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Global Exponential Stabilization for Several Classes of Uncertain Nonlinear Systems with Time-Varying Delay

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Abstract: In this paper, exponential stabilization for three classes of uncertain nonlinear systems with time-varying delay is investigated. A continuous feedback control is constructed for each class of systems, under which global exponential stability of the feedback-controlled system can be guaranteed. Our results are shown to be generalizations of several results reported in recent literature. A numerical example is provided to illustrate the use of our main results.

Keywords: *Global exponential stabilization; uncertain systems; time-varying delay; Lyapunov function.*

Mathematics Subject Classification (2000): 15A60, 93B52, 93C30, 93D05, 93D15.

1 Introduction

Any physical dynamic system inherently contains, more or less, some delay phenomena because energy in the system propagates with a finite speed. Typical systems with time delays include turbojet engine, microwave oscillator, control of epidemics, inferred grinding model, and population dynamics model [6, 7, 17]. It is noted that for many stable systems the introduction of arbitrarily small time delay into the loop of systems can cause instability [4, 10]. Furthermore, the system uncertainties could be present due to mathematical model errors, temperature varying, and element life. Thus, feedback control of uncertain time-delay systems is crucial for practical design of control systems; see, e.g., [2, 7–9, 13–17, 19, 20] and the references therein. We wish to point out that many systems whose dynamics contain a term that is affine-linear in control have been investigated in the past; see, e.g., [1, 3, 11, 14, 18–20]. The generalization allowing some systems whose dynamics contain a term which depends on the square of the control in addition to an affine term has been considered in [12].

Uncertain input nonlinearity considered in this paper is a general expression on uncertainties and control, that is a generalization of uncertain input nonlinearity for many time-delay systems. Moreover, we allow the dynamics of the system containing the term which depends on the control order up to $(r-1)$, where r is a positive integer greater than three.

The paper is organized as follows. In Section 2, some preliminaries are provided. Exponential stabilization for three classes of uncertain nonlinear systems are considered in Section 3 to Section 5, respectively. An illustrative example is provided in Section 6 to demonstrate the use of our main results. Finally, summary follows in Section 7.

2 Preliminaries

For convenience, we define some notation that will be used throughout this paper as follows:

R_+	– Set of all nonnegative reals.
R	– Set of all real numbers.
R^n	– n-dimensional real space.
$R^{n\times m}$	– Set of all real n by m matrices.
I	– Unit matrix.
A^{T}	– Transpose of matrix A.
$\|A\|$	– Spectral norm of matrix A.
$\|x\|$	– Euclidean norm of $x \in R^n$.
$\lambda_{\min}(P)$	– Minimal eigenvalue of symmetric matrix P.
$\lambda_{\max}(P)$	– Maximal eigenvalue of symmetric matrix P.
C	– Set of all continuous functions from $[-H,0]$ to R^n.
$\nabla_x V(t,x)$	– Gradient of smooth scalar function $V(t,x)$.
$\|a\|$	– Absolute value of real number a.
$\forall$	– Means “for every.”

Consider the following nonlinear time-delay dynamic system:

$$\dot{x}(t) = f(t, x_t), \quad \forall\, t \geq t_0 \geq 0, \tag{1}$$

$$x_{t_0}(t) = \theta(t), \quad t \in [-H,\, 0], \tag{2}$$

where $x \in R^n,\ x_t(s) = x(t+s),\ \forall\, s \in [-H,\, 0],\ H \geq 0$, with $\|x_t\|_s = \sup\limits_{-H\leq\tau\leq 0} \|x(t+\tau)\|$, and $\theta \in C$ is a given initial function. The function $f\colon R_+ \times C \to R^n$ is supposed to be completely continuous and to satisfy enough additional smoothness conditions to ensure the solution $x(t_0,\theta)(t)$ through (t_0,θ) is continuous in $(t_0,\theta,t),\ t \geq t_0 \geq 0$, in the domain of definition of the function [8].

Definition 2.1 *System* (1) is said to be *globally exponentially stable with convergence rate* $\alpha > 0$ if, for each $\theta \in C$ and $t_0 \in R_+$, we have

$$\|x(t_0,\theta)(t)\| \leq c(t_0, \|\theta\|_s)\, \exp(-\alpha(t-t_0)) \quad \text{for all} \quad t \geq t_0 \geq 0,$$

where $c(\cdot)$ is a bounded function depending on t_0 and $\|\theta\|_s$.

Lemma 2.1 *Assume there exist a sufficiently smooth function* $V(t,x)$ *and positive constants* λ_1, λ_2, λ_3, p, ε, *and* β, *with* $\beta > \lambda_3/\lambda_2$, *such that for all* $x \in R^n$, $t \ge t_0 \ge 0$,

$$\lambda_1 \|x\|^p \le V(t,x) \le \lambda_2 \|x\|^p, \tag{3}$$

and the derivative of V *along solutions of (1) satisfies*

$$\frac{dV(t,\, x(t))}{dt} = \nabla_t V(t,\, x(t)) + \nabla_x^{\mathrm{T}} V(t,\, x(t)) \cdot f(t, x_t) \le -\lambda_3 \|x(t)\|^p + \varepsilon e^{-\beta t}, \tag{4}$$

then system (1) is globally exponentially stable with guaranteed convergence rate

$$\eta = \lambda_3/(\lambda_2 p).$$

Proof See Appendix A.

Lemma 2.2 *For any nonnegative integers* i *and* j, *define* $0! = 1$, $i! = 1\times 2\times 3\times \ldots \times i$, *and* $C_j^i = i!/[j!(i-j)!]$, $i \ge j$. *Then, for any nonnegative integers* i, j, *and* r, *the following inequalities are true:*

(a) $C_{j-3}^{r-3} \ge C_{j-i}^{r-i}, \quad \forall\, 3 \le i \le j \le r,$

(b) $3\, C_{j-2}^{r-2} - 2\, C_{j-1}^{r-2} + C_j^{r-2} \ge 3\, C_{j-3}^{r-3}, \quad \forall\, 3 \le j \le r-2.$

Proof See Appendix B.

3 First Class of Systems

Consider the following uncertain system with a time-varying delay:

$$\begin{aligned} \dot{x}(t) &= F(t, x(t)) \\ &\quad + G(t, x(t), x(t-h(t)))\Delta\Psi(t, x(t), x(t-h(t)), u(t)), \quad t \ge t_0 \ge 0, \end{aligned} \tag{5}$$

$$x_{t_0}(t) = \theta(t), \quad t \in [-H,\, 0], \tag{6}$$

where $x \in R^n$, $h(t)$ is a delay argument with $0 \le h(t) \le H$, $u \in R^m$ is the input vector, and $\theta \in C$ is a given initial function. The functions F, G, and $\Delta\Psi$ (uncertain input nonlinearity) are assumed to be continuous with $F(t,0) = 0$, $t \ge t_0 \ge 0$.

Before presenting our main results, we make some assumptions as follows.

Assumption (A1) [18] There exist a sufficiently smooth function $W(t, x(t))$ and positive constants λ_1, λ_2, λ_3, and p such that, for all $x \in R^n$, $t \ge t_0 \ge 0$,

$$\lambda_1 \|x\|^p \le W(t,x) \le \lambda_2 \|x\|^p, \tag{7}$$

and the derivative of W along solutions of $\dot{x}(t) = F(t, x(t))$ satisfies

$$\frac{dW(t, x(t))}{dt} = \nabla_t W(t, x(t)) + \nabla_x^{\mathrm{T}} W(t, x(t)) \cdot F(t, x(t)) \le -\lambda_3 \|x(t)\|^p. \tag{8}$$

Assumption (A2) [12] There exist positive continuous functions $f_1(t,x,y)$, $f_2(t,x,y)$, and nonnegative continuous functions $f_3(t,x,y), \ldots, f_r(t,x,y)$ such that for all $t \geq t_0 \geq 0$, $x, y \in R^n$, and $u \in R^m$,

$$u^{\mathrm{T}} \cdot \Delta\Psi(t,x,y,u) \geq -f_1(t,x,y)\|u\| + f_2(t,x,y)\|u\|^2 - \sum_{j=3}^{r} f_j(t,x,y)\|u\|^j, \tag{9}$$

with

$$f_2^{r-1}(t,x,y) \geq \sum_{j=3}^{r} 2^{j-1} f_1^{j-2}(t,x,y) f_2^{r-j}(t,x,y) f_j(t,x,y). \tag{10}$$

Remark 3.1 For $r=3$, $r=4$, and $r=5$, inequality (10) becomes, respectively,

$$f_2^2 \geq 4 f_1 f_3, \tag{11}$$

$$f_2^3 \geq 4 f_1 f_2 f_3 + 8 f_1^2 f_4, \tag{12}$$

$$f_2^4(t,x,y) \geq 4 f_1 f_2^2 f_3 + 8 f_1^2 f_2 f_4 + 16 f_1^3 f_5. \tag{13}$$

It is interesting to note that (13) reduces to (12) by setting $f_5 = 0$ and (12) reduces to (11) by setting $f_4 = 0$. Similar statement can be made for higher r.

Theorem 3.1 *System (5) satisfying Assumptions (A1)∼(A2) is globally exponentially stabilizable with convergence rate* $\eta = \lambda_3/(\lambda_2 p)$ *under the control*

$$u(t) = -\gamma(t,\, x(t),\, x(t-h(t)))\, K(t,\, x(t),\, x(t-h(t))), \tag{14}$$

where

$$\gamma(t,\, x(t),\, x(t-h(t))) = \frac{2 f_1^2(t,\, x(t),\, x(t-h(t)))}{f_2(t,x(t),x(t-h(t)))[f_1(t,x(t),x(t-h(t)))\|K(t,x(t),x(t-h(t)))\| + \varepsilon^*(t)]}, \tag{15}$$

$$\varepsilon^*(t) = 3\exp(-\beta t), \tag{16}$$

$$K(t,x(t),x(t-h(t))) = G^{\mathrm{T}}(t,x(t),x(t-h(t)))\nabla_x W(t,x(t)), \tag{17}$$

with $\beta > \lambda_3/\lambda_2$.

Proof Let $W(t,x(t))$, satisfying (7)−(8), be a Lyapunov function candidate of the system (5) with (14)−(17). The time derivative of $W(t,x(t))$ along trajectories of the closed-loop system is given by

$$\dot{W} = \nabla_t W + \nabla_x^{\mathrm{T}} W(F + G\cdot\Delta\Psi) \leq -\lambda_3\|x\|^p + \nabla_x^{\mathrm{T}} W G \cdot \Delta\Psi. \tag{18}$$

From (9) and (14), we have

$$-\gamma K^{\mathrm{T}} \cdot \Delta\Psi \geqslant -f_1\gamma\|K\| + f_2\gamma^2\|K\|^2 - \sum_{j=3}^{r} f_j \gamma^j \|K\|^j,$$

which implies

$$K^{\mathrm{T}} \cdot \Delta\Psi \le f_1\|K\| - f_2\gamma\|K\|^2 + \sum_{j=3}^{r} f_j\gamma^{j-1}\|K\|^j. \tag{19}$$

Applying (19) to (18) with (15) – (17), we have

$$\dot W \le -\lambda_3\|x\|^p + f_1\|K\| - f_2\gamma\|K\|^2 + \sum_{j=3}^{r} f_j\gamma^{j-1}\|K\|^j.$$

In the following, the proof is made by setting $r = 3$, $r = 4$, and $r \ge 5$, respectively.

For $r = 3$, we have

$$\begin{aligned}
\dot W &\le -\lambda_3\|x\|^p + f_1\|K\| - f_2\gamma\|K\|^2 + f_3\gamma^2\|K\|^3 \\
&= -\lambda_3\|x\|^p + f_1\|K\| - \frac{2f_2f_1^2\|K\|^2}{f_2(f_1\|K\| + \varepsilon^*)} + \frac{4f_3f_1^4\|K\|^3}{f_2^2(f_1\|K\| + \varepsilon^*)^2} + (-e^{-\beta t} + e^{-\beta t}) \\
&= \frac{f_2^2(f_1\|K\| - e^{-\beta t})(f_1\|K\| + \varepsilon^*)^2 - 2f_2^2f_1^2\|K\|^2(f_1\|K\| + \varepsilon^*)}{f_2^2(f_1\|K\| + \varepsilon^*)^2} \\
&\quad + \frac{4f_3f_1^4\|K\|^3}{f_2^2(f_1\|K\| + \varepsilon^*)^2} - \lambda_3\|x\|^p + e^{-\beta t} \\
&= \frac{-\|K\|^3f_1^3(f_2^2 - 4f_1f_3) - \|K\|^2f_1^2f_2^2e^{-\beta t} + 3\|K\|f_1f_2^2e^{-2\beta t} - 9f_2^2e^{-3\beta t}}{f_2^2(f_1\|K\| + \varepsilon^*)^2} \\
&\quad - \lambda_3\|x\|^p + e^{-\beta t}.
\end{aligned} \tag{20}$$

By using the fact that $2ab \le a^2 + b^2$ for any $a,\, b \ge 0$, one has

$$3\|K\|f_1f_2^2e^{-2\beta t} = f_2^2e^{-\beta t}\left[2\, f_1\|K\|\, \frac{3e^{-\beta t}}{2}\right] \le f_2^2e^{-\beta t}\left[(f_1\|K\|)^2 + \frac{9e^{-2\beta t}}{4}\right]. \tag{21}$$

From (11), (20), and (21), we have

$$\dot W \le \frac{-\|K\|^3f_1^3(f_2^2 - 4f_1f_3) - (27/4)f_2^2e^{-3\beta t}}{f_2^2(f_1\|K\| + \varepsilon^*)^2} - \lambda_3\|x\|^p + e^{-\beta t} \le -\lambda_3\|x\|^p + e^{-\beta t}.$$

Next for $r = 4$, we have

$$\begin{aligned}
\dot W &\le -\lambda_3\|x\|^p + f_1\|K\| - f_2\gamma\|K\|^2 + f_3\gamma^2\|K\|^3 + f_4\gamma^3\|K\|^4 \\
&= -\frac{\|K\|^4f_1^4(f_2^3 - 4f_1f_2f_3 - 8f_1^2f_4) + \|K\|^3f_1^3e^{-\beta t}(4f_2^3 - 4f_1f_2f_3(3)) + 27f_2^3e^{-4\beta t}}{f_2^3(f_1\|K\| + \varepsilon^*)^3} \\
&\quad - \lambda_3\|x\|^p + e^{-\beta t}.
\end{aligned} \tag{22}$$

From (12) and (22), we have

$$\dot{W} \leq -\frac{\|K\|^4 f_1^4(f_2^3 - 4f_1f_2f_3 - 8f_1^2 f_4) + 4\|K\|^3 f_1^3 e^{-\beta t}(f_2^3 - 4f_1f_2f_3) + 27f_2^3 e^{-4\beta t}}{f_2^3(f_1\|K\| + \varepsilon^*)^3}$$
$$-\lambda_3\|x\|^p + e^{-\beta t} \leq -\lambda_3\|x\|^p + e^{-\beta t}.$$

Finally for $r \geq 5$, we have

$$\dot{W} \leq -\lambda_3\|x\|^p + f_1\|K\| - f_2\gamma\|K\|^2 + \sum_{j=3}^{r} f_j\gamma^{j-1}\|K\|^j$$
$$= -\lambda_3\|x\|^p + f_1\|K\| - \frac{2f_2f_1^2\|K\|^2}{f_2(f_1\|K\| + \varepsilon^*)} + \sum_{j=3}^{r}\frac{(2)^{j-1}f_jf_1^{2(j-1)}\|K\|^j}{f_2^{j-1}(f_1\|K\| + \varepsilon^*)^{j-1}} + (-e^{-\beta t} + e^{-\beta t})$$
$$= \frac{f_2^{r-1}(f_1\|K\| - e^{-\beta t})(f_1\|K\| + 3e^{-\beta t})^{r-1} - 2f_2^{r-1}f_1^2\|K\|^2(f_1\|K\| + 3e^{-\beta t})^{r-2}}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$
$$+ \frac{\sum_{j=3}^{r}(2)^{j-1}f_jf_1^{2(j-1)}\|K\|^jf_2^{r-j}(f_1\|K\| + 3e^{-\beta t})^{r-j}}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}} - \lambda_3\|x\|^p + e^{-\beta t}$$
$$= -\frac{\|K\|^rf_1^r\left(f_2^{r-1} - \sum_{j=3}^{r}2^{(j-1)}f_1^{j-2}f_2^{r-j}f_j\right)}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$
$$- \frac{\|K\|^{r-1}e^{-\beta t}f_1^{r-1}\left[f_2^{r-1}(3C_{r-3}^{r-2} - 2C_{r-2}^{r-2}) - \sum_{j=3}^{r-1}2^{(j-1)}f_1^{j-2}f_2^{r-j}f_j(3C_{r-1-j}^{r-j})\right]}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$
$$- \frac{\sum_{j=3}^{r-2}\|K\|^je^{-(r-j)\beta t}3^{r-j-1}f_1^j\left[f_2^{r-1}(3C_{j-2}^{r-2} - 2C_{j-1}^{r-2} + C_j^{r-2}) - \sum_{i=3}^{j}2^{(i-1)}f_1^{i-2}f_2^{r-i}f_i(3C_{j-i}^{r-i})\right]}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$
$$- \frac{\|K\|^2f_2^{r-1}f_1^23^{r-3}e^{-(r-2)\beta t}(3 - 2C_1^{r-2} + C_2^{r-2}) + \|K\|f_2^{r-1}f_1(3)^{r-2}e^{-(r-1)\beta t}(-2 + C_1^{r-2})}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$
$$- \frac{f_2^{r-2}(3)^{r-1}e^{-r\beta t}}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}} - \lambda_3\|x\|^p + e^{-\beta t}$$
$$= -\frac{\|K\|^rf_1^r\left(f_2^{r-1} - \sum_{j=3}^{r}2^{(j-1)}f_1^{j-2}f_2^{r-j}f_j\right)}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$
$$- \frac{\|K\|^{r-1}e^{-\beta t}f_1^{r-1}\left[f_2^{r-1}(3r - 8) - \sum_{j=3}^{r-1}2^{(j-1)}f_1^{j-2}f_2^{r-j}f_j(3(r-j))\right]}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$
$$- \frac{\sum_{j=3}^{r-2}\|K\|^je^{-(r-j)\beta t}3^{r-j-1}f_1^j\left[f_2^{r-1}(3C_{j-2}^{r-2} - 2C_{j-1}^{r-2} + C_j^{r-2}) - \sum_{i=3}^{j}2^{(i-1)}f_1^{i-2}f_2^{r-i}f_i(3C_{j-i}^{r-i})\right]}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}}$$

$$
\begin{aligned}
&- \frac{\|K\|^2 f_2^{r-1} f_1^2 (3)^{r-3} e^{-(r-2)\beta t}[(r-4)(r-5)] + \|K\| f_2^{r-1} f_1 (3)^{r-2} e^{-(r-1)\beta t}(r-4)}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}} \\
&- \frac{f_2^{r-2}(3)^{r-1}e^{-r\beta t}}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}} - \lambda_3\|x\|^p + e^{-\beta t}.
\end{aligned}
\tag{23}
$$

By (23), Lemma 2.2, and the fact that $3r-8 > 3(r-j),\ \forall\, 3 \le j \le r-1$, one has

$$
\begin{aligned}
\dot W \le &-\frac{\|K\|^r f_1^r \left(f_2^{r-1} - \sum\limits_{j=3}^{r} 2^{(j-1)} f_1^{j-2} f_2^{r-j} f_j \right)}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}} \\
&- (3r-8) \cdot \frac{\|K\|^{r-1} e^{-\beta t} f_1^{r-1} \left[f_2^{r-1} - \sum\limits_{j=3}^{r-1} 2^{(j-1)} f_1^{j-2} f_2^{r-j} f_j \right]}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}} \\
&- \frac{\sum\limits_{j=3}^{r-2} \|K\|^j e^{-(r-j)\beta t} 3^{r-j-1} f_1^j (3C_{j-2}^{r-2} - 2C_{j-1}^{r-2} + C_j^{r-2}) \left[f_2^{r-1} - \sum\limits_{i=3}^{j} 2^{(i-1)} f_1^{i-2} f_2^{r-i} f_i \right]}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}} \\
&- \frac{\|K\|^2 f_2^{r-1} f_1^2 (3)^{r-3} e^{-(r-2)\beta t}[(r-4)(r-5)] + \|K\| f_2^{r-1} f_1 (3)^{r-2} e^{-(r-1)\beta t}(r-4)}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}} \\
&- \frac{f_2^{r-2}(3)^{r-1}e^{-r\beta t}}{f_2^{r-1}(f_1\|K\| + 3e^{-\beta t})^{r-1}} - \lambda_3\|x\|^p + e^{-\beta t} \le -\lambda_3\|x\|^p + e^{-\beta t}.
\end{aligned}
$$

This completes our proof in view of Lemma 2.1.

Remark 3.2 In [18], exponential stability can be guaranteed via nonlinear state feedback control for system (4) with $G(t, x(t), x(t-h(t))) = \widetilde{G}(t, x(t)),\ \Delta\Psi(t, x(t), x(t-h(t)),\ u(t) = u(t) + \xi(t, x(t)),\ \|\xi(t, x(t))\| \le \rho(t, x(t)),\ p = 2$, where $\xi(\cdot,\cdot)\colon R \times R^n \to R^m$ and $\rho(\cdot,\cdot)$ is a nonnegative continuous function. In this case, there exists a positive continuous function $\widetilde{\rho}(t, x) \ge \rho(t, x),\ \forall\, t \ge t_0 \ge 0,\ x \in R^n$, such that

$$
u^{\mathrm{T}}\Delta\Psi(t, x, x(t-h), u) = \|u\|^2 + u^{\mathrm{T}}\xi(t, x) \ge -\widetilde{\rho}(t, x)\|u\| + \|u|^2.
$$

In view of (9), we have

$$
f_1 = \widetilde{\rho}(t, x), \quad f_2 = 1, \quad f_3 = 0, \quad \dots, \quad f_r = 0,
$$

and (10) is satisfied. Hence, global exponential stability can be guaranteed by memoryless controller (14) with $f_1 = \widetilde{\rho}(t, x)$ and $f_2 = 1$ by Theorem 3.1. However, our global exponential stability result holds for more general systems.

4 Second Class of Systems

Consider the following uncertain system with a time-varying delay:

$$
\dot x(t) = Ax(t) + B\Delta\Phi(t, x(t), x(t-h(t)), u(t)), \quad t \ge t_0 \ge 0, \tag{24}
$$

$$
x_{t_0}(t) = \theta(t), \quad t \in [-H, 0], \tag{25}
$$

where $x \in R^n,\ h(t)$ is a delay argument with $0 \le h(t) \le H,\ (A, B)$ is stabilizable, where $A \in R^{n\times n},\ B \in R^{n\times m}$, and $\Delta\Phi$, representing uncertain input nonlinearity, is assumed to be a continuous function satisfying the following assumption.

Assumption (A3) There exist positive continuous functions $\tilde{f}_1(t,x,y)$ and $\tilde{f}_2(t,x,y)$ such that for all $t \geq t_0 \geq 0$, $x, y \in R^n$, and $u \in R^m$,

$$u^{\mathrm{T}} \Delta\Phi(t,x,y,u) \geq -\tilde{f}_1(t,x,y)\|u\| + \tilde{f}_2(t,x,y)\|u\|^2.$$

Since the pair (A,B) is assumed to be stabilizable, we can find a matrix $M \in R^{m\times n}$ such that $\tilde{A} = A - BM$ is a Hurwitz matrix and the following Lyapunov equation

$$\tilde{A}^{\mathrm{T}} P + P\tilde{A} = -2I, \tag{26}$$

has a unique positive definite symmetric solution P.

Theorem 4.1 *System (24) satisfying Assumption (A3) is globally exponentially stabilizable with convergence rate* $\eta = \lambda_{\max}(P)^{-1}$ *under the control*

$$u(t) = -\gamma(t,\, x(t),\, x(t-h(t)))\, K(x(t)), \tag{27}$$

where

$$\begin{aligned}
&\gamma(t, x(t),\, x(t-h(t))) \\
&\quad = \frac{2[\tilde{f}_1(t,x(t),x(t-h(t))) + \|Mx(t)\|]^2}{\tilde{f}_2(t,x(t),x(t-h(t)))[(\tilde{f}_1(t,x(t),x(t-h(t))) + \|Mx(t)\|)\|K(x(t))\| + \varepsilon^*(t)]},
\end{aligned}$$

$$\varepsilon^*(t) = 3\exp(-\beta t),$$

$$K(x(t)) = 2B^{\mathrm{T}} P x(t),$$

with $\beta > 2/\lambda_{\max}(P)$, *and* P *is the solution of (26).*

Proof System (24) can be rewritten as

$$\dot{x}(t) = \tilde{A}x(t) + B[\Delta\Phi(t,\, x(t),\, x(t-h(t)),\, u(t)) + Mx(t)], \quad t \geq t_0 \geq 0.$$

In the following, we use Theorem 3.1 to prove this theorem. Choose

$$F(t,x) = \tilde{A}x, \quad G(t,\, x(t),\, x(t-h(t))) = B,$$

$$\Delta\Psi(t,\, x(t),\, x(t-h(t)),\, u) = Mx + \Delta\Phi(t,\, x(t),\, x(t-h(t)),\, u).$$

Let $W(t,x(t)) = x^{\mathrm{T}}(t)Px(t)$, then we have

$$\lambda_{\min}(P)\|x\|^2 \leq W(t,x) \leq \lambda_{\max}(P)\|x\|^2, \quad \forall\, t \geq t_0 \geq 0, \quad x \in R^n, \tag{28}$$

and the derivative of W along solutions of $\dot{x}(t) = \tilde{A}x(t)$ is given by

$$\frac{dW(t,x(t))}{dt} = x(t)^{\mathrm{T}}(\tilde{A}^{\mathrm{T}}P + P\tilde{A})x(t) = -2\|x(t)\|^2. \tag{29}$$

In view of Assumption (A3), we obtain

$$\begin{aligned}
u^{\mathrm{T}}\Delta\Psi(t,\, x,\, x(t-h(t)),\, u) &= u^{\mathrm{T}}Mx + u^{\mathrm{T}}\Delta\Phi(t,\, x,\, x(t-h(t)),\, u) \\
&\geq -[\|Mx\| + \tilde{f}_1(t,\, x(t),\, x(t-h(t)))]\cdot\|u\| + \tilde{f}_2(t,x(t),x(t-h(t)))\cdot\|u\|^2.
\end{aligned} \tag{30}$$

Comparing (28) – (30) with (7) – (9), one obtains

$$\lambda_1 = \lambda_{\min}(P), \quad \lambda_2 = \lambda_{\max}(P), \quad \lambda_3 = 2, \quad p = 2,$$
$$f_1 = \|Mx\| + \tilde{f}_1, \quad f_2 = \tilde{f}_2, \quad f_3 = 0, \quad \dots, \quad f_r = 0,$$

and (10) is satisfied. The rest follows immediately from Theorem 3.1.

In the following remarks, we show that the preceding result is a generalization of several results reported in recent literature.

Remark 4.1 In [1], practical stability can be guaranteed for system (20) with $\Delta\Phi(t,\, x,\, x(t-h(t)),\, u) = D(q)x + u + E(q)u + v(q),\ \|E(q)\| < 1,$ where A is a Hurwitz matrix, $D(\cdot)$, $E(\cdot)$, and $v(\cdot)$ depend continuously on their arguments, and the uncertainty q belongs to a compact set Q. In this case, we have

$$\begin{aligned} u^{\mathrm{T}}\Delta\Phi &= u^{\mathrm{T}}(D(q)x + u + E(q)u + v(q)) \geq -(\rho_D\|x\| + \rho_v)\cdot\|u\| + (1-\rho_E)\cdot\|u\|^2 \\ &\geq -(\rho_D\|x\| + \widetilde{\rho}_v)\cdot\|u\| + (1-\rho_E)\cdot\|u\|^2, \end{aligned}$$

where $\widetilde{\rho}_v > 0,\ \widetilde{\rho}_v \geq \rho_v = \max\limits_{q\in Q}\|v(q)\|,\ \rho_E = \max\limits_{q\in Q}\|E(q)\| < 1,$ and $\rho_D = \max\limits_{q\in Q}\|D(q)\|$. By the preceding theorem, global exponential stability can be guaranteed by memoryless controller (27) with $\tilde{f}_1 = \rho_D\|x\| + \widetilde{\rho}_v$ and $\tilde{f}_2 = 1 - \rho_E$.

Remark 4.2 In [3], global practical stability can be guaranteed for system (24) with $\Delta\Phi(t, x,\, x(t-h(t)),\, u) = u + e(t,x,u)$ and $\|e(t,x,u)\| \leq k_0 + k_1\|x\| + k_2\|u\|,$ where $k_0, k_1, k_2 \in R_+$ and $k_2 < 1$. In this case, we have

$$\begin{aligned} u^{\mathrm{T}}\Delta\Phi &= u^{\mathrm{T}}(u + e(t,x,u)) \geq -(k_0 + k_1\|x\|)\cdot\|u\| + (1-k_2)\cdot\|u\|^2 \\ &\geq -(k_3 + k_1\|x\|)\cdot\|u\| + (1-k_2)\cdot\|u\|^2, \end{aligned}$$

where $k_3 > 0$ and $k_3 \geq k_0$. By the preceding theorem, global exponential stability can be guaranteed by memoryless controller (27) with $\tilde{f}_1 = k_3 + k_1\|x\|$ and $\tilde{f}_2 = 1 - k_2$.

Remark 4.3 In [11], global exponential stability can be guaranteed by a linear control for system (24) with $\Delta\Phi = D(q)x + u + E(q)u,\ \|D(q)\| \leq \delta,\ \delta \in R_+,$ and $\delta_E = \lambda_{\min}(E^{\mathrm{T}}(q) + E(q)) > -1,\ \forall\, q \in Q,$ where $D(\cdot)$ and $E(\cdot)$ depend continuously on their arguments, and the uncertainty q belongs to a compact set Q. In this case, we have

$$\begin{aligned} u^{\mathrm{T}}\Delta\Phi &= u^{\mathrm{T}}(D(q)x + u + E(q)u) \\ &\geq -(\delta\|x\|)\cdot\|u\| + (1+\delta_E/2)\cdot\|u\|^2 \\ &\geq -(\delta_1 + \delta\|x\|)\cdot\|u\| + (1+\delta_E/2)\cdot\|u\|^2, \end{aligned}$$

where δ_1 is a any positive constant. By the preceding theorem, global exponential stability can be guaranteed by memoryless controller (27) with $\tilde{f}_1 = \delta_1 + \delta\|x\|$ and $\tilde{f}_2 = 1 + \delta_E/2$.

Remark 4.4 In [14], global practical stability can be guaranteed via a linear control for system (24) with $\Delta\Phi(t,\, x,\, x(t-h(t)),\, u) = \phi(u) + a(t,x,u),\ \gamma_1\|u\|^2 \leq u^{\mathrm{T}}\phi(u),$ and

$\|a(t,x,u)\| \le k_0 + k_1\|x\| + k_2\|u\|$, where $k_0, k_1, k_2 \in R_+$ and $k_2 < \gamma_1$. In this case, we have

$$\begin{aligned} u^{\mathrm{T}}\Delta\Phi &= u^{\mathrm{T}}(\phi(u) + a(t,x,u)) \\ &\ge -(k_0 + k_1\|x\|)\cdot\|u\| + (\gamma_1 - k_2)\cdot\|u\|^2 \\ &\ge -(k_3 + k_1\|x\|)\cdot\|u\| + (\gamma_1 - k_2)\cdot\|u\|^2, \end{aligned}$$

where $k_3 > 0$ and $k_3 \ge k_0$. By the preceding theorem, global exponential stability can be guaranteed by memoryless controller (27) with $\tilde{f}_1 = k_3 + k_1\|x\|$ and $\tilde{f}_2 = \gamma_1 - k_2$.

Remark 4.5 In [19], global exponential stability can be guaranteed by a composite control for system (24) with $\Delta\Phi = u + \xi(t,x) + \xi^h(t, x(t-h(t)))$, $\|\xi(t,x)\| \le \rho(t,x)$, and $\|\xi^h(t,x(t-h(t)))\| \le \delta\|x(t-h(t))\|$, where $\delta > 0$ and $\rho(\cdot,\cdot)\colon R_+ \times R^n \to R_+$ is a bounded continuous function. In this case, we have

$$\begin{aligned} u^{\mathrm{T}}\Delta\Phi &= u^{\mathrm{T}}(u + \xi(t,x) + \xi^h(t,x(t-h(t)))) \\ &\ge -(\rho(t,x) + \delta\|x(t-h(t))\|)\cdot\|u\| + \|u\|^2 \\ &\ge -(\rho_1(t,x) + \delta\|x(t-h(t))\|)\cdot\|u\| + \|u\|^2, \end{aligned}$$

where $\rho_1(t,x) \ge \rho(t,x)$ and $\rho_1(t,x) > 0$, $\forall\, t \ge t_0 \ge 0$, $x \in R^n$. By the preceding theorem, global exponential stability can also be guaranteed by controller (27) with $\tilde{f}_1 = \rho_1(t,x) + \delta\|x(t-h(t))\|$ and $\tilde{f}_2 = 1$.

5 Third Class of Systems

Consider the following uncertain system with a time-varying delay:

$$\begin{aligned} \dot{x}(t) = Ax(t) + A_1x(t-h(t)) + \Delta f(t, x(t), x(t-h(t))) \\ + B\Delta\Phi(t, x(t), x(t-h(t)), u(t)), \quad t \ge t_0 \ge 0, \end{aligned} \tag{31}$$

$$x_{t_0}(t) = \theta(t), \quad t \in [-H, 0], \tag{32}$$

where $x \in R^n$, $h(t)$ is a delay argument with $0 \le h(t) \le H$, $u \in R^m$ is the input vector, $\theta \in C$ is a given initial function, $A, A_1 \in R^{n\times n}$, and $B \in R^{n\times m}$ are constant matrices, (A,B) is stabilizable, $\mathrm{rank}(B) = n$, $\Delta\Phi$ is assumed to be continuous and satisfies Assumption (A3), and the mismatch uncertainty Δf is assumed to be continuous and satisfies the following assumption.

Assumption (A4) There exists a nonnegative continuous function $q(t,x,y)$ such that for all $t \ge t_0 \ge 0$ and $x, y \in R^n$,

$$\|\Delta f(t,x,y)\| \le q(t,x,y).$$

Theorem 5.1 *System (31) satisfying Assumptions (A3) and (A4) is globally exponentially stabilizable with convergence rate* $\eta = \lambda_{\max}(P)^{-1}$ *under the control*

$$u(t) = -\gamma(t,\, x(t),\, x(t-h(t)))\, K(x(t)), \tag{33}$$

where

$$\begin{gathered}
\gamma(t,\, x(t),\, x(t-h(t))) = \frac{2\hat{f}_1^2}{\tilde{f}_2(t, x(t), x(t-h(t)))[\hat{f}_1\|K(x(t))\| + \varepsilon^*(t)]}, \\
\hat{f}_1 = \|B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}A_1 x(t-h(t))\| + \|B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}\|q + \|Mx\| + \tilde{f}_1, \\
\varepsilon^*(t) = 3\exp(-\beta t), \\
K(x(t)) = 2B^{\mathrm{T}}Px(t),
\end{gathered}$$

with $\beta > 2/\lambda_{\max}(P)$, M *is a matrix such that* $\tilde{A} = A - BM$ *is Hurwitz, and* P *is the solution of (26).*

Proof Since $\mathrm{rank}(B) = n$, the matrix BB^{T} is nonsingular. System (31) can be rewritten as

$$\begin{aligned}
\dot{x}(t) = \tilde{A}x(t) &+ BB^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}[A_1 x(t-h(t)) + \Delta f(t, x(t), x(t-h(t)))] \\
&+ B[\Delta\Phi(t, x(t), x(t-h(t)), u(t)) + Mx(t)], \quad t \ge t_0 \ge 0.
\end{aligned}$$

Define

$$\begin{aligned}
\Delta\Psi(t, x(t), x(t-h(t)), u) = B^{\mathrm{T}}&(BB^{\mathrm{T}})^{-1}[A_1 x(t-h(t)) + \Delta f(t,\, x(t),\, x(t-h(t)))] \\
&+ Mx + \Delta\Phi(t,\, x(t),\, x(t-h(t)),\, u),
\end{aligned}$$

then we have

$$\begin{aligned}
u^{\mathrm{T}}\Delta\Psi(t,\, x,\, x(t-h(t)),\, u) \ge -\big[&\|B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}A_1 x(t-h(t))\| \\
&+ \|B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}\|\, q + \|Mx\| + \tilde{f}_1\big]\|u\| + \tilde{f}_2\|u\|^2.
\end{aligned}$$

Hence the result follows in view of Theorem 4.1.

Remark 5.1 In [13], global exponential stabilization has been considered for a class of uncertain systems with multiple time-varying delays and input deadzone nonlinearities. If they consider only single time-varying delay, their system can be put in the form of (26) with $q(t, x, y) = a_0\|x\| + a_1\|y\|$, $\Delta\Phi(t, x, y, u) = \Delta\Phi_3(t, x, y) + \phi(u)$, $\|\Delta\Phi_3(t, x, y)\| \le f(t, x, y)$, where $y = x(t-h(t))$, $a_0, a_1 \in R_+$, $\Delta\Phi_3(\cdot)$ and $f(\cdot)$ depend continuously on their arguments, $\phi(u) = [\phi_1(u_1), \dots, \phi_m(u_m)]^{\mathrm{T}}$ with each $\phi_i(u_i) \in D(u_i, d_1, d_2)$ representing the input deadzone nonlinearity, and $D(u_i, d_1, d_2)$ is defined in [13] with $d_1 \ge 0$, $d_2 > 0$. In this case, we have

$$\begin{aligned}
u^{\mathrm{T}}\Delta\Phi = u^{\mathrm{T}}[\Delta\Phi_3 + \phi(u) - d_2 u + d_2 u] &\ge -\|u\| \cdot [\|\Delta\Phi_3\| + \|\phi(u) - d_2 u\|] + d_2\|u\|^2 \\
\ge -\|u\| \cdot [f + m d_1 d_2] + d_2\|u\|^2 &\ge -\|u\| \cdot [f + m\tilde{d}_1 d_2] + d_2\|u\|^2,
\end{aligned}$$

where $\tilde{d}_1 > 0$, $\tilde{d}_1 \ge d_1$. By the preceding theorem, global exponential stability can also be guaranteed by controller (33) with $\tilde{f}_1 = f + m\tilde{d}_1 d_2$ and $\tilde{f}_2 = d_2$.

6 Example

Consider the following uncertain system with a time-varying delay:

$$\begin{aligned}\dot{x}(t) = &\begin{bmatrix} -2x_1 + 2x_1x_2^2 + 2x_1x_2\sqrt{|x_1x_2|} \\ -2x_2 - x_1^2x_2 - x_1^2\sqrt{|x_1x_2|} \end{bmatrix} \\ &+ \begin{bmatrix} x_2(t-h(t)) \\ -x_1^2 \end{bmatrix} [a(t) + (b(t) + c(t)|x_1(t-h(t))|)u + d(t)u^3],\end{aligned} \tag{34}$$

where $u \in R,\ x = [x_1, x_2]^{\mathrm{T}} \in R^2,\ h(t) = 2 + \cos(2t),\ -1 \le a(t) \le 1,\ 4 \le b(t) \le 4.5,$ $1 \le c(t) \le 2,\ -2 \le d(t) \le 2$ for all $t \ge t_0 \ge 0$. Comparing (34) with (5), one has

$$\begin{aligned} F(t, x(t)) &= \begin{bmatrix} -2x_1 + 2x_1x_2^2 + 2x_1x_2\sqrt{|x_1x_2|} \\ -2x_2 - x_1^2x_2 - x_1^2\sqrt{|x_1x_2|} \end{bmatrix}, \\ G(t,\, x(t),\, x(t-h(t))) &= \begin{bmatrix} x_2(t-h(t)) \\ -x_1^2 \end{bmatrix}, \\ \Delta\Psi(t, x(t), x(t-h(t)), u) &= a(t) + (b(t) + c(t)|x_1(t-h(t))|)u + d(t)u^3. \end{aligned}$$

Choose a simple quadratic functional

$$W(t,x) = x^{\mathrm{T}} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} x.$$

Then (7) and (8) are evidently satisfied with $\lambda_1 = 2,\ \lambda_2 = 4,\ p = 2,$ and $\lambda_3 = 8.$ In view of (9), we have

$$\begin{aligned} u^{\mathrm{T}}\Delta\Psi(t, x, x(t-h(t)), u) &= a(t)u + (b(t) + c(t)|x_1(t-h(t))|)u^2 + d(t)u^4 \\ &\ge -|u| + (4 + |x_1(t-h(t))|)|u|^2 - 2|u|^4. \end{aligned}$$

This suggests that in (9) we choose $f_1 = 1,\ f_2 = 4 + |x_1(t-h(t))|,\ f_3 = 0,$ and $f_4 = 2.$

It is easy to show that (10) is satisfied with $r = 4.$ According to (16) with $\beta = 2.1 > \lambda_3/\lambda_2 = 2,$ we have

$$\varepsilon^*(t) = 3\exp(-2.1t).$$

By (17) and (15), we obtain

$$\begin{aligned} K(t,\, x(t),\, x(t-h(t))) &= 4x_1(t)x_2(t-h(t)) - 8x_1^2(t)x_2(t), \\ \gamma(t, x, x(t-h(t))) &= \frac{2}{(4 + |x_1(t-h(t))|)(|K(t, x(t), x(t-h(t)))| + \varepsilon^*(t))}. \end{aligned}$$

Finally, owing to (14), it can be readily obtained that

$$u = -\gamma(t, x,\, x(t-h(t)))(4x_1(t)x_2(t-h(t)) - 8x_1^2x_2). \tag{35}$$

By Theorem 3.1, we conclude that system (34) with control (35) is globally exponentially stable with guaranteed convergence rate $\eta = 1.$ With, e.g., $a(t) = 1,\ b(t) = 4,$ $c(t) = 1,\ d(t) = 2,$ and $x_1(t) = 4,\ x_2(t) = -2,\ \forall\, t \in [-3, 0],$ state trajectories of the

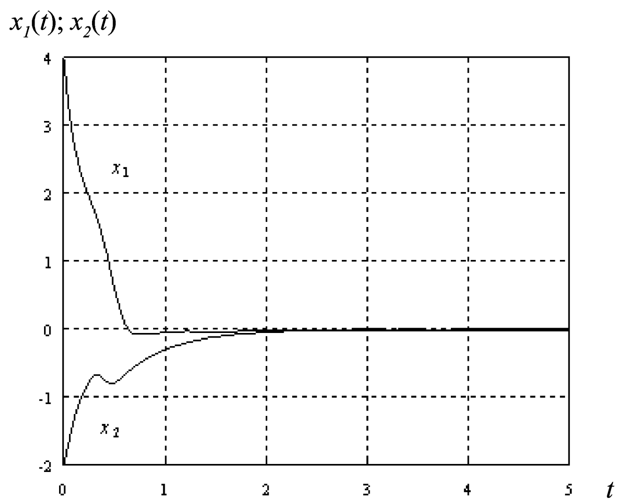

Figure 6.1. State trajectories of feedback-controlled system of (28).

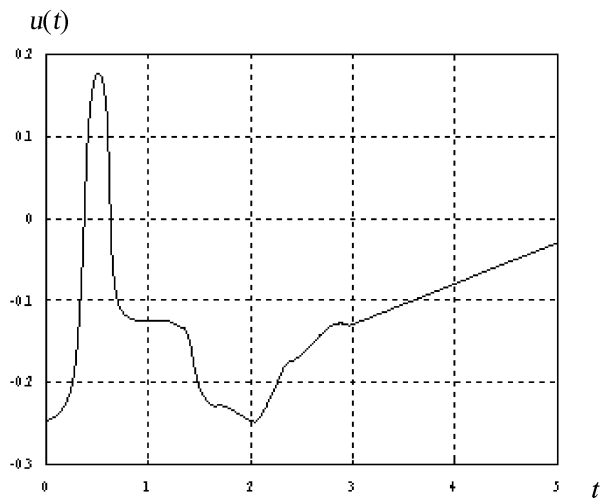

Figure 6.2. Typical control signal for system (28).

feedback-controlled system and control signal are depicted in Figure 6.1 and Figure 6.2, respectively.

7 Summary

In this paper, exponential stabilization for three classes of uncertain nonlinear systems with time-varying delay has been considered. A continuous state feedback control has been proposed in each case for exponential stability of feedback-controlled systems. Guaranteed convergence rate has also been provided. Our results have also been shown to be generalizations of several results reported in recent literature. Finally, a numerical example has been provided to illustrate the use of our main results. It is interesting to consider the problem of exponential stabilization for more general uncertain systems with time-varying delay.

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Appendix A

Proof of Lemma 2.1 Let

$$Q(t,x) = V(t,x)\exp(\lambda_3 t/\lambda_2). \tag{B1}$$

From (3), (4) and (B1), we have

$$\begin{aligned}\frac{d\,Q(t,x)}{dt} &= \frac{d\,V(t,x)}{dt}\exp(\lambda_3 t/\lambda_2) + \lambda_3 Q/\lambda_2 \\ &\le [(-\lambda_3/\lambda_2)V + \varepsilon\exp(-\beta t)]\exp(\lambda_3 t/\lambda_2) + \lambda_3 Q/\lambda_2 \\ &= \varepsilon\exp[-(\beta-\lambda_3/\lambda_2)t].\end{aligned} \tag{B2}$$

Set $\delta = \beta - \lambda_3/\lambda_2 > 0$. Integrating both sides of (B2), we have, for all $t \ge t_0 \ge 0$,

$$\begin{aligned}Q(t,x(t)) - Q(t_0,x(t_0)) &\le -\varepsilon\delta^{-1}[\exp(-\delta t) - \exp(-\delta t_0)] \\ &= \varepsilon\delta^{-1}[\exp(-\delta t_0) - \exp(-\delta t)] \le \varepsilon\delta^{-1}\exp(-\delta t_0) \le \varepsilon\delta^{-1} = \varepsilon(\beta-\lambda_3/\lambda_2)^{-1}.\end{aligned}$$

This implies that, for all $t \ge t_0 \ge 0$,

$$\begin{aligned}Q(t,x(t)) &\le Q(t_0,x(t_0)) + \varepsilon(\beta-\lambda_3/\lambda_2)^{-1} \\ &= V(t_0,x(t_0))\,\exp(\lambda_3 t_0/\lambda_2) + \varepsilon(\beta-\lambda_3/\lambda_2)^{-1} \\ &\le \lambda_2\|\theta\|_s^p\exp(\lambda_3 t_0/\lambda_2) + \varepsilon(\beta-\lambda_3/\lambda_2)^{-1} = a(t_0,\|\theta\|_s).\end{aligned} \tag{B3}$$

From (3), (B1), and (B3), we have, for all $t \ge t_0 \ge 0$,

$$\begin{aligned}\|x(t_0,\theta)(t)\| &\le [(1/\lambda_1)\,V(t,x(t))]^{1/p} = [(1/\lambda_1)\exp(-\lambda_3 t/\lambda_2)\cdot Q(t,x(t))]^{1/p} \\ &\le [(a/\lambda_1)\exp(-\lambda_3 t/\lambda_2)]^{1/p} = c(t_0,\|\theta\|_s)\,\exp(-\eta t) \\ &\le c(t_0,\|\theta\|_s)\,\exp[-\eta(t-t_0)],\end{aligned}$$

where $c(t_0,\|\theta\|_s) = [a(t_0,\|\theta\|_s)/\lambda_1]^{1/p}$ and $\eta = \lambda_3/(\lambda_2 p) > 0$.

This completes our proof.

Appendix B

Proof of Lemma 2.2 For any integers i, j, and r such that $3 \le i \le j < r$, one has

$$[(r-3)\times(r-4)\times\ldots\times(j-3+1)] \ge [(r-i)\times(r-i-1)\times\ldots\times(j-i+1)].$$

This implies

$$C_{j-3}^{r-3} = \frac{(r-3)!}{(j-3)!(r-j)!} \geq \frac{(r-i)!}{(j-i)!(r-j)!} = C_{j-i}^{r-i}, \quad \forall\, 3 \leq i \leq j < r,$$

and

$$C_{j-3}^{r-3} = 1 = C_{j-i}^{r-i}, \quad \forall\, 3 \leq i \leq j = r.$$

Hence statement (a) is true. Now for any integers j, r such that $r \geq 5$ and $3 \leq j \leq r-2$, one has

$$\begin{aligned}
&3\, C_{j-2}^{r-2} - 2\, C_{j-1}^{r-2} + C_{j}^{r-2} - 3\, C_{j-3}^{r-3} \\
&\quad = \frac{(r-2)!}{j!(r-j)!}\left[3j(j-1) - 2j(r-j) + (r-j)(r-j-1) - 3\frac{j(j-1)(j-2)}{(r-2)}\right] \\
&\quad = \frac{(r-2)!}{j!(r-j)!(r-2)}[r^3 + r^2(-4j-3) + r(6j^2+6j+2) - (3j^3+3j^2+2j)] \\
&\quad = \frac{(r-2)!}{j!(r-j)!(r-2)}[-3j^3 + (6r-3)j^2 - (4r^2-6r+2)j + (r^3-3r^2+2r)].
\end{aligned}$$

For any given $r \geq 5$, consider the following continuous function

$$g(y) = -3y^3 + (6r-3)y^2 - (4r^2-6r+2)y + (r^3-3r^2+2r), \quad y \in [3,\, r-2].$$

The derivative of $g(\cdot)$ is given by

$$\frac{d}{dy}g(y) = -9y^2 + (12r-6)y - (4r^2-6r+2).$$

Furthermore, the roots of the equation $\dot{g}(y) = 0$ is given by

$$a = \frac{2r-1-\sqrt{2r-1}}{3}, \quad b = \frac{2r-1+\sqrt{2r-1}}{3}.$$

With given $r \geq 5$, define

$$\begin{aligned}
g_1(r) &= g(a) = \frac{1}{9}\left(r^3 - 3r^2 + 4 - 2(2r-1)\sqrt{2r+1}\right), \\
g_2(r) &= g(b) = \frac{1}{9}\left(r^3 - 3r^2 + 4 + 2(2r-1)\sqrt{2r+1}\right), \\
g_3(r) &= g(3) = r^3 - 15r^2 + 74r - 114, \\
g_4(r) &= g(r-2) = 2r^2 - 12r + 16 = 2(r-2)(r-4).
\end{aligned}$$

Clearly we have

$$\begin{gathered}
g_1(5) = 0, \\
\frac{d}{dr}g_1(r) = \frac{1}{9}\left[3r^2 - 6r - 6\sqrt{2r-1}\right] > \frac{1}{9}\left[3r(r-5) + (9-6\sqrt{2})r\right] > 0, \quad \forall\, r \geq 5, \\
g_2(r) \geq g_1(r) \geq g_1(5) = 0, \quad \forall\, r \geq 5, \\
g_4(r) > 0, \quad \forall\, r \geq 5.
\end{gathered}$$

Moreover, by Sturm's theorem [5], it is easy to show that $g_3(r) > 0$, $\forall\, r \geq 5$. Consequently, $g(y) \geq 0$ for all $y \in [3,\, r-2]$ and for each $r \geq 5$. This completes the proof of statement (b).