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A Nonlinear Model of Composite Delaminated Beam with Piezoelectric Actuator, with Account of Nonpenetration Constraint for the Delamination Crack Faces

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Abstract: In this work, a new approach is developed for dynamic analysis of a composite beam with an inter-ply crack, in which a physically impossible interpenetration of the crack faces is prevented by imposing a special constraint, leading to nonlinearity of the formulated boundary value problem and to taking account of a contact interaction of the crack faces. A variational formulation of the problem and partial differential equations of motion with boundary conditions are developed, and solutions of example problems for a piezo-actuated cantilever beam are presented in a form of series in terms of eigenfunctions of the associated non-self-adjoint eigenvalue problem. A noticeable difference of forced vibrations of the delaminated and undelaminated beams due to the contact interaction of the crack faces is predicted by the developed model.

Keywords: *Composite beam; delamination; nonpenetration constraint for the crack faces; nonlinear dynamics; series solution; modal analysis.*

Mathematics Subject Classification (2000): 74A40, 74F15, 74S99.

1 Introduction

In this work, a new variational formulation and differential equations of motion with boundary conditions for a beam with through-width delamination are developed, in which a constraint is introduced that does not allow opposite faces of the crack to penetrate each other, leading to a nonlinear formulation of the problem and to taking account of contact interaction of the crack faces. An equation, which expresses this constraint, is written with the use of the Heaviside function in one of its analytical forms, and the constraint is imposed by the penalty function method. The longitudinal force resultants in the delaminated parts of the beam are taken into account, which are another source of the nonlinearity.

Besides, a variational formulation and a differential equation of motion with boundary conditions were developed for a beam without delamination and with a piezoelectric patch (actuator) on its upper surface. The two kinds of developed formulations, for the beam with the delamination and for the beam with the actuator, are combined to form a variational formulation and a system of differential equations with boundary conditions for a cantilever beam with the actuator and the delamination.

A solution for a transverse displacement as a function of time for the cantilever beam with the actuator and the delamination crack is found in a form of series of eigenfunctions of the differential eigenvalue problem, associated with the linearized differential equations of motion with boundary conditions. The series solution is found for both linearized and nonlinear formulations. The comparison of the two solutions is presented to emphasize the importance of using the nonlinear formulation to prevent the physically impossible interpenetration of the crack's faces. However, under small amplitudes of vibration, such interpenetration, as predicted by the solution based on the linearized formulation (without account of the nonpenetration constraint), is shown to be small in the example problem for the cantilever beam, excited by the piezoelectric actuator.

The rotary inertia terms in the differential equations of motion are taken into account (to produce more accurate results for frequencies), leading to non-self-adjoint differential operators for the linearized problem in case of clamped-free boundary conditions. The partial differential equations with the non-self-adjoint differential operators are solved by the Ritz method, with the use of the variational formulation of the problem. The solution for the transverse displacement is sought in the form of series of eigenfunctions of these non-self-adjoint differential operators, leading to the series solution of the linearized problem, which satisfies exactly both essential (displacement) and natural (force) boundary conditions, and a series solution of the nonlinearly formulated problem, which satisfies essential boundary conditions exactly and natural boundary conditions approximately.

In the example problems for the beam with the crack, excited by the piezoelectric actuator, with a voltage distributed uniformly along the length of the actuator, the time-dependent concentrated bending moment appears between the zones with the actuator and without the actuator, leading to nonhomogeneous time-dependent boundary condition between these two zones. The difficulty of solving the partial differential equations of motion with the time-dependent boundary condition is resolved by presenting the time-dependent bending moment in terms of the second spacial derivative of the Heaviside function and by including the bending moment into the equations of motion, as a forcing function, rather than into the boundary conditions.

Several types of models of delaminated beams have been proposed in the literature. In some models, for example, [1] and [2], the contact force between the delaminated parts is not taken into account, and the physically impossible mutual penetration of the delaminated parts is allowed. In other models, for example, [3], the delaminated parts are constrained to have the same transverse displacement, excluding the possibility of the delamination crack opening during the vibration. In the reference [4], the interaction between the delaminated parts is modeled with the use of a nonlinear (piecewise-linear) spring between the surfaces of the delaminated parts. Stiffness of the spring depends on the difference of displacements of the lower and upper delaminated parts. If the delamination crack is open, the stiffness of the spring is set equal to zero, making the distributed contact force equal to zero. When the delamination crack is closed, the stiffness of the spring is set either to infinity, or to some finite constant value. The authors set the spring stiffness equal to a constant (either zero, or 0.1, or infinity) before solving the problem, thus assuming that the crack remains either open or closed all the time during the vibration. So, the possibility for the crack to be open in some time intervals and closed in other time intervals during the vibration is not foreseen in this model.

In the paper [5], the contact force between the delaminated sublaminates is introduced as a function of the relative transverse displacement of the sublaminates, in such a way that the contact force automatically turns out to be zero, when the delamination crack is open, and takes on a non-zero value, if the crack is closed. So, this model does not require to specify in advance if the crack is open or closed, and allows for contact and separation of the crack faces during the vibration. However, the physically impossible interpenetration of the crack faces is not always prevented in this model. The interpenetration occurs because a constraint, preventing this phenomenon, is not introduced.

In the model of the delaminated composite beam, presented below, the constraint, preventing the mutual penetration (interpenetration, overlapping) of the delaminated sublaminates (of the crack's faces), is introduced with the use of the Heaviside function and the penalty function method, which is the main novelty of the presented approach to solving dynamic problems for beams with cracks. The longitudinal force resultants in the delaminated sublaminates and rotary inertia terms are taken into account also. The use of the constraint, which prevents the interpenetration of the crack faces, and taking account of the longitudinal force resultants lead to nonlinear partial differential equations of motion. Only thin beams are considered in this work, making it possible to develop a beam theory, based on assumption of negligibly small shear strains.

2 Model of Composite Beam with Delamination

2.1 Assumptions and notations

The x-coordinates of the delamination crack tips are denoted as α and β $(\alpha \le \beta)$, and z-coordinates of both crack tips are denoted as γ (Figure 2.1).

The transverse displacement of this beam is assumed to have the form

$$w(x,z,t) = W_0(x,t) + D^{\alpha}_{\beta}(x)H_{\gamma}(z)[W_1(x,t) - W_0(x,t)], \tag{1}$$

where $D^{\alpha}_{\beta}(x)$ is a double-sided unit step-function, defined by the formula

$$D^{\alpha}_{\beta}(x) \equiv \begin{cases} 1 & \text{for} \quad \alpha < x < \beta, \\ 0 & \text{for} \quad 0 \le x \le \alpha \quad \text{and} \quad \beta \le x \le L, \end{cases} \tag{2}$$

and $H_{\gamma}(z)$ is a Heaviside function (unit step-function), defined by the formula

$$H_{\gamma}(z) \equiv \begin{cases} 0 & \text{for} \quad -h/2 \le z \le \gamma, \\ 1 & \text{for} \quad \gamma < z \le h/2, \end{cases} \tag{3}$$

$W_0(x,t)$ is a transverse displacement at the beam's axis (at $z = 0$), and $W_1(x,t)$ is a transverse displacement of the upper sublaminate in the delaminated region $\alpha < x < \beta$. Equation (1) implies that the transverse displacement $w(x,z,t)$

(i) is equal to W_0 in the undelaminated regions, i.e. in the region $0 \le x \le \alpha$ (where it will be denoted as w_1) and in the region $\beta \le x \le L$ (where it will be denoted as w_4);

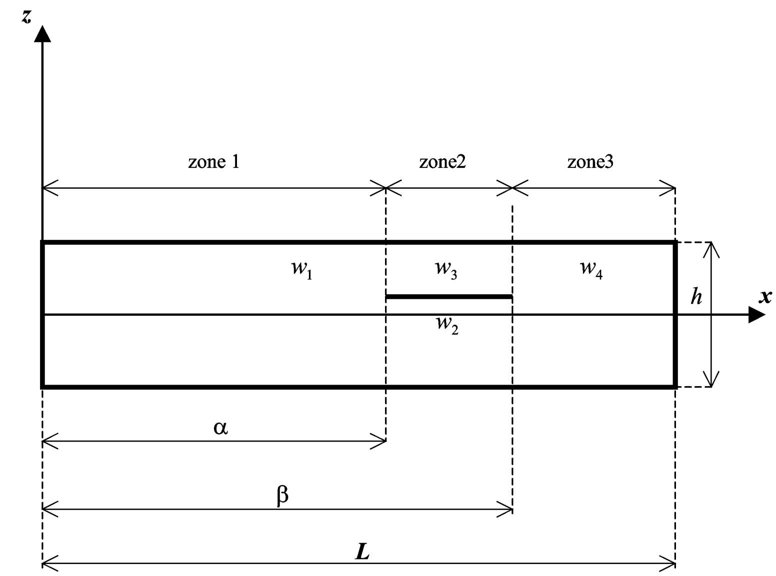

Figure 2.1. Beam with delamination.

α is x-coordinate of the left crack tip; β is x-coordinate of the right crack tip; γ is z-coordinate of the crack (distance from x-axis to crack); w_1 is transverse displacement of zone 1; w_2 is transverse displacement of lower part of zone 2 (under the crack); w_3 is transverse displacement of upper part of zone 2 (above the crack); w_4 is transverse displacement of zone 3.

(ii) is equal to W_0 in the lower sublaminate of the delaminated region (under the crack) i.e. in the region $\alpha < x < \beta$ and $-h/2 \leq z \leq \gamma$ (where it will be denoted as w_2);

(iii) is equal to W_1 in the upper sublaminate of the delaminated region, i.e. in the region $\alpha < x < \beta$ and $\gamma < z \leq h/2$ (where it will be denoted as w_3).

With the use of these notation, equation (1) can be written as follows (Figure 2.1):

$$w(x,z,t) = \begin{cases} w_1(x,t) & \text{in} \quad 0 \leq x \leq \alpha, \\ w_2(x,t) & \text{in} \quad \alpha < x < \beta \quad \text{and} \quad -h/2 \leq z \leq \gamma, \\ w_3(x,t) & \text{in} \quad \alpha \leq x \leq \beta \quad \text{and} \quad \gamma < z \leq h/2, \\ w_4(x,t) & \text{in} \quad \beta < x \leq L. \end{cases} \tag{4}$$

In the simplest beam theory, based on Euler-Bernoulli assumptions and with no longitudinal displacement at the middle surface $z = 0$, the longitudinal displacement (in the x-direction) can be assumed to have the form

$$u(x,z,t) = -\frac{\partial w}{\partial x} z = -\left[\frac{\partial W_0}{\partial x} + \left(\frac{\partial W_1}{\partial x} - \frac{\partial W_0}{\partial x}\right) DH\right] z. \tag{5}$$

From here on, the functions $D^{\alpha}_{\beta}(x)$ and $H_{\gamma}(z)$ are denoted as D and H, for brevity. Primes will denote differentiation with respect to the x-coordinate, and dots — differentiation with respect to time.

We have the following constraints at locations of the tips of the delamination crack (at $x = \alpha$ and at $x = \beta$):

$$\begin{aligned} &w_1(\alpha) = w_2(\alpha), \quad w_1(\alpha) = w_3(\alpha), \quad w_1'(\alpha) = w_2'(\alpha), \quad w_1'(\alpha) = w_3'(\alpha), \\ &w_4(\beta) = w_2(\beta), \quad w_4(\beta) = w_3(\beta), \quad w_4'(\beta) = w_2'(\beta), \quad w_4'(\beta) = w_3'(\beta). \end{aligned} \tag{6}$$

These constraints allow one to introduce the following notations:

$$\begin{aligned} &w(\alpha) \equiv w_1(\alpha) = w_2(\alpha) = w_3(\alpha), \quad w(\beta) \equiv w_2(\beta) = w_3(\beta) = w_4(\beta), \\ &w'(\alpha) \equiv w_1'(\alpha) = w_2'(\alpha) = w_3'(\alpha), \quad w'(\beta) \equiv w_2'(\beta) = w_3'(\beta) = w_4'(\beta). \end{aligned} \tag{7}$$

The constitutive equation for the stress σ_{xx} in a layer of the composite beam can be taken in the form [6]

$$\sigma_{xx} = \frac{1}{\overline{S}_{11}}\,\varepsilon_{xx}, \tag{8}$$

where

$$\overline{S}_{11} = \frac{1}{E_1}\cos^4\theta + \frac{1}{E_2}\sin^4\theta + \left(\frac{1}{G_{12}} - 2\frac{\nu_{12}}{E_1}\right)\sin^2\theta\cos^2\theta, \tag{9}$$

and θ is an angle between the fiber direction and the x-axis, measured counterclockwise, and E_1, E_2, G_{12} and ν_{12} are engineering elastic constants in the principal material coordinate system.

During the vibration of the delaminated beam, the upper and lower delaminated parts touch each other, and the force of their interaction needs to be taken into account. This force enters into the differential equations of motion as a reaction of constraint, which prevents overlapping of the upper and lower delaminated parts. A constraint of this nature can be expressed by a relationship between w_2 and w_3 (i.e. displacements of the lower and upper delaminated parts) that prevents the difference $w_3 - w_2$ to take on negative values:

$$f(w_2, w_3) \equiv (w_3 - w_2)[1 - H_0(w_3 - w_2)] = 0. \tag{10a}$$

If delaminated sublaminates “attempt” to overlap during the vibration (if $w_3 - w_2 < 0$), or if the crack is closed $(w_3 - w_2 = 0)$, then $H_0(w_3 - w_2) = 0$, and, therefore, due to equation (10a), the difference $w_3 - w_2$ is set equal to zero. If the crack is open $(w_3 - w_2 > 0)$, then $H_0(w_3 - w_2) = 1$, and no constraints are imposed on the difference $w_3 - w_2$. With the use of the analytical representation of the Heaviside function (Appendix A, equation (A-5)), the nonpenetration constraint, expressed by equation (10a), can be written as follows:

$$f(w_2, w_3) \equiv (w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}\right) = 0, \tag{10b}$$

where ϵ is some small number.

2.2 Differential equations of motion for delaminated beam

It is implied that the beam is under external distributed load q (force per unit length), applied on the upper surface of the beam, and the load does not depend on displacements. To derive differential equations of motion with boundary conditions, we use the Hamilton's principle:

$$\delta\int_{t_1}^{t_2} J(t)\,dt = 0, \tag{11}$$

where $J(t)$ is a modified Lagrangian function of the system, in which the nonpenetration constraint $f(w_2, w_3) = 0$, defined by equation (10b), is taken into account with the use of the method of Lagrange multipliers:

$$\begin{aligned} J(t) &= \iiint\limits_{(V)} (\widehat{U} - \widehat{T}) dV - \int_0^L \lambda(x,t) f(w_2, w_3)\, dx - \int_0^L qw\big|_{z=h/2}\, dx \\ &= b \int_0^\alpha \int_{-h/2}^{h/2} (\widehat{U} - \widehat{T})\, dz dx - \int_0^\alpha q w_1\, dx + b \int_\alpha^\beta \int_{-h/2}^{\gamma} (\widehat{U} - \widehat{T})\, dz \\ &\quad + b \int_\alpha^\beta \int_\gamma^{h/2} (\widehat{U} - \widehat{T})\, dz dx - \int_\alpha^\beta q w_3\, dx - \int_\alpha^\beta \lambda(x,t) f(w_2, w_3)\, dx \\ &\quad + b \int_\beta^L \int_{-h/2}^{h/2} (\widehat{U} - \widehat{T})\, dz dx - \int_\beta^L q w_4\, dx. \end{aligned} \tag{12}$$

In equation (12), $\widehat{U}$ is strain energy density, $\widehat{T}$ is kinetic energy density and $\lambda(x,t)$ is the Lagrange multiplier. Expressions for the kinetic energy density and strain energy density in terms of displacements are

$$\widehat{T} = \frac{1}{2} \rho(\dot{u}^2 + \dot{w}^2), \tag{13}$$

$$\widehat{U} = \frac{1}{2} \sigma_{xx}\varepsilon_{xx} = \frac{1}{2} \sigma_{xx} \left[u' + \frac{1}{2} (w')^2 \right]. \tag{14}$$

In the last equation, the nonlinear term $\frac{1}{2}(w')^2$ is included in the strain-displacement relation for the strain ε_{xx} to take account of longitudinal force resultants in the delaminated lower and upper sublaminates,

$$N_x^{(2)} = b \int_{-h/2}^{\gamma} \sigma_{xx}^{(2)} dz, \qquad N_x^{(3)} = b \int_\gamma^{h/2} \sigma_{xx}^{(3)} dz, \tag{15}$$

which may not be negligibly small even if there are no external longitudinal forces applied to the beam. If external longitudinal forces are not applied to the beam, the term $\frac{1}{2}\sigma_{xx}(w')^2$ need not be included into expression for strain energy density of the zones without delamination, $0 \le x \le \alpha$ and $\beta \le x \le L$. With the use of the assumed displacements (equations (1) and (5)), constitutive equation (8) and notations (4), the kinetic energy and the strain energy can be expressed in terms of the unknown functions $w_1(x,t)$, $w_2(x,t)$, $w_3(x,t)$ and $w_4(x,t)$, leading to the following expression for the Lagrangian function of the system:

$$J(t) = \int_0^\alpha \widetilde{J}_1(x,t)\, dx + \int_\alpha^\beta [\widetilde{J}_2(x,t) + \widetilde{J}_3(x,t)] dx + \int_\beta^L \widetilde{J}_4(x,t)\, dx, \tag{16}$$

where quantities $\widetilde{J}_1$, $\widetilde{J}_2$, $\widetilde{J}_3$ and $\widetilde{J}_4$ are linear densities of the Lagrangian in the corresponding parts of the beam,

$$\widetilde{J}_1(x,t) = \frac{1}{2}\,A_1(w_1'')^2 - \frac{1}{2}\,B_1(\dot w_1)^2 - \frac{1}{2}\,C_1(\dot w_1')^2 - q_1 w_1, \tag{17}$$

$$\widetilde{J}_2(x,t) = \frac{1}{2}\,A_2(w_2'')^2 - \frac{1}{2}\,B_2(\dot w_2)^2 - \frac{1}{2}\,C_2(\dot w_2')^2 + \frac{1}{4}\,N_x^{(2)}(w_2')^2 - \lambda(x,t)f(w_2,w_3), \tag{18}$$

$$\widetilde{J}_3(x,t) = \frac{1}{2}\,A_3(w_3'')^2 - \frac{1}{2}\,B_2(\dot w_3)^2 - \frac{1}{2}\,C_3(\dot w_3')^2 + \frac{1}{4}\,N_x^{(3)}(w_3')^2 - q_3 w_3, \tag{19}$$

$$\widetilde{J}_4(x,t) = \frac{1}{2}\,A_4(w_4'')^2 - \frac{1}{2}\,B_4(\dot w_4)^2\frac{1}{2}\,C_4(\dot w_4')^2 - q_4 w_4, \tag{20}$$

where q_1, q_3 and q_4 are external loads on the upper surface of the beam, acting on part 1 $(0 \le x \le \alpha)$, part 3 $\left(\alpha \le x \le \beta,\ \gamma < x \le \frac{h}{2}\right)$ and part 4 $(\beta \le x \le L)$ of the beam. Constants A_k, B_k, C_k $(k = 1,2,3,4)$ in equations (17)–(20) are defined as follows:

$$\begin{aligned}
&A_1 = b\int\limits_{-h/2}^{h/2} \frac{1}{\overline{S}_{11}^{(1)}}\, z^2\,dz, && B_1 = b\int\limits_{-h/2}^{h/2} \rho^{(1)} dz, && C_1 = b\int\limits_{-h/2}^{h/2} \rho^{(1)} z^2\,dz,\\
&A_2 = b\int\limits_{-h/2}^{\gamma} \frac{1}{\overline{S}_{11}^{(2)}}\, z^2\,dz, && B_2 = b\int\limits_{-h/2}^{\gamma} \rho^{(2)} dz, && C_2 = b\int\limits_{-h/2}^{\gamma} \rho^{(2)} z^2\,dz,\\
&A_3 = b\int\limits_{\gamma}^{h/2} \frac{1}{\overline{S}_{11}^{(3)}}\, z^2\,dz, && B_3 = b\int\limits_{\gamma}^{h/2} \rho^{(3)} dz, && C_3 = b\int\limits_{\gamma}^{h/2} \rho^{(3)} z^2\,dz,\\
&A_4 = b\int\limits_{-h/2}^{h/2} \frac{1}{\overline{S}_{11}^{(4)}}\, z^2\,dz, && B_4 = b\int\limits_{-h/2}^{h/2} \rho^{(4)} dz, && C_4 = b\int\limits_{-h/2}^{h/2} \rho^{(4)} z^2\,dz.
\end{aligned} \tag{21}$$

Upper index k in the notations $\overline{S}_{11}^{(k)}$ and $\rho^{(k)}$ $(k = 1,2,3,4)$ denotes that the material property is associated with the k-th part of the beam. Further we will consider beams for which $\overline{S}_{11}^{(1)} = \overline{S}_{11}^{(2)} = \overline{S}_{11}^{(3)} = \overline{S}_{11}^{(4)}$, $\rho^{(1)} = \rho^{(2)} = \rho^{(3)} = \rho^{(4)}$, and, therefore, $A_1 = A_4$, $B_1 = B_4$ and $C_1 = C_4$. But distinguishing between these last quantities will still be made to keep consistent index notations that allow for brief representation of subsequent equations.

In equations (18) and (19), the longitudinal force resultants are expressed in terms of displacements as follows:

$$N_x^{(2)} = b\int\limits_{-h/2}^{\gamma} \sigma_{xx}^{(2)}\,dz = b\int\limits_{-h/2}^{\gamma} \frac{1}{\overline{S}_{11}^{(2)}(z)}\,\varepsilon_{xx}^{(2)}\,dz = -H_2 w_2'' + \frac{1}{2}Q_2(w_2')^2, \tag{22a}$$

$$N_x^{(3)} = b\int\limits_{\gamma}^{h/2} \sigma_{xx}^{(3)}\,dz = b\int\limits_{\gamma}^{h/2} \frac{1}{\overline{S}_{11}^{(3)}(z)}\,\varepsilon_{xx}^{(3)}\,dz = -H_3 w_3'' + \frac{1}{2}\,Q_3(w_3')^2, \tag{22b}$$

where H_2, Q_2, H_3 and Q_3 are constants, defined as

$$\begin{aligned}
H_2 &= b\int_{-h/2}^{\gamma} \frac{1}{\overline{S}_{11}^{(2)}(z)}\, z\,dz, \qquad & Q_2 &= b\int_{-h/2}^{\gamma} \frac{1}{\overline{S}_{11}^{(2)}(z)}\, dz,\\
H_3 &= b\int_{\gamma}^{h/2} \frac{1}{\overline{S}_{11}^{(3)}(z)}\, z\,dz, \qquad & Q_3 &= b\int_{\gamma}^{h/2} \frac{1}{\overline{S}_{11}^{(3)}(z)}\, dz.
\end{aligned} \tag{23}$$

From the Hamilton's principle (11) with constraints (7), with account of expressions (16)–(20) and (22), and with the use of standard methods of calculus of variations, one can obtain the following differential equations, equation of constraint and boundary conditions.

Differential equations:

$$A_1 w_1'''' + B_1 \ddot{w}_1 - C_1 \ddot{w}_1'' = q_1 \quad \text{in} \quad 0 \le x \le \alpha, \tag{24}$$

$$\begin{gathered}
A_2 w_2'''' + B_2 \ddot{w}_2 - C_2 \ddot{w}_2'' - 3Q_2 (w_2')^2 w_2''\\
= \lambda(x,t)\left(\frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon} - \frac{1}{2}\right) \quad \text{in} \quad \alpha \le x \le \beta, \quad -h/2 \le z \le \gamma,
\end{gathered} \tag{25}$$

$$\begin{gathered}
A_3 w_3'''' + B_3 \ddot{w}_3 - C_3 \ddot{w}_3'' - 3Q_3 (w_3')^2 w_3''\\
= q_3 - \lambda(x,t)\left(\frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon} - \frac{1}{2}\right) \quad \text{in} \quad \alpha \le x \le \beta, \quad \gamma < z \le h/2,
\end{gathered} \tag{26}$$

$$A_4 w_4'''' + B_4 \ddot{w}_4 - C_4 \ddot{w}_4'' = q_3 \quad \text{in} \quad \beta \le x \le L. \tag{27}$$

Equation of constraint:

$$(w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\epsilon}\right) = 0 \tag{28}$$

(equation (28) is the same as equation (10b)).

Boundary conditions:
At $x = 0$:

$$\text{either} \quad A_1 w_1''' - C_1 \ddot{w}_1' = 0 \quad \text{or} \quad w_1 \;\text{ is constrained;} \tag{29a}$$

$$\text{either} \quad w_1'' = 0 \quad \text{or} \quad w_1' \;\text{ is constrained.} \tag{29b}$$

At $x = \alpha$:

$$\begin{aligned}
\text{either} \quad & \big(A_2 w_2''' - C_2 \ddot{w}_2' - Q_2 (w_2')^3\big) + \big(A_3 w_3''' - C_3 \ddot{w}_3' - Q_3 (w_3')^3\big)\\
& - (A_1 w_1''' - C_1 \ddot{w}_1') = 0 \quad \text{or } w \text{ is constrained;}
\end{aligned} \tag{30a}$$

$$\begin{aligned}
\text{either} \quad & A_1 w_1'' - \left(A_2 w_2'' - \frac{1}{2} H_2 (w_2')^2\right) - \left(A_3 w_3'' - \frac{1}{2} H_3 (w_3')^2\right) = 0\\
& \text{or } w' \text{ is constrained.}
\end{aligned} \tag{30b}$$

At $x = \beta$:

$$\text{either} \quad \left(A_2 w_2''' - C_2 \ddot{w}_2' - Q_2 (w_2')^3\right) + \left(A_3 w_3''' - C_3 \ddot{w}_3' - Q_3 (w_3')^3\right)$$
$$- \left(A_4 w_4''' - C_4 \ddot{w}_4'\right) = 0 \quad \text{or } w \text{ is constrained;} \tag{31a}$$

$$\text{either} \quad \left(A_2 w_2'' - \frac{1}{2} H_2 (w_2')^2\right) + \left(A_3 w_3'' - \frac{1}{2} H_3 (w_3')^2\right) - A_4 w_4'' = 0$$
$$\text{or } w' \text{ is constrained.} \tag{31b}$$

At $x = L$:

$$\text{either} \quad A_4 w_4''' - C_4 \ddot{w}_4' = 0 \quad \text{or } w \text{ is constrained,} \tag{32a}$$

$$\text{either} \quad w_4'' = 0 \quad \text{or } w' \text{ is constrained.} \tag{32b}$$

So, we obtained four differential equations (24) – (27) and one equation of constraint (28) for five unknown functions $w_1(x,t)$, $w_2(x,t)$, $w_3(x,t)$, $w_4(x,t)$ and $\lambda(x,t)$. The total order of these equations is 16. The number of boundary conditions is also 16. These boundary conditions are represented by equations (29) – (32) and (6).

3 Model of Composite Beam with Piezoelectric Actuator and Without Delamination

3.1 Assumptions and notations

In experiments and in structural health monitoring, it is convenient to excite and control vibrations of beams with the use of piezoelectric actuators, attached to them. Modeling such beams requires development of a differential equation of motion with boundary conditions for the beam's segment, covered with the piezoelectric actuator. This is the subject of the present paragraph. For simplicity, it is considered here that such a segment does not contain delaminations.

So, let us consider a *thin beam without delamination and with a piezoelectric layer*, attached to the beam's upper surface (Figure 3.1). In the subsequent text, the superscript (0) will denote quantities associated with the beam's composite layers without piezoelectric properties, and the superscript (p) will denote quantities associated with the piezoelectric patch (actuator). The distributed transverse load (force per unit length) on the surface of the beam, covered with the actuator, will be denoted as q_0.

The transverse and longitudinal displacements will be assumed to have the form of the Euler-Bernoulli theory:

$$w(x,z,t) = w_0(x,t), \tag{33}$$

$$u(x,z,t) = -\frac{\partial w_0(x,t)}{\partial x}\, z. \tag{34}$$

In equation (34), the axial longitudinal displacement $u\big|_{z=0}$ is assumed to be negligibly small, because we consider the case of no longitudinal external forces, applied to the beam, and small amplitudes of vibration. It is assumed that an electric field is applied to

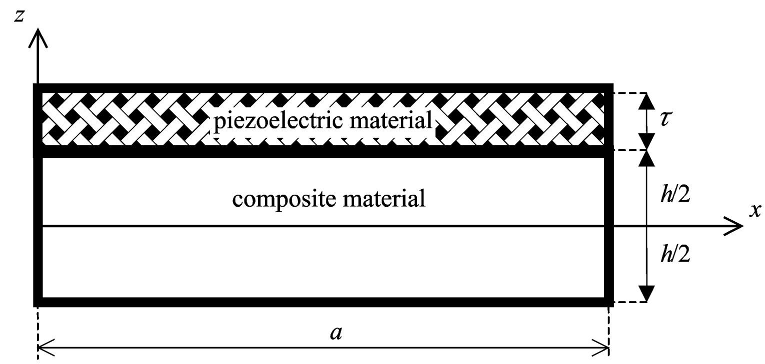

Figure 3.1. Composite beam with a layer of piezoelectric material attached to the upper surface.

the piezoelectric actuator in the direction of the beam's transverse direction, i.e. in the direction of the z-axis. It can be assumed that in a thin piezoelectric actuator, to which the external voltage $V(x,t)$ is applied, the electric potential $\varphi(x,z,t)$ varies linearly in the z-direction, therefore

$$\frac{\partial \varphi}{\partial z} \approx -\frac{V}{\tau}, \tag{35}$$

where τ is a thickness of the piezoelectric actuator. Then, from constitutive equations for the piezoelectric layer of the composite beam, with orthorhombic mm2 symmetry, such as polyvinylidene or lead-zirconate [6], we obtain the following constitutive equation for the stress $\sigma_{xx}^{(p)}$ in the piezoelectric layer

$$\sigma_{xx}^{(p)} = \frac{1}{\overline{S}_{11}^{(p)}}\,\varepsilon_{xx} - \frac{\overline{d}_{31}}{\overline{S}_{11}^{(p)}}\,\frac{V}{\tau}. \tag{36}$$

To derive the equation of motion and boundary conditions for the laminated composite beam with the piezoelectric actuator layer, we will use the virtual work principle for a piezoelectric deformable body [7],

$$\iiint\limits_{(V)} (\sigma_{ij}\delta\varepsilon_{ij} + D_i\delta\varphi_{,i})\,dV = \iiint\limits_{(V)} (\overline{F}_i - \rho\ddot{u}_i)\,\delta u_i + \iint\limits_{(S)} (\overline{t}_k\delta u_k - \overline{Q}\delta\varphi)\,dS, \tag{37}$$

where $\overline{Q}$ is a surface electric charge, $\overline{F}_i$ are components of body forces and $\overline{t}_k$ are components of surface forces. According to the assumption of equation (35), variations of the electric potential φ and the voltage V are related as

$$\delta\varphi = -\frac{\delta V}{\tau}\,z. \tag{38}$$

If the piezoelectric layer is used as the actuator, then the voltage $V(x,t)$, applied to this layer, is a known function of the coordinate x and time, and, therefore its variation δV is equal to zero. Then, according to equation (38), $\delta\varphi$ should be set to zero in the virtual work principle equation (37). So, if the piezoelectric layer is used as the actuator, then the electric field characteristics enter the virtual work principle only through the constitutive equations, and, therefore, equation (37) takes the form

$$\iiint\limits_{(V)} \sigma_{ij}\delta\varepsilon_{ij}\, dV = \iiint\limits_{(V)} (\overline{F}_i - \rho\ddot{u}_i)\,\delta u_i + \iint\limits_{(S)} \overline{t}_k \delta u_k \, dS. \tag{39}$$

Equation (39), applied to the beam with a rectangular cross-section and a piezoelectric layer of thickness τ, attached to the beam's upper surface, has the form

$$b\int\limits_0^a \int\limits_{-h/2}^{h/2+\tau} \sigma_{xx}\delta\varepsilon_{xx}\, dz\, dx = b\int\limits_0^a \int\limits_{-h/2}^{h/2+\tau} [(\overline{F}_x - \rho\ddot{u})\delta u + (\overline{F}_z - \rho\ddot{w}_0)\,\delta w_0]\, dz\, dx + \int\limits_0^a q_0\,\delta w_0\, dx. \tag{40}$$

The body force, acting on the beam, is the gravity force. Therefore,

$$\overline{F}_x = 0, \quad \overline{F}_z = -\rho g, \tag{41}$$

where ρ is mass density, and $g = 9.81\, m/s^2$ is intensity of the gravity field. With account of the constitutive equations (36), equations (41) and strain-displacement relation $\varepsilon_{xx} = u'$ (nonlinear terms are excluded), the virtual work principle (40) can be written in terms of the unknown displacements, material constants and voltage, applied to the piezoelectric actuator:

$$\begin{aligned} & b\int\limits_0^a \int\limits_{-h/2}^{h/2} \frac{1}{\overline{S}_{11}^{(0)}(z)}\, u'\,\delta u'\, dz\, dx + b\int\limits_0^a \int\limits_{h/2}^{h/2+\tau} \frac{1}{\overline{S}_{11}^{(p)}(z)} \left(u' - \overline{d}_{31}\frac{V}{\tau}\right)\delta u'\, dz\, dx \\ & + b\int\limits_0^a \int\limits_{-h/2}^{h/2} \rho^{(0)}(g + \ddot{w})\,\delta w_0\, dz\, dx + b\int\limits_0^a \int\limits_{h/2}^{h/2+\tau} \rho^{(p)}(g + \ddot{w}_0)\,\delta w_0\, dz\, dx \\ & + b\int\limits_0^a \int\limits_{-h/2}^{h/2} \rho^{(0)}\ddot{u}\,\delta u\, dz\, dx + b\int\limits_0^a \int\limits_{h/2}^{h/2+\tau} \rho^{(p)}\ddot{u}\,\delta u\, dz\, dx - \int\limits_0^a q_0\,\delta w_0\, dx = 0. \end{aligned} \tag{42}$$

3.2 Differential equation of motion for beam with piezoelectric actuator and without delamination

The virtual work principle (42) in conjunction with the simplifying assumptions (33) and (34), after applying standard methods of variational calculus, leads to the following differential equation of motion and boundary conditions:

$$A_0 w_0'''' + B_0\ddot{w}_0 - C_0\ddot{w}_0'' = q_0 - I_p V'' - B_0 g \quad \text{for} \quad 0 \le x \le a; \tag{43}$$

$$\text{either} \quad A_0 w_0'' + I_p V = 0 \quad \text{or } w_0 \text{ constrained at } x = 0 \text{ and } x = a; \tag{44}$$

$$\text{either} \quad A_0 w_0''' - C_0\ddot{w}_0' + I_p V' = 0 \quad \text{or } w_0 \text{ constrained at } x = 0 \text{ and } x = a, \tag{45}$$

where

$$A_0 = b\left(\int\limits_{-h/2}^{h/2} \frac{1}{\overline{S}_{11}^{(0)}}\, z^2\, dz + \int\limits_{h/2}^{h/2+\tau} \frac{1}{\overline{S}_{11}^{(p)}}\, z^2\, dz\right), \quad I_p = \frac{1}{\tau}\, b \int\limits_{h/2}^{h/2+\tau} \frac{\overline{d}_{31}}{\overline{S}_{11}^{(p)}}\, z\, dz,$$
$$B_0 = b\left(\int\limits_{-h/2}^{h/2} \rho^{(0)}\, dz + \int\limits_{h/2}^{h/2+\tau} \rho^{(p)}\, dz\right), \quad C_0 = b\left(\int\limits_{-h/2}^{h/2} \rho^{(0)} z^2\, dz \int\limits_{h/2}^{h/2+\tau} \rho^{(p)} z^2\, dz\right). \tag{46}$$

The differential equation (43) and the boundary conditions (44) and (45) imply that the voltage $V(x,t)$, applied to the piezoelectric actuator, produces the bending moment $I_p V(x,t)$ in a cross section of the beam.

If the voltage, applied to the piezoelectric actuator, is distributed uniformly along a region $x_1 \le x \le x_2$, i.e. if

$$V(x,t) = \begin{cases} V(t) & \text{in} \quad x_1 \le x \le x_2, \\ 0 & \text{for all other } x, \end{cases} \tag{47}$$

then this voltage can be presented as

$$V(x,t) = D_{x_2}^{x_1}(x) V(t) = (H_{x_1}(x) - H_{x_2}(x)) V(t), \tag{48}$$

and the quantity $I_p V''(x,t)$ in the right side of the differential equation of motion (43), takes the form

$$I_p V''(x,t) = I_p V(t) H''_{x_1}(x) - I_p V(t) H''_{x_2}(x). \tag{49}$$

If a concentrated external bending moment M is applied at a point $x = x_1$ of the beam, then this bending moment can be represented by an equivalent distributed load $M H''_{x_1}(x)$ in the differential equation of motion of the beam [8], where $H''_{x_1}(x)$ is the second derivative with respect to x of the Heaviside function, defined by equation (A–5) in Appendix A. Therefore, equation (49) implies that concentrated bending moments $I_p V(t)$ are applied at points $x = x_1$ and $x = x_2$, if the voltage, applied to the piezoelectric actuator, is distributed uniformly along the region $x_1 \le x \le x_2$. This fact will be used in the next paragraph to substitute the time-dependent bending moment in a boundary condition with the equivalent distributed load, entering into the differential equation of motion, thus allowing for elimination of nonhomogeneous time-dependent boundary condition and simplification of the problem.

4 Forced Vibration of Cantilever Beam with Delamination, under Effect of Voltage, Applied to Piezoelectric Actuator. Solution in the Form of Series in Terms Eigenfunctions. Linear Model

In this paragraph, we study solutions of vibration problems for a cantilever composite beam with the crack between its plies (Figure 4.1), with the nonlinear terms being discarded in the formulation, i.e. the non-penetration constraint and the longitudinal force resultants being not taken into account. Effects of neglecting the nonlinear terms are studied in the paragraph 5, by comparing results of linear and nonlinear analysis. The

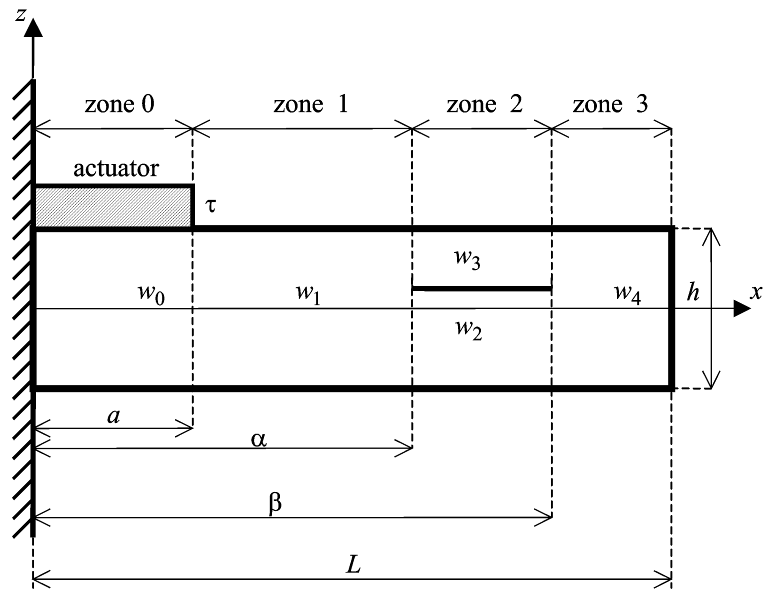

Figure 4.1. Cantilever beam with delamination and piezoelectric actuator.

a is length of the actuator; α is x-coordinate of the left crack tip; β is x-coordinate of the right crack tip; γ is z-coordinate of the crack (distance from x-axis to crack); τ is thickness of the actuator; w_0 is transverse displacement of zone 0; w_1 is transverse displacement of zone 1; w_2 is transverse displacement of lower part of zone 2 (under the crack); w_3 is transverse displacement of upper part of zone 2 (above the crack); w_4 is transverse displacement of zone 3.

voltage, applied to the piezoelectric actuator, is considered to be distributed uniformly along the length of the actuator. The partial differential equations of motion with boundary conditions, derived earlier in the general form, for this particular problem take the form presented below.

4.1 Formulation in terms of partial differential equations with boundary and initial conditions

Motion of the beam is described by the following system of five *partial differential equations*

$$A_0 w_0'''' + B_0 \ddot{w}_0 - C_0 \ddot{w}_0'' = I_p V(t) H_a''(x), \tag{50}$$

$$A_k w_k'''' + B_k \ddot{w}_k - C_k \ddot{w}_k'' = 0, \quad k = 1,2,3,4 \quad \text{(no summation with respect to } k). \tag{51}$$

The function $V(t)$ in equation (50) is the voltage, applied to the piezoelectric actuator and distributed uniformly along the region $0 \le x < a$, which does not include the point $x = a$:

$$V(x,t) = V(t) \quad \text{in} \quad 0 \le x < a.$$

The exclusion of the point $x = a$ from the region, where the voltage is applied, does not change the physics of the problem and allows to avoid having non-homogeneous time-dependent boundary condition at $x = a$, as in equation (44). The differential equation of motion (43) and the boundary condition (44) imply that the voltage $V(x,t)$, applied to the piezoelectric actuator, produces the bending moment $I_p V(x,t)$. If $V(x,t) = V(t)$ over an interval $0 \le x \le (a - \epsilon)$, where ϵ is some very small number, and if the beam's end $x = 0$ is clamped, then the external concentrated bending moment $I_p V(t)$ is applied at the point $x = a - \epsilon$, and this is taken into account by the term $I_p V(t) H_a''(x)$ in the right-hand side of the equation (50). The same result can be obtained from equation (43) directly. Indeed, the voltage $V(x,t) = V(t)$ in the interval $0 \le x < a$ can be written as

$$V(x,t) = V(t)(1 - H_a(x)).$$

Substitution of this expression into the expression $I_p V''(x,t)$ in the right side of equation (43) produces the result $I_p V(t) H_a''(x)$, i.e. the forcing function in the right side of equation (50).

The constants, entering into the differential equations (50) and (51), are defined by formulas (46) and (21).

Boundary conditions for the partial differential equations (50) and (51) are the following (see equations (29) – (32), (44) and (45)):

displacement boundary conditions:

$$\begin{gathered}
w_0(0) = 0, \qquad w_0'(0) = 0, \\
w_0(a) - w_1(a) = 0, \qquad w_0'(a) - w_1'(a) = 0, \\
w_1(\alpha) - w_2(\alpha) = 0, \qquad w_1'(\alpha) - w_2'(\alpha) = 0, \qquad w_2(\alpha) - w_3(\alpha) = 0, \\
w_2'(\alpha) - w_3'(\alpha) = 0, \qquad w_2(\beta) - w_4(\beta) = 0, \qquad w_2'(\beta) - w_4'(\beta) = 0, \\
w_2(\beta) - w_3(\beta) = 0, \qquad w_2'(\beta) - w_3'(\beta) = 0,
\end{gathered} \tag{52}$$

force boundary conditions:

$$\begin{gathered}
A_0 w_0''(a) - A_1 w_1''(a) = 0, \\
A_0 w_0'''(a) - C_0 \ddot{w}_0'(a) - [A_1 w_1'''(a) - C_1 \ddot{w}_1'(a)] = 0, \\
A_1 w_1''(\alpha) - A_2 w_2''(\alpha) - A_3 w_3''(\alpha) = 0, \\
[A_1 w_1'''(\alpha) - C_1 \ddot{w}_1'(\alpha)] - [A_2 w_2'''(\alpha) - C_2 \ddot{w}_2'(\alpha)] - [A_3 w_3'''(\alpha) - C_3 \ddot{w}_3'(\alpha)] = 0, \\
A_2 w_2''(\beta) + A_3 w_3''(\beta) - A_4 w_4''(\beta) = 0, \\
[A_2 w_2'''(\beta) - C_2 \ddot{w}_2'(\beta)] + [A_3 w_3'''(\beta) - C_3 \ddot{w}_3'(\beta)] - [A_4 w_4'''(\beta) - C_4 \ddot{w}_4'(\beta)] = 0, \\
A_4 w_4''(L) = 0, \\
A_4 w_4'''(L) - C_4 \ddot{w}_4'(L) = 0.
\end{gathered} \tag{53}$$

So, this problem is formulated in terms of five partial differential equations (50) and (51) and twenty boundary conditions (52) and (53). Each of the five partial differential equations is of the fourth order. So, the total order of the differential equations (twenty) is equal to the number of the boundary conditions.

Initial conditions for this problem are assumed to be

$$
\begin{aligned}
&w_0(0) = w_1(0) = w_2(0) = w_3(0) = w_4(0) = 0,\\
&\dot{w}_0(0) = \dot{w}_1(0) = \dot{w}_2(0) = \dot{w}_3(0) = \dot{w}_4(0) = 0.
\end{aligned}
\tag{54}
$$

4.2 Variational formulation of the problem

The partial differential equations (50) and (51) with boundary conditions (52) and (53) are equivalent to the condition of extremum of the functional

$$
\begin{aligned}
J = \frac{1}{2}\int_{t_1}^{t_2}\int_{0}^{a} \big[A_0(w_0'')^2 - B_0\dot{w}_0^2 - C_0(\dot{w}_0')^2 - 2I_pV(t)H_a''(x)w_0\big]\,dx\,dt\\
+\frac{1}{2}\int_{t_1}^{t_2}\int_{a}^{\alpha} \big[A_1(w_1'')^2 - B_1\dot{w}_1^2 - C_1(\dot{w}_1')^2\big]\,dx\,dt\\
+\frac{1}{2}\int_{t_1}^{t_2}\int_{\alpha}^{\beta} \big[A_2(w_2'')^2 - B_2\dot{w}_2^2 - C_2(\dot{w}_2')^2 + A_3(w_3'')^2 - B_3\dot{w}_3^2 - C_3(\dot{w}_3')^2\big]\,dx\,dt\\
+\frac{1}{2}\int_{t_1}^{t_2}\int_{\beta}^{L} \big[A_4(w_4'')^2 - B_4\dot{w}_4^2 - C_4(\dot{w}_4')^2\big]\,dx\,dt
\end{aligned}
\tag{55}
$$

with subsidiary conditions being the displacement boundary conditions (52).

With the use of standard methods of the calculus of variations, the partial differential equations (50) and (51) and natural (force) boundary conditions (53) follow from the condition of extremum of the functional J,

$$
\delta J = 0, \tag{56}
$$

with account of essential (displacement) boundary conditions (52). The same initial conditions (54) apply for the variational formulation.

4.3 Eigenvalue problem, associated with the partial differential equations and boundary conditions

To formulate the eigenvalue problem, we set the right side of equation (50) to zero and separate the variables:

$$
w_k(x,t) = X_k(x)T(t) \quad (k = 0,1,2,3,4). \tag{57}
$$

In the notation $X_k(x)$, the subscript k is a number of the beam's part, with which the function $X_k(x)$ are associated. A number of the eigenfunction, associated with a frequency ω_n, will be denoted by the second subscript n:

$$
\omega_n \to X_{kn} \quad (k = 0,1,2,3,4; \quad n = 1,2,\dots). \tag{58}
$$

The separation of the variables leads to equations

$$\ddot{T}(t) + \omega^2 T(t) = 0, \tag{59}$$

$$A_k \frac{d^4 X_k}{dx^4} + \omega^2 C_k \frac{d^2 X_k}{dx^2} = \omega^2 B_k X_k \qquad (k = 0, 1, 2, 3, 4), \tag{60}$$

where ω is a circular frequency (so far, the notation for frequency does not have an index).

General solution of the ordinary differential equations (60) has the form:

$$X_0(x) = a_1 \sin \mu_0 x + a_2 \cos \mu_0 x + a_3 \sinh \eta_0 x + a_4 \cosh \eta_0 x, \tag{61a}$$

$$X_1(x) = a_5 \sin \mu_1 x + a_6 \cos \mu_1 x + a_7 \sinh \eta_1 x + a_8 \cosh \eta_1 x, \tag{61b}$$

$$X_2(x) = a_9 \sin \mu_2 x + a_{10} \cos \mu_2 x + a_{11} \sinh \eta_2 x + a_{12} \cosh \eta_2 x, \tag{61c}$$

$$X_3(x) = a_{13} \sin \mu_3 x + a_{14} \cos \mu_3 x + a_{15} \sinh \eta_3 x + a_{16} \cosh \eta_3 x, \tag{61d}$$

$$X_4(x) = a_{17} \sin \mu_4 x + a_{18} \cos \mu_4 x + a_{19} \sinh \eta_4 x + a_{20} \cosh \eta_4 x, \tag{61e}$$

where

$$\mu_k = \sqrt{\frac{\omega}{2A_k}\Big(\omega C_k + \sqrt{\omega^2 C_k^2 + 4A_k B_k}\Big)}, \tag{62a}$$

$$\eta_k = \sqrt{\frac{\omega}{2A_k}\Big(-\omega C_k + \sqrt{\omega^2 C_k^2 + 4A_k B_k}\Big)} \qquad (k = 0, 1, 2, 3, 4). \tag{62b}$$

When equations (57), with account of equations (61), are substituted into the boundary conditions (52) and (53), one obtains a system of linear homogeneous algebraic equations, which can be written in the matrix form as

$$[D]_{(20\times 20)}\{a\}_{(20\times 1)} = \{0\}_{(20\times 1)}, \tag{63}$$

where the column-matrix $\{a\}$ consists of the coefficients $a_1, a_2, \ldots, a_{20}$ of the expressions (61), and components of the matrix $[D]$ depend on the unknown frequencies ω. Expressions for components of the matrix $[D]$ are written explicitly in reference [9]. Approximate values of frequencies $\omega \equiv \omega_n$ are computed numerically from equation

$$\det[D]_{(20\times 20)} = 0. \tag{64}$$

with the use of the bisection method. More accurate values of frequencies and the associated column-matrices $\{a\}_n$ of the dimensions (20×1) are computed by solving a nonlinear eigenvalue problem (63) by an iterative method described below, with initial approximations for the frequencies being the frequencies, computed from equation (64), by the bisection method.

4.4 Iterative solution of nonlinear eigenvalue problem

Let us consider a nonlinear eigenvalue problem of the type, represented by equations (63):

$$\mathbf{D}(\omega)\,\mathbf{a} = \mathbf{0}. \tag{65}$$

Let $\omega_n^{(0)}, \omega_n^{(1)}, \omega_n^{(2)}, \dots$ denote successive approximations of a frequency ω_n, which is one of the solutions of the nonlinear eigenvalue problem (65), the zeroth approximation, $\omega_n^{(0)}$, being an approximate value of the frequency ω_n, obtained by some other method. In the following presentation of the iterative procedure, the lower index, denoting a number of a frequency, will be omitted for simplicity of notation. Besides, let

$$\epsilon^{(k+1)} \equiv \omega^{(k+1)} - \omega^{(k)} \tag{66}$$

be a difference between successive approximations of the frequency ω. Then

$$\omega^{(k+1)} = \omega^{(k)} + \epsilon^{(k+1)}. \tag{67}$$

Assuming that the approximation with number $k+1$, i.e. $\omega^{(k+1)}$, satisfies equation (65) approximately, one can write

$$\mathbf{D}\big(\omega^{(k+1)}\big)\,\mathbf{a}^{(k+1)} \approx \mathbf{0}, \tag{68}$$

where $\mathbf{a}^{(k+1)}$ is an approximation with number $(k+1)$ of an eigenvector $\mathbf{a}$. With the use of the Taylor series expansion with two terms, we obtain

$$\mathbf{D}\big(\omega^{(k+1)}\big) \approx \mathbf{D}\big(\omega^{(k)} + \epsilon^{(k+1)}\big) \approx \mathbf{D}\big(\omega^{(k)}\big) - \epsilon^{(k+1)}\mathbf{B}\big(\omega^{(k)}\big), \tag{69}$$

where

$$\mathbf{B}(\omega) \equiv -\frac{d\mathbf{D}}{d\omega}\,. \tag{70}$$

Substitution of equation (69) into equation (68) yields

$$\Big(\mathbf{D}\big(\omega^{(k)}\big) - \epsilon^{(k+1)}\mathbf{B}\big(\omega^{(k)}\big)\Big)\mathbf{a}^{(k+1)} = \mathbf{0}, \tag{71}$$

which is an algebraic linear eigenvalue problem for computation of quantities $\epsilon^{(k+1)}$ as eigenvalues and vectors $\mathbf{a}^{(k+1)}$ as eigenvectors. In order for the Taylor series expansion in equation (69) to be as accurate as possible, the eigenvalue $\epsilon^{(k+1)}$ with the smallest absolute value should be chosen. The corresponding eigenvector $\mathbf{a}^{(k+1)}$ of the linear eigenvalue problem (71) is an approximation with number $(k+1)$ of the eigenvector $\mathbf{a}$ of the nonlinear eigenvalue problem (65). The updated $(k+1)$-st approximation for the frequency ω is computed by the formula (67). The iteration process continues until $\epsilon^{(k+1)} \equiv \omega^{(k+1)} - \omega^{(k)}$ becomes smaller than some chosen small number.

4.5 Forced vibration of delaminated beam with actuator (linear model)

The forced response is sought in the form

$$w_k(x,t) = \sum_{n=1}^{N} X_{kn}(x)\Theta_n(t) \quad (k = 0,1,2,3,4), \tag{72}$$

where X_{kn} are eigenfunctions (61), the subscript k denotes a number of a zone, and the subscript n denotes a number of an eigenfunction, corresponding to the frequency ω_n. Due to the fact that the shape functions in the series (72) are chosen to be the eigenfunctions $X_{kn}(x)$ of the differential operators of the problem, the series (72) satisfies not only essential (displacement) boundary conditions (52), as it is required by the Ritz method, but also the natural (force) boundary conditions (53). Therefore, the series (72) converges to the exact solution, if the unknown functions $\Theta(t)$ are computed from the condition of extremum of the functional J, defined by equation (55).

Substitution of equations (72) into the expression (55) for the functional J leads to the following result:

$$J = \int_{t_1}^{t_2} \mathcal{L}(\Theta_1(t), \ldots, \Theta_N(t);\ \dot{\Theta}_1(t), \ldots, \dot{\Theta}_N(t))\, dt, \tag{73}$$

where

$$\begin{aligned}\mathcal{L}(\Theta_1(t), \ldots, \Theta_N(t);\ \dot{\Theta}_1(t), \ldots, \dot{\Theta}_N(t)) &= \frac{1}{2}\sum_{i,j=1}^{N} K_{ij}\Theta_i(t)\Theta_j(t) \\ &- \frac{1}{2}\sum_{i,j=1}^{N} M_{ij}\dot{\Theta}_i(t)\dot{\Theta}_j(t) - \sum_{i=1}^{N} F_i(t)\Theta_i(t),\end{aligned} \tag{74}$$

and

$$\begin{aligned}K_{ij} = A_0 \int_0^a X''_{0i}X''_{0j}\, dx &+ A_1 \int_a^\alpha X''_{1i}X''_{1j}\, dx + A_2 \int_\alpha^\beta X''_{2i}X''_{2j}\, dx \\ &+ A_3 \int_\alpha^\beta X''_{3i}X''_{3j}\, dx + A_4 \int_\beta^L X''_{4i}X''_{4j}\, dx,\end{aligned} \tag{75}$$

$$\begin{aligned}M_{ij} &= B_0 \int_0^a X_{0i}X_{0j}\, dx + B_1 \int_a^\alpha X_{1i}X_{1j}\, dx + B_2 \int_\alpha^\beta X_{2i}X_{2j}\, dx \\ &+ B_3 \int_\alpha^\beta X_{3i}X_{3j}\, dx + B_4 \int_\beta^L X_{4i}X_{4j}\, dx + C_0 \int_0^a X'_{0i}X'_{0j}\, dx \\ &+ C_1 \int_a^\alpha X'_{1i}X'_{1j}\, dx + C_2 \int_\alpha^\beta X'_{2i}X'_{2j}\, dx + C_3 \int_\alpha^\beta X'_{3i}X'_{3j}\, dx + C_4 \int_\beta^L X'_{4i}X'_{4j}\, dx,\end{aligned} \tag{76}$$

$$F_i(t) = -I_p X'_{0i}(a)V(t). \tag{77}$$

The necessary condition of extremum of the functional $J = \int_{t_1}^{t_2} \mathcal{L}\, dt$ (equation 73),

$$\frac{\partial \mathcal{L}}{\partial \Theta_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Theta}_i} = 0, \tag{78}$$

produces the following system of ordinary differential equations

$$\sum_{j=1}^{N} M_{ij}\ddot{\Theta}_j(t) + \sum_{j=1}^{N} K_{ij}\Theta_j(t) = F_i(t), \tag{79}$$

or, in matrix form,

$$[M]_{(N\times N)}\{\ddot{\Theta}\}_{(N\times 1)} + [K]_{(N\times N)}\{\Theta\}_{(N\times 1)} = \{F\}_{(N\times 1)}. \tag{80}$$

Matrices $[K]$ and $[M]$ in equation (80) are symmetric, as follows from equations (75) and (76).

Example 4.1 As an example problem, we considered a clamped-free wooden beam with the following characteristics (Figure 4.1): length $L = 20 \times 10^{-2}\, m$, width $b = 2.76 \times 10^{-2}\, m$, thickness $h = 0.99 \times 10^{-2}\, m$, wood density $\rho^{(0)} = 418.02\, \dfrac{kg}{m^3}$, Young's modulus of the wood in the direction of fibers $E_1^{(0)} = 1.0897 \times 10^{10}\, \dfrac{N}{m^2}$. The piezoelectric actuator is QP10W (Active Control Experts). Thickness of the actuator is $\tau = 3.81 \times 10^{-4}\, m$, its length is $a = 5.08 \times 10^{-2}\, m$, the piezoelectric constant in the range of applied voltage (from $0\, V$ to $200\, V$) is $\overline{d}_{31} \approx -1.05 \times 10^{-9}\, \dfrac{m}{V}$, the Young's modulus of the actuator with its packaging is

$$E_1^{(p)} = 2.57 \times 10^{10}\, \frac{N}{m^2},$$

mass density of the actuator with its packaging is

$$\rho^{(p)} = 6151.1\, \frac{kg}{m^3}.$$

The voltage $V(t)$, applied to the piezoelectric actuator, is distributed uniformly along the length of the actuator and varies with time as

$$V(t) = V_a \sin(\Omega t + \phi_0), \tag{81a}$$

where

$$V_a = 200\, V, \quad \Omega = 600\, \frac{1}{s}, \quad \phi_0 = 0. \tag{81b}$$

The wooden beam is cut along its fibers, so that the angle θ in the formula (9) is equal to zero, and, therefore, the elastic compliance coefficient $\overline{S}_{11}$ for the wood is equal to

$$\overline{S}_{11}^{(0)} = \frac{1}{E_1^{(0)}} = 9.1768 \times 10^{-11}\, \frac{m^2}{N}.$$

For the piezoelectric actuator, the material coordinate system coincides with the problem coordinate system, so that the elastic compliance coefficient $\overline{S}_{11}$ for the material of the piezo-actuator is

$$\overline{S}_{11}^{(p)} = \frac{1}{E_1^{(p)}} = 3.8911 \times 10^{-11}\, \frac{m^2}{N}.$$

Results of calculation of circular frequencies for the **undelaminated** beam **with the actuator** are presented in the table below.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
without rotary inertia	1398.17	8249.5	22180.	42844.6	71127.6	1.06542×10^5	1.48245×10^5
with rotary inertia	1397.435	8217.9	21985.6	42205.0	69331	1.02371×10^5	1.40641×10^5

Now, let us consider frequencies of the same beam **with the delamination and with the actuator**.

In the next table, the results are presented for the coordinates of the crack tips $\alpha = 10 \times 10^{-2}\, m,\ \beta = 11 \times 10^{-2}\, m,\ \gamma = 1.65 \times 10^{-3}\, m$.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
with rotary inertia	1397.433	8217.909	21986.1	42204.9	69331.2	1.02371×10^5	1.40641×10^5

In the next table the results are presented for the coordinates of the crack tips $\alpha = 10 \times 10^{-2}\, m,\ \beta = 12 \times 10^{-2}\, m,\ \gamma = 1.65 \times 10^{-3}\, m$.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
with rotary inertia	1397.433	8217.90	21986.0	42200	69330	1.02368×10^5	1.40625×10^5

In the next table the results are presented for the coordinates of the crack tips $\alpha = 10 \times 10^{-2}\, m,\ \beta = 15 \times 10^{-2}\, m,\ \gamma = 1.65 \times 10^{-3}\, m$.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
with rotary inertia	1397.432	8217.62	21980.	42198	69094	1.01932×10^5	1.33019×10^5

Next, we will compare frequencies of the same cantilever beam **without the actuator**, obtained by different methods.

No delamination, no actuator:

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
without rotary inertia	1282.6	8037.9	22506.	44103.	72906.	1.08909×10^5	1.52113×10^5
with rotary inertia	1282.	8011.5	22330.9	43474.	71265.6	1.05385×10^5	1.51609×10^5

In the last table, the frequencies without account of rotary inertia were computed by a formula [10]

$$\omega_n = c_n^2 \frac{h}{L^2} \sqrt{\frac{E_1^{(0)}}{12\rho^{(0)}}},$$

where c_n are solutions of equation

$$\cos c_n \cosh c_n + 1 = 0.$$

With delamination, no actuator:

$$\alpha = 10 \times 10^{-2}\, m, \quad \beta = 11 \times 10^{-2}\, m, \quad \gamma = 1.65 \times 10^{-3}\, m$$

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
with rotary inertia	1282.0	8011.5	22330.9	43473.9	71265.6	1.05385×10^5	1.45467×10^5

$$\alpha = 10 \times 10^{-2}\, m, \quad \beta = 15 \times 10^{-2}\, m, \quad \gamma = 1.65 \times 10^{-3}\, m$$

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
with rotary inertia	1282.0	8011.2	22325.5	43468.0	70999.6	1.04969×10^5	1.33239×10^5

So, with the increase of the crack length, the frequencies decrease. This effect is more pronounced for higher frequencies.

4.6 Comparison of transverse displacements of cantilever beams with and without delamination at their free edges (linear analysis)

Plots of the transverse displacement as a function of time at the free end of the cantilever beam with delamination and of the same beam without delamination are presented in Figures 4.2a and 4.2b. The properties of the beams are the same as in the previous example problems (Figure 4.1 and the previous section of the text), coordinates of the crack tips are $\alpha = 10 \times 10^{-2}\, m$, $\beta = 15 \times 10^{-2}\, m$, $\gamma = 1.65 \times 10^{-3}\, m$. The beams are excited by the voltage, applied to the piezoelectric actuator. The difference in dynamic responses of the beams with and without delamination is not noticeable on the graphs, but this difference can be seen in the numerical data, used to plot the graph. This numerical data is presented below.

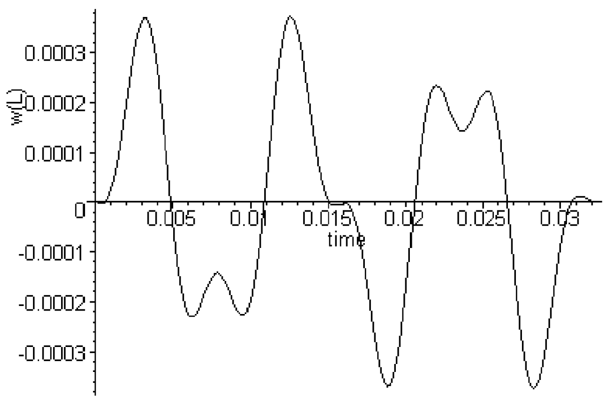

Figure 4.2a. Transverse displacement of free end of **delaminated** cantilever beam, excited by piezoelectric actuator. Beam length is $L = 0.2m$, x-coordinates of the crack tips are $\alpha = 0.1m$ and $\beta = 0.15m$, z-coordinate of the crack tips is $\gamma = 0.00165m$. **Linear analysis.**

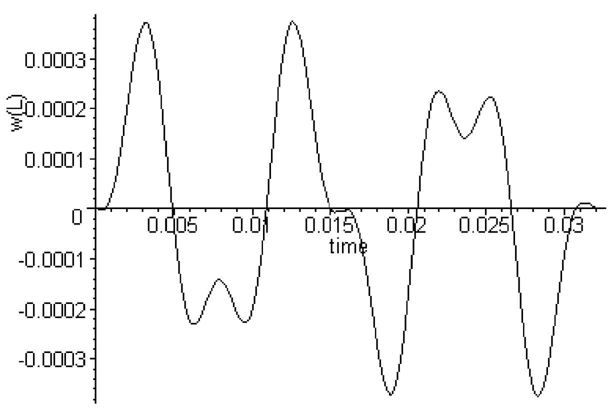

Figure 4.2b. Transverse displacement of free end of cantilever beam **without delamination**. Beam length is $L = 0.2m$. **Linear analysis.**

Displacement $w(0.2,\, t) = w(t)\big|_{x=0.2}$ **for beams with delamination and without delamination** (time is measured in seconds)

	with delamination	without delamination
$\omega(0.2, 0) =$	0	0
$\omega(0.2, 0.001) =$	4.089×10^{-5}	$4.088\,1 \times 10^{-5}$
$\omega(0.2, 0.002) =$	$2.268\,1 \times 10^{-4}$	$2.267\,8 \times 10^{-4}$
$\omega(0.2, 0.003) =$	$3.884\,6 \times 10^{-4}$	$3.884\,8 \times 10^{-4}$
$\omega(0.2, 0.004) =$	$2.740\,8 \times 10^{-4}$	$2.741\,9 \times 10^{-4}$
$\omega(0.2, 0.005) =$	$-3.944\,8 \times 10^{-5}$	-3.937×10^{-5}
$\omega(0.2, 0.006) =$	$-2.345\,3 \times 10^{-4}$	$-2.346\,5 \times 10^{-4}$
$\omega(0.2, 0.007) =$	$-2.081\,7 \times 10^{-4}$	$-2.083\,5 \times 10^{-4}$
$\omega(0.2, 0.008) =$	$-1.656\,2 \times 10^{-4}$	$-1.655\,8 \times 10^{-4}$
$\omega(0.2, 0.009) =$	$-2.242\,2 \times 10^{-4}$	$-2.240\,2 \times 10^{-4}$
$\omega(0.2, 0.010) =$	$-2.028\,5 \times 10^{-4}$	$-2.028\,5 \times 10^{-4}$
$\omega(0.2, 0.011) =$	$4.897\,5 \times 10^{-5}$	$4.871\,8 \times 10^{-5}$
$\omega(0.2, 0.012) =$	3.3710×10^{-4}	$3.370\,1 \times 10^{-4}$
$\omega(0.2, 0.013) =$	$3.663\,1 \times 10^{-4}$	$3.666\,1 \times 10^{-4}$
$\omega(0.2, 0.014) =$	$1.654\,4 \times 10^{-4}$	$1.657\,1 \times 10^{-4}$
$\omega(0.2, 0.015) =$	$1.269\,5 \times 10^{-5}$	$1.248\,9 \times 10^{-5}$
$\omega(0.2, 0.016) =$	$-5.818\,4 \times 10^{-6}$	$-6.182\,6 \times 10^{-6}$
$\omega(0.2, 0.017) =$	-8.037×10^{-5}	-8.028×10^{-5}
$\omega(0.2, 0.018) =$	$-2.855\,5 \times 10^{-4}$	$-2.851\,4 \times 10^{-4}$
$\omega(0.2, 0.020) =$	$-1.945\,2 \times 10^{-4}$	$-1.950\,2 \times 10^{-4}$

4.7 Crack opening, crack closure and interpenetration of crack faces in linear analysis

For $\alpha = 10 \times 10^{-2}\,m$, $\beta = 15 \times 10^{-2}\,m$, $\gamma = 1.65 \times 10^{-3}\,m$ at $x = 12.5 \times 10^{-2}\,m$ (at the middle of the crack's span), the difference of displacements of the upper and lower delaminated parts, $w_3(0.125, t) = w_3(t)\big|_{x=0.125}$ and $w_2(0.125, t) = w_2(t)\big|_{x=0.125}$, depends on time as shown in Figure 4.3a.

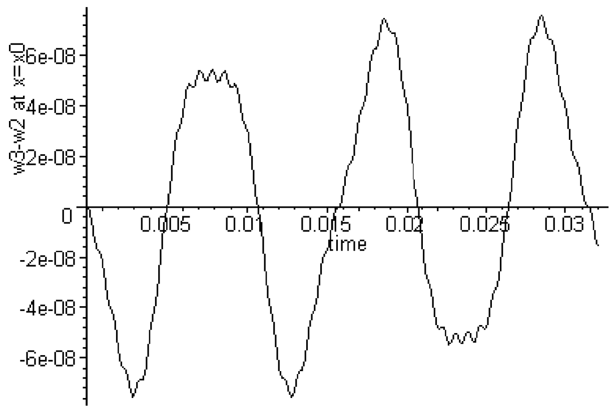

Figure 4.3a. Difference of transverse displacement of the upper and lower delaminated parts, at the middle of the crack's length, versus time, if the anti-overlapping constraint and the longitudinal force resultants are not taken into account. **Linear analysis.**

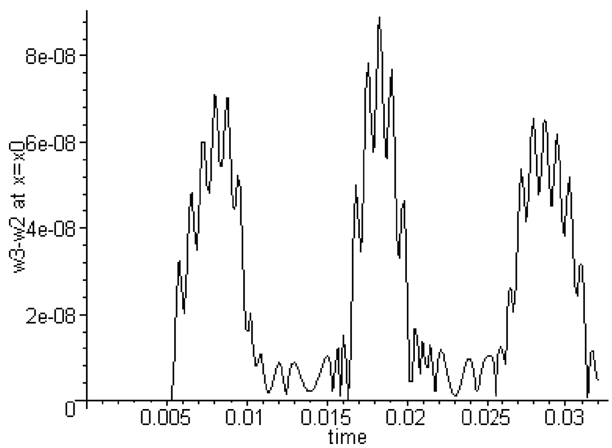

Figure 4.3b. Difference of transverse displacement of the upper and lower delaminated parts, at the middle of the crack's length, versus time, if the anti-overlapping constraint and the longitudinal force resultants are taken into account. **Linear analysis.**

Some of the numerical data, used for plotting this graph, is shown below:

$$w_3(0.125,0) = w_2(0.125,0) = 0;$$

$$\left.\begin{array}{l} w_3(0.125,0.001) = 1.9560 \times 10^{-5} \\ w_2(0.125,0.001) = 1.9564 \times 10^{-5} \end{array}\right\} \rightarrow \text{overlapping;}$$

$$\left.\begin{array}{l} w_3(0.125, 0.003) = 1.8582 \times 10^{-4} \\ w_2(0.125, 0.003) = 1.858, 6 \times 10^{-4} \end{array}\right\} \to \text{overlapping;}$$

$$\left.\begin{array}{l} w_3(0.125, 0.005) = -1.8870 \times 10^{-5} \\ w_2(0.125, 0.005) = -1.8874 \times 10^{-5} \end{array}\right\} \to \text{crack is open;}$$

$$\left.\begin{array}{l} w_3(0.125, 0.007) = -9.9578 \times 10^{-5} \\ w_2(0.125, 0.007) = -9.9599 \times 10^{-5} \end{array}\right\} \to \text{crack is open;}$$

$$\left.\begin{array}{l} w_3(0.125, 0.009) = -1.0726 \times 10^{-4} \\ w_2(0.125, 0.009) = -1.0728 \times 10^{-4} \end{array}\right\} \to \text{crack is open;}$$

$$\left.\begin{array}{l} w_3(0.125, 0.011) = 2.3427 \times 10^{-5} \\ w_2(0.125, 0.011) = 2.3432 \times 10^{-5} \end{array}\right\} \to \text{overlapping;}$$

$$\left.\begin{array}{l} w_3(0.125, 0.013) = 1.7522 \times 10^{-4} \\ w_2(0.125, 0.013) = 1.7526 \times 10^{-4} \end{array}\right\} \to \text{overlapping;}$$

$$\left.\begin{array}{l} w_3(0.125, 0.015) = 6.0726 \times 10^{-6} \\ w_2(0.125, 0.015) = 6.0738 \times 10^{-6} \end{array}\right\} \to \text{overlapping;}$$

So, in the dynamic response of the delaminated beam, computed from the linearly formulated problem, the overlapping of the upper and lower delaminated parts is present, which, of course, is physically impossible. However, the relative difference of displacements of the crack faces in the example problem is small, less than 0.01% of the transverse displacement.

5 Forced Vibration of Cantilever Beam with Delamination, under Effect of Voltage, Applied to Piezoelectric Actuator. Solution in the Form of Series in Terms of Eigenfunctions. Nonlinear Model

Analysis, based on the linear formulation, allows for interpenetration of the crack faces. A constraint, preventing such interpenetration, leads to the nonlinear formulation of the problem, as discussed previously. The additional source of nonlinearity is due to taking account of longitudinal force resultants in the delaminated parts of the beam.

In this chapter, a comparison is made between numerical results obtained without the constraint preventing the interpenetration of the crack faces (linear model) and with such constraint (nonlinear model). It is shown that the physically impossible interpenetration of the crack faces is prevented in the nonlinear model. Besides, the effect of the longitudinal force resultants on the solution for the transverse displacement is studied

In the example problem considered below, the same problem as in the previous paragraph is considered (Figure 4.1), but in nonlinear formulation, i.e. with account of the nonpenetration constraint and longitudinal force resultants in the delaminated parts.

5.1 Variational formulation of the problem

In the following text, the function $\lambda(t)$ will denote a Lagrange multiplier, used to impose the constraint that prevents interpenetration of the crack faces in the middle of the crack's span, i.e. at $x_0 = (\alpha + \beta)/2$. This constraint is expressed by the formulas

$$f(t) \equiv (w_3(x_0,t) - w_2(x_0,t))\left(\frac{1}{2} - \lim_{\epsilon \to 0} \frac{1}{\pi} \arctan \frac{w_3(x_0,t) - w_2(x_0,t)}{\epsilon}\right) = 0, \tag{82}$$

or

$$f(t) \equiv (w_3(x_0,t) - w_2(x_0,t))\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{w_3(x_0,t) - w_2(x_0,t)}{\epsilon}\right) = 0, \tag{83}$$

where ϵ is some small number. In our calculations, this number was chosen as $\epsilon = 5 \times 10^{-12}$. For explanation of formulas (82) and (83), see comments to formulas (10a) and (10b). It is assumed that if the interpenetration of the crack faces does not occur at the point $x_0 = (\beta + \alpha)/2$, then it does not occur anywhere along the crack, $\alpha < x < \beta$. This assumption is confirmed later by numerical data, obtained from the solution of the problem. The voltage $V(t)$, applied to the piezoelectric actuator, has the form

$$V(t) = V_a \sin(\Omega t + \phi_0). \tag{84}$$

The problem can be formulated in the form of the Hamilton's principle, i.e. in the form of the condition of extremum of the functional (see formulas (12) – (23) and comments to them)

$$\begin{aligned}
J = {} & \frac{1}{2}\int_{t_1}^{t_2}\int_{0}^{a} \left[A_0(w_0'')^2 - B_0(\dot{w}_0)^2 - C_0(\dot{w}_0')^2\right] dx\,dt - \int_{t_1}^{t_2}\int_{0}^{a} I_p V(t) H_a''(x) w_0 \,dx\,dt \\
& + \frac{1}{2}\int_{t_1}^{t_2}\int_{a}^{\alpha} \left[A_1(w_1'')^2 - B_1(\dot{w}_1)^2 - C_1(\dot{w}_1')^2\right] dx\,dt \\
& + \frac{1}{2}\int_{t_1}^{t_2}\int_{\alpha}^{\beta} \Bigg[A_2(w_2'')^2 - B_2(\dot{w}_2)^2 - C_2(\dot{w}_2')^2 + \frac{1}{2}N_x^{(2)}(w_2')^2 - 2\lambda(t)\, f(t) \\
& \qquad + A_3(w_3'')^2 - B_2(\dot{w}_3)^2 - C_3(\dot{w}_3')^2 + \frac{1}{2}N_x^{(3)}(w_3')^2\Bigg] dx\,dt \\
& + \frac{1}{2}\int_{t_1}^{t_2}\int_{\beta}^{L} \left[A_4(w_4'')^2 - B_4(\dot{w}_4)^2 - C_4(\dot{w}_4')^2\right] dx\,dt
\end{aligned} \tag{85}$$

with subsidiary conditions, represented by the following displacement (essential) boundary conditions:

$$\begin{gathered}
w_0(0) = 0, \qquad w_0'(0) = 0, \\
w_0(a) - w_1(a) = 0, \qquad w_0'(a) - w_1'(a) = 0, \\
w_1(\alpha) - w_2(\alpha) = 0, \qquad w_1(\alpha) - w_3(\alpha) = 0, \qquad w_1'(\alpha) - w_2'(\alpha) = 0, \\
w_1'(\alpha) - w_3'(\alpha) = 0, \qquad w_2(\beta) - w_4(\beta) = 0, \qquad w_3(\beta) - w_4(\beta) = 0, \\
w_2'(\beta) - w_4'(\beta) = 0, \qquad w_3'(\beta) - w_4'(\beta) = 0.
\end{gathered} \tag{86}$$

5.2 Forced vibration of delaminated beam with actuator (nonlinear model)

The forced dynamic response of the beam is sought in the form

$$w_k(x,t) = \sum_{n=1}^{N} X_{kn}(x)\,\Theta_n(t), \qquad k = 0,1,2,3,4, \tag{87}$$

where $X_{kn}(x)$ is an eigenfunction of the linearly formulated problem (equations (61)), in which the index k denotes a number of a beam's part (Figure 4.1), and the index n denotes a number of a natural frequency ω_n to which the eigenfunction $X_{kn}(x)$ corresponds.

Substitution of the series (87) into the expression for the functional (85) produces a result

$$\begin{aligned} J = &\int_{t_1}^{t_2} \mathcal{L}(\Theta_1(t),\ldots,\Theta_N(t);\, \dot\Theta_1(t),\ldots,\dot\Theta_N(t))dt \\ &+ \int_{t_1}^{t_2} S(\Theta_1(t),\ldots,\Theta_N(t))\,dt + \int_{t_1}^{t_2} \widetilde{\lambda}(t) f(\Theta_1(t),\ldots,\Theta_N(t))\,dt, \end{aligned} \tag{88}$$

where

$$\widetilde{\lambda}(t) = (\beta-\alpha)\lambda(t), \tag{89}$$

$$\begin{aligned} \mathcal{L}(\Theta_1(t),\ldots,\Theta_N(t);\, \dot\Theta_1(t),\ldots,\dot\Theta_N(t)) &= \frac{1}{2}\sum_{m,n=1}^{N} K_{mn}\Theta_m\Theta_n \\ &- \frac{1}{2}\sum_{m,n=1}^{N} M_{mn}\dot\Theta_m\dot\Theta_n - \sum_{n=1}^{N} F_n(t)\Theta_n(t), \end{aligned} \tag{90}$$

$$S(\Theta_1(t),\ldots,\Theta_N(t)) = \frac{1}{4}\sum_{k,l,m,n=1}^{N} \mathcal{A}_{klmn}\Theta_k\Theta_l\Theta_m\Theta_n - \frac{1}{4}\sum_{l,m,n=1}^{N} \mathcal{B}_{lmn}\Theta_l\Theta_m\Theta_n, \tag{91}$$

$$\begin{aligned} f(\Theta_1(t),\ldots,\Theta_N(t)) &= \left[\sum_{n=1}^{N}(X_{3n}(x_0) - X_{2n}(x_0))\,\Theta_n\right] \\ &\times \left[\frac{1}{\pi}\arctan\sum_{n=1}^{N}\frac{1}{\epsilon}(X_{3n}(x_0) - X_{2n}(x_0))\Theta_n - \frac{1}{2}\right] = 0. \end{aligned} \tag{92}$$

The constants K_{mn} and M_{mn} and components of the force vector $F_n(t)$, entering into equations (90), are defined by formulas (75)–(77). The constants $\mathcal{A}_{klmn}$ and $\mathcal{B}_{lmn}$ in equation (91) are defined as follows:

$$\begin{aligned}\mathcal{A}_{klmn} &= Q_2 \int_\alpha^\beta X'_{2k} X'_{2l} X'_{2m} X'_{2n}\, dx + Q_3 \int_\alpha^\beta X'_{3k} X'_{3l} X'_{3m} X'_{3n}\, dx,\\ \mathcal{B}_{lmn} &= H_2 \int_\alpha^\beta X''_{2l} X'_{2m} X'_{2n}\, dx + H_3 \int_\alpha^\beta X''_{3l} X'_{3m} X'_{3n}\, dx.\end{aligned} \tag{93}$$

In the equation (88), the last two terms, $\int_{t_1}^{t_2} S\, dt$ and $\int_{t_1}^{t_2} \widetilde{\lambda} f\, dt$, are due to the nonlinearity of the formulation of the problem. The term $\int_{t_1}^{t_2} S\, dt$ is due to taking into account the longitudinal force resultants in the delaminated parts, and the term $\int_{t_1}^{t_2} \widetilde{\lambda} f\, dt$ is due to taking account of the constraint that prevents the interpenetration of the crack faces.

The condition of extremum of the functional (88), $\delta J = 0$, leads to the following differential equations

$$\frac{\partial \mathcal{L}}{\partial \Theta_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Theta}_i}\right) + \frac{\partial S}{\partial \Theta_i} + \widetilde{\lambda}(t)\, \frac{\partial f}{\partial \Theta_i} = 0, \qquad i = 1, 2, \ldots N, \tag{94}$$

and the equation of constraint

$$f(\Theta_1, \ldots, \Theta_N) = 0. \tag{95}$$

The equation of constraint (95) is the same as the equation (92).

Following the penalty function method [11], the equation of constraint (95) can be written in the form

$$f(t) - \frac{1}{\mu}\, \widetilde{\lambda}(t) = 0, \tag{96}$$

where μ is some large number, or

$$\widetilde{\lambda}(t) = \mu f(t). \tag{97}$$

Then, substituting equation (97) into equation (94), we receive

$$\frac{\partial \mathcal{L}}{\partial \Theta_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Theta}_i}\right) + \frac{\partial S}{\partial \Theta_i} + \mu f\, \frac{\partial f}{\partial \Theta_i} = 0, \qquad i = 1, 2, \ldots N. \tag{98}$$

The substitution of equations (90) – (92) into equation (98) leads to the following ordinary differential equations

$$\sum_{m=1}^{N} M_{im} \ddot{\Theta}_m + \sum_{m=1}^{N} K_{im} \Theta_m + R_i(\Theta_1, \ldots, \Theta_N) = F_i, \qquad i = 1, \ldots, N, \tag{99}$$

where

$$R_i(\Theta_1,\dots,\Theta_N) = \sum_{k,l,m=1}^{N} \mathcal{A}_{iklm}\Theta_k\Theta_l\Theta_m + \sum_{l,m=1}^{N} \mathcal{C}_{ilm}\Theta_l\Theta_m + \mu G_i(\Theta_1,\dots,\Theta_N), \quad (100)$$

where

$$\mathcal{C}_{ilm} = -\frac{1}{4}(\mathcal{B}_{ilm} + \mathcal{B}_{lim} + \mathcal{B}_{mli}),$$

quantities $\mathcal{A}_{klmn}$ and $\mathcal{B}_{lmn}$ are defined by equations (93), and

$$G_i(\Theta_1,\dots,\Theta_N) = \big(X_{3i}(x_0) - X_{2i}(x_0)\big)\left[\frac{1}{\pi}\arctan\frac{\sum\limits_{m=1}^{N}\left(X_{3m}(x_0) - X_{2m}(x_0)\right)\Theta_m}{\epsilon} - \frac{1}{2}\right]^2 \times \sum_{n=1}^{N}\big(X_{3n}(x_0) - X_{2n}(x_0)\big)\Theta_n, \quad i = 1,\dots,N. \quad (101)$$

Equations (99) are a system of nonlinear ordinary differential equations, which can be written in matrix form as follows:

$$[M]_{(N\times N)}\{\ddot{\Theta}\}_{(N\times 1)} + [K]_{(N\times N)}\{\Theta\}_{(N\times 1)} + \{R\}_{(N\times 1)} = \{F\}_{(N\times 1)}. \quad (102)$$

In computation of the example problems, equations (102) were reduced to the system of first-order differential equations and solved by an implicit Adams method with direct iteration [12]. Some details on the method of the solution are presented in reference [9].

For the cantilever beam, excited by the piezoelectric actuator (Figure 4.1), with the same numerical values of material and geometric characteristics as in the previous paragraph, and with coordinates of the crack tips $\alpha = 10 \times 10^{-2}\,m$, $\beta = 15 \times 10^{-2}\,m$ and $\gamma = 1.65 \times 10^{-3}\,m$, the difference of the transverse displacements of the crack's faces at $x = 0.125\,m$, computed as the solution of the nonlinearly formulated problem, is presented in Figure 4.3b. The graph in this figure shows that interpenetration of the crack faces is prevented in the nonlinear analysis.

For the same beam, the transverse displacements of the free end of the delaminated beam, obtained from the linear and nonlinear analysis, are presented on graphs in Figure 5.1. As can be seen form these graphs, the results of the linear and nonlinear analysis are slightly different.

In the case of small amplitudes of vibration, neglecting the longitudinal force resultants in the delaminated parts (i.e. neglecting the nonlinear terms in the strain-displacement relations) does not produce a significant effect on results of the nonlinear analysis. This can be seen from graphs in Figure 5.2, obtained for the same beam as considered above.

At the free end of the beam, the transverse displacements of the delaminated and undelaminated beams, obtained from the nonlinear analysis, are presented by graphs in Figure 5.3. These graphs are noticeably different.

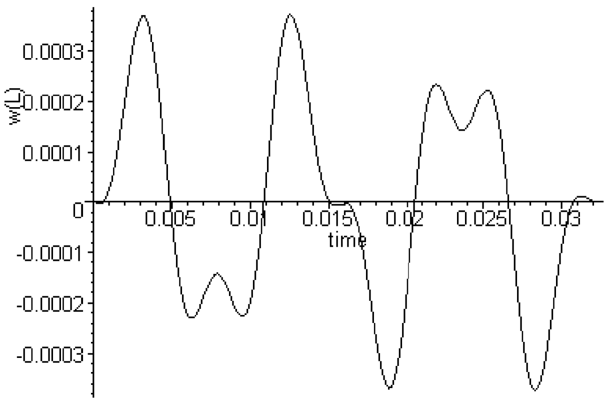

Figure 5.1a. Transverse displacement of free end of **delaminated** cantilever beam, excited by piezoelectric actuator. Beam length is $L = 0.2m,$ x-coordinates of the crack tips are $\alpha = 0.1m$ and $\beta = 0.15m,$ z-coordinate of the crack tips is $\gamma = 0.00165m$. **Linear analysis.**

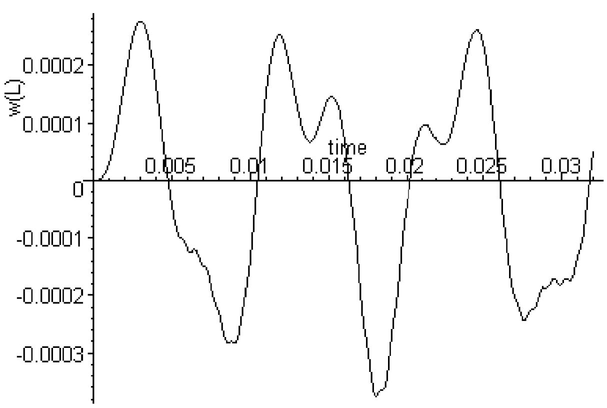

Figure 5.1b. Transverse displacement of free end of **delaminated** cantilever beam, excited by piezoelectric actuator. Beam length is $L = 0.2m,$ x-coordinates of the crack tips are $\alpha = 0.1m$ and $\beta = 0.15m,$ z-coordinate of the crack tips is $\gamma = 0.00165m$. **Nonlinear analysis.**

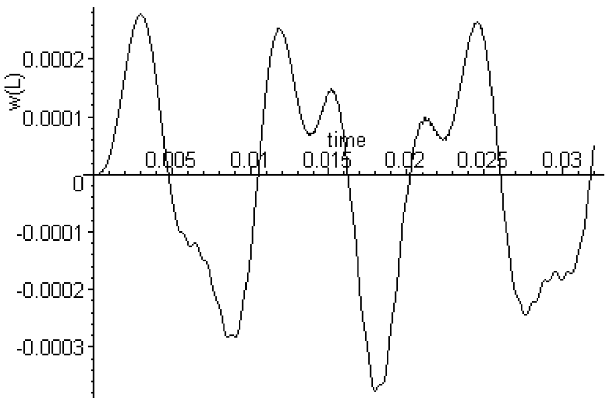

Figure 5.2a. Transverse displacement of free end of **delaminated** cantilever beam, excited by piezoelectric actuator. Beam length is $L = 0.2m$, x-coordinates of the crack tips are $\alpha = 0.1m$ and $\beta = 0.15m$, z-coordinate of the crack tips is $\gamma = 0.00165m$. Linear analysis. **Both types of nonlinearity** are taken into account: due to non-penetration constraint and due to longitudinal force resultants.

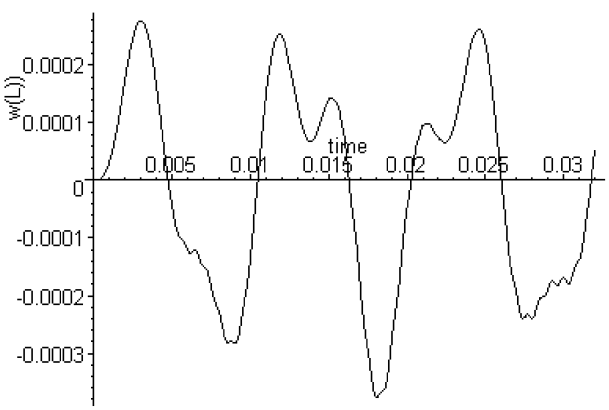

Figure 5.2b. Transverse displacement of free end of **delaminated** cantilever beam, excited by piezoelectric actuator. Beam length is $L = 0.2m$, x-coordinates of the crack tips are $\alpha = 0.1m$ and $\beta = 0.15m$, z-coordinate of the crack tips is $\gamma = 0.00165m$. Nonlinear analysis. **Only one types of nonlinearity** is take into account: due to non-penetration constraint.

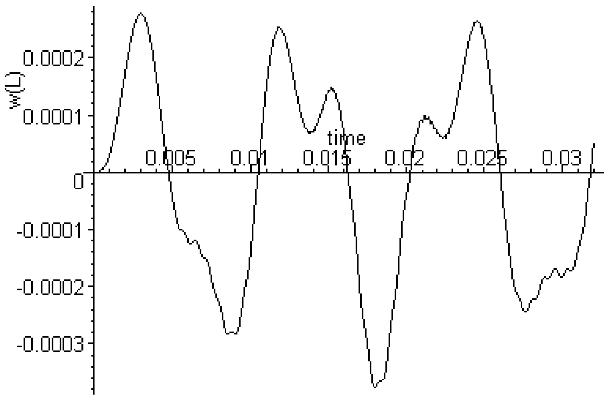

Figure 5.3a. Transverse displacement of free end of **delaminated** cantilever beam, excited by piezoelectric actuator. Beam length is $L = 0.2m,$ x-coordinates of the crack tips are $\alpha = 0.1m$ and $\beta = 0.15m,$ z-coordinate of the crack tips is $\gamma = 0.001165m$. **Nonlinear analysis.**

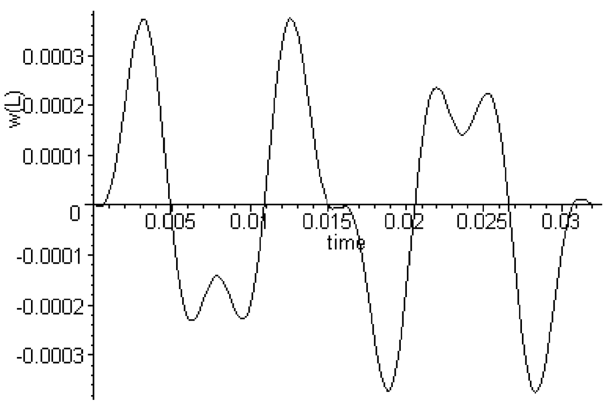

Figure 5.3b. Transverse displacement of free end of **undelaminated** cantilever beam, excited by piezoelectric actuator. Beam length is $L = 0.2m,$ x-coordinates of the crack tips are $\alpha = 0.1m$ and $\beta = 0.15m,$ z-coordinate of the crack tips is $\gamma = 0.001165m$. **Nonlinear analysis.**

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Appendix A. Properties of the Heaviside function

It can be shown [13] that the Heaviside function (unit step-function) $H_\alpha(x)$, defined by formula (3), has the following property

$$\frac{dH_\alpha(x)}{dx} = \delta_\alpha(x), \tag{A-1}$$

where $\delta_\alpha(x)$ is the Dirac's delta-function, defined as a function that has the following properties:

$$\delta_\alpha(x) = \begin{cases} 0 & \text{for} \quad x \neq \alpha, \\ \infty & \text{for} \quad x = \alpha \end{cases} \tag{A-2}$$

and

$$\int_{x_1}^{x_2} f(x)\delta_\alpha(x)\,dx = \begin{cases} f(\alpha) & \text{for} \quad x_1 < \alpha < x_2, \\ 0 & \text{for} \quad \alpha < x_1 \ \text{ and for } \ \alpha > x_2. \end{cases} \tag{A-3}$$

The delta-function has several analytical representations, one of which has the form [14]:

$$\delta_\alpha(x) = \lim_{\epsilon\to 0} \frac{1}{\pi}\,\frac{\epsilon}{\epsilon^2 + (x-\alpha)^2}. \tag{A-4}$$

According to formula (A-1), the analytical representation of the Heaviside function, corresponding to the analytical representation (A-4) of the delta-function is

$$H_\alpha(x) = \lim_{\epsilon\to 0} \frac{1}{\pi} \arctan\frac{x-\alpha}{\epsilon} + \frac{1}{2} = \begin{cases} 0 & \text{for} \quad x < \alpha, \\ \dfrac{1}{2} & \text{for} \quad x = \alpha, \\ 1 & \text{for} \quad x > \alpha. \end{cases} \tag{A-5}$$

We see that at the point $x = \alpha$ the Heaviside function, defined by the formula (A-5), is equal to $\frac{1}{2}$, while the Heaviside function, defined by the formula (3), is equal to 0. Such a change of the definition of the Heaviside function does not change a physical meaning and numerical solution of differential equations of motion, which contain the Heaviside function.

Carrying out the Heaviside function $H_\alpha(x)$ beyond the integral sign in an indefinite integral is done with the use of the formula

$$\int H_\alpha(x)\, f(x)\, dx = H_\alpha(x) \int_\alpha^x f(\eta)\, d\eta. \tag{A-6}$$

With the use of properties (A-1) and (A-3), it can be shown that

$$\int_{x_1}^{x_2} f(x) \frac{d^2 H_\alpha(x)}{dx^2}\, dx = \begin{cases} -\dfrac{df}{dx}(\alpha) & \text{for} \quad x_1 < \alpha < x_2, \\ 0 & \text{for} \quad \alpha < x_1 \ \text{ and for } \ \alpha > x_2. \end{cases} \tag{A-7}$$

The double-sided unit step-function $D_{\alpha}^{\beta}(x)$, defined by formula (2), can be expressed in terms of the Heaviside function $H_{\alpha}(x)$ as follows:

$$D_{\beta}^{\alpha}(x) = H_{\alpha}(x) - H_{\beta}(x). \tag{A-8}$$

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