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Adaptive Calculation of Lyapunov Exponents from Time Series Observations of Chaotic Time Varying Dynamical Systems

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Abstract: This paper considers the adaptive computation of Lyapunov Exponents (LEs) from time series observations based on the Jacobian approach. It is shown that the LEs can be calculated adaptively in the face of parameter variations of the dynamical system. This is achieved by formulating the regression vector properly and adaptively updating the parameter vector using the Recursive Least-Squares principles. In cases where the structure of the dynamical system is unknown, a general non-linear regression vector for local model fitting based on a locally adaptive algorithm is presented. In this case, the Recursive Least-Squares method is used to fit a suitable local model, then by state space realization in canonical form, the Jacobian matrices are computed which are used in the QR factorization method to calculate the LEs. This method essentially relies on recursive model estimation based on output data. Hence, this on-line dynamical modeling of the process will circumvent the computations typically required in the reconstructed state space. Therefore, difficulties such as the problem of large number of data and high computational effort and time are avoided. Finally, simulation results are presented for some well-known and practical chaotic systems with time varying parameters to show the effectiveness of the proposed adaptive methodology.

Keywords: *Time series; Lyapunov exponents; chaos; time varying chaotic systems; Jacobian matrices; QR factorization; non-linear regressive.*

Mathematics Subject Classification (2000): 37M10, 37L30, 37M25, 93C40.

1 Introduction

Chaos is defined based on its various characteristics [5], however, the Lyapunov exponents are conceptually the most basic and useful dynamical diagnostic for deterministic chaotic systems. The calculation of LEs for systems whose dynamical equations with constant parameters are known is straightforward. However, these methods cannot be applied directly to a set of measurement data. Two general approaches for computing the LEs from output time series are the *geometrical* and *Jacobian approaches.* In geometrical approaches, the long term evolution of an infinitesimal sphere of initial conditions is considered. [33] is one of the basic works on this approach whose idea has been modified for calculating the largest LE from short noisy data [27], and [19]. The extension of this approach for multiple time series has also been reported in [4]. On the other hand, in the Jacobian approach, local Jacobian matrices are estimated and the long term product of matrices is computed. This is presented in [29] and [15] and its idea has been extended in several references, e.g. [7], [11], and [25]. In this approach, the Jacobians are found by *locally linear mapping* the neighborhoods near the reference trajectory to neighborhoods at a subsequent time [8]. In [29] and [15], the linearized flow map from the neighbor data set into m step ahead of this set is considered as an approximation for the tangent map. In [7], it is shown that using the local neighborhood-to-neighborhood mappings with higher order Taylor series, can lead to superior results. But, all of these methods involve a state space reconstruction of the process and then finding the proper neighbors. In order to reconstruct the state space properly, the determination of embedding dimension and lag time is vital. To deal with these issues the *False Neighbor* [20] and *Singular Value Decomposition* [6] approaches are proposed which are modified and extended to multivariate time series cases [1], [2], and [24]. In addition, another problem associated with the methods based on the neighborhood approach is its high computational effort and time consuming procedures, which can be partially resolved by an adaptive reconstruction of the chaotic attractors from a single trajectory as presented in [34]. Four other methods for estimating the Jacobian have been referred to in [22], including the local thin-plate splines, radial basis functions, projection pursuit and neural nets. In [25], the Jacobians are estimated over boxes of the state space to speed up the algorithm of LEs computation.

However, in all the previous work associated with LE computation, it is generally assumed that the dynamical system under study has fixed parameters and is time invariant. But, in real applications, as it will be explained in Section 3, this is not always the case. Hence, in this paper, calculation of the LEs by an adaptive method is considered. Since the geometric approach is based on the evolution of neighbor trajectories in the reconstructed state space, it cannot be used adaptively for on-line calculation of LEs in the case of systems with time varying parameters. Therefore, the procedure adopted in this paper falls into the Jacobian approach category, which is shown to have the capability of on-line calculations. It is shown that in the proposed methodology, the LEs of an uncertain or time varying chaotic dynamical systems are computed adaptively. The important step in this approach is to estimate the Jacobian matrices. Since the LEs are derived from the eigenvalues of the Jacobians, any small error in the computation of Jacobians can cause major error in the LE computation. Some general perturbation results and error analysis in QR algorithms for computing LEs can be found in [14], [13]. In this paper, two main objectives are followed. The first goal is to use a known non-linear structure for the chaotic dynamical equations and recursively estimating the unknown parameters of the model, the procedure of the Jacobian estimation is performed on-line to overcome the problem of time variation and also uncertainty in the parameters of the dynamical system. Therefore, any variations in the unknown physical parameters of the system will appear on-line on the LEs, i.e., the LEs of the system for the current parameters is available. The advantages of on-line availability of LEs are discussed in Section 3. The second objective of the paper is to consider a general non-linear structure for the completely unknown chaotic dynamical system. This is a local model that is fitted to the system by using the Recursive Least-Squares method. Then by realizing the derived difference equation in state space canonical form, the Jacobians are estimated in each point of the trajectory. These Jacobians are then used to calculate the LEs in the QR algorithm. In this method, a general time varying non-linear model is proposed for the unknown dynamical system.

This paper is organized as follows. The background materials are given in Section 2. Some practical time varying chaotic systems and the problems associated with the LE computations for such systems are outlined in Section 3. An adaptive algorithm for calculation of the LEs is presented in Section 4. In Section 5, by considering the general non-linear regression vector, a locally adaptive algorithm for calculating the LEs is presented. Finally, simulation results are provided to show the effectiveness of the proposed methodology in well known and practical chaotic dynamical systems in Section 6.

2 Background Materials

To present the adaptive LE estimation based on the Jacobian approach, some basic definitions and algorithms are provided as follows. Consider the autonomous discrete-time dynamical system described in the following form:

$$X_{k+1} = F(X_k), \quad k = 0, 1, \dots \tag{1}$$

where X_k is the state vector in the R^m space and $F(\cdot)$ is a continuously differentiable non-linear function. Linearization of the system for a small range around the operational trajectory in the phase space can be written as:

$$\delta X_{k+1} \cong J_k \, \delta X_k, \quad k = 0, 1, \dots \tag{2}$$

where $J_k = \dfrac{\partial F}{\partial X} \,|_{X_k} \in R^{m\times m}$ is the Jacobian matrix in point k. The LEs are defined as [14].

Definition 1 Let $Y^k = J_{k-1}J_{k-2}\cdots J_0$, then the following symmetric positive definite $m\times m$ matrix exists:

$$\Lambda = \lim_{k\to\infty} \Big((Y^k)^{\mathrm{T}} Y^k\Big)^{\frac{1}{2k}} \tag{3}$$

and the logarithms of their eigenvalues are called the *Lyapunov Exponents.*

However, computation of the LEs by using this definition has some problems. The first problem is that for large value of k, the fundamental solution Y^k may take very large values and the calculation of Λ is therefore not feasible. Further, the computation of Y^k should be such that the linear independence of the columns is maintained. Otherwise, this computation leads only to the largest LE. To deal with these problems, the QR factorization algorithm is used for approximation of LEs [15], [7], [11], [25], [14], and [13]. The steps involved in this method can be summarized as follows [14]:

1. Consider the orthogonal $m \times m$ matrix Q_0 such that $Q_0^{\mathrm{T}} Q_0 = I_{m\times m}$.
2. Solve $Z_{k+1} = J_k Q_k$, $k = 0, 1, \dots,$ and obtain the decomposition: $Z_{k+1} = Q_{k+1} R_{k+1}$ where Q_{k+1} is an orthogonal $m \times m$ matrix and R_{k+1} is an upper triangular $m \times m$ matrix with positive diagonal elements.
3. $\lambda_i = \lim_{k\to\infty} \dfrac{1}{k} \log((R_-\{k\})_{ii} \cdots (R_-\{1\})_{ii}) = \lim_{k\to\infty} \dfrac{1}{k} \sum_{j=1}^{k} \log((R_-\{j\})_{ii})$, $i = 1, \dots, m$.

3 Practical Motivations

Analysing the chaotic motion has become an active field of research, due to its wide applications and chaos theory has been successfully applied to many engineering systems such as pulse combustors, internal combustion engines and power plant pulverized coal burners.

The evolutionary motion of each system is described by its dynamical equations. In practical cases, some of the parameters in the system model may not be completely known or may vary in time. In the following, two practical time varying chaotic systems are provided

3.1 Power electronics circuits

Power electronic is a discipline spawned by real life applications in industrial, commercial, residential and aerospace environments. Much of the developments of the field of the power electronics evolve around some immediate needs for solving specific power conversion problems. Power electronics circuits can be described as piecewise switched circuits, which assume different topologies at different times. The result is a *non-linear time varying* operation, which naturally demands the use of non-linear methods for analysis and design. On the other hand, most power supply engineers would have experienced chaos in switching regulators when some parameters like input voltage and feedback gain are varied [31]. Also, in [9], the bifurcation behaviour under variation of a range of circuit parameters including storage inductance, load resistance, output capacitance is examined. Further attempts to derive the related maps for power electronics circuits and the demonstration of the occurrence of chaos under variation of parameters can be found in [12] and references therein.

3.2 Plasma-dust grain system

Researchers on plasma-dust grain systems are developing new research fields in plasma physics. In [28] a plasma-dust grain system, which is spatially one dimensional and has no external electric and magnetic field is considered. The charge of each dust grain, q, is a time dependent variable and continuously changes with time. It is assumed that the density fluctuation depends only on time and the dust charge varies temporally as:

$$q = q_0(\delta - \varepsilon \cos(\omega t))^{1/2}, \tag{4}$$

where q_0, δ, ε, and ω are determined as the fixed parameters. Equation of motion in this case is as follows:

$$\ddot{x} - (\alpha - \beta x^2)\dot{x} + x\omega_0^2(\delta - \varepsilon\cos(\omega t)) = 0, \tag{5}$$

where x is the average velocity of the dust grains. The coefficients α, β, and ω_0 correspond to production rate, loss rate, and the plasma frequency of the dust grains, respectively. The second term of the left-hand side of (5) is similar to that of the van der Pol equation, and the third term to the Mathieu one. Henceforth, this equation is called *van der Pol–Mathieu* equation.

In practice, the parameters α and β are time varying which causes large fluctuations in the behaviour of the system for different values. Two typical behaviour of the system are considered as follows. It should be noted that the values of the other parameters are assumed fixed.

Case I. The limit cycle-like behaviour For the fixed value of β, by examining the shape of the attractor for different values of α, it is seen that for some values the attractor is similar to a limit cycle. The LEs of the system for $\beta = 100$ and $\alpha = 0.78, 20$ have been computed which are summarized in Table 3.1. It is seen that there is no positive LE which confirms the non-chaotic behaviour.

Table 3.1 The calculated LEs of plasma dust-grain system for different parameters.

Parameters		Lyapunov exponents	
$\alpha = 0.78$	$\beta = 100$	$\lambda_1 = -0.1711$	$\lambda_2 = -0.7781$
$\alpha = 20$	$\beta = 100$	$\lambda_1 = -0.0491$	$\lambda_2 = -35.3375$
$\alpha = 1$	$\beta = 10$	$\lambda_1 = 0.0164$	$\lambda_2 = -1.0857$

Case II. The chaotic behaviour In this case the parameters are selected as $\alpha = 1$, $\beta = 10$. The computed LEs are provided in Table 3.1. It is seen that one of the LEs is positive which corresponds to the case of chaos.

The time varying nature of the parameters of a chaotic system can be observed in many other applications. For example, in [23], the chaotic instability behaviour of a spacecraft for a range of forcing amplitudes and frequencies when a sinusoidally varying torque is applied to the spacecraft is found. Such a torque may arise in practice from an unbalanced rotor or from vibrations in appendages. In [10], two-axis rate gyro with feedback control mounted on a space vehicle is considered and chaos is detected in the non-autonomous case in which there is an sinusoidal angular velocity about the spin of gyro. These results are of importance to spacecraft designers as any instabilities in the attitude dynamics of spacecraft could have disastrous effects on its normal operation. For example, chaotic motion in the attitude motion of communication satellite would be seriously detrimental to the high pointing accuracies required by antennae providing the desired coverage on the earth's surface. It is thus prudent for designers to avoid the region of chaotic instability via parameter design [23]. In the power electronics circuits which was explained in Section 3.1, the usual reaction is to avoid the occurrence of chaos by adjusting the component values and parameters. Thus, knowing how and when chaos occurs will be of prime importance [31]. In addition, control of chaos is the other important related subject in the field of chaotic systems [16], [26], and [32]. In mechanical systems which chaos may lead to irregular motions, it has to be reduced or suppressed. In this case, a feedback constant control torque with the assistance of the LEs calculations is used to bring the system from a chaotic regime to a regular one [10].

Therefore, the adaptive computation of quantitative LEs in parametric space as a common tool to determine chaos onset and different operational regions can be of vital importance in many engineering applications.

4 Adaptive Calculation of LEs

This section presents the adaptive calculation of the LEs. It is supposed that the output data of the dynamical system is available as a univariate time series. The dynamical behaviour of system is described by the following non-linear difference equation:

$$y(k+1) = f(X(k)), \tag{6}$$

where $f(\cdot)$ is a continuously differentiable function and $X(k)$ is a delayed vector as:

$$X(k) = [y(k-m+1),\, y(k-m+2),\, \dots\, y(k)]. \tag{7}$$

In this section, it is supposed that the dynamical structure of the system is known. Hence, m also has a definite value. However, it is assumed that the parameters of the dynamical equations are not known or they have variations with time. Here, a definite structure for the non-linear autoregressive function (6) is assumed as follows, which is linear in the unknown parameters:

$$y(k+1) = \sum_i \theta_i \phi_i(X(k)), \tag{8}$$

where ϕ_i are definite basis functions and θ_i are unknown and time varying parameters.

By considering $X(k)$ in (7) as the state vector, a canonical state space representation of the system is obtained as follows:

$$X(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_m(k) \end{bmatrix} = \begin{bmatrix} y(k-m+1) \\ y(k-m+2) \\ \vdots \\ y(k) \end{bmatrix} \Longrightarrow X(k+1) = \begin{bmatrix} x_2(k) \\ x_3(k) \\ \vdots \\ f(X(k)) \end{bmatrix}. \tag{9}$$

The Jacobian $m \times m$ matrix J_k in each point k of the typical trajectory for this canonical representation is as:

$$J_k = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Df_1 & Df_2 & \dots & Df_{m-1} & Df_m \end{bmatrix}, \tag{10}$$

where $Df_i = \dfrac{\partial f}{\partial x_i}$.

Assuming the structure given by equation (8), the Df_i, $i = 1, \dots, m$, are known expressions in terms of the parameters of the model. Therefore, to have the Jacobians in each point of the trajectory, only the recursive estimation of unknown parameters in equation (8) is required. To achieve this, the Recursive Least-Squares algorithm is used. By defining the regression vector as:

$$\phi(k) = [\phi_1(k), \phi_2(k), \dots, \phi_q(k)]^{\mathrm{T}}, \tag{11}$$

where q is number of basis functions and by considering the parameter vector θ, Recursive Least-Squares method is used to estimate the vector θ as follows [21]:

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k+1)\phi(k)\varepsilon^0(k+1), \tag{12}$$

where:

$$\begin{aligned} F(k+1) &= F(k) - \frac{F(k)\phi(k)\phi^{\mathrm{T}}(k)F(k)}{1+\phi^{\mathrm{T}}(k)F(k)\phi(k)}, \\ \varepsilon^0(k+1) &= y(k+1) - \hat{\theta}^{\mathrm{T}}(k)\phi(k). \end{aligned} \tag{13}$$

Now, by using the estimated parameters, the QR method for calculation of the LEs can be modified as an *adaptive algorithm* for the computation of the LEs as follows:

Algorithm 1:

1. In step k using the relation (12), the unknown parameters are estimated. Therefore, the function $f(\cdot)$ according to the difference equation (8) is known.
2. The Jacobian $m \times m$ matrix J_k is computed and the decomposition $J_k Q_k = Q_{k+1} R_{k+1}$ is obtained where Q_k is an orthogonal $m \times m$ matrix and R_{k+1} is an upper triangular matrix with positive diagonal elements.
3. The LEs are calculated adaptively as:

$$\lambda_i(k+1) = \frac{1}{k+1}\Big(k\lambda_i(k) + \log\big((R_-\{k+1\})_{ii}\big)\Big), \quad i = 1, \dots, m, \tag{14}$$

for $k \geq M$, where M is large enough.

In the face of system parameter variations, the LEs of the system will change and the proposed adaptive algorithm shall identify the new LEs. Therefore, calculating the LEs adaptively makes it possible to have the estimated value in each time step.

5 Adaptive Calculation of LEs for Systems with Unknown Structure

In Section 4, it was assumed that the model for the evolutionary motion of the time varying dynamical system is known. However, in some practical applications the structure of the underlying dynamical system, which generates the data is unknown. The process output signal $y(t)$ of a causal non-linear process, whose dynamic behaviour is described by the differential equation of the form

$$L(D)\{y(t)\} + F\{y(t), \dot{y}(t), \dots\} = L(D)\{u(t)\} \tag{15}$$

can be calculated by the Volterra (functional) series of infinite order. In equation (15), $L(D)$ are differential operators with $D = \dfrac{d}{dt}$, $u(t)$ is the input signal, and $F(\cdot)$ is called a multinomial in $y(t)$. The discrete Volterra series is approximated by a parametric non-linear model. By the use of the discrete parametric Volterra model, the static and dynamic input/output behaviour of all non-linear processes whose differential equations belong to the class of non-linear systems given by equation (15) can be described. Therefore, a non-linear process model with a finite number of parameters and linear in the unknown parameters will be derived from the discrete Volterra series for the use in the adaptive computation loop. For each non-linear differential equation of the form given by equation (15), a static and dynamically equivalent input/output relation difference equation model can be derived. This difference equation can be formulated in a general expression as follows [18]:

$$\begin{aligned} y(k+1) &+ \sum_{i=0}^{m-1} \theta_{1i} y(k-i) + \sum_{\beta=0}^{h} \sum_{i=0}^{m-1} \theta_{2\beta i} y(k-i) y(k-i-\beta) + \dots \\ &+ \sum_{\beta_1=0}^{h} \sum_{\beta_2=\beta_1}^{h} \dots \sum_{\beta_{p-1}=\beta_{p-2}}^{h} \sum_{i=0}^{m-1} \theta_{p\beta_1\dots\beta_{p-1}i} y(k-i) y(k-1-\beta_1) \dots y(k-1-\beta_{p-1}) \\ &= \sum_{i=0}^{m-1} \varphi_i u(k-i) + \theta_0, \end{aligned} \tag{16}$$

where p is the degree of non-linearity of the difference equation, m is the dynamic order, and h is an integer time-shift operator.

Since the solutions of the non-linear differential equations of the chaotic systems are strongly depend on the parameters and initial conditions, the idea of using the general expression (16) for locally modeling the systems is considered. As, only the output time series is assumed available, $u = 0$ is supposed. To present the adaptive calculation of the LEs, consider the following time series:

$$y(t_0),\, y(t_0+t_s),\, y(t_0+2t_s),\, \dots,\, y(t_0+(N-1)t_s) \equiv y_1,\, y_2,\, \dots,\, y_N, \tag{17}$$

where t_s is the sampling time, t_0 is the starting point of observation and N is the total number of data. The proposed algorithm, which we call it a *Locally Adaptive Algorithm* can be summarized as follows:

Algorithm 2:

1. Consider the points with indices $j = d, 2d, 3d, \dots, \left(\left[\frac{N}{d}\right] - 1\right)d$, where d is an integer value. Note that, the Jacobians will be computed in these points.
2. For each value of j, consider the last r data as $Y_j = (y_{j-r+1}, \dots, y_j)$, where r is an integer value and $r \le d$.
3. Employ the Recursive Least-Squares algorithm to estimate the unknown parameters of the general non-linear autoregressive model.
4. Compute the Jacobian $m \times m$ matrix J_j from equation (10) and the decomposition $J_j Q_j = Q_{j+d} R_{j+d}$ is obtained where Q_j is an orthogonal $m \times m$ matrix, and R_{j+d} is an upper triangular $m \times m$ matrix with positive diagonal elements.
5. The LEs are calculated adaptively as:

$$\lambda_i(j+d) = \frac{1}{\left(\dfrac{j}{d}\right)+1} \left(\left(\frac{j}{d}\right) \lambda_i(j) + \log((R_{-}\{j+d\})_{ii}) \right), \quad i = 1, \dots, m. \tag{18}$$

In the implementation of algorithm (2), the following remarks should be taken into account:

Remark 1 In this case the number of elements in the delay vector (7), which is also the order of polynomial model, is generally not known a priori. In fact, these delay vectors construct the embedding vector space of the original state space of the chaotic system. Therefore, the embedding dimension and the order of the polynomial model, have the same role. For different values of model order, various polynomials are achieved which lead to different LEs. It is therefore important to have an appropriate criteria for model order selection, if it is not known in a practical problem. A criterion for choosing the suitable model order or embedding dimension by using polynomial modelling has been presented in [3]. In addition, since this is a relevant problem in computing the LEs, many other methods are available which provide the minimum embedding dimension, as stated in the introduction.

Remark 2 Here, the term "local" is used in the sense of time, i.e., the points which are used for the parameter estimation in each index j, are neighbors in time not in the position in the reconstructed state space. Therefore, computing the Jacobians do not rely on finding the local map between neighbors of any reference point in the reconstructed state space and their subsequences as is done in [29] and [15]. This requires much computational efforts and a large number of data and is time consuming. Note that all these have been avoided in this adaptive methodology. In addition, the local map concept in the previous work can not be followed in an adaptive methodology, since many points of the attractor are required to find the local neighbors and after any parameter change the new attractor must be found.

Remark 3 It is assumed that, the system dynamics is observable through the available time series (17). This is a generic property, which is assumed for state space reconstruction from time series [30]. In [2], it is shown that the determination of optimum embedding dimension, sometimes fails for some time series and multiple time series are required in this case. This may occur due to lack of observability condition from a single time series. This problem can also occur for the estimation of Jacobians.

Remark 4 Selection of r is based on the convergence of model parameters and the initial vector of the unknown parameter and significantly effects the rate of convergence. The choice of estimated parameter in j is a good initial vector for stage $j + d$ in the Recursive Least-Squares algorithm.

Remark 5 If the number of samples in the Y_j, are not enough for the parameter convergence, by using the re-sampling method, the number of data in this group can be increased. And, convergence of the parameters can be achieved during sufficient values of iterations.

6 Simulation Results

To show the effectiveness of the proposed adaptive calculation of LEs, the algorithms are applied to some well-known chaotic systems. The dissipative systems, which can be described either by flows or maps are considered. The flows and maps denote to a set of autonomous first-order differential and difference equations, respectively.

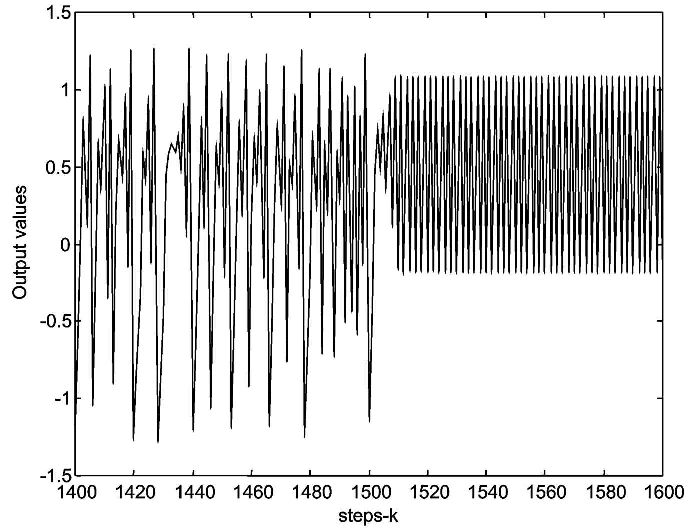

Figure 6.1. Data of Henon map with changes in parameters in step $k = 1500$.

6.1 Henon map

To illustrate the application of the Algorithm 1, first the Henon map is considered. This map can be considered as a two dimensional extension of the logistic map. It is described by the following equation:

$$y(k+1) = 1 - ay^2(k) + by(k-1), \tag{19}$$

where a and $|b| \leq 1$ are unknown time varying parameters. Suppose that the nominal parameters are $a = 1.4,\ b = 0.3$ which after some steps, change to the new values as $a = 1,\ b = 0.1$. Figure 6.1 shows the graph of the output data around the region that the nominal parameters have been changed to their new values. In practical systems, this kind of changes in nominal parameters is a common phenomenon, which occurs due to time dependent variations in the physical quantities of the system and causes variations in the LEs of the system. To see the effect of these variations, the algorithm (1) is applied to this data. In this example, the regression vector is a polynomial of order 2 and degree of non-linearity 2, which is the same as the structure of system which is known. The calculated LEs are shown in Figures 6.2a and 6.2b. It is seen that in the first stage of simulation, after a few iterations the LEs have converged to the true values, which are $\lambda_1 = 0.42,\ \lambda_2 = -1.62$. In $k = 1500$, which the parameters change to the new values $a = 1,\ b = 0.1,$ the calculated LEs converge to the correct LEs for these parameters. It is shown that, the estimated LEs converge to the true values given by, $\lambda_1 = -0.306$, $\lambda_2 = -1.99$.

6.2 Plasma-dust grain system

In this part, the plasma-dust grain system, which was explained in Section 3.2, is considered. We rewrite the equation (5) as a set of two first-order ordinary differential

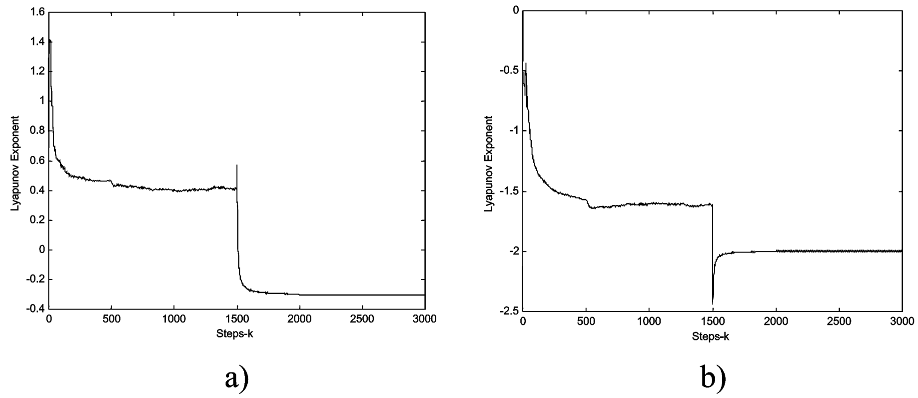

Figure 6.2. The calculated LEs of Henon map for the data with changes in parameters in $k = 1500$. a) First LE; b) Second LE.

equations:

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= (\alpha - \beta x^2)y - \omega_0^2 x(\delta - \varepsilon \cos(\omega t)), \end{aligned} \tag{20}$$

where in the simulations the parameters δ, ε, ω_0 and ω are assumed fixed but, α and β are time variant. In the first stage, the values of parameters are considered as $\alpha = 1$ and $\beta = 10$. Then after 500 sec., they change to the new values given by $\alpha = 0.78$ and $\beta = 100$. The corresponding LEs of these two regions have been calculated by using equation (5), which is shown in Table 3.1. As it was discussed in Section 3.2, chaotic and limit cycle-like behaviours are expected for these two operational regions, respectively. Now, let the time series observations of the variable x with a sampling time of 0.05 sec be available. By considering a second order polynomial model with degree of non-linearity equal to 3, the algorithm (1) is applied to calculate the LEs. As it is shown in Figures 6.3a and 6.3b, after a number of iterations the LEs have converged near true values, which are $\lambda_1 = 0.0161$, $\lambda_2 = -0.9089$.

In $k = 10000$, after a change in the parameters, the calculated LEs converge to the new LEs, which after 10000 iterations are $\lambda_1 = -0.0104$, $\lambda_2 = -0.7907$.

6.3 Ikeda map

In order to show the effectiveness of the proposed *Locally Adaptive Algorithm*, the Algorithm 2 is applied to the laser ring cavity problem. In quantum optics, the behaviour of the laser ring cavity is described by the following equation

$$z_{t+1} = \alpha e^{i\rho} z_t + \beta, \tag{21}$$

which is known as the Ikeda map. In this equation, the complex variable $z_t = x_t + iy_t$ represents the electric field at the beginning of the t^{th} passage around the ring, α is the coefficient of reflectivity of the partially reflecting output mirror, while β is related to

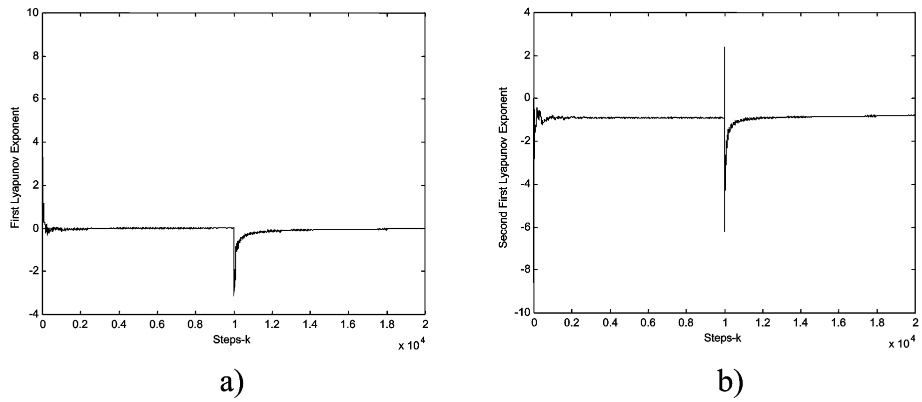

Figure 6.3. The calculated LEs of plasma-dust grain flow for the data with changes in parameters in $k = 10000$. a) First LE; b) Second LE.

the laser input amplitude. The quantity ρ is a relatively complicated functional of the laser field inside the cavity and can be considered as

$$\rho = \Delta - \frac{\delta}{1 + |z_t|^2}, \tag{22}$$

where without any loss of generality it is assumed that, $\delta = 6$, and $\Delta = 0.4$ [17]. By selecting the definite values $\alpha = 0.9$ and $\beta = 1$, the Ikeda map can be rewritten as follows:

$$\begin{aligned} \rho &= 0.4 - \frac{6}{(1 + x^2(k) + y^2(k))}, \\ x(k+1) &= 1 + 0.9(x(k)\cos(\rho) - y(k)\sin(\rho)), \\ y(k+1) &= 0.9(x(k)\sin(\rho) + y(k)\cos(\rho)), \end{aligned} \tag{23}$$

where, the corresponding LEs are $\lambda_1 = 0.505$, $\lambda_2 = -0.715$ [19].

It is assumed that the system difference equations are not available. Therefore, the *Locally Adaptive Algorithm* is used to calculate the LEs. For this, the total number of $N = 5000$ data of the Ikeda map was considered in a time series. A second order polynomial model with a degree of non-linearity equal to 2, is considered as the non-linear autoregressive model. Then, by selecting $d = 5$, for computing the Jacobian matrix in each step, and by considering $r = 5$, all the available data in the interval $Y_j = (y_{j-r+1}, \ldots, y_j)$ were used for the Recursive Least-Squares algorithm to estimate the unknown parameters of the model. Then, by continuing the Algorithm 2 the LEs were calculated which are shown in Figures 6.4a and 6.4b. It is clearly shown that, the estimated LEs converge to $\lambda_1 = 0.5196$, $\lambda_2 = -0.6615$.

In the second test, the parameter α was considered to change from 0.9 to 0.55. The calculated LEs by using the differential equations in the second region are $\lambda_1 = 0.0921$, $\lambda_2 = -0.9537$. The *Locally Adaptive Algorithm* is then applied to the time series data of x variable, by using a polynomial model with order 2 and degree of non-linearity equal 2 and $r = d = 10$. As it is shown in Figures 6.5a and 6.5b, after a change in the parameters, the calculated LEs converge to the new LEs $\lambda_1 = 0.0933$, $\lambda_2 = -1.6205$.

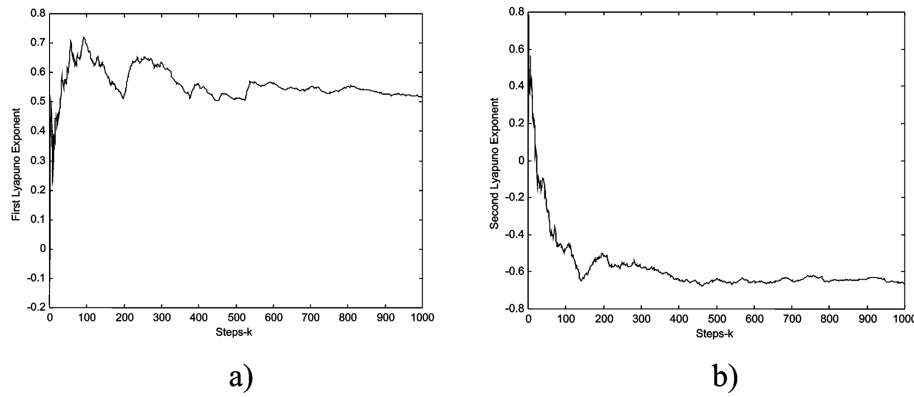

Figure 6.4. The calculated LEs of Ikeda map by using Locally Adaptive Algorithm. a) First LE; b) Second LE.

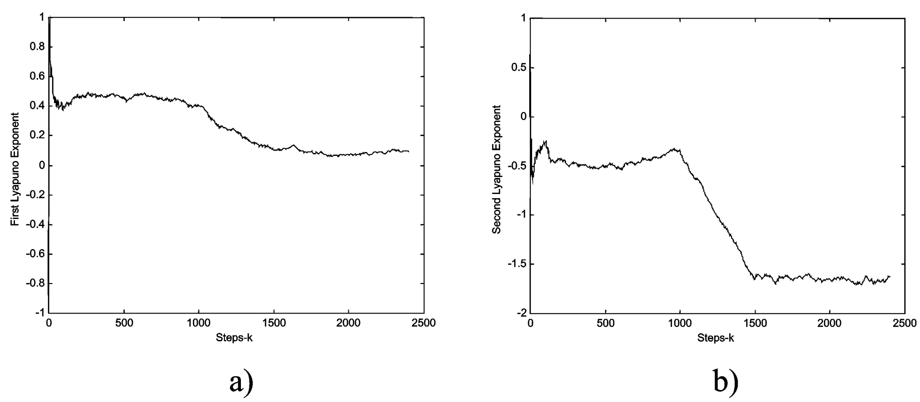

Figure 6.5. The calculated LEs of Ikeda map for the data with changes in parameters in $k = 1000$. a) First LE; b) Second LE.

7 Conclusions

In this paper, an adaptive approach for the calculation of LEs is proposed. This ensures the effective calculation of LEs in the face of system parameter variations. The adaptive methodology is based on a non-linear regression vector and the Recursive Least-Squares algorithm for the on line parameter update. This requires a prior knowledge of the structure of the system. However, in some practical applications this structure is unknown and therefore by using a general non-linear regression vector for the local model fitting, a locally adaptive algorithm is also presented. The adaptive methodology not only solves the problem of LEs calculation for time varying and unknown chaotic dynamical systems, but also circumvents the requirement for computations in the reconstructed state space and the problem of large data number for finding the neighbours in the local mapping procedure for LE computation. Finally, to show the effectiveness of the proposed adaptive methodology, it is applied to the well-known Henon and Ikeda chaotic systems, and also the plasma dust-grain system. Simulation results are provided to present the main points of the paper.

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