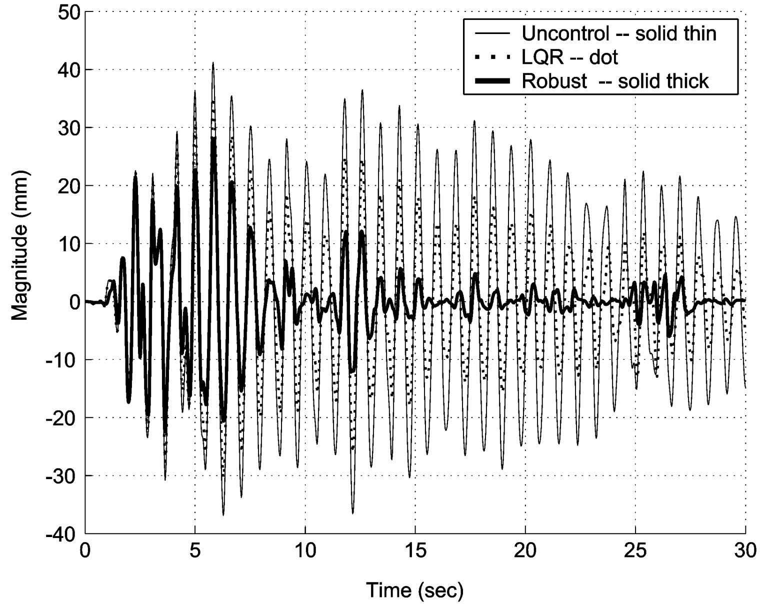


**Figure 7.1**. Nominal model: Uncontrolled and controlled interstory drift of the 4-th floor.

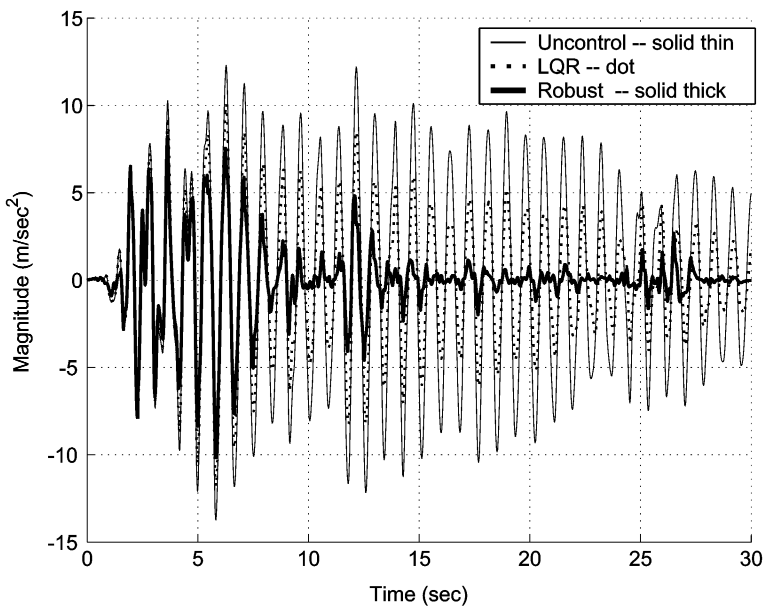


**Figure 7.2**. Nominal model: Uncontrolled and controlled absolute acceleration of the 4-th floor.

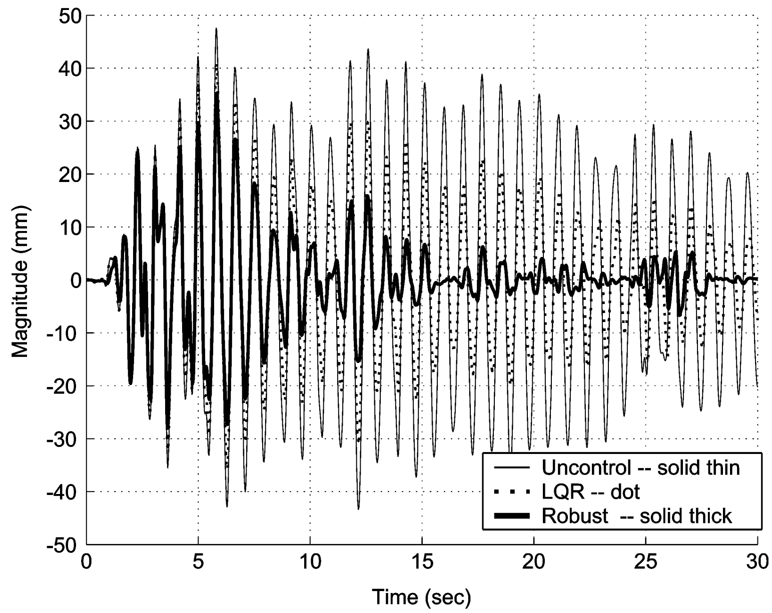


**Figure 7.3**. Perturbed model: Uncontrolled and controlled interstory drift of the 4-th floor.

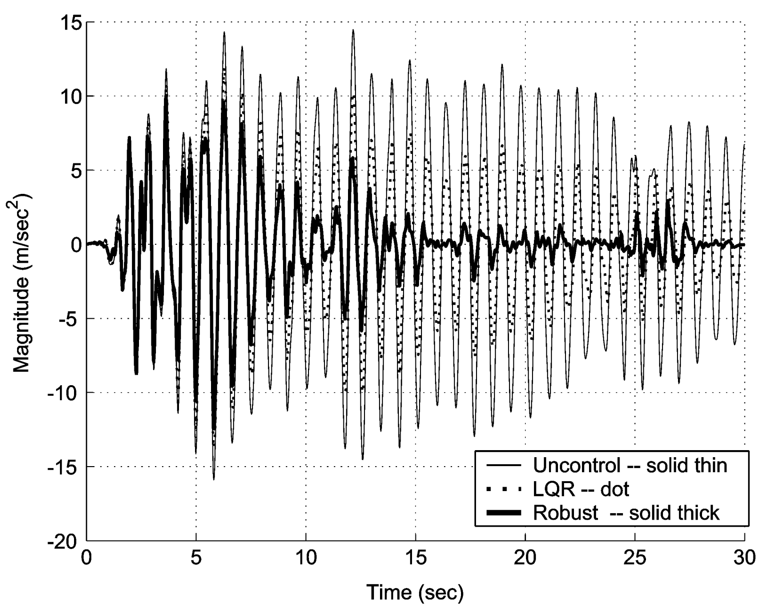


**Figure 7.4**. Perturbed model: Uncontrolled and controlled absolute acceleration of the 4-th floor.

Quantitative results from numerical simulations for the nominal and perturbed structure are listed in Tables 7.1 and 7.2, respectively. Cases presented include uncontrolled, LQR-controlled, robust controlled ($r = 1.04$), small gain robust controlled ($r = 1.637 \times 10^{-2}$), and clipped robust controlled ($r = 1.04$, $u_{max} = 810\,\text{kN}$). Simulation results in this paper and [13] appear to bode well for experimental implementation.

**Table 7.1** Comparison of simulation performance: Nominal model.

| Performance Index | Uncontrolled Model | LQR Control $Q = 1\times10^{12} I$, $R = I$ | Robust Control $r = 1.04$ | Small Gain Robust Control $r = 1.637\times10^{-2}$ | Clipped Robust Control $r = 1.06$ |
|---|---|---|---|---|---|
| Max Peak Absolute Accel. (m/s$^2$) | 13.72 | 11.79 | 3.58 | 11.72 | 10.18 |
| Max RMS Absolute Accel. (m/s$^2$) | 5.84 | 3.82 | 0.68 | 3.81 | 2.15 |
| Max Peak Interstory Drift. (mm) | 73.0 | 63.1 | 1.2 | 62.8 | 49.9 |
| Max RMS Interstory Drift. (mm) | 31.8 | 20.6 | 0.3 | 20.5 | 11.2 |
| Max Peak Force (kN) | — | 711 | 12,274 | 706 | 810 |
| Max RMS Force (kN) | — | 251.7 | 2,194.4 | 247.6 | 720.7 |

**Table 7.2** Comparison of simulation performance: Perturbed model.

| Performance Index | Uncontrolled Model | LQR Control $Q = 1\times10^{12} I$, $R = I$ | Robust Control $r = 1.04$ | Small Gain Robust Control $r = 1.637\times10^{-2}$ | Clipped Robust Control $r = 1.06$ |
|---|---|---|---|---|---|
| Max Peak Absolute Accel. (m/s$^2$) | 15.90 | 13.70 | 3.98 | 13.61 | 12.47 |
| Max RMS Absolute Accel. (m/s$^2$) | 7.13 | 4.63 | 0.75 | 4.61 | 2.77 |
| Max Peak Interstory Drift. (mm) | 82.8 | 72.5 | 1.4 | 72.0 | 61.4 |
| Max RMS Interstory Drift. (mm) | 38.9 | 25.0 | 0.3 | 24.9 | 14.5 |
| Max Peak Force (kN) | — | 811 | 14,398 | 810 | 810 |
| Max RMS Force (kN) | — | 305.4 | 2,559.8 | 300.4 | 734.9 |

## 8 Conclusions

In this paper, a general structural system model based on Lagrange's equation has been introduced. Its form is that of a special structural block companion matrix form, and an active robust controller for the uncertain structural system is described. General structured uncertainties and matched structured uncertainties are described and considered for uncertain structural systems. Considered structured uncertainties include those in the system, control input, and especially disturbance input matrices. In addition, special weighted SVD decomposition is applied to all structured uncertainties. An approach to design robust state-feedback algorithms for matched and general uncertain structural systems has been proposed. The active robust controller has robust $\alpha$-degree relative stability, robust $H_\infty$ $\delta$-degree disturbance rejection, and robust $H_2$ optimality for a family of uncertain systems. Settling time of the controlled system is always less than $4/\alpha$. Moreover, the $H_\infty$-norm of the transfer function from the disturbance vector $\mathbf{w}$ to the observed output vector $\mathbf{z}$ is not greater than $\delta$, i.e., $\|\mathbf{T}_{zw}(s)\|_\infty \le \delta$. Thus, hazardous effects of disturbances such as earthquakes and strong winds to the structural system are controlled and attenuated due to robust $H_\infty$ $\delta$-degree disturbance rejection. In addition, response to the disturbance is quickly reduced due to robust $\alpha$-degree relative stability and a judicious selection of the gain parameter $r$. The proposed controller is also $H_2$ optimal with a special performance index that is shown in Section 4. Thus, the designed robust controller provides infinity gain margin and at least a 60°-phase margin for entire uncertain structural system with all admissible uncertainties. A set of adjustable parameters provides flexibility in design of the robust controller. An example of an uncertain four-story building is used to illustrate results. Numerical simulations are carried on the building excited by the 1940 El Centro earthquake data and compared with the LQR controller by the six performance evaluation indices. Results show that the performance of the robust controller is very good.

## References

[1] Anderson, B.D.O. and Moore, J.B. *Linear Optimal Control*. Prentice Hall, Englewood Cliffs, NJ, 1971.

[2] Fujino,Y., Soong, T.T., and Spencer, B.F. Jr. Structural control: Basic concepts and applications. *Proc. ASCE Struct. Cong. XIV*, Chicago, IL, 1996. P. 1277–1287.

[3] Grandhi, R.V., Haq, I. and Khot, N.S. Enhanced robustness in integrated structural/control system design. *AIAA Journal* **29** (1991) 1168–1173.

[4] Housner, G.W., Bergman, L.A., Caughey, T.K., Chassiakos, A.G., Claus, R.O., Masri, S.F., Skelton, R.E., Soong, T.T., Spencer, B.F. and Yao, J.T.P. Structural control: Past and present. *J. Engin. Mech.* **123** (1997) 897–971.

[5] Housner, G.W., Soong, T.T. and Masri, S.F. Second generation of active structural control in civil engineering. *First World Conference on Structural Control*, Los Angeles, CA, 1994, P.3–18.

[6] James, M.L., Smith, G.M., Wolford, J.C. and Whaley, P.W. *Vibration of Mechanical and Structural Systems*. Harper Collins College Publishers, 1994.

[7] Kwakernaak, H. Robust control and $H_\infty$-optimization: Tutorial paper. *Automatica* **29** (1987) 255–273.

[8] Leitmann, G. Guaranteed asymptotic stability for some linear systems with bounded uncertainties. *Dynamics Syst., Measurement and Contr.* **101** (1979) 212–216.

[9] Soong, T.T. *Active Structural Control: Theory and Practice*. Longman Scientific and Technical, Essex, England, 1990.

[10] Spencer, B.F. Jr. and Sain, M.K. Controlling buildings: A new frontier in feedback. *IEEE Control Systems* **17** (1997) 19–35.

[11] Spencer, B.F. Jr., Sain, M.K., Kaspari, D.C. and Kantor, J.C. Reliability-based design of active control strategies. *Proc. ATC 17–1 Seminar on Seismic Isolation, Passive Energy Dissipation and Active Control*, San Francisco, Vol. 2, 1994. P. 761–772.

[12] Wang, S.-G., Lin, S., Shieh, L.S. and Sunkel, J.W. Observer-based controller for robust pole clustering in a vertical strip and disturbance rejection in structured uncertain systems. *Int. J. Robust and Nonlinear Control* **8** (1998) 1073–1084.

[13] Wang, S.-G., Roschke, P.N. and Yeh, H.Y. Robust control for structural systems with uncertainties. *Proc. of SPIE's 6th Annual Int. Symp. on Smart Structures and Materials, Conf. of Smart Systems for Bridges, Structures, and Highways, (SS06)*, Vol. 3671, 1999. P. 206–216.

[14] Wang, S.-G., Shieh, L.S. and Sunkel, J.W. Optimal pole-clustering in a vertical strip and disturbance rejection for Lagrange's systems. *Int. J. Dynamics and Control* **5** (1995) 295–312.

[15] Wang, S.-G., Shieh, L.S. and Sunkel, J.W. Robust optimal pole-placement in a vertical strip and disturbance rejection. *Int. J. of Systems Science* **26** (1995) 1839–1853.

[16] Wang, S.-G., Yeh, H.Y. and Roschke, P.N. Robust control for structural systems with parametric and unstructured uncertainties. *J. Vibration and Control* **7** (2001) 753–772.

[17] Yang, J.N., Wu, J.C., Reinhorn, A.M., Riley, M., Schmitendorf, W.E. and Jabbari, F. Experimental verifications of $H_\infty$ and sliding mode control for seismic-excited buildings. *J. of Structural Engin.* **122** (1996) 69–75.

[18] Yao, J.T.P. Concept of structural control. *J. Struct. Div.* **98** (1972) 1567–1574.

[19] Zhou, K., Doyle, J.C. and Glover, K. *Robust and Optimal Control*. Prentice Hall, Upper Saddle River, NJ, 1996.