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Robust Control for a Class of Dynamical Systems with Uncertainties

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Abstract: In this paper, a new robust control is proposed for a class of dynamical systems with uncertainties. The considered dynamical systems may be nonminimum phase systems. The designed controller requires only input output measurement of the system. First, by using least square approximation technique, nonminimum phase systems are approximated by minimum phase systems. Then, the uncertainty is approximately estimated. Finally, based on the approximate minimum phase system and the estimate of the uncertainty, the robust control input is synthesized. Example and simulation results are presented to show the effectiveness of the proposed algorithm.

Keywords: *Robust control; nonminimum phase systems; uncertainties; approximate inverse systems; least square method.*

Mathematics Subject Classification (2000): 93C15, 93C80, 93D09, 93E10, 93E12.

1 Introduction

In recent years, the robust control for uncertain dynamical systems has been a topic of considerable interest. It is well known that all the practical control systems are subjected to uncertainties. Various robust design methodologies have been proposed for minimum phase dynamical systems until now [7, 10, 11]. For the systems with uncertainties, robust controllers are proposed in [3 − 5, 9, 13] recently. The overall systems can be ensured to be globally uniformly ultimately bounded (GUUB) which can be made arbitrarily close to exponential stability if the control energy permits. However, these approaches cannot be extended to the robust control for nonminimum phase dynamical systems with uncertainties.

It has long been known that the output tracking control of nonminimum phase plants is very difficult [8] even though the systems are perfectly known, and there is a fundamental limitation to the control performance, because the boundedness of all signals is not assured due to the unstable pole-zero cancellation. For discrete time nonminimum phase dynamical systems, one considerable method proposed by Clarke [6], in which the perfect output tracking is given up, is to minimize the control input and the difference between the plant output and the desired output. Chen and Fukuda [1] give a robust control for the continuous time systems with uncertainties by also minimizing the control input and the output error. The shortcoming of this kind of approach is that the difference between the plant output and the desired output still remains.

This paper tries to consider the robust control for a class of uncertain systems which may be nonminimum phase systems. By applying least square techniques, the class of nonminimum phase systems are approximated by minimum phase dynamical systems. Then, based on the approximated minimum phase system, the uncertainties are estimated. Finally, the robust control, which assures that the system input and output remain bounded in the closed-loop system, is synthesized. The output tracking error is controlled by the design parameters. This paper is organized as follows. Section 2 gives the problem formulation. In Section 3, approximate inverse system is introduced, the class of nonminimum phase systems are approximated by minimum phase systems. In Section 4, based on the approximated minimum phase systems, the uncertainties are estimated. In Section 5, the robust controller is synthesized. In Section 6, design example and simulations are presented to show the effectiveness of the proposed algorithm. Section 7 concludes this paper.

2 Problem Statement

Consider an uncertain system of the form

$$a(s)y(t) = b(s)u(t) + k(s)v(t), \tag{1}$$

where s denotes the differential operator; $u(t)$ and $y(t)$ are scalar input and output, respectively; $v(t)$ is an unknown signal composed of model uncertainties, nonlinearities and disturbances; $a(s)$ and $b(s)$ are described by

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n, \tag{2}$$

$$b(s) = b_r s^{n-r} + b_{r-1} s^{n-r+1} + \cdots + b_{n-1}s + b_n, \tag{3}$$

$$k(s) = k_m s^{n-m} + k_{m-1} s^{n-m+1} + \cdots + k_{n-1}s + k_n. \tag{4}$$

In this paper, we make the following assumptions:

(A1) The parameters in $a(s)$ and $b(s)$ are known; $b_r \neq 0$; $a(s)$ and $b(s)$ are coprime.

(A2) The real parts of the roots of $b(s)$ are smaller than 1.

This paper attempts to construct a robust controller to drive the system output to track a desired uniformly bounded signal $y_d(t)$ for the uncertain system, where $y_d(t)$ is differentiable to a necessary order and the derivatives are also uniformly bounded.

Even though assumption (A2) looks somewhat strict, it is meaningful to consider the output tracking problem for the formulated system because many practical control systems meet this assumption.

3 Approximate Inverse Systems

Express $b(s)$ as

$$b(s) = b_r \kappa_1(s)\kappa_2(s), \tag{5}$$

where $\kappa_1(s)$ is a υ-th order monic polynomial with no root lying in the left half plane, $\kappa_2(s)$ is an $(n-r-\upsilon)$-th order monic Hurwitz polynomial. Furthermore, we suppose

$$\kappa_1(s) = (s-\phi_1)\dots(s-\phi_\tau)(s-\alpha_1)(s-\bar{\alpha}_1)\dots(s-\alpha_\iota)(s-\bar{\alpha}_\iota), \tag{6}$$

where ϕ_i $(i=1,\dots,\tau)$ are real numbers satisfying $1 > \phi_i \geq 0$; α_j $(j=1,\dots,\iota)$ are complex numbers satisfying $1 > \mathrm{Re}\,(\alpha_j) \geq 0$; $\tau + 2\iota = \upsilon$.

Now, we introduce the next polynomial

$$\xi(s) = \left\{\left\{\prod_{i=1}^{\tau}(s+\chi_i)\right\}\left\{\prod_{j=1}^{\iota}(s+\beta_j)(s+\bar{\beta}_j)\right\}\right\}^{p+1}, \tag{7}$$

where p is a positive integer, χ_i's are positive real numbers, β_j's are complex numbers, $j=1,\dots,\iota$. Let

$$\begin{aligned}(s+\chi_i)^{p+1} &= s^{p+1} + g_{i1}s^p + \dots + g_{ip}s + g_{i,p+1},\\ (s+\beta_j)^{p+1} &= s^{p+1} + l_{j1}s^p + \dots + l_{jp}s + l_{j,p+1}.\end{aligned} \tag{8}$$

Furthermore, we introduce the following polynomials

$$\begin{aligned}\theta_i(s) &= s^p + \theta_{i1}s^{p-1} + \dots + \theta_{i,p-1}s + \theta_{ip},\\ \vartheta_j(s) &= s^p + \vartheta_{j1}s^{p-1} + \dots + \vartheta_{j,p-1}s + \vartheta_{jp},\\ \bar{\vartheta}_j(s) &= s^p + \bar{\vartheta}_{j1}s^{p-1} + \dots + \bar{\vartheta}_{j,p-1}s + \bar{\vartheta}_{jp}.\end{aligned} \tag{9}$$

The coefficients of $\theta_i(s)$ and $\vartheta_j(s)$ are determined by

$$\theta_i = (N_i^T N_i)^{-1} N_i^T g_i, \quad \vartheta_j = (K_j^* K_j)^{-1} K_j^* l_j, \tag{10}$$

where

$$N_i = \begin{bmatrix} 1 & 0 & \dots & 0\\ -\phi_i & 1 & \dots & \dots\\ 0 & -\phi_i & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\\ 0 & 0 & \dots & -\phi_i \end{bmatrix}, \quad \theta_i = \begin{bmatrix} 1\\ \theta_{i1}\\ \vdots\\ \theta_{ip} \end{bmatrix}, \quad g_i = \begin{bmatrix} 1\\ g_{i1}\\ \vdots\\ g_{i,p+1} \end{bmatrix}, \quad i=1,\dots,\tau, \tag{11}$$

$$K_j = \begin{bmatrix} 1 & 0 & \dots & 0\\ -\alpha_j & 1 & \dots & \dots\\ 0 & -\alpha_j & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\\ 0 & 0 & \dots & -\alpha_j \end{bmatrix}, \quad \vartheta_j = \begin{bmatrix} 1\\ \vartheta_{j1}\\ \vdots\\ \vartheta_{jp} \end{bmatrix}, \quad l_j = \begin{bmatrix} 1\\ l_{j1}\\ \vdots\\ l_{j,p+1} \end{bmatrix}, \quad j=1,\dots,\iota. \tag{12}$$

Define

$$\zeta(s)=\left\{\prod_{i=1}^{\tau}\theta_i(s)\right\}\left\{\prod_{j=1}^{\iota}\vartheta_j(s)\bar{\vartheta}_j(s)\right\}, \tag{13}$$

and

$$\bar{u}(t)=\frac{\kappa_1(s)\zeta(s)}{\xi(s)}\,u(t). \tag{14}$$

Remark 1 It should be pointed out that $\xi(s)$ is a monic $(p+1)\upsilon$-th order Hurwitz polynomial. $\kappa_1(s)\zeta(s)$ is a monic $(p+1)\upsilon$-th order polynomial.

Let

$$\Delta(s)=\xi(s)-\kappa_1(s)\zeta(s)=\psi_1 s^{(p+1)\upsilon-1}+\cdots+\psi_{(p+1)\upsilon-1}s+\psi_{(p+1)\upsilon}. \tag{15}$$

We have the next theorem to describe the coefficients of $\Delta(s)$.

Theorem 1 *If the parameters $\chi_i,\ i=1,\dots,\tau,$ and $\beta_j,\ j=1,\dots,\iota,$ are chosen such that $1-\phi_i>\chi_i>0,\ 0<\mathrm{Re}(\beta_j)<1-\mathrm{Re}\,(\alpha_j),\ \mathrm{Im}\,(\beta_j)=-\,\mathrm{Im}\,(\alpha_j),$ then*

$$(p+1)^2\upsilon^2\sum_{i=1}^{(p+1)\upsilon}|\psi_i|^2\to 0 \tag{16}$$

as $p\to\infty$.

Proof The proof is given in the Appendix.

Remark 2 In general, the parameter p should not be chosen to be very large, since a very large p may result in complicated computation, slow and long transients, etc.

Theorem 2 *For a uniformly bounded signal $\sigma(t)$, the next relation uniformly holds*

$$\sigma(t)-\frac{\kappa_1(s)\zeta(s)}{\xi(s)}\sigma(t)\to 0 \tag{17}$$

for all t as $p\to\infty$.

Proof Express $\sigma(t)-\frac{\kappa_1(s)\zeta(s)}{\xi(s)}\,\sigma(t)$ as

$$\begin{aligned}\sigma(t)-\frac{\kappa_1(s)\zeta(s)}{\xi(s)}\,\sigma(t)=\psi_1\frac{s^{(p+1)\upsilon-1}}{\xi(s)}\,\sigma(t)+\ldots+\psi_{(p+1)\upsilon-1}\frac{s}{\xi(s)}\,\sigma(t)\\+\psi_{(p+1)\upsilon}\frac{1}{\xi(s)}\,\sigma(t).\end{aligned} \tag{18}$$

Since $\frac{s^i}{\xi(s)}\,\sigma(t)$ are bounded for $i=0,1,\dots,(p+1)\upsilon-1,$ relation (17) can be easily concluded by using (18) and Theorem 1.

Remark 3 The difference $\sigma(t)-\frac{\kappa_1(s)\zeta(s)}{\xi(s)}\,\sigma(t)$ also depends on the frequency of the signal $\sigma(t)$.

Define

$$\bar{v}(t)=\frac{k(s)\zeta(s)}{b_r\kappa_2(s)\xi(s)}\,v(t). \tag{19}$$

Then, by employing the definition (14) and (19), (1) can be rewritten as

$$a(s)\zeta(s)y(t) = b_r\kappa_2(s)\xi(s)\left\{\bar{u}(t)+\bar{v}(t)\right\}, \tag{20}$$

where $a(s)\zeta(s)$ is a monic $(n+\upsilon p)$-th order polynomial, $\kappa_2(s)\xi(s)$ is a monic $(n+\upsilon p-r)$-th order Hurwitz polynomial with real coefficients.

For simplicity, the signal $\bar{v}(t)$ is called "disturbance" of the system in the following sections of this paper.

4 Disturbance Identifier Formulation

In this section, by estimating the filters of $\bar{v}(t)$, the signal $\bar{v}(t)$ is eventually estimated, based on our proposed formulation in [2]. For the disturbance $\bar{v}(t)$, we make the following assumption.

(A3) The disturbance $\bar{v}(t)$ and its first order derivative are bounded. However, the bounds are unknown.

Because $\bar{v}(t)$ is bounded, it is easy to see that its filters are also bounded, i.e.

$$\left|\frac{1}{(s+\lambda)^i}\,\bar{v}(t)\right| \le C_i \tag{21}$$

for $i \ge 0$, where λ is a positive constant, C_i's are unknown positive constants.

Now, we introduce a monic $(n+\upsilon p)$-th order Hurwitz polynomial

$$f(s) = \kappa_2(s)\xi(s)(s+\lambda)^r. \tag{22}$$

Then, (20) can be rewritten as

$$\dot{y}(t)+\lambda y(t) = \frac{f(s)-a(s)\zeta(s)}{\kappa_2(s)\xi(s)(s+\lambda)^{r-1}}\,y(t) + \frac{b_r}{(s+\lambda)^{r-1}}\,\bar{u}(t) + \frac{b_r}{(s+\lambda)^{r-1}}\,\bar{v}(t). \tag{23}$$

As $f(s)-a(s)\zeta(s)$ is an $(n+\upsilon p-1)$-th order polynomial, it is easy to know that $\dfrac{f(s)-a(s)\zeta(s)}{\kappa_2(s)\xi(s)(s+\lambda)^{r-1}}\,y(t)$ is a signal which can be calculated.

The next proposition gives an estimate of the signal $\bar{v}(t)$.

Proposition 1 For small positive constants $\delta_i > 0$ $(i = 1,\dots,r)$, construct the dynamical systems described by

$$\begin{gathered}\dot{\hat{y}}(t)+\lambda\hat{y}(t) = \frac{f(s)-a(s)\zeta(s)}{\kappa_2(s)\xi(s)(s+\lambda)^{r-1}}\,y(t) + \frac{b_r}{(s+\lambda)^{r-1}}\,\bar{u}(t) + b_r w_1(t),\\ \hat{y}(t_0) = y(t_0),\end{gathered} \tag{24}$$

$$\dot{\hat{w}}_{i-1}(t)+\lambda\hat{w}_{i-1}(t) = w_i(t)\,, \quad \hat{w}_{i-1}(t_0) = 0, \quad i = 2,\dots,r, \tag{25}$$

where $w_i(t),\ i = 1,\dots,r$, are given as

$$w_1(t) = \frac{b_r\{y(t)-\hat{y}(t)\}\widehat{C}_{r-1}(t)}{|b_r\{y(t)-\hat{y}(t)\}|+\delta_1} \tag{26}$$

and

$$w_i(t) = \frac{\{w_{i-1}(t) - \hat{w}_{i-1}(t)\}\widehat{C}_{r-i}(t)}{|w_{i-1}(t) - \hat{w}_{i-1}(t)| + \delta_i}, \quad i = 2, \dots, r, \tag{27}$$

respectively; $\widehat{C}_i(t)$'s are updated by the following adaptive algorithm

$$\dot{\widehat{C}}_{r-1}(t) = \begin{cases} o_{r-1}|y(t) - \hat{y}(t)| & \text{if } |b_r\{y(t) - \hat{y}(t)\}| > \delta_1, \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

$$\dot{\widehat{C}}_{r-i}(t) = \begin{cases} o_{r-i}|w_{i-1}(t) - \hat{w}_{i-1}(t)| & \text{if } |w_{i-1}(t) - \hat{w}_{i-1}(t)\}| > \delta_i, \\ 0 & \text{otherwise,} \end{cases} \tag{29}$$

$$(i = 2, \dots, r)$$

where o_{r-i}'s are positive constants. It can be concluded that, when $\sum_{j=1}^{r} \delta_j$ is very small, $w_i(t)$'s are all bounded for $i = 1, \dots, r$. Furthermore, $w_i(t)$'s are the corresponding approximate estimates of $\frac{1}{(s+\lambda)^{r-i}}\bar{v}(t)$, i.e. there exist $\epsilon_i(\delta_1, \dots, \delta_i) > 0$ and $T_i > 0$ such that

$$\left| \frac{1}{(s+\lambda)^{r-i}}\bar{v}(t) - w_i(t) \right| \le \epsilon_i(\delta_1, \dots, \delta_i) \tag{30}$$

as $t > T_i$, where $\epsilon_i(\delta_1, \dots, \delta_i)$ has the property that $\epsilon_i(\delta_1, \dots, \delta_i) \to 0$ as $\sum_{j=1}^{i} \delta_j \to 0$ for $i = 1, \dots, r$.

Proof The proposition can be similarly proved by referring to [2].

Remark 4 The design parameter $\lambda > 0$ determines the estimating speed. The design parameters $\delta_i > 0$ $(i = 1, \dots, r)$ determine the estimating precision.

5 The Robust Control Input

Now, we introduce monic Hurwitz polynomials $d(s)$ and $h(s)$ of orders $(n + \upsilon p)$ and r, respectively. Consider the following equation

$$d(s)h(s) = \eta(s)\{\zeta(s)a(s)\} + \mu(s), \tag{31}$$

where $\eta(s)$ is a monic r-th order polynomial, $\mu(s)$ is a $(n + \upsilon p - 1)$-th order polynomial. It is very clear that the solutions $\eta(s)$ and $\mu(s)$ exist uniquely. Multiplying (31) by $y(t)$ and applying (20) yields

$$d(s)h(s)y(t) = b_r\eta(s)\kappa_2(s)\xi(s)\{\bar{u}(t) + \bar{v}(t)\} + \mu(s)y(t) \tag{32}$$

i.e.

$$h(s)y(t) = b_r\{\bar{u}(t) + \bar{v}(t)\} + b_r \frac{\eta(s)\kappa_2(s)\xi(s) - d(s)}{d(s)}\{\bar{u}(t) + \bar{v}(t)\} + \frac{\mu(s)}{d(s)}y(t). \tag{33}$$

Based on the above preparation, we have the next theorem.

Theorem 3 *If $\bar{u}(t)$ is set as*

$$\bar{u}(t) = -w_r(t) - \frac{\eta(s)\kappa_2(s)\xi(s) - d(s)}{d(s)}\{\bar{u}(t) + w_r(t)\} + \frac{1}{b_r}\left\{-\frac{\mu(s)}{d(s)}y(t) + h(s)y_d(t)\right\}, \quad (34)$$

in which $w_r(t)$ is the estimate of $\bar{v}(t)$ obtained in Theorem 2, then there exist $T' > t_0$ and $\varepsilon'(t, \delta_1, \dots, \delta_r) > 0$ such that

$$|y(t) - y_d(t)| < \varepsilon'(t, \delta_1, \dots, \delta_r) \quad (35)$$

for all $t > T'$, where $\varepsilon'(t, \delta_1, \dots, \delta_r)$ has the property that $\varepsilon'(t, \delta_1, \dots, \delta_r) \to 0$ as $t \to \infty$ and $\sum_{i=1}^{r} \delta_i \to 0$.

Proof By combining (33) and (34), the result is obvious by applying Proposition 1.

By the definition of $\bar{u}(t)$, it can be seen that it is a filter of $u(t)$. Further, from Theorem 2, it can be known that the difference between $\bar{u}(t)$ and $u(t)$ is very small if $u(t)$ is uniformly bounded. Thus, we are inspired to choose the real control input $u(t)$ as

$$u(t) = -w_r(t) - \frac{\eta(s)\kappa_2(s)\xi(s) - d(s)}{d(s)}\{u(t) + w_r(t)\} + \frac{1}{b_r}\left\{-\frac{\mu(s)}{d(s)}y(t) + h(s)y_d(t)\right\}. \quad (36)$$

The next theorem is derived to describe the stability of the closed-loop system.

Theorem 4 *If the control $u(t)$ is chosen as (36), then all the signals in the loop remain uniformly bounded for a sufficiently large p. Furthermore, there exist $T > t_0$ and $\varepsilon(t, p, \delta_1, \dots, \delta_r) > 0$ such that*

$$|y(t) - y_d(t)| < \varepsilon(t, p, \delta_1, \dots, \delta_r) \quad (37)$$

for all $t > T$, where $\varepsilon(t, p, \delta_1, \dots, \delta_r)$ has the property that $\varepsilon(t, p, \delta_1, \dots, \delta_r) \to 0$ as $t \to \infty$, $p \to \infty$ and $\sum_{i=1}^{r} \delta_i \to 0$.

Proof By using (20) and the definition of $\bar{u}(t)$, system (1) can be rewritten as

$$a(s)\zeta(s)y(t) = b_r\kappa_2(s)\xi(s)\{u(t) + \bar{v}(t)\} - b_r\kappa_2(s)\Delta(s)u(t). \quad (38)$$

From (36) and (38), the closed-loop system can be expressed as

$$\begin{aligned} &\begin{bmatrix} a(s)\zeta(s) & -b_r\kappa_2(s)\{\xi(s) - \Delta(s)\} \\ [\mu(s) & b_r\eta(s)\kappa_2(s)\xi(s) \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \\ &\quad = \begin{bmatrix} b_r\kappa_2(s)\xi(s) \\ 0 \end{bmatrix} \bar{v}(t) - \begin{bmatrix} 0 \\ b_r\eta(s)\kappa_2(s)\xi(s) \end{bmatrix} w_r(t) + \begin{bmatrix} 0 \\ h(s) \end{bmatrix} y_d(t). \end{aligned} \quad (39)$$

Since

$$\det \begin{bmatrix} a(s)\zeta(s) & -b_r\kappa_2(s)\xi(s) \\ \mu(s) & b_r\eta(s)\kappa_2(s)\xi(s) \end{bmatrix} = b_r\kappa_2(s)\xi(s)d(s)h(s) \quad (40)$$

is a Hurwitz polynomial and the order of $\Delta(s)$ is lower than that of $\xi(s)$, by Theorem 1, it can be concluded that

$$\det\begin{bmatrix} a(s)\zeta(s) & -b_r\kappa_2(s)\{\xi(s)-\Delta(s)\} \\ \mu(s) & b_r\eta(s)\kappa_2(s)\xi(s)\end{bmatrix} = b_r\kappa_2(s)\left(\xi(s)d(s)h(s)-\mu(s)\Delta(s)\right) \tag{41}$$

is also a Hurwitz polynomial if p is chosen to be large enough. Therefore, based on (39), it can be seen that all the signals in the closed-loop remain uniformly bounded for a sufficiently large p.

By the definition of $\bar{u}(t)$, (32) can be rewritten as

$$\begin{aligned} d(s)h(s)y(t) = {} & b_r\eta(s)\kappa_2(s)\xi(s)\{u(t)+\bar{v}(t)\} \\ & + \mu(s)y(t) - b_r\kappa_2(s)\eta(s)(\xi(s)-\kappa_1(s)\zeta(s))u(t). \end{aligned} \tag{42}$$

Substituting (36) into (42) gives

$$\begin{aligned} h(s)(y(t)-y_d(t)) = {} & \frac{b_r\eta(s)\kappa_2(s)\xi(s)}{d(s)}\{\bar{v}(t)-w_r(t)\} \\ & - \frac{b_r\eta(s)\kappa_2(s)\xi(s)}{d(s)}\left\{u(t)-\frac{\kappa_1(s)\zeta(s)}{\xi(s)}u(t)\right\}. \end{aligned} \tag{43}$$

Since $\dfrac{b_r\eta(s)\kappa_2(s)\xi(s)}{d(s)}$ is proper, by Theorem 2 and the above discussions, it can be seen that $\dfrac{b_r\eta(s)\kappa_2(s)\xi(s)}{d(s)}\left\{u(t)-\dfrac{\kappa_1(s)\zeta(s)}{\xi(s)}u(t)\right\}$ approaches zero as $p\to\infty$. Furthermore, by using the fact that $w_r(t)$ is the approximate estimate of $\bar{v}(t)$, (37) can be proved based on (43). Thus, the theorem is proved.

Remark 5 As p increases, the computation may become complicated. On the other hand, as p is large enough, $u(t)$ is uniformly bounded and good tracking performance may be obtained. Therefore, the choice of the parameter p depends on the requirement of the considered system.

6 Design Example and Simulation Results

In this section, a nonminimum phase system will be presented to show the design procedure of the proposed output tracking algorithm. Consider the system described by

$$(s-1)^3y(t) = (4s-0.5)\,u(t) + (2s-1)\,v(t), \tag{44}$$

where $y(t)$ is the output; $u(t)$ is the input; $v(t)$ is the unknown disturbance governed by

$$v(t) = \cos(5t)\left(\frac{\{\dot{y}(t)+u(t)\}}{|\dot{y}(t)+u(t)|+0.5}\right)\left(\frac{y(t)}{|y(t)|+1}\right).$$

The purpose of the control is to drive the output to follow the signal $y_d(t) = 2\sin(t)$.

As $b(s) = 4(s-0.125)$ is a first order polynomial, for simplicity, we use the inverse system proposed for $s-\alpha$ in (58)–(63). The parameter β is chosen as $\beta = 0.3$. The accuracy of the approximate inverse system depends on the choice of the parameter p. However, when p is chosen too large, the computation may become complicated. In the presented example, p is chosen as $p = 7$. Under the above choice, the value of J is $J = 1.1153 \times 10^{-6}$. The least square approximate solution c of (63) is obtained as $c_1 = 2.5250$, $c_2 = 2.8356$, $c_3 = 1.8665$, $c_4 = 0.8003$, $c_5 = 0.2369$, $c_6 = 0.0499$, $c_7 = 0.0079$.

Corresponding to (20), system (44) can be rewritten as

$$(s-1)^3 c(s) y(t) = 4(s+0.3)^8 \{\bar{u}(t) + \bar{v}(t)\}, \tag{45}$$

where

$$\bar{u}(t) = \frac{(s-0.125)c(s)}{(s+1)^8}\, u(t), \qquad \bar{v}(t) = \frac{(2s-1)c(s)}{4(s+0.3)^8}\, v(t). \tag{46}$$

Choose the Hurwitz polynomial $f(s)$ in (22) as $f(s) = (s+0.3)^8(s+2)^2$, where λ is chosen as $\lambda = 2$. Corresponding to (23), we have

$$\dot{y}(t) + 2y(t) = \frac{f(s) - (s-1)^3 c(s)}{(s+0.3)^8 (s+2)}\, y(t) + \frac{4}{s+2}\, \bar{u}(t) + \frac{4}{s+2} \bar{v}(t). \tag{47}$$

From Proposition 1, we construct the following dynamical systems

$$\dot{\hat{y}}(t) + 2\hat{y}(t) = \frac{f(s) - (s-1)^3 c(s)}{(s+0.3)^8 (s+2)} y(t) + \frac{4}{s+2} \bar{u}(t) + 4w_1(t), \qquad \hat{y}(0) = 0, \tag{48}$$

$$\dot{\hat{w}}_1(t) + 2\hat{w}_1(t) = w_2(t), \quad \hat{w}_1(0) = 0, \tag{49}$$

where $w_1(t)$ and $w_2(t)$ are respectively determined by

$$w_1(t) = \frac{4\{y(t) - \hat{y}(t)\}\hat{C}_1(t)}{4|y(t) - \hat{y}(t)| + \delta_1}, \tag{50}$$

$$w_2(t) = \frac{\{w_1(t) - \hat{w}_1(t)\}\hat{C}_0(t)}{|w_1(t) - \hat{w}_1(t)| + \delta_2}, \tag{51}$$

and $\widehat{C}_1(t)$, $\widehat{C}_0(t)$ are respectively determined as

$$\dot{\widehat{C}}_1(t) = \begin{cases} o_1|y(t) - \hat{y}(t)| & \text{if } 4|y(t) - \hat{y}(t)| > \delta_1, \\ 0 & \text{otherwise}, \end{cases} \qquad \widehat{C}_1(0) = 0.1, \tag{52}$$

$$\dot{\widehat{C}}_0(t) = \begin{cases} o_0|w_1(t) - \hat{w}_1(t)| & \text{if } |w_1(t) - \hat{w}_1(t)\}| > \delta_2, \\ 0 & \text{otherwise}, \end{cases} \qquad \widehat{C}_0(0) = 0.1. \tag{53}$$

Therefore, $w_2(t)$ can be regarded as the approximate estimate of the disturbance $\bar{v}(t)$.

Choose the polynomials $h(s)$ and $d(s)$ as

$$h(s) = (s+1)^2, \quad d(s) = (s+1)^{10}. \tag{54}$$

Solving (31) yields

$$\eta(s) = s^2 + 12.4750s + 73.6650, \tag{55}$$

$$\begin{aligned} \mu(s) = {} & 276.7552s^9 + 622.7649s^8 + 781.4575s^7 + 829.5264s^6 + 749.6166s^5 \\ & + 507.3089s^4 + 238.4558s^3 + 74.5042s^2 + 14.0286s + 1.5820. \end{aligned} \tag{56}$$

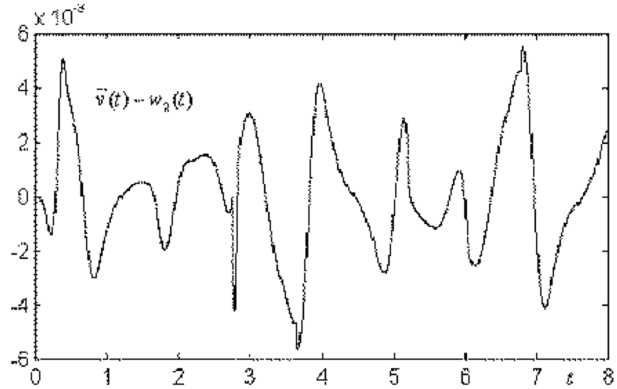

Figure 6.1. The difference between $\bar{v}(t)$ and its estimate $w_2(t)$.

Therefore, the control should be chosen as

$$\begin{aligned} u(t) = &- w_2(t) - \frac{\eta(s)(s+0.3)^8 - d(s)}{d(s)}\{u(t) + w_2(t)\} \\ &+ \frac{1}{4}\Big\{ - \frac{\mu(s)}{d(s)} y(t) + 2(s+1)^2 \sin(t) \Big\}. \end{aligned} \tag{57}$$

In the simulation process, the sampling period is chosen as 1×10^{-4} second. The parameters are chosen as $\delta_1 = \delta_2 = 2 \times 10^{-4}$, $o_1 = o_0 = 0.5$. The starting time is $t_0 = 0$. Figure 6.1 shows the difference $\bar{v}(t) - w_2(t)$. Figure 6.2 shows the output tracking control input. It can be seen the control input remains uniformly bounded. Figure 6.3 shows the difference between the output and the desired output. It can be seen that the proposed control works very well. If the parameters δ_2 and δ_2 are chosen to be much smaller, and the parameter p is chosen to be much larger, the output tracking performance may become much better.

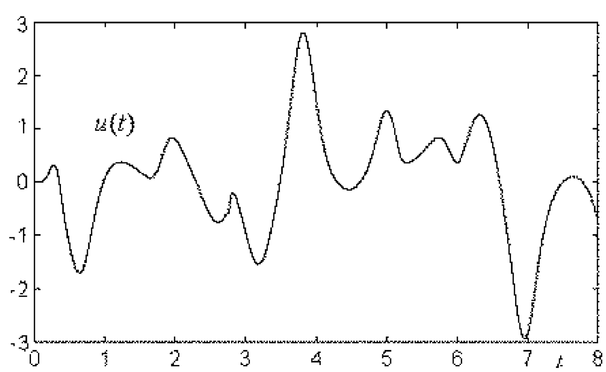

Figure 6.2. The output tracking control input $u(t)$.

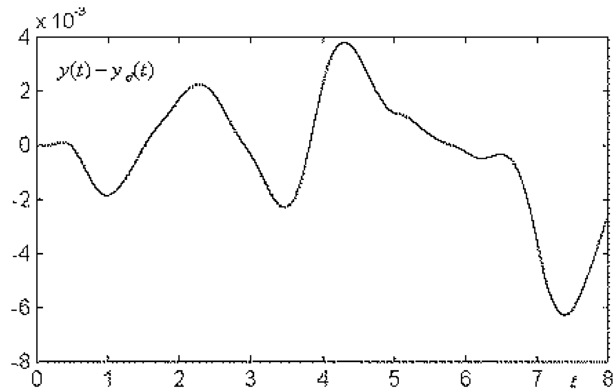

Figure 6.3. The difference between the output and the desired output.

7 Conclusions

In this paper, a new robust controller is formulated for a class of uncertain systems by using only the input output information. The disturbance, which is composed of the nonlinearities, the model uncertainties, etc., is assumed bounded with unknown bound. First, based on the least square approximate inverse systems method, the class of non-minimum phase systems is approximated by minimum phase systems. The approximate error can be made to be as small as necessary by choosing large p. Then, the disturbance is estimated. Finally, the robust controller is formulated based on the approximated minimum phase systems and the disturbance error. The output tracking error is controlled by the design parameters. Simulation results of the robust control for a nonminimum phase system show the effectiveness of the proposed method.

Appendix: Proof of Theorem 1

First, we consider the approximate inverse system of $s-\alpha$, where $\alpha \in C$ (C denotes the set of complex numbers), $\mathrm{Re}(\alpha) \geq 0$. Consider the equation

$$(s-\alpha)c(s) = (s+\beta)^{p+1}, \tag{58}$$

$$\begin{aligned} c(s) &= s^p + c_1 s^{p-1} + \cdots + c_{p-1}s + c_p, \\ (s+\beta)^{p+1} &= s^{p+1} + l_1 s^p + \cdots + l_p s + l_{p+1}, \end{aligned} \tag{59}$$

where $\mathrm{Re}(\beta) > 0$, $\beta \in C$ can be assigned in advance; p is a positive integer. The problem is finding $c(s)$ such that (58) holds. The parameter p is introduced so that the accuracy of the approximate inverse system becomes better.

It is easy to see that solving (58) is equivalent to solving the following equation

$$Kc = l, \tag{60}$$

where

$$K = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -\alpha & 1 & \dots & \dots \\ 0 & -\alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & -\alpha \end{bmatrix}_{(p+2)\times(p+1)}, \quad c = \begin{bmatrix} 1 \\ c_1 \\ \vdots \\ c_p \end{bmatrix}, \quad l = \begin{bmatrix} 1 \\ l_1 \\ \vdots \\ l_{p+1} \end{bmatrix}. \tag{61}$$

Since (60) cannot be satisfied exactly, the solution of c which may minimize the following criterion

$$J = (Kc - l)^*(Kc - l) \tag{62}$$

will be derived, where A^* denotes the complex conjugate of the transpose of A. It is well known that the least square approximate solution is given by [12]

$$c = (K^*K)^{-1}K^*l. \tag{63}$$

Lemma A.1 *If* β *is chosen such that* $0 < \mathrm{Re}(\beta) < 1 - \mathrm{Re}(\alpha)$ *and* $\mathrm{Im}(\beta) = -\mathrm{Im}(\alpha)$, *then*

$$(p+1)^2 J \to 0 \tag{64}$$

as $p \to \infty$.

Proof It is well-known that there exists a unitary matrix $U \in C^{(p+2)\times(p+2)}$ such that

$$U^*K = \begin{bmatrix} Q \\ 0 \end{bmatrix}, \quad \text{i.e.,} \quad K = U \begin{bmatrix} Q \\ 0 \end{bmatrix}, \tag{65}$$

where $Q \in C^{(p+1)\times(p+1)}$ is an upper triangular matrix. Thus, combining (62), (63) and (65) yields

$$J = l^*U \begin{bmatrix} 0_{(p+1)\times(p+1)} & 0 \\ 0 & 1 \end{bmatrix} U^*l. \tag{66}$$

Now, express U^* and K as

$$U^* = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad , K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \tag{67}$$

where

$$U_{11} \in C^{(p+1)\times(p+1)}, \quad U_{12} \in C^{1\times(p+1)}, \quad K_1 \in C^{(p+1)\times(p+1)},$$
$$U_{21} \in C^{(p+1)\times 1}, \quad U_{22} \in C, \quad K_2 \in C^{1\times(p+1)}.$$

From (65) and (67), we can also get $U_{21}K_1 + U_{22}K_2 = 0$, i.e.,

$$\begin{aligned} U_{21} = -U_{22}K_2K_1^{-1} &= -U_{22}\left[0 \quad \dots \quad 0 \quad -\alpha\right] \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \alpha & 1 & 0 & \dots & 0 \\ \alpha^2 & \alpha & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha^p & \alpha^{p-1} & \alpha^{p-2} & \dots & 1 \end{bmatrix} \\ &= \alpha U_{22}\left[\alpha^p \quad \dots \quad \alpha \quad 1\right]. \end{aligned} \tag{68}$$

Thus, from (66), (68) and (59), it gives

$$J = |[U_{21} \quad U_{22}]l|^2 = |U_{22}(\alpha^{p+1} + l_1\alpha^p + \cdots + l_p\alpha + l_{p+1})|^2 = |U_{22}|^2|\alpha + \beta|^{2(p+1)}. \quad (69)$$

It should be pointed out that $0 < |U_{22}| \le 1$. Since $\mathrm{Re}(\beta) > 0$, it can be seen that a necessary condition to make J to be very small is that $\mathrm{Re}(\alpha) < 1$. This is why we make the assumption that the real parts of the unstable zeros of $b(s)$ are smaller than 1. Under this assumption, it is very clear that $(p+1)^2 J \to 0$ if $p \to \infty$ and β is chosen such that $0 < \mathrm{Re}(\beta) < 1 - \mathrm{Re}(\alpha)$ and $\mathrm{Im}(\beta) = -\mathrm{Im}(\alpha)$.

Now, define $\bar{c}(s) = [s^p, \dots, s, 1]\bar{c}$, a similar result about the coefficients of $(s+\bar{\beta})^{p+1} - (s-\bar{\alpha})\bar{c}(s)$ can be derived as in Lemma A.1. Let

$$\begin{aligned} \left\{(s+\beta)(s+\bar{\beta})\right\}^{p+1} &- (s-\alpha)(s-\bar{\alpha})c(s)\bar{c}(s) \\ &= \varpi_1 s^{2(p+1)-1} + \cdots + \varpi_{2(p+1)-1}s + \varpi_{2(p+1)}. \end{aligned} \quad (70)$$

It can be easily proved that $4(p+1)^2 \sum_{i=1}^{2(p+1)} |\varpi_i|^2 \to 0$ as $p \to \infty$.

Therefore, the theorem can be proved by considering all the factors of $\kappa_1(s)$.

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