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A Modified LQ-Optimal Control Problem for Causal Functional Differential Equations

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Abstract: This paper continues some ideas from a preceding paper of the author, in which only point-wise initial data are considered. Here, the constraints on state variables and control involve functional initial data, leading to a modified control problem.

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1 Introduction

The following problem will be considered in this paper.

Given the functional differential equations, with causal operators A and B

$$\frac{dx}{dt} = (Ax)(t) + (Bu)(t), \quad t \in [t_0, T], \tag{1}$$

with $x\colon [t_0, T] \to R^n$, $u\colon [t_0, T] \to R^m$, $A\colon L^2([0, T], R^n) \to L^2([t_0, T], R^n)$ and $B\colon L^2([t_0, T], R^m) \to L^2([t_0, T], R^n)$, one attaches the initial value condition

$$x(t) = \varphi(t), \quad t \in [0, t_0), \quad x(t_0) = \theta, \tag{2}$$

and considers the minimization of the cost functional

$$C(x; \varphi, u) = \int_0^{t_0} \langle (P\varphi)(t), \varphi(t)\rangle\, dt + \int_{t_0}^{T} \big(\langle (Qx)(t), x(t)\rangle \; + \; \langle (Ru)(t), u(t)\rangle\big) dt, \tag{3}$$

under certain conditions to be specified below. Our main interest will be in proving the existence of an optimal triplet $(\bar{x}; \bar{\varphi}, \bar{u})$, such that

$$C(\bar{x}; \bar{\varphi}, \bar{u}) = \min C(x; \varphi, u), \tag{4}$$

the minimum being taken with respect to $\varphi \in \Phi \subset L^2([0,t_0],\, R^n)$ and $u \in U \subset L^2([t_0,T[,\, R^m)$, where Φ and U are the admissible sets for φ and u, respectively.

Remark 1 The case of the point data initial value problem has been discussed in our preceding paper [1], as well as in our book [2]. In that case, the first integral in the right-hand side of (3) is missing, since the only initial condition was $x(t_0) = \theta$. This particular case of the initial value is not restrictive. Indeed, if we substitute to $x(t_0) = \theta$ the more general condition $x(t_0) = x^0 \in R^n$, then letting $y(t) = x(t) - \bar{x}(t)$, one finds instead of (1) the equation

$$\frac{dy}{dt} = (Ay)(t) + (Bu)(t),$$

if $\bar{x}(t)$ is the (unique) solution of the homogeneous equation $dx/dt = (Ax)(t)$, such that $\bar{x}(t_0) = x^0$. Obviously, $y(t_0) = \theta$ is the null element of R^n, which agrees with the second condition in (2).

Remark 2 The nature of the functional (3) suggests the following interpretation of the control problem formulated above.

Namely, once we obtain the optimal triplet $(\bar{x};\bar{\varphi},\bar{u})$, then imposing on the dynamical system described by (1) the dynamics resulting from (2), and then applying the control u on $[t_0,T]$, we will obtain the optimal trajectory on $[t_0,T]$.

This feature of the problem illustrates the possibility of achieving a certain objective by acting on the initial interval $[0,t_0]$, first in accordance with (2), and then implementing the control u as resulting from the optimal problem.

Remark 3 It is possible to formulate a more general problem than the one described above, by considering nonlinear equations instead of (1), such as

$$\frac{dx}{dt} = (Fx)(t) + (Gu)(t), \tag{5}$$

under some initial data (2), and with a nonlinear cost functional of the general form

$$C(x;\varphi,u) = \int_0^{t_0} (K\varphi)(t)\,dt + \int_{t_0}^{T} L(x;u)(t)\,dt. \tag{6}$$

We shall not attempt to deal with problems of this type, in which F, G, K and L stand for some nonlinear operators, with adequate properties.

Remark 4 Once proven the existence of the optimal triplet $(\bar{x};\bar{\varphi},\bar{u})$, the next important problem consists in achieving the synthesis of the control problem. In other words, to express the variable u in terms of φ and x. Or, maybe it is more adequate to express both φ and u in terms of the (desired) trajectory x, if at all possible. Of course, these feedback relations should also contain causal operators. A paper by A.J. Pritchard and Yuncheng You [3], in which only classical Volterra operator are considered, seems to be promising in this regard.

2 The Main Result

We shall now formulate a set of sufficient conditions assuring the existence of the optimal triplet $(\bar{x};\bar{\varphi},\bar{u})$.

This will be achieved by reducing the problem to an elementary result in the theory of Hilbert spaces. Namely, *in a Hilbert space, every closed convex set contains a unique element of minimal norm.* We have applied this result in [1, 2], when only the case of point-wise data was considered, which meant that only the second integral appeared in the right-hand side of (3).

We shall take as underlying space the Hilbert space

$$H = L^2([0,t_0], R^n) \times L^2([t_0,T],\, R^m), \tag{7}$$

in which the scalar product is given by the sum of the scalar products in the factor spaces. This implies the fact that the norm in H is the square root of the sum of squares of norms in the factor spaces.

Let us now state the basic conditions under which we shall be able to prove the existence and uniqueness of the optimal triplet $(\bar{x}; \bar{\varphi}, \bar{u})$.

(1) The operators A and B appearing in the equation (1) are linear, continuous and causal on the space $L^2([0,T],\, R^n)$, resp. from $L^2([t_0,T],\, R^m)$ into $L^2([t_0,T],\, R^n)$.

(2) The linear operators P, Q and R appearing in the cost functional (2) are bounded and self-adjoint; moreover, P and R are positive definite, while Q is nonnegative definite.

(3) The initial set $\Phi \subset L^2([0,t_0],\, R^n)$, and the control set $U \subset L^2([t_0,T], R^m)$ are convex closed sets.

The following result can be stated.

Theorem *Consider the modified LQ-optimal control problem of minimizing the cost functional $C(x;\varphi,u)$ given by* (3)*, under the constraints (1), (2) and $\varphi \in \Phi$, $u \in U$. If conditions (1), (2) and (3) formulated above are satisfied, then there exists a unique optimal triplet $(\bar{x}; \bar{\varphi}, \bar{u})$, i.e., such that (4) takes place.*

Proof First of all, it is necessary to show that the cost functional $C(x;\varphi,u)$, given by (3), has a meaning for any $\varphi \in \Phi$ and $u \in U$. In other words, we need to prove that $x(t)$ from (1), under initial condition (2), is defined on the whole interval $[t_0, T]$.

We notice that (1) has the form

$$\frac{dx}{dt} = (Ax)(t) + f(t), \qquad t \in [t_0, T], \tag{8}$$

with $f \in L^2([t_0,T],\, R^n)$, because $Bu \in L^2([t_0,T],\, R^n)$ when $u \in U$. Hence, the solution of (8) under condition (2), can be represented by the variation of parameter formula

$$x(t) = \int_{t_0}^{t} X(t,s) f(s)\, ds + \int_{0}^{t_0} \widetilde{X}(t,s;t_0)\varphi(s)\, ds, \qquad t \in [t_0, T], \tag{9}$$

with $X(t,s)$ the Cauchy matrix attached to the homogeneous system $dx/dt = (Ax)(t)$ on the interval $[t_0, T]$, and $\widetilde{X}(t,s;t_0)$ a matrix whose definition and significance are given in [2]. The last integral in (9) represents the solution of the homogenous system, with initial condition (2).

Returning to the equation (1), and taking (8) into account, the solution of (1), for given $u \in U$, under initial condition (2), is given by

$$x(t) = \int_{t_0}^{t} X(t,s)(Bu)(s)\, ds + \int_{0}^{t_0} \widetilde{X}(t,s;t_0)\varphi(s)\, ds, \tag{10}$$

on $[t_0, T]$. The formula (10) shows that the solution $x(t)$ of (1), (2) is defined on $[t_0, T]$. It is absolutely continuous on that interval. The integrals in (3) obviously make sense.

Following the same lines as in $[1, 2]$ and taking into account the fact that the new scalar product in H is given by

$$\langle\langle(x;\varphi,u),(y;\psi,v)\rangle\rangle = \int_0^{t_0} \langle (P\varphi)(t), \psi(t)\rangle\, dt + \int_{t_0}^{T} (\langle (Qx)(t), y(t)\rangle + \langle (Ru)(t), v(t)\rangle)\, dt, \quad (11)$$

one can easily see that the cost functional $C(x;\varphi,u)$, given by (3), can be represented in the form

$$C(x;\varphi,u) = \langle\langle(x;\varphi,u),(x;\varphi,u)\rangle\rangle = |||(x;\varphi,u)|||^2. \quad (12)$$

In (12), the triple bar stands for the new norm in H. Therefore, the problem of minimizing the cost functional $C(x;\varphi,u)$ in (3), has been reduced to the problem of minimum norm in the Hilbert space H.

Since the product $\Phi\times U$ is a convex set in H, we need to show that it is also closed in the topology of H, induced by the norm $|||\cdot|||$, derived from the scalar product defined by (11). Using estimates established in [2], as well as a similar one for $x(t)$ given by (10),

$$\int_{t_0}^{T} |x(t)|^2\, dt \le C_1 \int_{t_0}^{T} |u(t)|^2\, dt + C_2 \int_0^{t_0} |\varphi(t)|^2\, dt,$$

one obtains for some positive constants λ, $\Lambda > 0$,

$$\lambda\left(\int_0^{t_0} |\varphi(t)|^2\, dt + \int_{t_0}^{T} |u(t)|^2\, dt\right) \le |||(x;\varphi,u)|||^2 \le \Lambda\left(\int_0^{t_0} |\varphi(t)|^2\, dt + \int_{t_0}^{T} |u(t)|^2\, dt\right), \quad (13)$$

which proves the equivalence of the two topologies on H; that induced by the L^2–norms in the factor spaces and the new norm $|||\cdot|||$.

Consequently, by applying the minimum norm property of Hilbert spaces quoted above, we derive the existence and uniqueness of an element $(\overline{\varphi},\overline{u}) \in \Phi\times U$, such that the triplet $(\bar{x};\overline{\varphi},\overline{u})$, with $\bar{x}$ determined from (1), (2) when $u=\overline{u}$, $\varphi=\overline{\varphi}$, is the unique optimal triplet for the problem considered above.

This ends the proof of the theorem stated in this section.

Remark 1 Some properties of the matrices $X(t,s)$ and $\widetilde{X}(t,s;t_0)$ are mentioned in [2]. For instance, noticing that the integral

$$\int_0^{t_0} \widetilde{X}(t,s;t_0)\varphi(s)\, ds$$

represents an absolutely continuous function of $t\in[t_0,T]$, with values in R^n, for each $\varphi\in L^2([0,t_0],R^n)$, enables us to derive estimates appearing in (13).

Remark 2 The relationship between the elements of the optimal triplet is given by the formula (10), i.e.,

$$\bar{x}(t) = \int_{t_0}^{t} X(t,s)(B\overline{u})(s)\,ds + \int_{0}^{t_0} \widetilde{X}(t,s;t_0)\,\overline{\varphi}(s)\,ds. \tag{14}$$

It is useful to notice that the first integral in the right-hand side of (14) can be expressed in the form

$$\int_{t_0}^{t} X(t,s)(Bu)(s)\,ds = \int_{t_0}^{t} X_1(t,s)u(s)\,ds, \tag{15}$$

where $X_1(t,s)$, $t_0 \le s \le t \le T$, is completely determined by $X(t,s)$ and the operator B. The existence of $X_1(t,s)$, which is a matrix of type n by m, follows from the fact that the first term in (15) represents a continuous operator from $L^2([t_0,T],\, R^m)$ into $L^2([t_0,T],\, R^n)$ (actually, each $u \in L^2$ is taken into an absolutely continuous function).

Therefore, (14) can be rewritten as

$$\bar{x}(t) = \int_{t_0}^{t} X_1(t,s)\overline{u}(s)\,ds + \int_{0}^{t_0} \widetilde{X}(t,s;t_0)\,\overline{\varphi}(s)\,ds \tag{16}$$

which shows that in order to determine the feedback equation, one has to solve (16) with respect to $\overline{u}(t)$. When this is possible, the feedback equation will be of the form $\overline{u}(t) = F(\bar{x},\overline{\varphi})(t)$. Equation (16) is a first kind Volterra integral equation not always solvable.

3 Feedback Control

It is always important to establish the feedback relationship in any control problem. This will allow to apply the control in such a manner that the desired trajectory, and finally the target, are obtained.

Let us notice that the equation (16) has the form

$$y(t) = f(t) + \int_{t_0}^{t} K(t,s)u(s)\,ds, \qquad t_0 \le t \le T, \tag{17}$$

which expresses the input–output relation. Identifying (16) and (17) is an elementary operation. For instance, $K(t,s)$ is given by

$$K(t,s) = X_1(t,s), \tag{18}$$

with $X_1(t,s)$ resulting from (15). It is determined by $X(t,s)$ and the operator B, but we do not have a constructive way to obtain $X_1(t,s)$, $t_0 \le s \le t \le T$.

In regard to the equation (17), the study of A.J. Pritchard and Yuncheng You [3], in which R^n or R^m are substituted by arbitrary Hilbert spaces, brings substantial contributions when the cost functional is chosen of the form

$$J(u,f) = \langle (Gy)(T), y(T)\rangle + \int_{t_0}^{T} (\langle\langle (Qy)(t), y(t)\rangle + \langle (Ry)(t), y(t)\rangle) dt. \tag{19}$$

While (19) is similar to (3), there is a difference because of the modified form of the optimal control problem we have dealt with in preceding sections of this paper.

It would be interesting to see if the modified problem can be treated by the method developed in [3]. The existence of the optimal control can be proven using a similar scheme as above.

A more general input-output equation than (17) is also considered in [3]. Namely,

$$y(t) = f(t) + \int_{t_0}^{t} \Lambda(t,s) y(s)\, ds + \int_{t_0}^{t} N(t,s) u(s)\, ds$$

is reduced to the form (17), the same way our modified control problem is reduced to the form (17).

Extending the treatment of the problem, from the case when the cost functional (19) is replaced by the functional (3), constitutes, we believe, a new type of problem in LQ-optimal control.

As a byproduct of the solution of the above formulated problem, will be the causal character of the feedback relation. This property is examined in detail in [3], where a truncation procedure is exposed and connection with some Fredholm integral equations is emphasized.

We are not attempting here to get in more detail in respect to the above mentioned problems and the procedures of their solution.

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