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On H_∞ Control Design for Singular Continuous-Time Delay Systems with Parametric Uncertainties

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Abstract: In this paper we study the problem of H_∞ control of singular linear continuous-time systems with parametric uncertainty. The system under consideration is subjected to time-delay in state, and norm-bounded parametric uncertainty entering all matrices of the system and output equations. First, the problem of robust stabilization of the underlying system is investigated. Next, we address the problem of robust H_∞ state feedback control in which both robust stability and a prescribed H_∞ performance are required to be achieved irrespective of the uncertainty and time-delay. It is shown that the above control problem can be solved in terms of solutions of some linear matrix inequalities.

Keywords: *Singular continuous-time systems; parameter uncertainty; time-delay; linear matrix inequality (LMI).*

Mathematics Subject Classification (2000): 37N35, 55N10.

1 Introduction

Time delay is commonly encountered in various engineering systems, which often occurs in the transmission of information or material between different parts of a system and is frequently a source of instability and poor performance (Malek-Zavarei and Jamshidi [15]. Transportation systems, communications systems, chemical process, power systems are typical examples of time-delay systems. During the past years, the study of time-delay systems has received considerable interest, see, e.g., Suh and Bien [30]. In the work of Gutman and Palmor [8], nonlinear state feedback controllers have been considered whereas Basher, *et al.* [9] has focused on memoryless linear state feedback. Recently, memoryless stabilization and H_∞ control of uncertain continuous-time delay systems have been extensively investigated. For some representative prior work on this general topic, we refer the reader to Shen, *et al.* [21], Lee, *et al.* [12], Mahmoud and Al-Muthairi [14], Nguang [18], Benjelloun, *et al.* [1], Kim, *et al.* [10], Moheimani and Petersen [16], Li and de Souza [13] and the very recent book by Boukas and Liu [2]. The problem of robust stabilization for a class of time varying delay systems with both Lipschitz and non-Lipschitz bounded uncertainties has been studied by Nguang [18] via Riccati equation approach, and a memoryless state feedback controller is designed. In the research conducted by Mahmoud and Al-Muthairi [14], quadratic stabilization of continuous time systems with state-delay and norm-bounded time-varying uncertainties has been considered. More recently, optimal quadratic guaranteed cost control for a class of uncertain linear time-delay systems with norm-bounded uncertainty has been designed by Moheimani and Petersen [16]. The issue of delay-dependent robust stability and stabilization of uncertain linear delay systems has been tackled by Li and de Souza [13] via a linear matrix inequality approach. However, to the best of authors' knowledge, the problems of robust stability and H_∞ control of singular continuous-time delay uncertain systems has not been fully investigated yet. In this paper, the problems of robust stability and control of a class of singular uncertain systems with unknown time delays in both system state and output equations are addressed. We consider uncertain systems with norm-bounded time-varying parameter uncertainty in all system matrices. We deal with the problems of robust stabilization and robust H_∞ control, where in the latter the controller is required to guarantee both the robust stability and a prescribed robust H_∞ performance, irrespective of the uncertainty and unknown time delay.

Notation. The notation in this paper is quite standard. $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix of appropriate dimension. $L_2[0,\infty)$ is the space of square integrable functions over $[0,\infty)$. $\|\cdot\|$ will refer to the Euclidean vector norm.

2 Problem Formulation and Preliminaries

The system considered in this paper is assumed to be a state-space model as follows:

$$\begin{aligned} E\dot{x}(t) &= [A + \Delta A(t)]x(t) + [A_d + \Delta A_d(t)]x(t - d_1(t)) \\ &\quad + [B + \Delta B(t)]u(t) + [B_w + \Delta B_w(t)]w(t), \\ z(t) &= [C + \Delta C(t)]x(t) + [C_d + \Delta C_d(t)]x(t - d_1(t)) \\ &\quad + [D + \Delta D(t)]u(t) + [D_w + \Delta D_w(t)]w(t), \\ x(t) &= \phi_1(t),\ \forall t \in [-d_1(t), 0], \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the control, $w(t) \in \mathbf{R}^p$ is the disturbance from $L_2[0,\infty)$, i.e., square-integrable, $z(t) \in \mathbf{R}^q$ is the controlled output, A, A_d, B, B_w, C, C_d, D and D_w are real-valued constant matrices of appropriate dimensions that describe the nominal system, $\Delta A(t)$, $\Delta A_d(t)$, $\Delta B(t)$, $\Delta B_w(t)$, $\Delta C(t)$, $\Delta C_d(t)$, $\Delta D(t)$, and $\Delta D_w(t)$ are real time-varying matrix functions representing parameter uncertainties, and the matrix E is a singular matrix with $\operatorname{rank}(E) = r \le n$. $d_1(t) \ge 0$ is an unknown time-varying time delay in state, $\phi_1(t)$, $t \in [-d_1(t), 0]$, is continuous vector valued initial function. $d_1(t)$ satisfies the following condition:

$$0 \le d_1(t) < \infty, \qquad \dot{d}_1(t) \le \beta_1 < 1. \tag{2.2}$$

The *admissible parameter uncertainties* in this paper is assumed to be modeled as

$$\begin{bmatrix} \Delta A(t) & \Delta A_d(t) & \Delta B(t) & \Delta B_w(t) \\ \Delta C(t) & \Delta C_d(t) & \Delta D(t) & \Delta D_w(t) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F(t)[E_1 \ \ E_2 \ \ E_3 \ \ E_4], \tag{2.3}$$

where H_1, H_2, E_1, E_2, E_3, E_4 and E_5 are known real constant matrices, and $F(t)$ is an unknown time-varying matrix function satisfying

$$\|F(t)\| \le 1, \qquad \forall\, t \in [0, \infty). \tag{2.4}$$

Remark 2.1 It should be noted that (2.1) encompasses many state space models of delay systems and can be used to represent many important physical systems; for example, power systems [29], singular space perturbation theory [31], circuits theory [17], and also cold rolling mills, wind tunnel and water resources systems (see, e.g., [15] and the references therein).

Remark 2.2 The parameter uncertainty structure as in (2.3) and (2.4) is an extension of the so-called "matching condition" which has been widely used in the problems of robust control and robust filtering of uncertain systems (see, e.g., [3, 19, 22 − 27, 33] and the references therein), and many practical systems possess parameter uncertainties which can be either exactly modeled, or overbounded by (2.4). The matrices H_1, H_2, E_1, E_2, E_3 and E_4 specify how the uncertain parameters in $F(t)$ affect the nominal matrices of system (2.1). Observe that the unknown matrix $F(t)$ in (2.3) can even be allowed to be state-dependent, i.e., $F(t) = F(t, x(t))$, as long as (2.4) is satisfied. It also should be noted that the unit overbound for $F(t)$ does not cause any loss of generality. Indeed, $F(t)$ can be always normalized, in the sense of (2.4), by appropriately choosing the matrices H_1, H_2, E_1, E_2, E_3 and E_4. Furthermore, we may consider the more general structure of the uncertainties in system (2.1), that is,

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + A_d x(t - d_1(t)) + B_w w(t) + \Delta_1(t, x, u), \\ z(t) &= Cx(t) + Du(t) + C_d x(t - d_1(t)) + D_w w(t) + \Delta_2(t, x, u), \end{aligned}$$

where

$$\begin{gathered} \|\Delta_i(t, x, u)\| \le a_i\|x\| + b_i\|u\|, \\ i = 1, 2, \quad \forall\, t \in [0, \infty), \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m, \end{gathered} \tag{2.5}$$

where $a_i \ge 0$ and $b_i \ge 0$, $i = 1, 2$, are known constant numbers. In the work of Shi and Shue [28], it has been shown that the set of the uncertainties satisfying (2.3) and (2.4) is equivalent to the set of the uncertainties satisfying (2.5) after appropriately choosing the constants a_i, b_i and the matrices H_1, H_2, E_1, E_2, E_3 and E_4.

Definition 2.1 For any given two matrices $E \in \mathbf{R}^{n \times n}$ and $A \in \mathbf{R}^{n \times n}$, the *pencil* (E, A) is said *regular* if there exists a constant number α such that $|\alpha E + A| \ne 0$ or the polynomial $|sE - A| \ne 0$.

In this paper, we assume that the nominal system (2.1) is regular, i.e., the pair $(E, A + A_d e^{-sd_1})$ is regular, where $d_1 = \max_t d_1(t)$. This condition will guarantee the existence and uniqueness of the solution for the nominal system (2.1). In addition, we assume that the nominal system (2.1) is impulse free, which ensures the delay system has no infinite poles. Throughout this paper, it is also assumed that the state is measurable for feedback. In this paper, we are concerned with the problem of robust state feedback control for the singular uncertain time-delay system (2.1) for all admissible uncertainties. Our attention is to design a state feedback controller $\mathcal{G}$:

$$u(t) = Kx(t), \tag{2.6}$$

such that for a given scalar $\gamma > 0$, for all non-zero $w(t) \in L_2[0, \infty)$ and for all parameter uncertainties satisfying (2.3) and (2.4)

$$\sup_{0 \neq w \in L_2[0,\infty)} \left(\frac{\|z\|_2}{\|w\|_2} \right) < \gamma. \tag{2.7}$$

In this situation, the system of (2.1) with the controller (2.6) is said to have a *robust H_∞ performance* (2.7). More specifically, our objective is to *design a state feedback controller $\mathcal{G}$ such that: the system of (2.1) with $\mathcal{G}$ is robustly stable and has a robust H_∞ performance (2.7).* Here, *robustly stable* means that the uncertain system (2.1) is asymptotically stable about the origin for all admissible uncertainties. In the remainder of this section, we will establish stability and H_∞ control results associated with the nominal system of (2.1), i.e., the case when $F(t) = 0$.

$$\left\{ \begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t - d_1(t)) + Bu(t) + B_w w(t), \\ z(t) &= Cx(t) + Du(t) + C_d x(t - d_1(t)) + D_w w(t), \\ x(t) &= \phi(t), \quad \forall\, t \in [-d_1(t), 0]. \end{aligned} \right. \tag{2.8}$$

First we recall the following lemma.

Lemma 2.1 (Schur Complements) *Given constant matrices M, L and Q of appropriate dimensions with M and Q are symmetric and $Q > 0$, then $M + L^T Q L < 0$ if and only if*

$$\begin{bmatrix} M & L^T \\ L & -Q^{-1} \end{bmatrix} < 0 \qquad or \qquad \begin{bmatrix} -Q^{-1} & L \\ L^T & M \end{bmatrix} < 0.$$

Lemma 2.2 *Let $T_0, \cdots, T_p \in R^{n \times n}$ be symmetric matrices. If there exists $0 \leq \tau_i$, $1 \leq \tau \leq p$ such that*

$$T_0 - \sum_{i=1}^{p} \tau_i T_i > 0,$$

then we have

$$\xi^\top T_0 \xi > 0 \tag{2.9}$$

holds for all $\xi \neq 0$ satisfying $T_0 - \sum_{i=1}^{p} \tau_i T_i > 0$.

Lemma 2.3 *Let H be a symmetric matrix and D, E be matrices with appropriate dimensions. Then, $H+DF(t)E+E^TF^T(t)D^T<0$ holds for any $F^T(t)F(t)\le I$ if and only if there exists a constant scalar $\varepsilon>0$ satisfying $H+\varepsilon DD^T+\frac{1}{\varepsilon}E^TE<0$.*

Lemma 2.4 *Let G, U, V be given matrices with G being symmetric. Then there exists matrix X such that*

$$G+UXV^\top+VX^\top U^\top>0 \tag{2.10}$$

if and only if

$$U_\perp^\top GU_\perp>0, \qquad V_\perp^\top GV_\perp>0 \tag{2.11}$$

hold, where $U_\perp$, $V_\perp$ are orthogonal complements of U and V respectively. $U_\perp^\top GU_\perp>0$ holds if and only if there exists a scalar σ such that

$$G-\sigma UU^\top>0.$$

Throughout this paper we will make the following assumption.

Assumption 2.1 There exist a positive scalar ρ such that

$$\|x(t)\|^2\le\rho\|x(t-d_1(t))\|^2. \tag{2.12}$$

Theorem 2.1 *Consider the singular time-delay system (2.8) with all uncertainties disturbance input $w(t)=0$. Under Assumption 2.1, system (2.1) is asymptotically stable for all $d_1(t)\ge 0$ satisfying (2.2), if there exist matrices $P>0$ and $R_1>0$ such that the following inequality holds*

$$\begin{bmatrix} E^TPA+A^TPE+R_1-\rho\tau I & E^TPA_d \\ A_d^TPE & -\tilde{R}_1+\tau I\end{bmatrix}<0, \tag{2.13}$$

where $\tilde{R}_1=(1-\beta_1)R_1>0$.

Remark 2.3 Before proving Theorem 2.1, we have the following observations:

1) If system (2.8) is stable, then (2.12) is satisfied with $\rho=1$. If system (2.8) is instable, assumption (2.12) means that increase rate of the system trajectory can not be greater than ρ.
2) With no loss of generality, we can assume E is of form $E=\begin{pmatrix} I & 0\\ 0 & N\end{pmatrix}$, where I is the identity and N is a nilpotent. In this case, $E^TPA+A^\top PE+R_1$ is always infeasible. The purpose of introducing Assumption 2.1 is to overcome this infeasibility.

Proof of Theorem 2.1 Let $\mathbf{x}_t\in C[-d_1,0]$ be defined by $\mathbf{x}_t(s)=x(t+s)$, $s\in[-d_1,0]$. Let us consider a Lyapunov functional candidate as

$$V(\mathbf{x}_t)\stackrel{\Delta}{=}x^T(t)E^TPEx(t)+\int\limits_{t-d_1(t)}^{t}x^T(\tau)R_1x(\tau)\,d\tau. \tag{2.14}$$

The derivative of the Lyapunov functional (2.14) along the trajectory of (2.1) is

$$\begin{aligned}\dot{V}(\mathbf{x}_t) &= \dot{x}^T(t)E^TPEx(t) + x^T(t)E^TPE\dot{x}(t)\\ &\quad + x^T(t)R_1x(t) - (1-\dot{d}_1(t))x^T(t-d_1(t))R_1x(t-d_1(t))\\ &= [Ax(t) + A_dx(t-d_1(t))]^TPEx(t) + x^T(t)E^TP[Ax(t) + A_dx(t-d_1(t))]\\ &\quad + x^T(t)R_1x(t) - (1-\dot{d}_1(t))x^T(t-d_1(t))R_1x(t-d_1(t))\\ &\le \begin{bmatrix} x(t) \\ x(t-d_1(t)) \end{bmatrix}^T \begin{bmatrix} E^TPA + A^TPE + R_1 & E^TPA_d \\ A_d^TPE & -\tilde{R}_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d_1(t)) \end{bmatrix}.\end{aligned}$$

Using Lemma 2.2, together with (2.13), implies that

$$\begin{bmatrix} x(t) \\ x(t-d_1(t)) \end{bmatrix}^T \begin{bmatrix} E^TPA + A^TPE + R_1 & E^TPA_d \\ A_d^TPE & -\tilde{R}_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d_1(t)) \end{bmatrix} < 0,$$

holds for any $x(t) \neq 0$ satisfying (2.12), which concludes the stability of system (2.8).

Remark 2.4 Theorem 2.1 provides a delay independent stability criteria since inequality (2.13) does not include the unknown delay $d_1(t)$. However, we have used the information β_1 on $d_1(t)$, which seems the best we can do. It should be noted that when $E = I$, Theorem 2.1 will reduce to the result in [4]. Furthermore, when $A_d = 0$ and $E = I$, the inequality (2.13) becomes the standard necessary and sufficient condition of stability for the non-singular systems without time-delay.

Next, we conduct the H_∞ analysis for the nominal system (2.1) (setting $F(t) = 0$).

Theorem 2.2 *Consider the singular time-delay system (2.1) with all uncertainties being zero. Under Assumption 2.1, for a given constant $\gamma > 0$, system (2.1) is asymptotically stable and has an H_∞ performance γ for all $d_1(t) \ge 0$ satisfying (2.2), if there exist matrices $P > 0$, $R_1 > 0$ and a scalar $\tau > 0$ such that the following inequality holds*

$$\begin{bmatrix} E^TPA + A^TPE + R_1 - \rho\tau I & E^TPA_d & E^TPB_w & C^T \\ A_d^TPE & -\tilde{R}_1 + \tau I & 0 & C_d^T \\ B_w^TPE & 0 & -\gamma^2 I & D_w^T \\ C & C_d & D_w & -I \end{bmatrix} < 0, \tag{2.15}$$

where $\tilde{R}_1 = (1-\beta_1)R_1 > 0$.

Proof We first show that the stability of the closed-loop system (2.1) under the condition of (2.15). Again, let us define a Lyapunov functional candidate as

$$V(\mathbf{x}_t) \triangleq x^T(t)E^TPEx(t) + \int_{t-d_1(t)}^{t} x^T(\tau)R_1x(\tau)\,d\tau. \tag{2.16}$$

Note that the negativeness of (2.15) implies

$$\begin{bmatrix} E^TPA + A^TPE + R_1 - \rho\tau I & E^TPA_d \\ A_d^TPE & -\tilde{R}_1 + \tau I \end{bmatrix} < 0, \tag{2.17}$$

which combined with Theorem 2.1 implies that the system is internally asymptotically stable, i.e., system (2.8) is asymptotically stable with $w(t) \equiv 0$. Next, we analyze the H_∞ performance of the closed-loop system (2.1). Without loss of generality, we assume the system has a zero initial condition. Taking the derivative of the Lyapunov functional (2.16) along the trajectory of (2.1), we have

$$\begin{aligned}
\dot V(\mathbf{x}_t) &= \dot x^T(t)E^TPEx(t) + x^T(t)E^TPE\dot x(t) + x^T(t)R_1x(t) \\
&\quad - (1-\dot d_1(t))x^T(t-d_1(t))R_1x(t-d_1(t)) \\
&= [Ax(t) + A_dx(t-d_1(t)) + B_ww(t)]^TPEx(t) \\
&\quad + x^T(t)E^TP[Ax(t) + A_dx(t-d_1(t)) + B_ww(t)] \\
&\quad + x^T(t)R_1x(t) - (1-\dot d_1(t))x^T(t-d_1(t))R_1x(t-d_1(t)) \\
&\le [Ax(t) + A_dx(t-d_1(t)) + B_ww(t)]^TPEx(t) \\
&\quad + x^T(t)E^TP[Ax(t) + A_dx(t-d_1(t)) + B_ww(t)] \\
&\quad + x^T(t)R_1x(t) - x^T(t-d_1(t))\tilde R_1x(t-d_1(t)) \triangleq \dot{\overline{V}}(\mathbf{x}_t).
\end{aligned}$$

Let us define performance function

$$J = \int_0^\infty [z^T(t)z(t) - \gamma^2w^T(t)w(t)]\,dt. \tag{2.18}$$

Then for any $0 \not\equiv w(t) \in L_2[0,\infty)$, one has

$$\begin{aligned}
J &\le \int_0^\infty [z^T(t)z(t) - \gamma^2w^T(t)w(t) + \dot V(\mathbf{x}_t)]\,dt \\
&\le \int_0^\infty [z^T(t)z(t) - \gamma^2w^T(t)w(t) + \dot{\overline{V}}(x(t))]\,dt.
\end{aligned} \tag{2.19}$$

Substituting $\dot{\overline{V}}(x(t))$ into (2.19), we obtain

$$J \le \int_0^\infty \xi^T(t)Z\xi(t)\,dt,$$

where

$$\xi(t) = [x^T(t) \ \ x^T(t-d_1(t)) \ \ w^T(t)]^T$$

$$Z = \begin{bmatrix} H & E^TPA_d + C^TC_d & E^TPB_w + C^TD_w \\ A_d^TPE + C_d^TC & C_d^TC_d - (1-\beta_1)R_1 & C_d^TD_w \\ B_w^TPE + D_w^TC & D_w^TC_d & -\gamma^2I + D_w^TD_w \end{bmatrix},$$

where

$$H = E^TPA + A^TPE + C^TC + R_1.$$

Therefore, using Lemma 2.1, (2.15) implies $J < 0$, that is, $\|z(t)\|_2 < \gamma\|w(t)\|_2$. Therefore, system (2.8) is internally asymptotically stable and has an H_∞ disturbance attenuation γ. The proof ends.

3 Robust Controller Design

Substituting (2.6) into (2.1) yields the dynamics of the closed-loop system as follows:

$$\begin{cases} E\dot{x}(t) = A_c(t)x(t) + A_d(t)x(t-d_1(t)) + [B_w + \Delta B_w(t)]w(t), \\ z(t) = C_c(t)x(t) + [C_d + \Delta C_d(t)]x(t-d_1(t)) + [D_w + \Delta D_w(t)]w(t), \end{cases} \tag{3.1}$$

where $A_c(t) = A_c + H_1F(t)E_c$, $C_c(t) = C_c + H_2F(t)E_c$ with $A_c = A + BK$, $C_c = C + DK$, $E_c = E_1 + E_3K$. By the same arguments as in the proof of Theorem 2.2, we have the following result.

Proposition 3.1 *Consider the singular time-delay system (3.1) with all uncertainties being zero. Under Assumption 2.1, for a given constant $\gamma > 0$, system (3.1) is asymptotically stable and has an H_∞ performance γ for all $d_1(t) \geq 0$ satisfying (2.2), if there exist matrices $P > 0$, $R_1 > 0$ and a scalar $\tau > 0$ such that the following inequality holds*

$$\begin{bmatrix} E^TPA_c(t) + A_c^T(t)PE + R_1 - \rho\tau I & E^TPA_d(t) & E^TPB_w(t) & C_c^T(t) \\ A_d^T(t)PE & -\tilde{R}_1 + \tau I & 0 & C_d^T(t) \\ B_w^T(t)PE & 0 & -\gamma^2 I & D_w^T(t) \\ C_c(t) & C_d(t) & D_w(t) & -I \end{bmatrix} < 0. \tag{3.2}$$

Noting that the left hand side of (3.2) can be rewritten as

$$\begin{bmatrix} E^TPA_c + A_c^TPE + R_1 - \rho\tau I & E^TPA_d & E^TPB_w & C_c^T \\ A_d^TPE & -\tilde{R}_1 + \tau I & 0 & C_d^T \\ B_w^TPE & 0 & -\gamma^2 I & D_w^T \\ C_c & C_d & D_w & -I \end{bmatrix}$$
$$+ \begin{pmatrix} E^TPH_1 \\ 0 \\ 0 \\ H_2 \end{pmatrix} F(t)\begin{pmatrix} E_c & E_2 & E_4 & 0 \end{pmatrix}$$
$$+ \left[\begin{pmatrix} E^TPH_1 \\ 0 \\ 0 \\ H_2 \end{pmatrix} F(t)\begin{pmatrix} E_c & E_2 & E_4 & 0 \end{pmatrix}\right]^T.$$

Using Lemma 2.3, we conclude that (3.2) holds if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} E^TPA_c + A_c^TPE + R_1 - \rho\tau I & E^TPA_d & E^TPB_w & C_c^T \\ A_d^TPE & -\tilde{R}_1 + \tau I & 0 & C_d^T \\ B_w^TPE & 0 & -\gamma^2 I & D_w^T \\ C_c & C_d & D_w & -I \end{bmatrix}$$
$$+ \varepsilon\begin{pmatrix} E^TPH_1 \\ 0 \\ 0 \\ H_2 \end{pmatrix}\begin{pmatrix} H_1^TPE & 0 & 0 & H_2^T \end{pmatrix} + \frac{1}{\varepsilon}\begin{pmatrix} E_c^T \\ E_2^T \\ E_4^T \\ 0 \end{pmatrix}\begin{pmatrix} E_c & E_2 & E_4 & 0 \end{pmatrix} < 0. \tag{3.3}$$

We therefore get the following proposition.

Proposition 3.2 *For a given matrix K, if μ_0 is the solution of the following optimization problem*

$$\min_{K,\,P>0} \mu, \tag{3.4}$$

$$s.t.\ (3.3)\ \text{with}\ \gamma^2\ \text{replaced by}\ \mu, \tag{3.5}$$

then controller (2.6) robustly stabilizes system (2.1) and the closed-loop system has noise attenuation level $\sqrt{\mu_0}$.

Since (3.2) in nonlinear with respect to design parameters K, P, it can not be used to design a controller directly. To solve (3.3) using LMI toolbox, we will use an iterative algorithm. For this purpose, let's give the following equivalent forms of (3.3). Using Schur complement, (3.3) hold if and only if

$$\Phi_1 \triangleq \begin{bmatrix} J_1 & E^TPA_d+\varepsilon E_c^TE_2 & E^TPB_w+\varepsilon E_c^TE_4 & C_c^T & E^TPH_1 \\ A_d^TPE+\varepsilon E_2^TE_c & -\tilde{R}_1+\tau I+\varepsilon E_2^TE_2 & \varepsilon E_2^TE_4 & C_d^T & 0 \\ B_w^TPE+\varepsilon E_4^TE_c & \varepsilon E_4^TE_2 & -\mu I+\varepsilon E_4^TE_4 & D_w^T & 0 \\ C_c & C_d & D_w & -I & H_2 \\ H_1^TPE & 0 & 0 & H_2^T & -\varepsilon I \end{bmatrix} < 0, \tag{3.6}$$

where $J_1 = E^TPA_c + A_c^TPE + R_1 - \rho\tau I + \varepsilon E_c^TE_c$, ε and τ are positive scalars or

$$\Phi_2 = \begin{bmatrix} J_2 & E^TPA_d & E^TPB_w & C_c^T+\eta E^TPH_1H_2^T & E_c^T \\ A_d^TPE & -\tilde{R}_1+\tau I & 0 & C_d^T & E_2^T \\ B_w^TPE & 0 & -\mu I & D_w^T & E_4^T \\ C_c+\eta H_2H_1^TPE & C_d & D_w & -I+\eta H_2H_2^T & 0 \\ E_c & E_2 & E_4 & 0 & -\eta I \end{bmatrix} < 0, \tag{3.7}$$

where $J_2 = E^TPA_c + A_c^TPE + R_1 - \rho\tau I + \eta E^TPH_1H_1^TPE$, and η is a positive scalar. Since Φ_2 can be rewritten as

$$\Phi_2 = \begin{bmatrix} \tilde{J}_2 & E^TPA_d & E^TPB_w & C_c^T+\eta E^TPH_1H_2^T & E^T \\ A_d^TPE & -\tilde{R}_1+\tau I & 0 & C_d^T & E_2^T \\ B_w^TPE & 0 & -\mu I & D_w^T & E_4^T \\ C+\eta H_2H_1^TPE & C_d & D_w & -I+\eta H_2H_2^T & 0 \\ E_1 & E_2 & E_4 & 0 & -\eta I \end{bmatrix}$$

$$+\begin{pmatrix} E^TPB \\ 0 \\ 0 \\ B \\ E_3 \end{pmatrix} K\Big(I \quad 0 \quad 0 \quad 0 \quad 0\Big) + \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} K^T\Big(B^TPE \quad 0 \quad 0 \quad B^T \quad E_3^T\Big),$$

where $\tilde{J}_2 = E^TPA + A^TPE + R_1 - \rho\tau I + \eta E^TPH_1H_1^TPE$, it follows from Lemma 2.4 that (3.2) is equivalent to

$$\begin{bmatrix} J_3 & E^TPA_d & E^TPB_w & C_c^T+\eta E^TPH_1H_2^T & E^T-\varepsilon_1E^TPBE_3^T \\ A_d^TPE & -\tilde{R}_1+\tau I & 0 & C_d^T & E_2^T \\ B_w^TPE & 0 & -\mu I & D_w^T & E_4^T \\ C+\eta H_2H_1^TPE & C_d & D_w & -I+\eta H_2H_2^T & 0 \\ E_1-\varepsilon_1E_3B^TPE & E_2 & E_4 & 0 & -\eta I-\varepsilon_1E_3E_3^T \end{bmatrix} < 0, \tag{3.8}$$

where $J_3 = E^T PA + A^T PE + R_1 - \rho\tau I + \eta E^T PH_1 H_1^T PE - \varepsilon_1 E^T PBB^T PE$, and ε_1 is a positive scalar. Using Proposition 3.1 and noting that (3.2) is equivalent to (3.7), we obtain the following theorem.

Theorem 3.1 *If there exist matrix $P > 0$ positive scalars η, ε_1, τ, μ satisfying (3.8), then there exists gain matrix K such that controller (2.6) internally stabilizes system (2.1) and guarantees that the closed-loop system verifies noise attenuation level $\sqrt{\mu}$.*

This theorem shows that (3.8) provides a LMI for the existence of linear memoryless state feedback controller (2.6) that internally stabilizes system (2.1) and guarantees the closed-loop system verifies noise attenuation level $\sqrt{\mu}$. However, since the present of E the conventional method to solve LMI can not be directly used here. The following algorithm establishes an iterative algorithm to handle the controller design problem.

Algorithm 3.1 (Robust Controller Design Algorithm)

Step 1 Set an error bound $\varrho_0 > 0$ and give an initial $P_0 > 0$.

Step 2 With P given, solve the following optimization problem K and denote the optimal v by v_1,

$$\min_{\mu>0,\,\eta>0,\,\tau>0,\,K} v, \qquad s.t. \quad \Phi_2 < vI;$$

Step 3 With K obtained in Step 2, solve the following optimization problem to get P and denote the optimal performance by v_2

$$\min_{\varepsilon>0,\,\tau>0,\,P>0} v, \qquad s.t. \quad \Phi_1 < vI.$$

If $\|v_1 - v_2\| < \varrho_0$ and $v_1 < 0$, $v_2 < 0$, stop, else go to Step 2.

4 Illustrative Example

To illustrate the validness of the algorithm developed in previous section, this section gives a numerical example. Let us consider a system described by (2.1) with the following system parameters:

$$R_1 = I, \qquad \rho = 2, \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 2 & -0.1 & 0.1 & 0 \\ 0 & 0.1 & 1 & 0.1 \\ 0.1 & 0 & -1 & 0.1 \\ 0.2 & 0.1 & -0.1 & -1 \end{pmatrix}, \qquad A_d = \begin{pmatrix} 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1 & -0.1 & 0 \\ -0.1 & 0 & 0 & 0.1 \\ 0 & 0 & -0.1 & -0.1 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.3 \\ -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}, \qquad B_w = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$D_w = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & -0.1 & 0 & 0.1 \end{pmatrix}^T, \quad H_2 = -0.1,$$

$$E_1 = \begin{pmatrix} 0.1 & 0.1 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0.1 & 0 & 0 & 0.1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0.1 & -0.1 \end{pmatrix}, \qquad E_4 = \begin{pmatrix} 0.1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}, \qquad C_d = \begin{pmatrix} 0.4 & 0 & -0.1 & 0 \end{pmatrix},$$

$$D = (\, 0 \quad -0.1 \,), \qquad D_w = (\, 0.4 \quad -0.1 \,),$$

$$\beta = 0.2, \qquad \eta = 1, \qquad \tau = 0.1, \qquad \mu = 2.$$

With this set of data and choosing initial $P = 0.4 * I$, $\varepsilon_0 = 0.01$, using Algorithm 3.1 yields $K = \begin{pmatrix} 15.9056 & 3.1213 & 9.7273 & 4.2990 \\ 12.0251 & 2.5436 & 9.5698 & -0.0953 \end{pmatrix}$, then the corresponding controller (2.6) stabilizes system (2.1) with a guaranteed disturbance attanuation $\sqrt{\mu}$.

5 Conclusion

This paper dealt with the class of singular continuous-time systems with delay. Under the norm bounded uncertainties, the problems of asymptotic stability, stabilizability, H_∞ control and their robustness have been studied. Delay independent sufficient conditions provided to solve all the problems. *These conditions are in some sense restrictive.* Presently we are working on the more general delay-dependent conditions for the above problems.

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