Volume 3, Number 2, 2003 ISSN 1562-8353

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

Academic Periodical Press

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

ISSN 1562-8353 Printed in Ukraine

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys
Published since 2001

Volume 3 Number 2 2003

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Founded by A.A.Martynyuk in 2001.
Registered in Ukraine Number: KB №5267 / 04.07.2001.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Nonlinear Dynamics and Systems Theory (ISSN 1562-8353) is an international journal published under the auspices of the S.P.Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and the Laboratory for Industrial and Aplied Mathematics (LIAM) at York University (Toronto, Canada). It is aimed at publishing high quality original scientific papers and surveys in area of nonlinear dynamics and systems theory and technical reports on solving practical problems. The scope of the journal is very broad covering:

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PUBLICATION AND SUBSCRIPTION INFORMATION

The Nonlinear Dynamics and Systems Theory (ISSN 1562-8353) is published two times per year in 2003.

Base list subscription price per volume: US$99.00. This price is available only to individuals whose library subscribes to the journal OR who warrant that the Journal is for their own use and provide a home address for mailing. Separate rates apply to academic and corporate/ government institutions. Postage and handling charges are extra. Our charge includes postage, packing, handling and airmail delivery of all issues.

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ABSTRACTING INFORMATION

Nonlinear Dynamics and Systems Theory is currently abstracted/indexed by Zentralblatt MATH and Mathematical Reviews.

Nonlinear Dynamics and Systems Theory, **3**(2) (2003) 119–137

Stochastic Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control of Time-Varying Delay Systems

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Received: October 18, 2002; Revised: July 7, 2003

Abstract: This paper deals with the class of continuous-time linear systems with Markovian jumps and time-delay. The delay in the system dynamics is assumed to be time-varying. Under norm-bounded uncertainties and based on the Lyapunov method, a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller that minimizes the $\mathcal{H}_2$ performance measure when satisfying a prescribed $\mathcal{H}_\infty$ norm bound on the closed-loop system is proposed. LMI-based sufficient conditions for the existence of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller and the upper bound of the performance measure are developed.

Keywords: *Jump linear system; linear matrix inequality; stochastic stability; stochastic stabilizability; norm bounded uncertainty; state feedback.*

Mathematics Subject Classification (2000): 93B36, 93C05, 93E03; 93E15.

1 Introduction

During the last decades the state feedback control that meets desired performance and/or robustness specifications has attracted a lot of researchers from the control community and different types of controllers were proposed. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state feedback controller belongs to this class of controllers and it consists of determining a state feedback gain that achieves a certain nominal (suboptimal) performance measure subject to a robustness constraint. This feedback controller satisfying simultaneously the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ specifications is interesting since it gives robust stability and nominal performance.

Bernstein and Haddad [1] were the first to introduc the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem. Their approach consists of minimizing an auxiliary cost function subject to the $\mathcal{H}_\infty$ norm constraint and this cost provides an upper bound on the $\mathcal{H}_2$ norm. The work of Berstein and Haddad has been extended to other mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem (see for instance the work in [2, 3]). For other related works on the design of $\mathcal{H}_2/\mathcal{H}_\infty$ controllers by state feedback or output feedback, we refer the reader to Haddad, *et al.* [4], Limbeer, *et al.* [5], Mustapha [6], Rotea and Khargonekar [7] and Saberi, *et al.* [8] and Leibfritz [14]. For results using the LMI formalism we quote the works of Geromel, *et al.* [9], Giusto, *et al.* [10], Kaminer, *et al.* [11], Khargonekar and Rotea [12], and Rotea and Khargonekar [13].

For the time-delay system there exists only one reference that deals with the robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$. This work was done by Kim [15]. The paper considers the norm bounded uncertainties. The time was considered to be time-varying. Kim developed some LMI-based sufficient conditions that solve the robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for uncertain linear systems with time-delay.

As it was mentioned by different papers reported in the literature, there exist some plants that can not be modelled by deterministic time invariant model as it is the case in the work of Kim due maybe to abrupt changes in the dynamics for instance or to any equivalent phenomena that makes the dynamics switches instantaneously and randomly between some finite number of models. This behavior was shown to be adequately represented by the class of Markovian jumping parameters that has recently attracted a lot of researchers due to its power to model different practical situations that the standard time-invariant linear model doesn't do. For more details of this class of systems we refer the reader to Mariton [16] and the references therein. For the class of systems with time-delay and all the connected works we refer the reader to Boukas and Liu [17].

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for the class of linear systems with Markovian Jumping parameters was studied by Costa and Marques for the discrete-time case [18] and Aliyu and Boukas [19] for the continuous-time case. In these references, the given results are not in LMI-based. The problem of $\mathcal{H}_\infty$ control of the class of Markovian jumping parameters systems with time-delay has been tackled by some authors among them we quote the works of [17, 20, 21].

To the best of our knowledge, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of the class of systems we are considering in this paper has never been studied. The extension of the results on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ to the class of Markovian jumping parameters is of great interest for the control community due to the importance of this class of systems in practice. The problem we are addressing here consists of determining a mean-square stabilizing controller that minimizes the upper bound of the $\mathcal{H}_2$ performance measure under the restriction that the $\mathcal{H}_\infty$ performance measure is less than a prescribed value $\gamma > 0$ for all $\omega \in \mathcal{L}_2[0, \infty)$. We are interested by LMI-based conditions that can be easily solved using the existing LMI tools. In this paper we will address the design of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller with or without uncertainties in the dynamics of the class of Markovian jumping parameters with time-varying delay.

The rest of this paper is organized as follows. In Section 2, the problem is stated and the goal of the paper is presented. In Section 3, the main results are given and they include results on stochastic stabilizability and its robustness. A memoryless controller is used in this paper and a design algorithm in terms of the solutions of linear matrix inequalities is proposed to synthesize the controller gains we are using.

Notation. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "$^\top$" denotes matrix transposition and the notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrices with compatible dimensions. Y is a constant matrix associated with the controller. $\mathbb{E}\{\cdot\}$ denotes the expectation operator with respective to some probability measure $\mathcal{P}$. L_2 is the space of integral vector over $[0, \infty)$. $\|\cdot\|$ will refer to the Euclidean vector norm whereas $\|\cdot\|$ denotes the L_2-norm over $[0,\infty)$ defined as $\|f\|^2 = \mathbb{E}\Big[\int\limits_0^\infty f^\top(t)f(t)\,dt\Big]$.

2 Problem Statement

Consider a hybrid linear continuous-time system with N modes, i.e., $\mathbf{S} = \{1,2,\cdots,N\}$ and assume that the mode switching is governed by a continuous-time Markov process $\{r_t, t \ge 0\}$ taking values in the state space $\mathbf{S}$ and having the following infinitesimal generator:

$$\Lambda = (\lambda_{ij}), \quad i,j \in \mathbf{S},$$

where $\lambda_{ij} \ge 0$, $\forall\, j \ne i$, $\lambda_{ii} = -\sum\limits_{j\ne i} \lambda_{ij}$.

The mode transition probabilities are described as follows:

$$P[r_{t+\Delta} = j \mid r_t = i] = \begin{cases} \lambda_{ij}\Delta + o(\Delta), & j \ne i, \\ 1 + \lambda_{ii}\Delta + o(\Delta), & j = i, \end{cases} \tag{1}$$

where $\lim\limits_{\Delta\to 0} o(\Delta)/\Delta = 0$.

Let $x(t) \in \mathbb{R}^n$ be the physical state of the system, which satisfies the following dynamics:

$$\begin{cases} \dot{x}(t) = A(r_t,t)x(t) + A_1(r_t,t)x(t-h(t)) + B(r_t,t)u(t) + B_1(r_t)\omega(t), \\ x(s) = \phi(s), -\tau \le s \le 0, \\ z_1(t) = C_1(r_t)x(t) + D_1(r_t)u(t), \\ z_2(t) = C_2(r_t)x(t) + D_2(r_t)u(t), \end{cases} \tag{2}$$

where $u(t) \in \mathbb{R}^m$ is the control input system, $\omega(t) \in \mathbb{R}^l$ is the disturbance to be rejected and/or reference to be tracked, which we assume to belong to $L_2[0,\infty)$, $z_i(t) \in \mathbb{R}^p$, $i = 1,2$ is the controlled (regulated) signal, $A(r_t,t) = A(r_t) + D_A(r_t)F_1(r_t,t)E_A(r_t) \in \mathbb{R}^{n\times n}$, $A_1(r_t,t) = A_1(r_t) + D_{A1}(r_t)F_2(r_t,t)E_{A1}(r_t) \in \mathbb{R}^{n\times n}$, and $B(r_t,t) = B(r_t) + D_B(r_t)F_3(r_t,t)E_B(r_t) \in \mathbb{R}^{n\times m}$ with $A(r_t)$, $A_1(r_t)$, $B(r_t)$, $B_1(r_t)$, $D_A(r_t)$, $D_{A1}(r_t)$, $D_B(r_t)$, $E_A(r_t)$, $E_{A1}(r_t)$, and $E_B(r_t)$, are known real matrices with appropriate dimensions for each $r_t \in \mathbf{S}$, and $F_k(r_t)$, $k = 1,2,3$ are unknown real time-varying matrices with appropriate dimensions satisfying the following:

$$F_k^\top(r_t,t)F_k(r_t,t) \le I, \quad \forall\, r_t \in \mathbf{S}, \tag{3}$$

$h(t) > 0$ represents the system delay, that satisfies $0 \le h(t) \le \tau$, $\dot{h}(t) \le \beta < 1$, and $\phi(t)$ is a smooth vector-valued initial function in $[-\tau, 0]$.

The initial condition of the system is specified as $(r_0, \phi(\cdot))$ with r_0 is the initial mode and $\phi(.)$ is the initial functional such that

$$x(s) = \phi(s) \in L_2[-\tau,0] \stackrel{\Delta}{=} \{f(\cdot) | \int\limits_0^\infty f^\top(t)f(t)\,dt < \infty\}.$$

Remark 2.1 The uncertainties that satisfies the conditions (3) are referred to be admissible. The uncertainties we are considering here are time and mode system dependent only. The results we are developing here will remain valid for systems with uncertainties that may depend on time, modes and states systems.

For system (2) with $u(.) \stackrel{\Delta}{=} 0$ for $t \geq 0$, we have the following definitions:

Definition 2.1 *System* (2) with $u(.) \stackrel{\Delta}{=} 0$, $\forall\, t \geq 0$ and all the uncertainties equal to zero, is said to be

(i) *stochastically stable* (SS) if there exists a positive constant $T(r_0, \phi(\cdot))$ such that the following holds for any initial condition $(r_0, \phi(.))$:

$$\mathbb{E}\left[\int_0^\infty \|x(t)\|^2\, dt \,\middle|\, r_0,\ x(s) = \phi(s),\ s \in [-\tau, 0]\right] \leq T(r_0, \phi(\cdot)); \tag{4}$$

(ii) *mean square stable* (MSS) if the following holds for any initial condition $(r_0, \phi(.))$:

$$\lim_{t\to\infty} \mathbb{E}\|x(t)\|^2 = 0;$$

(iii) *mean exponentially stable* (MES) if there exist constants $\alpha(r_0, \phi(\cdot)) > 0$, $\beta > 0$ such that the following holds for any initial condition $(r_0, \phi(.))$:

$$\mathbb{E}[\|x(t)\|^2] \leq \alpha(r_0, \phi(\cdot)) e^{-\beta t}. \tag{5}$$

Obviously, MES implies MSS and SS.

Definition 2.2 *System* (2) with $u(.) \stackrel{\Delta}{=} 0$ for $t \geq 0$, is said to be

(i) *robustly stochastically stable* (RSS) if there exists a positive constant $T(r_0, \phi(\cdot))$ such that (4) holds for any initial condition $(r_0, \phi(.))$ and for all admissible uncertainties;

(ii) *robustly mean exponentially stable* (RMES) if there exist constants $\alpha(r_0, \phi(\cdot)) > 0$, $\beta > 0$ such that (5) holds for any initial condition $(r_0, \phi(.))$ and for all admissible uncertainties.

Obviously, we can show that RMES implies RSS.

In the rest of this paper, we will be interested by the design of a state feedback control law in the following form:

$$u(t) = K(r_t)x(t), \tag{6}$$

where $x(t)$ is the system state, and $K(i)$, $i \in \mathbf{S}$ is a constant gain matrix that has to be determined and which constitutes one of our main goal in this paper.

In the rest of this paper, we will assume that we have complete access to the state vector, $x(t)$, and to the mode, r_t at nay time $t \geq 0$.

Definition 2.3 *System* (2) with all the uncertainties equal to zero, is said to be *stabilizable in the stochastic sense* if there exists a control law of the form (6) such that the closed-loop system is stochastically stable for any initial condition $(r_0, \phi(.))$.

Definition 2.4 *System* (2) is said to be *robustly stabilizable in the stochastic sense* if there exists a state feedback controller of the form (6) such that the closed-loop system is robustly stochastically stable for any initial condition $(r_0, \phi(.))$ and for all admissible uncertainties.

Remark 2.2 Notice that the stability in each mode doesn't imply the stochastic stability of the global system. It is the same for the stabilization problem. The stability and the stabilization problems of the class of system we are considering is not a trivial one and more care should be taken when working with this class of systems.

The $\mathcal{H}_2$ performance and $\mathcal{H}_\infty$ performance measures used in the rest of this paper are defined as follows:

$$J_{\mathcal{H}_2} = \mathbb{E}\left[\int_0^\infty z_1^\top(t) z_1(t)\, dt\right]: \ \mathcal{H}_2 \text{ performance measure,} \tag{7}$$

$$J_{H_\infty} = \mathbb{E}\left[\int_0^\infty z_2^\top(t) z_2(t) - \gamma^2 \omega^\top(t)\omega(t)\, dt\right]: \ \mathcal{H}_\infty \text{ performance measure.} \tag{8}$$

The goal of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control can be summarized as follows: Given the dynamical system (2) find a controller (6) that achieves the minimization of $\mathcal{H}_2$ performance measure and satisfying $\mathcal{H}_\infty$ norm bound within γ (a given real positive constant) for all $\omega(t) \in \mathcal{L}_2[0,\infty)$. In other words, the aim of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control is to minimize the output energy of $z_1(t)$ and at the same time satisfy the prescribed $\mathcal{H}_\infty$ norm bound of the closed-loop system from $\omega(t)$ to $z_2(t)$.

Plugging the controller (6) in the dynamics (2) we get:

$$\begin{cases} \dot{x}(t) = A_K(r_t,t)x(t) + A_1(r_t,t)x(t-h(t)) + B_1(r_t,t)\omega(t), \\ x(s) = \phi(s), \quad -\tau \le s \le 0, \\ z_1(t) = C_{1K}(r_t)x(t), \\ z_2(t) = C_{2K}(r_t)x(t), \end{cases} \tag{9}$$

where $A_K(r_t,t) = A(r_t,t) + B(r_t,t)K(r_t)$, $C_{1K}(r_t) = C_1(r_t) + D_1(r_t)K(r_t)$ and $C_{2K}(r_t) = C_{2K}(r_t) + D_2(r_t)K(r_t)$.

Let us now give the following lemmas that we will use extensively in proving our results in the rest of this paper. The proofs of the results of these lemmas can be found in Boukas and Liu [17] or any equivalent reference.

Lemma 2.1 *Let Y be a given symmetric and positive-definite matrix, $x(t)$ and $y(t)$ be two given vectors of appropriate dimensions, and $F(t)$ a matrix with appropriate dimension satisfying $F^\top(t)F(t) \le I$. Then, for any $\epsilon > 0$ we have:*

$$pm2x^\top(t)F(t)y(t) \le \epsilon x^\top(t)Yx(t) + \epsilon^{-1}y^\top(t)Y^{-1}y(t), \quad \forall\, r_t \in \mathbf{S}.$$

Lemma 2.2 *Let A, D, F, E be real matrices of appropriate dimensions with $\|F\| \le 1$. Then, we have*

(i) *for any matrix $P > 0$ and scalar $\varepsilon > 0$ satisfying $\varepsilon I - EPE^\top > 0$,*

$$(A+DFE)P(A+DFE)^\top \le APA^\top + APE^\top(\varepsilon I - EPE^\top)^{-1}EPA^\top + \varepsilon DD^\top; \tag{10}$$

(ii) *for any matrix $P > 0$ and scalar $\varepsilon > 0$ satisfying $P - \varepsilon DD^\top > 0$,*

$$(A+DFE)^\top P^{-1}(A+DFE) \le A^\top(P - \varepsilon DD^\top)^{-1}A + \frac{1}{\varepsilon}E^\top E. \tag{11}$$

Lemma 2.3 *The linear matrix inequality*

$$\begin{bmatrix} H & S^\top \\ S & R \end{bmatrix} > 0$$

is equivalent to

$$R > 0, \quad H - S^\top R^{-1} S > 0,$$

where $H = H^\top$, $R = R^\top$ and S is a constant matrix.

3 Main Results

The main goal of this paper is to develop an LMI-based design procedure for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller for the class of systems we are considering. The rest of this section will treat the nominal system first and then consider the case of uncertain systems with norm bounded uncertainties. In both cases, we will establish LMI-based conditions for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller design.

3.1 Nominal system

Let us now assume that the uncertainties in the dynamics (2) are equal to zero for all time and for all modes. In this case, the previous closed-loop dynamics becomes:

$$\begin{cases} \dot{x}(t) = A_K(r_t)x(t) + A_1(r_t)x(t-h(t)) + B_1(r_t)\omega(t), \\ x(s) = \phi(s), \quad -\tau \le s \le 0, \\ z_1(t) = C_{1K}(r_t)x(t), \\ z_2(t) = C_{2K}(r_t)x(t), \end{cases} \tag{12}$$

where $A_K(r_t) = A(r_t) + B(r_t)K(r_t)$, $C_{1K}(r_t) = C_1(r_t) + D_1(r_t)K(r_t)$ and $C_{2K}(r_t) = C_{2K}(r_t) + D_2(r_t)K(r_t)$.

When the external disturbance $\omega(t)$ is equal to zero for all $t \ge 0$, the following theorem gives the conditions that controller (6) should satisfy to stabilize the class of systems under consideration.

Theorem 3.1 *Let the disturbance input be equal to zero, i.e. $\omega(t) = 0$ for $t \ge 0$. The controller (6) is an $\mathcal{H}_2$ optimal controller satisfying the minimization of the $\mathcal{H}_2$ performance measure (7) if there exist symmetric and positive-definite matrices $P = (P(1), \dots, P(N))$, Q and a controller gain $K = (K(1), \dots, K(N))$ that the following holds for every mode $i \in \mathbf{S}$:*

$$\Theta(i) \triangleq \begin{bmatrix} J(i) & P(i)A_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q \end{bmatrix} < 0 \tag{13}$$

with $J(i) = A_K^\top(i)P(i) + P(i)A_K(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) + Q + C_{1K}^\top(r_t)C_{1K}(r_t)$. The $\mathcal{H}_2$ performance measure is bounded by a positive scalar, i.e.:

$$J_{H_2} \le J^\star \triangleq \left[x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds \right]^{\frac{1}{2}}. \tag{14}$$

Proof Let $\mathbb{C}[-\tau, 0]$ be a space of continuous functions on the interval $[-\tau, 0]$ and for any $x(t)$, $t \in \mathbb{C}[-\tau, 0]$, define $\|x\| = \sup_{-\tau \le s \le 0} \|x(s)\|$. Obviously, the evolution of $x(t)$ depends on $x(s)$, $t-\tau \le s \le t$, which means that $\{(x(t), r_t), t \ge 0\}$ is not a Markov process. To cast our model into the framework of Markov system, let us define a process $\mathbf{x}(t)$ taking values in $\mathbb{C}[-\tau, 0]$ by

$$\mathbf{x}_s(t) = x(s+t), \quad t-\tau \le s \le t$$

then, $\{(\mathbf{x}(t), r_t), t \ge 0\}$ is a strong Markov process. Consider now the Lyapunov functional candidate with the following form:

$$V(\mathbf{x}(t), r_t) = x^\top(t) P(r_t) x(t) + \int\limits_{t-h(t)}^{t} x^\top(\theta) Q x(\theta)\, d\theta, \tag{15}$$

where $P(r_t)$ and Q are symmetric and positive-definite matrices.

Let $\mathcal{A}$ be the infinitesimal generator of the process $\{(\mathbf{x}(t), r_t), t \ge 0\}$. Then, we get:

$$\begin{aligned}
\mathcal{A}V(\mathbf{x}(t), r_t) &= \dot{x}^\top(t) P(r_t) x(t) + x^\top(t) P(r_t) \dot{x}(t) + x^\top(t) Q x(t) \\
&\qquad - (1-\dot{h}(t)) x^\top(t-h(t)) Q x(t-h(t)) + \sum_{j=1}^{N} \lambda_{r_t j} x^\top(t) P(j) x(t) \\
&= [(A(r_t) + B(r_t)K(r_t))x(t) + A_1(r_t)x(t-h(t))]^\top P(r_t) x(t) \\
&\quad + x^\top(t) P(r_t)[(A(r_t) + B(r_t)K(r_t))x(t) + A_1(r_t)x(t-h(t))] \\
&\quad + x^\top(t) Q x(t) - (1-\dot{h}(t)) x^\top(t-h(t)) Q x(t-h(t)) \\
&\quad + \sum_{j=1}^{N} \lambda_{r_t j} x^\top(t) P(j) x(t)
\end{aligned}$$

which gives the following:

$$\begin{aligned}
\mathcal{A}V(\mathbf{x}(t), r_t) \le x^\top(t) \Big[A_K^\top(r_t) P(r_t) + P(r_t) A_K(r_t) + Q + \sum_{j=1}^{N} \lambda_{r_t j} P(j) \Big] x(t) \\
+ 2x^\top(t) P(r_t) A_1(r_t) x(t-h(t)) - (1-\beta) x^\top(t-h(t)) Q x(t-h(t))
\end{aligned}$$

Notice that (13) can be rewritten as follows:

$$\begin{bmatrix} A_K^\top(i)P(i) + P(i)A_K(i) + Q + \sum\limits_{j=1}^{N} \lambda_{ij} P(j) & P(i)A_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q \end{bmatrix} + \begin{bmatrix} C_{1K}^\top(i) \\ 0 \end{bmatrix} [C_{1K}(i) \;\; 0] < 0$$

which gives in turn:

$$\begin{bmatrix} A_K^\top(i)P(i) + P(i)A_K(i) + Q + \sum\limits_{j=1}^{N} \lambda_{ij} P(j) & P(i)A_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q \end{bmatrix} < 0.$$

This implies that the system is stochastically stable under the control law (6) (see Boukas and Liu [17] for the details of the proof).

Using now Dynkin's formula, we get:

$$\mathbb{E}[V(\mathbf{x}(t), r_t)] - V(\mathbf{x}(0), r_0) = \mathbb{E}\left[\int_0^t \mathcal{A}V(\mathbf{x}(s), r_s)\, ds\right].$$

Combining this with (13) we have:

$$\mathbb{E}[V(\mathbf{x}(t_f), r_{t_f})] - V(\mathbf{x}(0), r_0) \leq \mathbb{E}\left[\int_0^{t_f} \zeta^\top(s)\Theta(r_s)\zeta(s)\, ds\right]$$

with $\zeta(s) = \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}$.

Using the fact that system is stable, this implies the following when letting t_f goes to infinity:

$$\mathbb{E}\left[\int_0^\infty z_1^\top(s) z_1(s)\, ds\right] \leq V(\mathbf{x}(0), r_0)$$

$$= x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} x^\top(\theta)Qx(\theta)\, d\theta,$$

i.e.:

$$\|z_1\| \leq \left[x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} x^\top(\theta)Qx(\theta)\, d\theta\right]^{\frac{1}{2}}$$

which gives an upper bound for the $\mathcal{H}_2$ performance measure for the class of systems we are dealing with. This ends the proof of Theorem 3.1.

Let us now put the condition of Theorem 3.1 in the LMI formalism since it is now nonlinear in $P(r_t)$ and $K(R_t)$. From (13) we get the following using Schur complement:

$$\begin{bmatrix} A_K^\top(i)P(i) + P(i)A_K(i) + \sum_{j=1}^{N} \lambda_{ij} P(j) & P(i)A_1(i) & I & C_{1K}^\top(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q & 0 & 0 \\ I & 0 & -Q^{-1} & 0 \\ C_{1K}(i) & 0 & 0 & -I \end{bmatrix} < 0.$$

Letting $\bar{Q} = (1-\beta)Q$, the previous condition becomes:

$$\begin{bmatrix} A_K^\top(i)P(i)+P(i)A_K(i)+\sum_{j=1}^{N} \lambda_{ij} P(j)+P(i)A_1(i)\bar{Q}^{-1}A_1^\top(i)P(i) & I & C_{1K}^\top(i) \\ I & -Q^{-1} & 0 \\ C_{1K}(i) & 0 & -I \end{bmatrix} < 0.$$

Letting now $X(i) = P^{-1}(i)$ and pre and post-multiplying the previous condition by $\operatorname{diag}(X(i), I, I)$ we get:

$$\begin{bmatrix} J_0(i) & X(i) & X(i)K^\top(i)D_1^\top(i) + X(i)C_1^\top(i) \\ X(i) & -Q^{-1} & 0 \\ D_1(i)K(i)X(i) + C_1^\top(i)X(i) & 0 & -I \end{bmatrix} < 0$$

with $J_0(i) = X(i)A_K^\top(i) + A_K(i)X(i) + X(i)\Big[\sum_{j=1}^{N} \lambda_{ij} X^{-1}(j)\Big]X(i) + A_1(i)\bar{Q}^{-1}A_1^\top(i)$.

Putting

$$\begin{aligned} U &= Q^{-1} \\ Y(i) &= K(i)X(i) \\ S_i(X) &= \Big(\sqrt{\lambda_{i1}}X(i), \dots \sqrt{\lambda_{ii-1}}X(i), \sqrt{\lambda_{ii}}X(i), \dots, \sqrt{\lambda_{iN}}X(i)\Big) \\ \mathcal{X}_i &= \operatorname{diag}(X(1), \dots, X(i-1), X(i+1), \dots, X(N)) \end{aligned}$$

and noticing that:

$$X(i)A_K^\top(i) = X(i)(A(i) + B(i)K(i))^\top = X(i)A^\top(i) + Y^\top(i)B^\top(i),$$

$$X(i)\Bigg[\sum_{j=1}^{N} \lambda_{ij} X^{-1}(j)\Bigg]X(i) = \lambda_{ii}X(i) + S_i(X)\mathcal{X}_i^{-1}S_i^\top(X)$$

the previous condition becomes:

$$\begin{bmatrix} J_1(i) & X(i) & Y^\top(i)D_1^\top(i) + X(i)C_1^\top(i) & S_i(X) \\ X(i) & -U & 0 & 0 \\ D_1(i)Y(i) + C_1(i)X(i) & 0 & -I & 0 \\ S_i^\top(X) & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0$$

with

$$\begin{aligned} J_1(i) &= X(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) \\ &+ B(i)Y(i) + \lambda_{ii}X(i) + (1-\beta)^{-1}A_1(i)UA_1^\top(i). \end{aligned}$$

This condition can be solved using the LMI toolbox of Matlab or any equivalent tool to get the controller gain, $K(r_t)$ for every $r_t \in \mathbf{S}$.

Let us now consider that the external disturbance is not equal to zero. The controller (6) in this case is a $\mathcal{H}_\infty$ controller and the following theorem gives the associated results.

Theorem 3.2 *Let γ be a given positive constant. The controller (6) will stabilize the system and guarantee the disturbance rejection of level γ for all $\omega(t) \in \mathcal{L}_2[0,\infty)$ if there exist symmetric and positive-definite matrices $P = (P(1), \dots, P(N))$ and Q, and a controller gain $K = (K(1), \dots, K(N))$ such that the following holds for every $i \in \mathcal{S}$:*

$$\Theta_{H_\infty} = \begin{bmatrix} \tilde{J}_2(i) & P(i)A_1(i) & P(i)B_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q & 0 \\ B_1^\top(i)P(i) & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{16}$$

with $\tilde{J}_2(i) = A_K^\top(i)P(i) + P(i)A_K(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) + Q + C_{2K}^\top(i)C_{2K}(i)$. *In this case we have:*

$$\|z_2\| = \left[\gamma^2\|\omega\|^2 + x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds\right]^{\frac{1}{2}}. \tag{17}$$

Proof To prove this theorem, let us assume that the controller exists and show that it stochastically stabilizes the class of system we are considering. For this purpose notice that (16) implies the following:

$$\begin{bmatrix} \tilde{J}_2(i) & P(i)A_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q \end{bmatrix} < 0. \tag{18}$$

This inequality can be rewritten as:

$$\begin{bmatrix} A_K^\top(i)P(i) + P(i)A_K(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) + Q & P(i)A_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q \end{bmatrix} + \begin{bmatrix} C_2^\top(i) \\ 0 \end{bmatrix} [C_2(i) \;\; 0] < 0$$

which gives in turn:

$$\begin{bmatrix} A_K^\top(i)P(i) + P(i)A_K(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) + Q & P(i)A_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q \end{bmatrix} < 0.$$

This implies in turn that the system is stochastically stable under the controller (6) (for more details on the rest of the proof, we refer the reader to Boukas and Liu [17]).

Let us now, show that the $\mathcal{H}_{H_\infty}$ performance measure is bounded. For this purpose, let us define the performance function:

$$\mathcal{J}_T = \mathbb{E}\left[\int_0^T \left(z_2^\top(t)z_2(t) - \gamma^2\omega^\top(t)\omega(t)\right) dt\right].$$

To prove that $\mathcal{H}_{H_\infty}$ performance measure is bounded, it suffices to establish

$$\mathcal{J}_\infty \le V(\mathbf{x}(0), r_0) = x_0^\top P(r_0)x_0 + \int_{-h(0)}^{0} \phi(s)Q\phi(s)\,ds.$$

Using Dynkin's formula, we have

$$\mathbb{E}\left[\int_0^T \mathcal{A}V(\mathbf{x}(t), r_t)]\,dt\right] = \mathbb{E}[V(\mathbf{x}(T), r_T)] - V(\mathbf{x}(0), r_0),$$

where $V(\mathbf{x}(t), r_t)$ is given by equation (15).

Noticing that:

$$z_2^\top(t)z_2(t) - \gamma^2\omega^\top(t)\omega(t) = x^\top(t)C_{2K}^\top(r_t)C_{2K}(r_t)x(t) - \gamma^2\omega^\top(t)\omega(t)$$

and

$$\begin{aligned}\mathcal{A}V(\mathbf{x}(t), r_t)] \le x^\top(t)\Big[A_K^\top(r_t)P(r_t) + P(r_t)A_K(r_t) + Q + \sum_{j=1}^N \lambda_{ij}P(j)\Big]x(t) \\ + 2x^\top(t)P(r_t)A_1(r_t)x(t-h(t)) + 2x^\top(t)P(r_t)B_1(r_t)\omega(t) \\ - (1-\beta)x^\top(t-h(t))Qx(t-h(t))\end{aligned}$$

we get:

$$z_2^\top(t)z_2(t) - \gamma^2\omega^\top(t)\omega(t) + \mathcal{A}V(\mathbf{x}(t), r_t) \le \eta^\top(t)\Theta_{H_\infty}(r_t)\eta(t),$$

where $\eta^\top(t) = \big(x^\top(t) \;\; x^\top(t-h(t)) \;\; \omega^\top(t)\big)$. Therefore,

$$\begin{aligned}\mathcal{J}_T &= \mathbb{E}\Big[\int_0^T [z_2^\top(t)z_2(t) - \gamma^2\omega^\top(t)\omega(t) + \mathcal{A}V(\mathbf{x}(t), r_t)]\,dt\Big] \\ &\quad - \mathbb{E}\Big[\int_0^T \mathcal{A}V(\mathbf{x}(t), r_t)]\,dt\Big] \\ &\le \mathbb{E}\Big[\int_0^T \eta^\top(t)\Theta_{H_\infty}(r_t)\eta(t)\,dt\Big] - \mathbb{E}[V(\mathbf{x}(T), r_T)] + V(\mathbf{x}(0), r_0).\end{aligned} \tag{19}$$

Since $\Theta_{H_\infty}(i) < 0$ and $\mathbb{E}[V(\mathbf{x}(T), r_T)] \ge 0$, (19) implies

$$\mathcal{J}_T \le V(\mathbf{x}(0), r_0),$$

yielding

$$\mathcal{J}_\infty \le V(\mathbf{x}(0), r_0),$$

i.e.,

$$\|z_2\|_2^2 - \gamma^2\|w\|_2^2 \le x_0^\top P(r_0)x_0 + \int_{-h(0)}^0 \phi^\top(s)Q\phi(s)\,ds.$$

This yields

$$\|z_2\|_2^2 \le \gamma^2\|w\|_2^2 + x_0^\top P(r_0)x_0 + \int_{-h(0)}^0 \phi^\top(s)Q\phi(s)\,ds,$$

which gives the bound we are looking for.

This ends the proof of Theorem 3.2.

In a similar way we can put the condition (16) in the LMI formalism in the design parameters. The new conditions becomes:

$$\begin{bmatrix} J_2(i) & X(i) & B_1(i) & Y^\top(i)D_2^\top(i) + X(i)C_2^\top(i) & S_i(X) \\ X(i) & -U & 0 & 0 & 0 \\ B^\top(i) & 0 & -\gamma^{-2}I & 0 & 0 \\ C_2(i)X(i) + D_2(i)Y(i) & 0 & 0 & -I & 0 \\ S_i^\top(X) & 0 & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0 \quad (20)$$

with

$$J_2(i) = X(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) + B(i)Y(i) + \lambda_{ii}X(i) + (1-\beta)^{-1}A_1(i)UA_1^\top(i).$$

Notice that in (14) and (31) we have the following common term:

$$x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds$$

that we should minimize to guarantee good performances. By doing so, simultaneously we will guarantee a minimum upper bound for the $\mathcal{H}_2$ performance measure and a good disturbance rejection with a level γ for all $\omega(t) \in \mathcal{L}_2[0,\infty)$. Before giving the optimization problem that will allow us to reach our goal, let us formulate the cost function.

First of all, notice that $x^\top(0)P(i)x(0)$ for all $i \in \mathbf{S}$ can be bounded by a real positive constant that we should minimize:

$$x^\top(0)P(i)x(0) \le \alpha$$

with $\alpha = \max(x^\top(0)P(1)x(0), \ldots, x^\top(0)P(N)x(0))$; which we can rewrite as follows:

$$-\alpha + \phi^\top(0)X^{-1}(i)\phi(0) < 0,$$

where $X(i) = P^{-1}(i)$.

This can be rewritten in matrix form as:

$$\begin{bmatrix} -\alpha & \phi^\top(0) \\ \phi(0) & -X(i) \end{bmatrix} < 0. \quad (21)$$

For the second term of the common term for the two performance measures, notice that:

$$\int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds = \int_{-h(0)}^{0} \mathrm{tr}\,\left(\phi^\top(s)U^{-1}\phi(s)\right) ds$$
$$= \mathrm{tr}\,\left(\mathcal{N}\mathcal{N}^\top U^{-1}\right) = \mathrm{tr}\,\left(\mathcal{N}^\top U^{-1}\mathcal{N}\right) < \mathrm{tr}\,(\mathcal{Q})$$

with $\mathcal{N}\mathcal{N}^\top = \int_{-h(0)}^{0} \phi(s)\phi^\top(s)\,ds$. This gives:

$$-Q_1 + \mathcal{N}^\top U^{-1}\mathcal{N} < 0.$$

In matrix form we get:

$$\begin{bmatrix} -Q_1 & \mathcal{N}^\top \\ \mathcal{N} & -U \end{bmatrix} < 0. \quad (22)$$

The following theorem gives the optimization that we could solve to get the controller gain.

Theorem 3.3 *Let γ be a given positive constant. If there exist symmetric and positive-definite matrices $X = (X(1), \dots, X(N))$, U and $\mathcal{Q}$ and a positive scalar α, and a matrix $Y = (Y(1), \dots, Y(N))$, solution of the following optimization problem:*

$$\min\big(\alpha + \operatorname{tr}(\mathcal{Q})\big)$$

$$s.t:$$

$$\begin{bmatrix} J_1(i) & X(i) & Y^\top(i)D_1^\top(i) + X(i)C_1(i) & S_i(X) \\ X(i) & -U & 0 & 0 \\ C_1(i)X(i) + D_1(i)Y(i) & 0 & -I & 0 \\ S_i^\top(X) & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} J_2(i) & X(i) & B_1(i) & Y^\top(i)D_2^\top(i) + X(i)C_2^\top(i) & S_i(X) \\ X(i) & -U & 0 & 0 & 0 \\ B_1^\top(i) & 0 & -\gamma^{-2}I & 0 & 0 \\ C_2(i)X(i) + D_2(i)Y(i) & 0 & 0 & -I & 0 \\ S_i^\top(X) & 0 & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0, \tag{24}$$

$$\begin{bmatrix} -\alpha & \phi^\top(0) \\ \phi(0) & -X(i) \end{bmatrix} < 0, \tag{25}$$

$$\begin{bmatrix} -Q_1 & \mathcal{N}^\top \\ \mathcal{N} & -U \end{bmatrix} < 0, \tag{26}$$

then the controller (6) is a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller satisfying the control objective (8). The controller gain is given by $K(r_t) = Y(r_t)X^{-1}(r_t)$, for every mode $r_t \in \mathbf{S}$.

This theorem gives a procedure to design the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller for the nominal class of systems we are dealing with. The optimization that we propose is a convex one that we can solve using the existing tools like the one of Matlab or any equivalent one.

In the next subsection we will see how we can modify the results on this subsection to handle the case of uncertain systems.

3.2 Uncertain system

Let us now assume that uncertainties are not equal to zero and suppose that they satisfy the conditions (3). In this case the closed-loop dynamics becomes:

$$\begin{cases} \dot{x}(t) = A_K(r_t)x(t) + A_1(r_t,t)x(t-h(t)) + B_1(r_t)\omega(t), \\ x(s) = \phi(s), -\tau \le s \le 0, \\ z_1(t) = C_{1K}(r_t)x(t), \\ z_2(t) = C_{2K}(r_t)x(t), \end{cases} \tag{27}$$

where $A_K(r_t) = A(r_t,t) + B(r_t,t)K(r_t)$, $C_{1K}(r_t) = C_1(r_t) + D_1(r_t)K(r_t)$ and $C_{2K}(r_t) = C_{2K}(r_t) + D_2(r_t)K(r_t)$.

If we apply the results of Theorem 3.1 to the uncertain system (27), we get:

$$\begin{bmatrix} J(r_t,t) & P(r_t)A_1(r_t,t) \\ A_1^\top(r_t,t)P(r_t) & -(1-\beta)Q \end{bmatrix} < 0$$

with $J(r_t,t) = A_K^\top(r_t,t)P(r_t) + P(r_t)A_K(r_t,t) + \sum_{j=1}^{N} \lambda_{r_t j}P(j) + Q + C_{1K}^\top(r_t)C_{1K}(r_t)$.

This gives in turn the following:

$$J(r_t,t) + (1-\beta)^{-1}P(r_t)A_1(r_t,t)Q^{-1}A_1^\top(r_t,t)P(r_t) < 0 \tag{28}$$

since

$$\begin{aligned} A_K^\top(r_t,t)P(r_t) + P(r_t)A_K(r_t,t) &= A_K^\top(r_t)P(r_t) + P(r_t)A_K(r_t) \\ &\quad + 2P(r_t)\Delta A(r_t,t) + 2P(r_t)\Delta B(r_t,t)K(r_t) \end{aligned}$$

using Lemma 2.1, we get:

$$\begin{aligned} A_K^\top(r_t,t)P(r_t) + P(r_t)A_K(r_t,t) &\le A_K^\top(r_t)P(r_t) + P(r_t)A_K(r_t) \\ &\quad + \varepsilon_A P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t) + \varepsilon_A^{-1}E_A^\top(r_t)E_A(r_t) \\ &\quad + \varepsilon_B P(r_t)D_B(r_t)D_B^\top(r_t)P(r_t) + \varepsilon_B^{-1}K^\top(r_t)E_B^\top(r_t)E_B(r_t)K(r_t). \end{aligned}$$

For the term $(1-\beta)^{-1}P(r_t)A_1(r_t,t)Q^{-1}A_1^\top(r_t,t)P(r_t)$ notice that using Lemma 2.2, we have:

$$\begin{aligned} A_1(r_t,t)Q^{-1}A_1^\top(r_t,t) &\le A_1(r_t)Q^{-1}A_1^\top(r_t) + A_1(r_t)Q^{-1}E_{A_1}^\top(r_t) \\ &\times \left(\varepsilon_{A1}I - E_{A_1}(r_t)Q^{-1}E_{A_1}^\top(r_t)\right)^{-1}E_{A_1}(r_t)Q^{-1}A_1^\top(r_t) + \varepsilon_{A1}D_{A_1}(r_t)D_{A_1}^\top(r_t) \end{aligned}$$

which gives the following when we replace Q^{-1} by $(1-\beta)^{-1}Q^{-1}$:

$$\begin{aligned} A_1(r_t,t)(1-\beta)^{-1}Q^{-1}A_1^\top(r_t,t) &\le A_1(r_t)(1-\beta)^{-1}Q^{-1}A_1^\top(r_t) \\ &+ A_1(r_t)(1-\beta)^{-1}Q^{-1}E_{A_1}^\top(r_t)\left(\varepsilon_{A1}I - E_{A_1}(r_t)(1-\beta)^{-1}Q^{-1}E_{A_1}^\top(r_t)\right)^{-1} \\ &\times E_{A_1}(r_t)(1-\beta)^{-1}Q^{-1}A_1^\top(r_t) + \varepsilon_{A1}D_{A_1}(r_t)D_{A_1}^\top(r_t). \end{aligned}$$

Based on all these transformations the condition (28) becomes:

$$\begin{aligned} &A_K^\top(r_t)P(r_t) + P(r_t)A_K(r_t) + \varepsilon_A P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t) + Q + \sum_{j=1}^{N}\lambda_{r_t j}P(j) \\ &+ C_{1K}^\top(r_t)C_{1K}(r_t) + \varepsilon_A^{-1}E_A^\top(r_t)E_A(r_t) + \varepsilon_B P(r_t)D_B(r_t)D_B^\top(r_t)P(r_t) \\ &+ \varepsilon_B^{-1}K^\top(r_t)E_B^\top(r_t)E_B(r_t)K(r_t) + P(r_t)A_1(r_t)(1-\beta)^{-1}Q^{-1}A_1^\top(r_t)P(r_t) \\ &+ P(r_t)A_1(r_t)(1-\beta)^{-1}Q^{-1}E_{A_1}^\top(r_t)\left(\varepsilon_{A1}I - E_{A_1}(r_t)(1-\beta)^{-1}Q^{-1}E_{A_1}^\top(r_t)\right)^{-1} \\ &\times E_{A_1}(r_t)(1-\beta)^{-1}Q^{-1}A_1^\top(r_t)P(r_t) + \varepsilon_{A1}P(r_t)D_{A_1}(r_t)D_{A_1}^\top(r_t)P(r_t) < 0. \end{aligned}$$

In matrix form we get:

$$\begin{bmatrix} \tilde{J}_3(r_t) & E_A^\top(r_t) & K^\top(r_t)E_B^\top(r_t) & \frac{P(r_t)A_1(r_t)Q^{-1}E_{A1}(r_t)}{(1-\beta)} & I & C_{1K}^\top(r_t) \\ E_A(r_t) & \varepsilon_A I & 0 & 0 & 0 & 0 \\ E_B(r_t)K(r_t) & 0 & \varepsilon_B I & 0 & 0 & 0 \\ \frac{E_{A1}^\top(r_t)Q^{-1}A_1^\top(r_t)P(r_t)}{(1-\beta)} & 0 & 0 & -\varepsilon_{A1}I + \frac{E_A(r_t)Q^{-1}E_A^\top(r_t)}{(1-\beta)} & 0 & 0 \\ I & 0 & 0 & 0 & -Q^{-1} & 0 \\ C^{1K}(r_t) & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0$$

with

$$\tilde{J}_3(r_t) = A_K^\top(r_t)P(r_t) + P(r_t)A_K^\top(r_t) + \sum_{j=1}^{N} \lambda_{r_t j} P(j)$$
$$+ \varepsilon_A P(r_t) D_A(r_t) D_A^\top(r_t) P(r_t) + \varepsilon_B P(r_t) D_B(r_t) D_B^\top(r_t) P(r_t)$$
$$\times \varepsilon_{A1} P(r_t) D_{A1}(r_t) D_{A1}^\top(r_t) P(r_t) + (1-\beta)^{-1} P(r_t) A_1(r_t) Q^{-1} A_1^\top(r_t) P(r_t).$$

Now if we pre and post-multiplying the right hand side term by diag $(X(i), I, I, I, I)$ with $X(i) = P^{-1}(i)$ and by following the same steps as we followed to transform (13) in LMI form, we get:

$$\begin{bmatrix} J_3(r_t) & X(r_t)E_A^\top(r_t) & Y^\top(r_t)E_B^\top(r_t) & \frac{A_1(r_t)UE_{A1}^\top(r_t)}{(1-\beta)} & X(r_t) & Y^\top(r_t)D_1^\top(r_t) + X(r_t)C_1^\top(r_t) & S_{r_t}(X) \\ E_A(r_t)X(r_t) & \varepsilon_A I & 0 & 0 & 0 & 0 & 0 \\ E_B(r_t)Y(r_t) & 0 & \varepsilon_B I & 0 & 0 & 0 & 0 \\ \frac{E_{A1}^\top(r_t)UA_1(r_t)}{(1-\beta)} & 0 & 0 & -\varepsilon_{A1}I + \frac{E_A(r_t)UE_A^\top(r_t)}{(1-\beta)} & 0 & 0 & 0 \\ X(r_t) & 0 & 0 & 0 & -U & 0 & 0 \\ C_1(r_t)X(r_t) + D_1(r_t)Y(r_t) & 0 & 0 & 0 & 0 & -I & 0 \\ S_{r_t}^\top(X) & 0 & 0 & 0 & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0 \qquad (29)$$

with

$$J_3(r_t) = A(r_t)X(r_t) + X(r_t)A^\top(r_t) + \lambda_{r_t r_t} X(r_t) + B(r_t)Y(r_t) + Y^\top(r_t)B^\top(r_t)$$
$$+ \varepsilon_A D_A(r_t) D_A^\top(r_t) + \varepsilon_B D_B(r_t) D_B^\top(r_t) + \varepsilon_{A1} D_{A1}(r_t) D_{A1}^\top(r_t)$$
$$+ (1-\beta)^{-1} A_1(r_t) U A_1^\top(r_t).$$

If this condition is satisfied, we can easily prove following the steps of Theorem 3.1's proof that the system is stable under the control law (6) when the external disturbance is equal to zero and that the $\mathcal{H}_2$ performance measure is bounded, i.e.:

$$\|z_1\| \le \left[x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} \phi^\top(s) Q \phi(s)\, ds \right]^{\frac{1}{2}}.$$

The following theorem summarizes the corresponding results.

Theorem 3.4 *Let the disturbance input be equal to zero, i.e. $\omega(t) = 0$ for $t \geq 0$. The controller (6) is an $\mathcal{H}_2$ optimal controller satisfying the minimization of the $\mathcal{H}_2$ performance measure (7) if there exist symmetric and positive-definite matrices $P = (P(1), \dots, P(N))$, Q and a matrix $Y = (Y(1), \dots, Y(N))$ that (29) holds for every mode $i \in \mathbf{S}$. The $\mathcal{H}_2$ performance measure is bounded by a positive scalar, i.e.:*

$$J_{H_2} \leq J^\star \triangleq \left[x^\top(0) P(r_0) x(0) + \int_{-h(0)}^{0} \phi^\top(s) Q \phi(s)\, ds \right]^{\frac{1}{2}}.$$

The controller gain $K(i) = Y(i) X^{-1}(i)$ for every $i \in \mathbf{S}$.

When the external disturbance is not equal to zero we can easily follow the same step as for Theorem 3.4 to establish the results of Theorem 3.5.

Theorem 3.5 *Let γ be a given positive constant. The controller (6) will stabilize the system and guarantee the disturbance rejection of level γ if there exist symmetric and positive-definite matrices $P = (P(1), \dots, P(N))$, Q and positive constants ε_A, ε_B and ε_{A_1}, and a matrix $Y = (Y(1), \dots, Y(N))$ such that the following holds for every $i \in \mathbf{S}$:*

$$\begin{bmatrix} J_5(r_t) & X(r_t)E_A^\top(r_t) & Y^\top(r_t)E_B^\top(r_t) & \frac{A_1(r_t)UE_{A1}^\top(r_t)}{(1-\beta)} & X(r_t) & B_1(r_t) & Y^\top(r_t)D_2^\top(r_t) + X(r_t)C_2^\top(r_t) & S_{r_t}(X) \\ E_A(r_t)X(r_t) & \varepsilon_A I & 0 & 0 & 0 & 0 & 0 & 0 \\ E_B(r_t)Y(r_t) & 0 & \varepsilon_B I & 0 & 0 & 0 & 0 & 0 \\ \frac{E_{A1}^\top(r_t)UA_1(r_t)}{(1-\beta)} & 0 & 0 & -\varepsilon_{A1} I + \frac{E_A(r_t)UE_A^\top(r_t)}{(1-\beta)} & 0 & 0 & 0 & 0 \\ X(r_t) & 0 & 0 & 0 & -U & 0 & 0 & 0 \\ B^\top(r_t) & 0 & 0 & 0 & 0 & -\gamma^{-2} I & 0 & 0 \\ C_2(r_t)X(r_t) + D_2(r_t)Y(r_t) & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ S_{r_t}^\top(X) & 0 & 0 & 0 & 0 & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0 \qquad (30)$$

with

$$\begin{aligned} J_5(i) &= X(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) + B(i)Y(i) + \lambda_{ii}X(i) \\ &\quad + \varepsilon_A D_A(i)D_A^\top(i) + \varepsilon_B D_B(i)D_B^\top(i) + \varepsilon_{A1} D_{A1}(i)D_{A1}^\top(i) \\ &\quad + (1-\beta)^{-1} A_1(i)UA^\top(i) + \gamma^{-2} B_1(i)B_1^\top(i). \end{aligned}$$

In this case we have:

$$\|z_2\| = \left[\gamma^2 \|\omega\|^2 + x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} \phi^\top(s) Q \phi(s)\, ds \right]^{\frac{1}{2}}.$$

For the same reasons as before, if we combine the two previous theorems we get the following one that gives the optimization that we could solve to get the controller gains in each mode for the uncertain class of systems we are dealing with.

Theorem 3.6 *Let γ be a given positive constant. If there exist symmetric and positive-definite matrices $X = (X(1),\dots,X(N))$, U and $\mathcal{Q}$ and positive scalars α, ε_A, ε_B and ε_{A_1}, and a matrix $Y = (Y(1),\dots,Y(N))$, solution of the following optimization problem:*

$$\min(\alpha + \mathrm{tr}\,(\mathcal{Q}))$$
$$s.t:$$

$$\begin{bmatrix} J_4(r_t) & X(r_t)E_A^\top(r_t) & Y^\top(r_t)E_B^\top(r_t) & \frac{A_1(r_t)UE_{A1}^\top(r_t)}{(1-\beta)} & X(r_t) & Y^\top(r_t)D_1^\top(r_t) + X(r_t)C_1^\top(r_t) & S_{r_t}(X) \\ E_A(r_t)X(r_t) & \varepsilon_A I & 0 & 0 & 0 & 0 & 0 \\ E_B(r_t)Y(r_t) & 0 & \varepsilon_B I & 0 & 0 & 0 & 0 \\ \frac{E_{A1}^\top(r_t)UA_1(r_t)}{(1-\beta)} & 0 & 0 & -\varepsilon_{A1}I + \frac{E_A(r_t)UE_A^\top(r_t)}{(1-\beta)} & 0 & 0 & 0 \\ X(r_t) & 0 & 0 & 0 & -U & 0 & 0 \\ C_1(r_t)X(r_t) + D_1(r_t)Y(r_t) & 0 & 0 & 0 & 0 & -I & 0 \\ S_{r_t}^\top(X) & 0 & 0 & 0 & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} J_5(r_t) & X(r_t)E_A^\top(r_t) & Y^\top(r_t)E_B^\top(r_t) & \frac{A_1(r_t)UE_{A1}^\top(r_t)}{(1-\beta)} & X(r_t) & B_1(r_t) & Y^\top(r_t)D_2^\top(r_t) + X(r_t)C_2^\top(r_t) & S_{r_t}(X) \\ E_A(r_t)X(r_t) & \varepsilon_A I & 0 & 0 & 0 & 0 & 0 & 0 \\ E_B(r_t)Y(r_t) & 0 & \varepsilon_B I & 0 & 0 & 0 & 0 & 0 \\ \frac{E_{A1}^\top(r_t)UA_1(r_t)}{(1-\beta)} & 0 & 0 & -\varepsilon_{A1}I + \frac{E_A(r_t)UE_A^\top(r_t)}{(1-\beta)} & 0 & 0 & 0 & 0 \\ X(r_t) & 0 & 0 & 0 & -U & 0 & 0 & 0 \\ B^\top(r_t) & 0 & 0 & 0 & 0 & -\gamma^{-2}I & 0 & 0 \\ C_2(r_t)X(r_t) + D_2(r_t)Y(r_t) & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ S_{r_t}^\top(X) & 0 & 0 & 0 & 0 & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} -\alpha & \phi^\top(0) \\ \phi(0) & -X(i) \end{bmatrix} < 0, \tag{33}$$

$$\begin{bmatrix} -\mathcal{Q} & \mathcal{N}^\top \\ \mathcal{N} & -U \end{bmatrix} < 0, \tag{34}$$

then the controller (6) is a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller satisfying the control objective (8). The controller gain is given by $K(r_t) = Y(r_t)X^{-1}(r_t)$, for every mode $r_t \in \mathbf{S}$.

This theorem provides a procedure to design a memoryless state feedback controller of the form (6) that stabilizes system (2) in the robust SS sense. The advantage of these results is that we can use the LMI tools to solve it for any dynamical system of the class we are considering in this paper.

4 Conclusion

This paper deals with the class of continuous-time linear systems with Markovian jumps and time-delays. The time-delay is assumed to be time-varying. Results on stochastic stabilizability and its robustness are developed. The LMI framework is used to establish the different results on stabilizability. The conditions we developed can easily be solved using any LMI toolbox like the one of Matlab or the one of Scilab. These results we can be extended to other type of controller and also to the case where the time-delay is mode dependent as it was developed in Boukas and Liu [17]. This will be the subject of our future research.

References

[1] Bernstein, D.S. and Haddad, W.M. LQG control with an $\mathcal{H}_\infty$ performance bound: A Riccati equation approach. *IEEE Trans. Automat. Control* **34** (1989) 293–305.

[2] Zhou, K., Glover, K., Bodenheimer, B. and Doyle, J. Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives I: Robust performance analysis. *IEEE Trans. Automat. Control* **39** (1004) 1564–1574.

[3] Doyle, J., Zhou, K., Glover, K. and Bodenheimer, B. Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives II: Optimal control. *IEEE Trans. Automat. Control* **39** (1994) 1575–1587.

[4] Haddad, W.M., Kapila, V. and Bernstein, D.S. Robust $\mathcal{H}_\infty$ stabilization via parameterized Lyapunov bounds. *IEEE Trans. Automat. Control* **42** (1997) 241–248.

[5] Limbeer, D.J.N., Anderson, B.D.O. and Hendel, B. A Nash game approach to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control. *IEEE Trans. Automat. Control* **39** (1994) 69–82.

[6] Mustapha, D. Combined $\mathcal{H}_\infty$/LQG control via the optimal projection equations: On minimizing the LQG cost bound. *Int. Journal Robust Nonlinear Control* **1** (1991) 99–109.

[7] Rotea, M.A. and Khargonekar, P.P. $\mathcal{H}_2$-optimal control with $\mathcal{H}_\infty$-constraint: The state feedback case. *Automatica* **27** (1991) 307–316.

[8] Saberi, A., Chen, B.M., Sannuti, P. and Ly, U.-L. Simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control: The state feedback case. *Automatica* **29**(6) (1993) 1611–1614.

[9] Geromel, J.C., Peres, P.L.D. and Souza, S.R. A convex approach to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for discrete-time uncertain systems. *SIAM J. Control Optim.* **33** (1995) 1816–1833.

[10] Giusto, A., Trofino-Neto, A. and Castelan, E.B. $\mathcal{H}_\infty$ and $\mathcal{H}_2$ design techniques for a class of prefilters. *IEEE Trans. Automat. Control* **41** (1996) 864–871.

[11] Kaminer, I., Khargonekar, P.P. and Rotea, M.A. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for discrete-time systems via convex optimization. *Automatica* **36** (1991) 824–837.

[12] Khargonekar, P.P. and Rotea, M.A. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control: A convex optimization approach. *IEEE Trans. Automat. Control* **36** (1991) 824–837.

[13] Rotea, M.A. and Khargonekar, P.P. Generalized $\mathcal{H}_2/\mathcal{H}_\infty$ control problem. In: *Robust Control Theory*, (Eds.: B.A.Fancis and P.P.Khargonekar), IMA, Vol. Math. Appl., 66, Springer-Verlag, New York, 1995, P.81–103.

[14] Leibfritz, F. An LMI-based algorithm for designing suboptimal static feedback $\mathcal{H}_2/\mathcal{H}_\infty$ output feedback controllers. *Siam J. Control Otim.* **39**(6) (2001) 1711–1735.

[15] Kim, J. H. Robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of time-delay systems. *Int. J. of Systems Science* **32**(11) (2001) 1345–1351.

[16] Mariton, M. *Jump Linear Systems in Automatic Control*. Marcel Dekker Inc., New York, 1990.

[17] Boukas, E.K. and Liu, Z.K. *Deterministic and Stochastic Systems with Time-Delay*. Birkhäser, Boston, 2002.

[18] Costa, O.L.V. and Marques, R. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of discrete-time Markovian jump linear systems. *IEEE Trans. Automat. Control* **43** (1998) 95–100.

[19] Aliyu, M.D.S. and Boukas, E.K. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ stochastic control problem. *14th IFAC 99*, Beijing, China, July 5-9, 1999, P.223–228.

[20] Benjelloun, K., Boukas, E. K. and Yang, H. Robust stabilizability of uncertain linear time-delay with Markovian jumping parameters. *Journal of Dynamic Systems, Measurement and Control* **118** (1996) 776–783.

[21] Cao, Y.Y. and Lam, J. Robust H_∞ control of uncertain Markovian jump systems with time-delay. *IEEE Transactions on Automatic Control* **45**(1) (2000) 77–83.

Nonlinear Dynamics and Systems Theory, **3**(2) (2003) 139–150

Subharmonic Solutions of a Class of Hamiltonian Systems

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Received: March 14, 2003; Revised: July 15, 2003

Abstract: In this paper, we prove the existence of subharmonic solutions for the non autonomous Hamiltonian system: $\dot{u}(t) = J\nabla H(t, u(t))$ when H is convex and non coercive.

Keywords: *Subharmonic solution; Hamiltonian system; minimal period.*

Mathematics Subject Classification (2000): 34-XX, 34C25.

1 Introduction and Statement of Results

Let $G \in C^1(\mathbb{R}^n, \mathbb{R})$ be a convex function, A, $B \in C(\mathbb{R}, \mathcal{M}_n(\mathbb{R}))$ be periodic with minimal period T $(T > 0)$, $B(t)$ be invertible for all $t \in \mathbb{R}$ and $h = (f, g) \in C(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^n)$ be T-periodic with mean value zero.

Let $H(t,(r,p)) = G(A(t)r + B(t)p) + \prec h(t), (r,p) \succ$, $\forall\,(r,p) \in \mathbb{R}^n \times \mathbb{R}^n$, $\forall\, t \in \mathbb{R}$.

In this paper we consider the Hamiltonian system of ordinary differential equations

$$\dot{u}(t) = J\nabla H(t, u(t)), \tag{$\mathcal{H}_h$}$$

where ∇H is the first derivative of the Hamiltonian H with respect to (r,p) and J is the standard symplectic $(2n \times 2n)$-matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The motion of a relativist particle submitted to an electromagnetic field is governed by a noncoercive Hamiltonian system. However, most of results proving the existence of solutions to systems like $(\mathcal{H}_h)$ have been made use of a coercivity assumption on H, i.e., $\lim\limits_{|x| \to +\infty} H(t,x) = \infty$, see for example [5, 8, 9, 12] and references therein.

Timoumi investigates the case of non coercivity when H is convex (see [10, 11]). The purpose of this paper is to improve and complete the results obtained in [10, 11] dealing with this problem.

In the first theorem we establish the existence of subharmonic solutions, i.e., periodic solutions with minimal period in the set $\{kT,\, k \in \mathbb{N},\, k \geq 2\}$ for the Hamiltonian system of ordinary differential equations $(\mathcal{H}_0)$.

The problem of search for subharmonics is classical, it has been dealt with using various methods, especially index theories in different settings, see [3, 5, 6, 12].

In [10], Timoumi studied the question when the Hamiltonian has the form

$$H(t,(r,p)) = f(|p - A(t)r|),$$

where $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that:

$$\exists\, \lambda,\, \mu > 0 / f(t) \leq \lambda t + \mu \quad \forall\, t \geq 0$$

and the matrix $A(t)$ satisfies

1. $A^*(t) = -A(t)\ \forall\, t \in \mathbb{R}$
2. $\int_0^T A(t)\, dt \neq 0$.

Here, we try to conserve the same results when the Hamiltonian is subquadratic and $A(t)$ belongs to a larger set of matrices.

Precisely, we assume

(H_1) $\lim\limits_{|x|\to+\infty} G(x) = +\infty$;

(H_2) $\lim\limits_{|x|\to\infty} \frac{G(x)}{|x|^2} = 0$;

(H_3) G' is one to one;

(H_4) $C_0 = \int_0^T B^{-1}(t)A(t)\, dt$ is non symmetric.

Theorem 1.1 *Under the above assumptions, for all $k \in \mathbb{N}^*$, $(\mathcal{H}_0)$ possesses a kT periodic solution $u_k = (r_k, p_k)$ satisfying*

(i) $\lim\limits_{k\to+\infty} \|Ar_k + Bp_k\|_\infty = +\infty$.

(ii) *The minimal period of u_k is kT for any sufficiently large and prime integer k.*

Corollary 1.1 *Under the assumptions (H_2), (H_4) and*

(H_5) *G is strictly convex;*

(H_6) $\lim\limits_{|x|\to\infty} \frac{G(x)}{|x|} = +\infty$

the conclusion of Theorem 1.1 holds.

The second result concerns the forced case ($h \neq 0$), where h is interpreted as exterior forcing term. Here we prove the existence of a non constant T-periodic solution for $(\mathcal{H}_h)$ without the following assumption, needed in [11]

$$\forall\, r \in \mathbb{R}^n \backslash \{0\} \quad t \longmapsto A(t)r \quad \text{is non constant.}$$

Assume that

(H_7) $G(x) > G(0)$, $\forall\, x \in \mathbb{R}^n \backslash \{0\}$;

(H_8) $(B^{-1}A)^* g \neq f$.

Theorem 1.2 *Under assumptions* (H_1), (H_2), (H_7), (H_8), *the problem* $(\mathcal{H}_h)$ *possesses a non constant* T*-periodic solution.*

Remark 1.1 The assumption (H_8) is technical, it will be used only to guarantee the non constancy of solution for $(\mathcal{H}_h)$.

2 Proof of Theorem 1.1

Proof of the first part:

We use the dual action of Clarke-Ekeland.

Denote $H_0(t,r,p) = G(A(t)r + B(t)p)$. H_0 is convex with respect to (r,p) and its Fenchel's conjugate H_0^* is given by

$$\forall\,(s,q) \in \mathbb{R}^n \times \mathbb{R}^n, \quad H_0^*(t,s,q) = \begin{cases} G^*(B^{-1*}q) & \text{if } s = (B^{-1}A)^*q, \\ +\infty & \text{otherwise.} \end{cases}$$

For all $k \in \mathbb{N}^*$ we consider the functional

$$\Phi_k(w) = \frac{1}{2}\int_0^{kT} \prec Jw, \pi w \succ\, dt + \int_0^{kT} H_0^*(t,w)\,dt$$

defined on the space

$$L_0^2(0,kT,\mathbb{R}^{2n}) = \left\{ w \in L^2(0,kT,\mathbb{R}^{2n}) \Big/ \int_0^{kT} w(t)\,dt = 0 \right\},$$

where πw is the primitive of w with mean value zero.

Also, for all $v \in L_0^2(0,kT,\mathbb{R}^n)$ we define

$$\Psi_k(v) = \int_0^{kT} \prec B^{-1}A\pi v, v \succ\, dt + \int_0^{kT} G^*(B^{-1*}v)\,dt.$$

Obviously, we have $\Phi_k(w) = \Psi_k(v)$ for all $w = ((B^{-1}A)^*v, v) \in L_0^2(0,kT,\mathbb{R}^{2n})$.

Hence, we use the functional Ψ_k on the space $E_k = L_0^2(0,kT,\mathbb{R}^n)$.

For $v \in E_k$ we set

$$g(v) = \int_0^{kT} G^*(B^{-1*}v)\,dt$$

and

$$Q(v) = \int_0^{kT} \prec B^{-1}A\pi v, v \succ\, dt.$$

Lemma 2.1 *Ψ_k has a global minimum on E_k attained in $\bar{v}_k$.*

Proof Using Wirtinger's inequality, there exists a constant $\alpha_0 > 0$ such that

$$Q(v) \geq -\alpha_0 \|v\|_{L^2}^2, \quad \forall\, v \in E_k. \tag{1}$$

By (H_2), for all $\alpha > 0$ there exists $\beta > 0$ such that

$$G(x) \leq \alpha |x|^2 + \beta, \quad \forall\, x \in \mathbb{R}^n \tag{2}$$

and by going to the conjugate, we get

$$G^*(y) \geq \frac{1}{4\alpha}\, |y|^2 - \beta, \quad \forall\, y \in \mathbb{R}^n$$

so

$$g(v) \geq \frac{1}{4\alpha} \|B^{-1*} v\|_{L^2}^2 - \beta k T, \quad \forall\, v \in E_k. \tag{3}$$

From (1) and (3) there exists a constant $\gamma > 0$ such that

$$\Psi_k(v) \geq \gamma \|v\|_{L^2}^2 - \beta k T, \quad \forall\, v \in E_k. \tag{4}$$

Let $(v_n) \in E_k$ be a minimizing sequence of Ψ_k. From (4), (v_n) is bounded and since E_k is reflexive, there exists a subsequence (v_{n_j}) weakly convergent to $\bar{v}_k$.

Moreover, g is weakly lower semi-continuous, so

$$\underline{\lim} \int_0^{kT} G^*(B^{-1*} v_{n_j})\, dt \geq \int_0^{kT} G^*(B^{-1*} \bar{v}_k)\, dt.$$

Since the operator π is compact then

$$\pi v_{n_j} \longrightarrow \pi \bar{v}_k$$

and so

$$\lim_{j \to +\infty} \int_0^{kT} \prec B^{-1} A \pi v_{n_j}, v_{n_j} \succ \; dt = \int_0^{kT} \prec B^{-1} A \pi \bar{v}_k, \bar{v}_k \succ \; dt.$$

Consequently

$$\min_{E_k} \Psi_k = \Psi_k(\bar{v}_k).$$

Lemma 2.2 *For all $v \in E_k$ on which g is finite we have*

$$\bar{\partial} g(v) = \Big\{ u \in L^2(0, kT, \mathbb{R}^n) / \exists\, \xi \in \mathbb{R}^n \colon B(t)(u(t) + \xi) \in \partial G^*(B^{-1*} v) \ a.e. \Big\},$$

where $\bar{\partial} g$ denotes the restriction of g on E_k.

Proof Let $u \in L^2(0, kT, \mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$ such that

$$B(t)(u(t) + \xi) \in \partial G^*(B^{-1*} v) \quad a.e.$$

so it's easy to show that $u \in \bar{\partial} g(v)$.

Conversely, it's clear that for $v \in E_k$

$$\bar{\partial} g(v) = \partial(g + \delta_{E_k})(v),$$

where

$$\delta_{E_k}(v) = \begin{cases} 0 & \text{if } v \in E_k, \\ +\infty & \text{otherwise.} \end{cases}$$

Since

$$\partial g(v) = \{u \in L^2(0, kT, \mathbb{R}^n)/B(t)u(t) \in \partial G^*(B^{-1*}v) \ a.e.\}$$

and

$$\partial \delta_{E_k} = \mathbb{R}^n$$

the result will be proved if

$$\partial(g + \delta_{E_k}) = \partial g + \partial \delta_{E_k}.$$

The functionals g and δ_{E_k} are proper convex and l.s.c., it suffices to prove that the inf-convolute $g^* \nabla \delta^*_{E_k}$ is exact (i.e., the infimum is attained).

Indeed, we have

$$(g^* \nabla \delta^*_{E_k})(v) = \inf_{x \in \mathbb{R}^n} \int_0^{kT} G(B(t)v + B(t)x)\, dt.$$

The function

$$F(x) = \int_0^{kT} G(B(t)v + B(t)x)\, dt, \quad \forall\, x \in \mathbb{R}^n$$

is continuous on $\mathbb{R}^n$, so by (H_1) and the fact that $B(t)$ is invertible it's clear that $\lim\limits_{|x| \to +\infty} F(x) = +\infty$ and consequently F attains its minimum on $\mathbb{R}^n$.

Conclusion of the first part:

Let $\bar{v}_k \in E_k$, where Ψ_k attains its minimum, we have

$$0 \in Q'(\bar{v}_k) + \bar{\partial} g(\bar{v}_k)$$

which implies that

$$-Q'(\bar{v}_k) \in \bar{\partial} g(\bar{v}_k).$$

By Lemma 2.2, there exists $\xi_k \in \mathbb{R}^n$ such that

$$B(-B^{-1}A\pi\bar{v}_k + \pi(B^{-1}A)^*\bar{v}_k + \xi_k) \in \partial G^*(B^{-1*}\bar{v}_k) \quad a.e.$$

Setting

$$r_k = -\pi\bar{v}_k, \quad p_k = \pi(B^{-1}A)^*\bar{v}_k + \xi_k, \quad u_k = (r_k, p_k). \tag{5}$$

We get, by Fenchel's reciprocity

$$B^{-1^*}\bar{v}_k = \nabla G(Ar_k + Bp_k) \tag{6}$$

and

$$\begin{cases} \dot{r}_k = -\bar{v}_k = -B^*\nabla G(Ar_k + Bp_k) = -\dfrac{\partial H_0}{\partial p}(t, u_k(t)) \\ \dot{p}_k = (B^{-1}A)^*\bar{v}_k = A^*\nabla G(Ar_k + Bp_k) = \dfrac{\partial H_0}{\partial r}(t, u_k(t)). \end{cases}$$

Therefore u_k is a solution of $(\mathcal{H}_0)$, moreover since $\bar{v}_k \in E_k$, r_k is kT periodic.

In the other hand r_k is C^1 so $\dot{r}_k$ is kT periodic. By (H_3) and (6), we have

$$p_k = B^{-1}[\nabla G^{-1}(-B^{-1*}\dot{r}_k) - Ar_k]$$

so p_k is kT periodic and then u_k is kT periodic.

Proof of the second part:

By (H_1) and the convexity assumption of G there exist two constants $m, M > 0$ such that

$$G(x) \geq m|x| - M, \quad \forall\, x \in \mathbb{R}^n \tag{7}$$

so for all $y \in \mathbb{R}^n$ such that $|y| \leq m$ we have

$$-G(0) \leq G^*(y) \leq M. \tag{8}$$

Let

$$q(t) = a\cos\left(\frac{2\pi}{kT}\,t\right) + b\sin\left(\frac{2\pi}{kT}\,t\right)$$

with any $(a, b) \in \mathbb{R}^{2n}$.

It's clear that $q \in E_k$ and a simple computation gives for all $k \geq 3$

$$Q(q) = \frac{k^2T^2}{4\pi} \prec (C_0 - C_0^*)a, b \succ .$$

By the assumption (H_4), we can choose (a, b) such that

$$\begin{cases} \prec (C_0 - C_0^*)a, b \succ < 0 \\ \|B^{-1^*}q\|_\infty \leq m. \end{cases} \tag{9}$$

Setting $\delta = -\frac{T}{4\pi} \prec (C_0 - C_0^*)a, b \succ$, we have

$$Q(q) = -\delta T k^2, \text{ with } \delta > 0 \text{ independent of } k.$$

Now, by (8) and (9) we have

$$\Psi_k(\bar{v}_k) \leq \Psi_k(q) \leq -\delta T k^2 + MkT, \quad \forall\, k \geq 3 \tag{10}$$

and

$$Q(\bar{v}_k) \leq -\delta T k^2 + MkT + G(0)kT \leq 0 \tag{11}$$

for all $k \geq k_0$ sufficiently large.

In the other hand, by duality we have

$$G(Ar_k+Bp_k)+G^*(B^{-1*}\bar v_k)=\prec Ar_k+Bp_k, B^{-1*}\bar v_k\succ$$

and by integration, we obtain

$$\int_0^{kT} G(Ar_k+Bp_k)\,dt+\int_0^{kT} G^*(B^{-1^*}\bar v_k)\,dt=-2\int_0^{kT}\prec B^{-1}A\pi\bar v_k,\bar v_k\succ\ dt.$$

Then it follows from (10) and (11) that

$$\int_0^{kT} G(Ar_k+Bp_k)\,dt=-Q(\bar v_k)-\Psi_k(\bar v_k)\ge \delta T k^2-MkT,\quad \forall\, k\ge k_0$$

which gives

$$\frac{1}{kT}\int_0^{kT} G(Ar_k+Bp_k)\,dt\ge \delta k-M,\quad \forall\, k\ge k_0.$$

Hence by (2) we obtain

$$\delta k-M\le \frac{\alpha}{kT}\int_0^{kT}|Ar_k+Bp_k|^2dt+\beta\le \alpha\|Ar_k+Bp_k\|_\infty^2+\beta,\quad \forall\, k\ge k_0$$

and consequently

$$\lim_{k\to+\infty}\|Ar_k+Bp_k\|_\infty=+\infty.$$

To prove (ii) of Theorem 1.1, we need the following lemma:

Lemma 2.3 *For all T-periodic solution $u=(r,p)$ of $(\mathcal{H}_0)$ we have*

1. $\int_0^T |\dot u|^2dt\le \frac{2\alpha(\beta+M)\pi T}{\pi-\alpha T}$,
2. $\frac{1}{T}\int_0^T |Ar+Bp|\,dt\le \frac{(\beta+M)\pi}{m(\pi-\alpha T)}$.

Proof By (H_2) and (7), for all $\alpha\in\left]0,\frac{\pi}{T}\right[$ there exists $\beta>0$ only dependent on α such that

$$-M\le H_0(t,x)\le\frac{\alpha}{2}\,|x|^2+\beta,\quad \forall\, x\in\mathbb{R}^{2n},\quad \forall\, t\in[0,T].$$

A result of convex analysis gives

$$\frac{1}{2\alpha}\,|\nabla H_0(t,x)|^2\le \prec\nabla H_0(t,x),x\succ+\beta+M,\quad \forall\, x\in\mathbb{R}^{2n}.$$

It follows from $(\mathcal{H}_0)$ that

$$\frac{1}{2\alpha}\int_0^T|\dot u|^2dt+\int_0^T\prec J\dot u,u\succ\ dt\le(\beta+M)T$$

so

$$\left(\frac{1}{2\alpha} - \frac{T}{2\pi}\right) \int_0^T |\dot{u}|^2 dt \leq (\beta + M)T$$

and therefore

$$\int_0^T |\dot{u}|^2 dt \leq \frac{2\alpha(\beta + M)\pi T}{\pi - \alpha T}. \tag{12}$$

By convexity and (7), for all T-periodic solution $u = (r, p)$ of $(\mathcal{H}_0)$ we have

$$m \int_0^T |Ar + Bp|\, dt - MT \leq TG(0) + \frac{T}{2\pi} \int_0^T |\dot{u}|^2 dt. \tag{13}$$

By (12) and (13), we deduce the desired result.

Now, we shall prove that the minimal period of u_k tends to $+\infty$ as k tends to $+\infty$. If not, there exists $\tau > 0$ and a subsequence (k_n) such that the minimal period T_{k_n} of u_{k_n} satisfies $T_{k_n} \leq \tau$, $\forall\, n \in \mathbb{N}$. By Lemma 2.3, with T replaced by T_{k_n}, we get

$$\int_0^{T_{k_n}} |\dot{u}_{k_n}|^2 dt \leq \frac{2\alpha(\beta + M)\pi T_{k_n}}{\pi - \alpha T_{k_n}} \leq \frac{2\alpha(\beta + M)\pi\tau}{\pi - \alpha\tau} \tag{14}$$

and

$$\frac{1}{T_{k_n}} \int_0^{T_{k_n}} |Ar_{k_n} + Bp_{k_n}|\, dt \leq \frac{\pi(\beta + M)}{m(\pi - \alpha\tau)}. \tag{15}$$

Writing $u_k = \bar{u}_k + \tilde{u}_k$ with $\bar{u}_k = \frac{1}{T_k} \int_0^{T_k} u_k(t)\, dt$.

By Sobolev's inequality and (14), we obtain

$$\|\tilde{u}_{k_n}\|_\infty^2 \leq \frac{\tau}{12}\left(\frac{2\alpha(\beta + M)\pi\tau}{\pi - \alpha\tau}\right)$$

thus $\|\tilde{u}_{k_n}\|_\infty$ is bounded. By (5) we have

$$\bar{u}_{k_n} = (\bar{r}_{k_n}, \bar{p}_{k_n}) = (0, \xi_{k_n}).$$

Since $\|u_{k_n}\|_\infty \longrightarrow +\infty$ and $\|\widetilde{u}_{k_n}\|_\infty$ is bounded so $|\xi_{k_n}| \longrightarrow +\infty$.

In the other hand, by (15) we deduce that

$$\frac{1}{T} \int_0^T |B(t)\xi_{k_n}|\, dt = \frac{1}{T_{k_n}} \int_0^{T_{k_n}} |A(t)\bar{r}_{k_n} + B(t)\bar{p}_{k_n}|\, dt$$

is bounded, but this is in contradiction with the fact that

$$|B(t)\xi_{k_n}| \longrightarrow +\infty, \quad \forall\, t \in [0, T].$$

Then, the minimal period T_k of u_k tends to $+\infty$ as k tends to $+\infty$ and so for sufficiently large prime integer k, the minimal period of u_k is kT.

3 Proof of Theorem 1.2

We consider the functional Φ defined on the space $L_0^2 = L_0^2(0, T, \mathbb{R}^{2n})$ by

$$\Phi(w) = \frac{1}{2}\int_0^T \prec Jw, \pi w \succ\, dt + \int_0^T H_0^*(t, w - h)\, dt.$$

Let for $w \in L_0^2$

$$Q(w) = \frac{1}{2}\int_0^T \prec Jw, \pi w \succ\, dt \quad \text{and} \quad \psi(w) = \int_0^T H_0^*(t, w - h)\, dt.$$

We follow the same ideas of the proof of Theorem 1.1.

Lemma 3.1 *Φ achieves its minimum over L_0^2 in $\bar{v}$.*

Proof By (H_2), for all $\alpha \in]0, \frac{2\pi}{T}[$ there exists $\beta > 0$ such that

$$H_0(t, x) \le \frac{\alpha}{2}\,|x|^2 + \beta, \quad \forall\, x \in \mathbb{R}^{2n}, \quad \forall\, t \in [0, T],$$

and by going to the conjugate, we get

$$H_0^*(t, y) \ge \frac{1}{2\alpha}\,|y|^2 - \beta, \quad \forall\, y \in \mathbb{R}^{2n}, \quad \forall\, t \in [0, T]$$

so

$$\int_0^T H_0^*(t, w)\, dt \ge \frac{1}{2\alpha}\,\|w\|_{L^2}^2 - \beta T, \quad \forall\, w \in L_0^2.$$

Moreover, by Wirtinger's inequality, we get for all $w \in L_0^2$

$$\Phi(w) \ge \frac{1}{2}\left(\frac{1}{\alpha} - \frac{T}{2\pi}\right)\|w\|_{L^2}^2 + \frac{1}{2\alpha}\|h\|_{L^2}^2 - \frac{1}{\alpha}\|w\|_{L^2}\|h\|_{L^2} - \beta T. \tag{16}$$

Let $(v_n) \in L_0^2$ be a minimizing sequence of Φ. From (16), (v_n) is bounded and since L_0^2 is reflexive, there exists a subsequence (v_{n_k}) weakly convergent to $\bar{v}$.

Moreover, ψ is weakly l.s.c., so

$$\underline{\lim} \int_0^T H_0^*(t, v_{n_k} - h)\, dt \ge \int_0^T H_0^*(t, \bar{v} - h)\, dt$$

and

$$\lim_{k \to +\infty} \int_0^T \prec Jv_{n_k}, \pi v_{n_k} \succ\, dt = \int_0^T \prec J\bar{v}, \pi\bar{v} \succ\, dt.$$

Consequently

$$\min_{L_0^2} \Phi = \Phi(\bar{v}).$$

Lemma 3.2 *For every $v \in L_0^2$ on which ψ is finite, we have*

$$\bar{\partial}\psi(v) = \left\{u \in L^2 / \exists \xi \in \mathbb{R}^{2n} : u(t) + \xi \in \partial H_0^*(t, v(t) - h(t)) \ a.e.\right\}.$$

Proof Let $I(v) = \int_0^T H_0^*(t, v)\, dt, \ \forall\, v \in L^2$, then $\psi(v) = I(v - h)$.

For $u, v \in L_0^2$ and $\xi \in \mathbb{R}^{2n}$ such that

$$u(t) + \xi \in \partial H_0^*(t, v(t)) \ a.e.,$$

we can prove easily that $u \in \bar{\partial} I(v)$.

Conversely, it's clear that for $v \in L_0^2$ we have

$$\bar{\partial} I(v) = \partial(I + \delta_{L_0^2})(v),$$

where

$$\delta_{L_0^2}(v) = \begin{cases} 0 & \text{if } v \in L_0^2, \\ +\infty & \text{otherwise.} \end{cases}$$

Arguing as in proof of Lemma 2.2, it suffices to prove that the inf-convolution $I^* \nabla \delta_{L_0^2}^*$ is exact.

In fact, for $u = (r, p) \in L^2$ we have

$$(I^* \nabla \delta_{L_0^2}^*)(u) = \inf_{x \in \mathbb{R}^{2n}} \int_0^T H_0(t, u(t) + x)\, dt$$

$$= \inf_{(a,b) \in \mathbb{R}^{2n}} \int_0^T G[A(t)r + B(t)p + A(t)a + B(t)b]\, dt.$$

We need the following lemma:

Lemma 3.3 *The function*

$$F(a, b) = \int_0^T G(A(t)r + B(t)p + A(t)a + B(t)b)\, dt, \quad \forall\, (a, b) \in \mathbb{R}^{2n}$$

attains its minimum on $\mathbb{R}^{2n}$.

Proof Let

$$E = \left\{a \in \mathbb{R}^n / B^{-1}(t)A(t)a = B^{-1}(0)A(0)a, \ \forall\, 0 \le t \le T\right\},$$

E is a linear subspace of $\mathbb{R}^n$, so for all $a \in \mathbb{R}^n$ there exists $a_0 \in \mathbb{R}^n$ such that $a - a_0 \in E^{\perp}$. Notice that

$$F(a, b) = F(a - a_0, b + B^{-1}A(0)a_0) \in F(E^{\perp} \times \mathbb{R}^n)$$

so
$$\inf_{\mathbb{R}^{2n}} F = \inf_{E^\perp \times \mathbb{R}^n} F.$$
Arguing by contradiction, we suppose that $\inf\limits_{E^\perp \times \mathbb{R}^n} F$ is not attained so there exists a sequence $(a_n, b_n) \in E^\perp \times \mathbb{R}^n$ such that
$$\lim_{n\to+\infty} (a_n^2 + b_n^2) = +\infty \quad \text{and} \quad \lim_{n\to+\infty} F(a_n, b_n) = \inf F.$$
It follows that
$$\lim_{n\to+\infty} \frac{F(a_n, b_n)}{\sqrt{a_n^2 + b_n^2}} = 0.$$
In the other hand, by convexity of G, we have for n large enough
$$\int_0^T G\left(\frac{A(t)r + B(t)p + A(t)a_n + B(t)b_n}{\sqrt{a_n^2 + b_n^2}}\right) dt \le \frac{F(a_n, b_n)}{\sqrt{a_n^2 + b_n^2}} + \left(1 - \frac{1}{\sqrt{a_n^2 + b_n^2}}\right) G(0)T.$$
The sequence
$$\left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}}, \frac{b_n}{\sqrt{a_n^2 + b_n^2}}\right) \in E^\perp \times \mathbb{R}^n$$
is bounded, then by going to the limit in the above inequality through a subsequence, we obtain
$$\int_0^T G(A(t)a + B(t)b)\, dt \le G(0)T$$
for some $(a, b) \in E^\perp \times \mathbb{R}^n$ such that $a^2 + b^2 = 1$. Then
$$\int_0^T [G(A(t)a + B(t)b) - G(0)]\, dt \le 0$$
and by (H_7) we obtain
$$A(t)a + B(t)b = 0, \quad \forall\, t \in [0, T]$$
which is equivalent to
$$B^{-1}(t)A(t)a + b = 0, \quad \forall\, t \in [0, T],$$
but this is in contradiction with $a \in E^\perp$ and $a^2 + b^2 = 1$.

Conclusion of the proof

Let $\bar{v} \in L_0^2$ where Φ attains its minimum so
$$0 \in J\pi\bar{v} + \bar{\partial}\psi(\bar{v}).$$
By Lemma 3.2, there exists $\xi \in \mathbb{R}^{2n}$ such that
$$J\pi\bar{v} + \xi \in \partial H_0^*(t, \bar{v}(t) - h(t)) \ \ a.e.$$

Let $u = J\pi\bar{v} + \xi$, by Fenchel's reciprocity, we get

$$\dot{u} = J\bar{v} = J\nabla H(t, u(t))$$

and it's clear that $u(0) = u(T)$.

It remains to prove that u is not constant.

Setting $u = (r, p), (\mathcal{H}_h)$ is equivalent to

$$\dot{u}(t) = \begin{pmatrix} \dot{r} \\ \\ \dot{p} \end{pmatrix} = J \left[\begin{pmatrix} A^* \\ \\ B^* \end{pmatrix} \nabla G(Ar + Bp) + \begin{pmatrix} f \\ \\ g \end{pmatrix} \right]$$

but $\dot{u} = 0$ gives

$$-\begin{pmatrix} f \\ \\ g \end{pmatrix} = \begin{pmatrix} A^* \\ \\ B^* \end{pmatrix} \nabla G(Ar + Bp)$$

and then $(B^{-1}A)^* g = f$, which is in contradiction with the assumption (H_8).

References

[1] Aubin, J.P. and Ekeland, I. *Applied Non Linear Analysis.* Wiley, New York, 1986.

[2] Aze, D. and Blot, J. *Systèmes Hamiltoniens: Leurs Solutions Pé riodiques.* CEDIC-Nodhan, Paris, 1982. [France].

[3] Conley, C. and Zehnder, E. Subharmonic solutions and Morse theory. *Physica* **124A** (1984) 649–658.

[4] Ekeland, I. and Hofer, H. Subharmonics for convex non autonomous Hamiltonian systems. *Com. Pure Appl. Math.* **40** (1987) 1–36.

[5] Fonda, A. and Lazer, A.C. Subharmonic solutions of conservative systems with non convex potentials. *Proceeding of the Amer. Math. Soc.* **115-1** (1992) 183–190.

[6] Fonda, A. and Willem, M. Subharmonic oscillations of forced pendulum type equations. *J. Diff. Equations* **81** (1989) 215–220.

[7] Mawhin, J. and Willem, M. *Critical Point Theory and Hamiltonian Systems.* Springer-Verlag, Berlin, 1987.

[8] Michalek, R. and Tarantello, G. Subharmonic solutions with prescribed minimal period for non autonomous Hamiltonian systems. *J. Diff. Equations* **72** (1988) 28–55.

[9] Rabinowitz, P.H. On subharmonic solutions of Hamiltonian systems. *Com. Pure Appl. Math.* **33** (1980) 609–633.

[10] Timoumi, M. Subharmonics of convex non coercive Hamiltonian systems. *Collect. Math.* **43**(1) (1992) 63–69.

[11] Timoumi, M. Solutions périodiques des systèmes Hamiltoniens convexes non coèrcives. *Bulletin de la classe des sciences* 5^{eme} *série*, Tome L XXV, No. 12, 1989.

[12] Willem, M. Subharmonic oscillations of convex Hamiltonian systems. *Nonlinear Analysis* **9-11** (1985) 1303–1311.

Nonlinear Dynamics and Systems Theory, **3**(2) (2003) 151–161

Set Differential Equations and Monotone Flows

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Received: November 7, 2002; Revised: June 30, 2003

Abstract: Monotone iterative technique is extended to set differential equations. The nonlinear function involved is allowed to be difference of two monotone functions, which takes care of several results known and new.

Keywords: *Set differential equations; monotone flows.*

Mathematics Subject Classification (2000): 34A60, 49K24, 93B03, 94D05.

1 Preliminaries

Let $K(R^n)(K_c(R^n))$ denote the collection of all nonempty, compact (compact, convex) subsets of R^n. Define the Hausdorff metric

$$D[A,B] = \max\big[\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\big], \tag{1.1}$$

where $d[x,A] = \inf[d(x,y)\colon y \in A]$, A, B are bounded sets in R^n. We note that $K(R^n)$, $(K_c(R^n))$, with the metric is a complete metric space.

It is known that if the space $K_c(R^n)$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, then $K_c(R^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space [1, 9].

The Hausdorff metric (1.1) satisfies the following properties.

$$D[A+C, B+C] = D[A,B] \quad \text{and} \quad D[A,B] = D[B,A], \tag{1.2}$$

$$D[\lambda A, \lambda B] = \lambda D[A,B], \tag{1.3}$$

$$D[A,B] \le D[A,C] + D[C,B], \tag{1.4}$$

for all A, B, $C \in K_c(R^n)$ and $\lambda \in R^+$.

Let $A, B \in K_c(R^n)$. The set $C \in K_c(R^n)$ satisfying $A = B + C$ is known as the geometric difference of the sets A and B and is denoted by the symbol $A - B$. We say that the mapping $F\colon I \to K_c(R^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if there exists an element $D_H F(t_0) \in K_c(R^n)$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h}, \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist in the topology of $K_c(R^n)$ and are equal to $D_H F(t_0)$. Here I is any interval in R.

By embedding $K_c(R^n)$ as a complete cone in a corresponding Banach space and taking into account the result on differentiation of Bochner integral, we find that if

$$F(t) = X_0 + \int_0^t \Phi(s)\, ds, \qquad X_0 \in K_c(R^n), \tag{1.5}$$

where $\Phi\colon I \to K_c(R^n)$ is integrable in the sense of Bochner, then $D_H F(t)$ exists and the equality

$$D_H F(t) = \Phi(t), \quad \text{a.e on} \quad I, \tag{1.6}$$

holds. Also, the Hukuhara integral

$$\int_I F(s)\, ds = \left[\int_I f(s)\, ds \colon \ f \text{ is a continuous selector of } F \right],$$

for any compact set $I \subset R_+$. With these preliminaries, we consider the set differential equation

$$D_H U = F(t, U), \quad U(t_0) = U_0 \in K_c(R^n), \quad t_0 \geq 0, \tag{1.7}$$

where $F \in C[R_+ \times K_c(R^n), K_c(R^n)]$.

The mapping $U \in C^1[J, K_c(R^n)]$, $J = [t_0, t_0 + a]$ is said to be a solution of (1.7) on J if it satisfies (1.7) on J. Since $U(t)$ is continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^t D_H U(s)\, ds, \qquad t \in J. \tag{1.8}$$

Thus we associate with the initial value problem (IVP) (1.7) the following

$$U(t) = U_0 + \int_{t_0}^t F(s, U(s))\, ds, \qquad t \in J, \tag{1.9}$$

where the integral is the Hukuhara integral. Observe also that $U(t)$ is a solution of (1.7) iff it satisfies (1.9) on J. The investigation of set differential equation (1.7) as an independent subject has some advantages. For example, when $U(t)$ is a singlevalued mapping, it is easy to see that Hukuhara derivative and the integral reduce to the ordinary vector derivative and the integral, and therefore, the results obtained in this new framework of (1.7) become the corresponding results of ordinary differential systems. Also, we have only semilinear complete metric space to work with, in the present setup, compared to the complete normed linear space one employs in the study of ordinary differential systems. Furthermore, set differential equations, that are generated by multivalued differential inclusions, when the multivalued functions involved do not possess convex values, can be used as a tool for studying multivalued differential inclusions. See Tolstonogov [9]. Moreover one can utilize set differential equations of the type (1.7) to investigate profitably fuzzy differential equations, since the original formulation of which suffers from grave disadvantages and does not reflect the rich behavior of corresponding differential equations without fuzziness [2, 3, 6]. This is due to the fact that the diameter of any solution of fuzzy differential equation increases as time increases because of the necessity of the fuzzification of the derivative involved.

It is well known that the ideas embedded in the interesting and fruitful method of monotone iterative technique have proved to be of immense value and have played a crucial role in unifying a wide variety of nonlinear problems [4]. In this paper, we shall develop this technique to set differential equations (1.7) in a unified way following the work in [5]. In [7], we initiated the study of set differential equations of the type (1.7) as an independent subject and in [8] an interconnection between fuzzy differential equations and set differential equation is investigated.

2 Comparison Results

Let us introduce a partial ordering in the metric space $(K_c(R^n), D)$ which is needed in order to prove a basic comparison result that is required for our discussion.

We denote by $K(K^0)$ the subfamily of $K_c(R^n)$ consisting of sets $X \in K_c(R^n)$ such that any $x \in X$ is a nonnegative (positive) vector of n-components satisfying $x_i \geq 0$ $(x_i > 0)$ for $i = 1, 2, ..., n$. Thus K is a cone in $K_c(R^n)$ and K^0 is the nonempty interior of K. We can therefore induce a partial ordering in $K_c(R^n)$. See [1] for this approach.

Definition 2.1 For any X and $Y \in K_c(R^n)$, if there exists a $Z \in K_c(R^n)$ such that $Z \in K(K^0)$ and

$$X = Y + Z,$$

then we write $X \geq Y (X > Y)$ respectively. Similarly, one can define $X \leq Y (X < Y)$.

We can now prove the following basic result on set differential inequalities.

Theorem 2.1 *Assume that*

(i) $V, W \in C^1[R_+, K_c(R^n)]$, $F \in C[R_+ \times K_c(R^n), K_c(R^n)]$, $F(t, X)$ *is monotone nondecreasing in* X *for each* $t \in R_+$ *and*

$$D_H V \leq F(t, V), \quad D_H W \geq F(t, W), \quad t \in R_+;$$

(ii) *for any* X, $Y \in K_c(R^n)$ *such that* $X \geq Y$, $t \in R_+$,

$$F(t, X) \leq F(t, Y) + L(X - Y)$$

for some $L > 0$.

Then $V(t_0) \leq W(t_0)$ *implies*

$$V(t) \leq W(t), \quad t \geq t_0. \tag{2.1}$$

Proof Let $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) > 0$ and define $\tilde{W} = W + \epsilon e^{2Lt}$. Since $V(t_0) \le W(t_0) < \tilde{W}(t_0)$, it is enough to prove that

$$V(t) < \tilde{W}(t), \quad t \ge t_0, \tag{2.2}$$

to prove the conclusion (2.1) in view of the fact $\epsilon > 0$ is arbitrary.

Let $t_1 > 0$ be the supremum of all positive numbers $\delta > 0$ such that $V(t_0) < \tilde{W}(t_0)$ implies $V(t) < \tilde{W}(t)$ on $[t_0, \delta]$. It is clear that $t_1 > t_0$ and $V(t_1) \le \tilde{W}(t_1)$. From this follows, using the nondecreasing nature of F and condition (ii), that

$$\begin{aligned} D_H V(t_1) &\le F(t_1, V(t_1)) \le F(t_1, \tilde{W}(t_1)) \le F(t_1, W(t_1)) + L(\tilde{W} - W) \\ &\le D_H W(t_1) + L \in e^{2Lt_1} < D_H W(t_1) + 2L \in e^{2Lt_1} = D_H \tilde{W}(t_1). \end{aligned}$$

Consequently, it follows that there exists an $\eta > 0$ satisfying

$$V(t) - \tilde{W}(t) > V(t_1) - \tilde{W}(t_1), \quad t_1 - \eta < t < t_1.$$

This implies that $t_1 > t_0$ cannot be the supremum in view of the continuity of the functions involved and therefore the relation (2.2) is true, which, in turn, leads to the conclusion (2.1). The proof is complete.

The following corollary is useful.

Corollary 2.1 *Let V, $W \in C^1[R_+, K_c(R^n)]$, $\sigma \in C[R_+, K_c(R^n)]$. Suppose that*

$$D_H V \le \sigma, \quad D_H W \ge \sigma \quad \text{for} \quad t \ge t_0.$$

Then $V(t) \le W(t)$, $t \ge t_0$, provided $V(t_0) \le W(t_0)$.

3 Monotone Flows

In this section, we shall consider the following set differential equation

$$D_H U = F(t, U) + G(t, U), \quad U(0) = U_0 \in K_c(R^n), \tag{3.1}$$

where F, $G \in C[J \times K_c(R^n), K_c(R^n)]$ and $J = [0, T]$. We need the following definition which gives various possible notions of lower and upper solutions relative to (3.1).

Definition 3.1 Let V, $W \in C^1[J, K_c(R^n)]$. Then V, W are said to be

(a) *natural lower and upper solutions* of (3.1) if

$$D_H V \le F(t, V) + G(t, V), \quad D_H W \ge F(t, W) + G(t, W), \quad t \in J; \tag{3.2}$$

(b) *coupled lower and upper solutions* of type I of (3.1) if

$$D_H V \le F(t, V) + G(t, W), \quad D_H W \ge F(t, W) + G(t, V), \quad t \in J; \tag{3.3}$$

(c) *coupled lower and upper solutions* of type II of (3.1) if

$$D_H V \le F(t, W) + G(t, V), \quad D_H W \ge F(t, V) + G(t, W), \quad t \in J; \tag{3.4}$$

(d) *coupled lower and upper solutions* of type III of (3.1) if

$$D_H V \le F(t, W) + G(t, W), \quad D_H W \ge F(t, V) + G(t, V), \quad t \in J. \tag{3.5}$$

We observe that whenever we have $V(t) \le W(t)$, $t \in J$, if $F(t, X)$ is nondecreasing in X for each $t \in J$ and $G(t, Y)$ is nonincreasing in Y for each $t \in J$, the lower and upper solutions defined by (3.2) and (3.5) reduce to (3.4) and consequently, it is sufficient to investigate the cases (3.3) and (3.4).

We are now in a position to prove the following result.

Theorem 3.1 *Assume that*

(A_1) *V, $W \in C^1[J, K_c(R^n)]$ are coupled lower and upper solutions of type I relative to (3.1) with $V(t) \le W(t)$, $t \in J$;*

(A_2) *F, $G \in C[J \times K_c(R^n), K_c(R^n)]$, $F(t,X)$ is nondecreasing in X and $G(t,Y)$ is nonincreasing in Y, for each $t \in J$.*

Then there exist monotone sequences $\{V_n(t)\}$, $\{W_n(t)\} \in K_c(R^n)$ such that $V_n(t) \to \rho(t)$, $W_n(t) \to R(t)$ in $K_c(R^n)$ and (ρ, R) are the coupled minimal and maximal solutions of (3.1) respectively, that is, they satisfy

$$D_H\rho = F(t,\rho) + G(t,R), \quad \rho(0) = U_0,$$
$$D_H R = F(t,R) + G(t,\rho), \quad R(0) = U_0, \text{ on } J.$$

Proof For each $n \ge 0$, define the unique solutions $V_{n+1}(t)$, $W_{n+1}(t)$ by

$$D_H V_{n+1} = F(t, V_n) + G(t, W_n), \quad V_{n+1}(0) = U_0, \tag{3.6}$$
$$D_H W_{n+1} = F(t, W_n) + G(t, V_n), \quad W_{n+1}(0) = U_0, \quad t \in J, \tag{3.7}$$

where $V(0) \le U_0 \le W(0)$. We set $V_0 = V$, $W_0 = W$. Our aim is to prove

$$V_0 \le V_1 \le V_2 \le \dots \le V_n \le W_n \le \dots \le W_2 \le W_1 \le W_0, \quad t \in J. \tag{3.8}$$

Since V_0 is the coupled lower solutions of type I of (3.1), we have using the fact $V_0 \le W_0$ and the nondecreasing character of F,

$$D_H V_0 \le F(t, V_0) + G(t, W_0).$$

Also from (3.6), we get for $n = 0$,

$$D_H V_1 = F(t, V_0) + G(t, W_0).$$

Consequently, we arrive at $V_0 \le V_1$ on J. A similar argument shows that $W_1 \le W_0$ on J. We next prove $V_1 \le W_1$ on J. For this purpose consider

$$D_H V_1 = F(t, V_0) + G(t, W_0) \quad \text{and}$$
$$D_H W_1 = F(t, W_0) + G(t, V_0), \quad V_1(0) = W_1(0) = U_0.$$

Then, the monotone nature of F and G respectively yield

$$D_H V_1 \le F(t, W_0) + G(t, W_0), \quad D_H W_1 \ge F(t, W_0) + G(t, W_0), \quad t \in J.$$

We therefore have, by Corollary 2.1, $V_1 \le W_1$ on J. As a result, we obtain

$$V_0 \le V_1 \le W_1 \le W_0 \quad \text{on} \quad J. \tag{3.9}$$

Assume that for some $j > 1$, we have

$$V_{j-1} \le V_j \le W_j \le W_{j-1} \quad \text{on} \quad J. \tag{3.10}$$

Then we show that

$$V_j \le V_{j+1} \le W_{j+1} \le W_j \quad \text{on} \quad J. \tag{3.11}$$

To do this, consider

$$\begin{aligned} &D_H V_j = F(t, V_{j-1}) + G(t, W_{j-1}), \quad V_j(0) = U_0, \\ &D_H V_{j+1} = F(t, V_j) + G(t, W_j) \ge F(t, V_{j-1}) + G(t, W_{j-1}), \quad t \in J. \end{aligned}$$

Here we have employed (3.10) and the monotone nature of F and G. Corollary 2.1 now gives $V_j \le V_{j+1}$ on J. Similarly, we can get $W_{j+1} \le W_j$ on J. Next we show that $V_{j+1} \le W_{j+1}$, $t \in J$. We have from (3.6) and (3.7)

$$\begin{aligned} &D_H V_{j+1} = F(t, V_j) + G(t, W_j), \quad V_{j+1}(0) = U_0, \\ &D_H W_{j+1} = F(t, W_j) + G(t, V_j), \quad W_{j+1}(0) = U_0, \quad t \in J. \end{aligned}$$

Using (3.10) and the monotone character of F and G, we arrive at

$$\begin{aligned} &D_H V_{j+1} \le F(t, W_j) + G(t, W_j), \\ &D_H W_{j+1} \ge F(t, W_j) + G(t, W_j), \quad t \in J, \end{aligned}$$

and therefore Corollary 2.1 implies that $V_{j+1} \le W_{j+1}$, $t \in J$. Hence (3.11) follows and consequently, by induction (3.8) is valid for all n. Clearly the sequences $\{V_n\}$, $\{W_n\}$ are uniformly bounded on J. To show that they are equicontinuous, consider for any $s < t$, where t, $s \in J$,

$$\begin{aligned} D[V_n(t), V_n(s)] = D\Big[&U_0 + \int_0^t \{F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi))\}\, d\xi, \\ &U_0 + \int_0^s \{F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi))\}\, d\xi \Big] \\ = D\Big[\int_0^t \{F(\xi, V_{n-1}(\xi)) &+ G(\xi, W_{n-1}(\xi)\}\, d\xi, \int_0^s \{F(\xi, V_{n-1}(\xi) + G(\xi, W_{n-1}(\xi))\}\, d\xi \Big] \\ \le \int_s^t D[F(\xi, V_{n-1}(\xi)) &+ G(\xi, W_{n-1}(\xi)), \theta]\, d\xi \le M|t - s|. \end{aligned}$$

Here we have utilized the properties of integral and the metric D, together with the fact $F + G$ are bounded since $\{V_n\}$, $\{W_n\}$ are uniformly bounded. Hence $\{V_n(t)\}$ is equicontinuous on J. The corresponding Ascoli's Theorem [9] now gives a subsequence $\{V_{n_k}(t)\}$ which converges uniformly to $\rho(t) \in K_c(R^n)$, $t \in J$, and since $\{V_n(t)\}$ is monotone nondecreasing sequence, the entire sequence $\{V_n(t)\}$ converges uniformly to $\rho(t)$ on J. Similar arguments apply to the sequence $\{W_n(t)\}$ and $W_n(t) \to R(t)$ uniformly on J. It therefore follows, using the integral representations of (3.6) and (3.7) that $\rho(t)$, $R(t)$ satisfy

$$\begin{cases} D_H \rho(t) = F(t, \rho(t)) + G(t, R(t)), \quad \rho(0) = U_0, \\ D_H R(t) = F(t, R(t)) + G(t, \rho(t)), \quad R(0) = U_0, \quad t \in J, \end{cases} \tag{3.12}$$

and that

$$V_0 \le \rho \le R \le W_0, \quad t \in J. \tag{3.13}$$

Next we claim that (ρ, R) are coupled minimal and maximal solution of (3.1), that is, if $U(t)$ is any solution of (3.1) such that

$$V_0 \le U \le W_0, \quad t \in J, \tag{3.14}$$

then

$$V_0 \le \rho \le U \le R \le V_0, \quad t \in J. \tag{3.15}$$

Suppose that for some n,

$$V_n \le U \le W_n \quad \text{on} \quad J. \tag{3.16}$$

Then we have using monotone nature of F, G and (3.16),

$$\begin{aligned} &D_H U = F(t,U) + G(t,U) \ge F(t,V_n) + G(t,W_n), \quad U(0) = U_0, \\ &D_H V_{n+1} = F(t,V_n) + G(t,W_n), \quad V_{n+1}(0) = U_0. \end{aligned}$$

Corollary 2.1 yields $V_{n+1} \le U$ on J. Similarly $W_{n+1} \ge U$ on J. Hence by induction (3.16) is true for all $n \ge 1$. Now taking limit as $n \to \infty$, we get (3.15) proving the claim. The proof is therefore complete.

Corollary 3.1 *If, in addition to the assumptions of Theorem 3.1, F and G satisfy whenever $X \ge Y$, $X, Y \in K_c(R^n)$,*

$$F(t,X) \le F(t,Y) + N_1(X - Y)$$

and

$$G(t,X) + N_2(X - Y) \ge G(t,Y),$$

where N_1, $N_2 > 0$. Then $\rho = R = U$ is the unique solution of (3.1).

Proof Since $\rho \le R$ on J, it is enough to prove that $R \le \rho$ on J. We know that

$$\begin{aligned} &D_H \rho = F(t,\rho) + G(t,R), \quad \rho(0) = U_0, \\ &D_H R = F(t,R) + G(t,\rho), \quad R(0) = U_0, \quad t \in J. \end{aligned}$$

Using the assumptions, we then get

$$D_H(R - \rho) \le (N_1 + N_2)(R - \rho),$$

which leads to by Theorem 2.1, $R \le \rho$ on J, proving the claimed uniqueness of $\rho = R = U$, completing the proof.

Several remarks are now in order.

Remark 3.1

(1) In Theorem 3.1, if $G(t,Y) \equiv 0$, then we get a result when F is nondecreasing.
(2) In (1) above, suppose that F is not nondecreasing but $\tilde{F}(t,X) = F(t,X) + MX$ is nondecreasing in X for each $t \in J$, for some $M > 0$, then one can consider the IVP

$$D_H U + MU = \tilde{F}(t,U), \quad U(0) = U_0,$$

where $\tilde{F}(t,X) = F(t,X) + MX$ to obtain the same conclusion as in (1). To see this, use the transformation $\tilde{U} = Ue^{Mt}$ so that

$$\begin{gathered} D_H\tilde{U} = [D_HU + MU]e^{Mt} = \tilde{F}(t, \tilde{U}e^{-Mt})e^{Mt} \equiv F_0(t,\tilde{U}), \\ \tilde{U}(0) = U_0. \end{gathered} \tag{3.17}$$

Clearly (3.17) has $\tilde{V} = Ve^{Mt}$ as a lower solution and $\tilde{W} = We^{Mt}$ as an upper solution and therefore we have the same conclusion as in (1).

(3) If $f(t,X) \equiv 0$ in Theorem 3.1, then we obtain the result for G nonincreasing.

(4) If in (3) above, G is not monotone but $\tilde{G}(t,Y) = G(t,Y) - MY$, $M > 0$ is nonincreasing in Y for each $t \in J$, then one can consider the IVP

$$D_HU - MU = \tilde{G}(t,U), \quad U(0) = U_0.$$

The transformation $\tilde{U} = Ue^{-Mt}$ gives the IVP

$$D_H\tilde{U} = G_0(t,\tilde{U}), \quad \tilde{U}(0) = U_0, \tag{3.18}$$

where $G_0(t,\tilde{U}) = \tilde{G}(t,\tilde{U}e^{Mt})e^{-Mt}$. In this case, we need to assume that (3.18) has coupled lower and upper solutions of (3.18) to get the same conclusion as in (3).

(5) Suppose that in Theorem 3.1, $G(t,Y)$ is nonincreasing in Y and $F(t,X)$ is not monotone but $\tilde{F}(t,X) = F(t,X) + MX$, $M > 0$ is nondecreasing in X. Then we consider the IVP

$$D_HU + MU = \tilde{F}(t,U) + G(t,U), \quad U(0) = U_0. \tag{3.19}$$

The transformation as in (2) yields the conclusion by Theorem 3.1 in this case as well.

(6) If F in Theorem 3.1 is nondecreasing and G is not monotone but $\tilde{G}_0(t,Y) = G(t,Y) - MY$, $M > 0$ is nonincreasing in Y for each $t \in J$, then we consider the IVP

$$D_HU - MU = F(t,U) + \tilde{G}(t,U), \quad U(0) = U_0,$$

and employ the same transformation as in (4) to obtain

$$D_H\tilde{U} = F_0(t,\tilde{U}) + G_0(t,\tilde{U}), \quad \tilde{U}(0) = U_0, \tag{3.20}$$

where $F_0(t,\tilde{U}) = F(t,\tilde{U}e^{Mt})e^{-Mt}$ and $G_0(t,\tilde{U}) = \tilde{G}(t,\tilde{U}e^{Mt})e^{-Mt}$. If we assume that (3.20) has coupled lower and upper solutions of type I then we get by Theorem 3.1 the same result in this case also.

(7) If both F and G are not monotone in Theorem 3.1 but $\tilde{F}(t,X) = F(t,X) + MX$, $M > 0$, $\tilde{G}(t,Y) = G(t,Y) - NY$, $N > 0$ are nondecreasing and nonincreasing respectively, then we consider the IVP

$$D_HU + (M - N)U = \tilde{F}(t,U) + \tilde{G}(t,U), \quad U(0) = U_0,$$

one can utilize a similar transformation to obtain

$$D_H\tilde{U} = F_0(t,\tilde{U}) + G_0(t,\tilde{U}), \quad \tilde{U}(0) = U_0, \tag{3.20*}$$

where F_0, G_0 are defined suitably as before. Assuming that (3.20*) has coupled lower and upper solutions of type I, one gets the same conclusion by Theorem 3.1.

Let us next consider utilizing the coupled lower and upper solutions of type II. In this case, we don't need to assume the existence of coupled lower and upper solutions of type II of (3.1) since one can construct them under the given assumptions. However, we have to pay a price to get monotone flows, by assuming certain conditions on the second iterates. Also, we get alternative sequences which are monotone but complicated.

Theorem 3.2 *Assume that* (A_2) *of Theorem 3.1 holds. Then for any solution* $U(t)$ *of (3.1) with* $V_0 \le U \le W_0$ *on* J*, we have the iterates* $\{V_n\}$, $\{W_n\}$ *satisfying*

$$V_0 \le V_2 \le \dots \le V_{2n} \le U \le V_{2n+1} \le \dots \le V_3 \le V_1 \quad \text{on} \quad J, \tag{3.21}$$

$$W_1 \le W_3 \le \dots \le W_{2n+1} \le U \le W_{2n} \le \dots \le W_2 \le W_0 \quad \text{on} \quad J, \tag{3.22}$$

Provided $V_0 \le V_2$, $W_2 \le W_0$ *on* J*, where the iterative schemes are given by*

$$D_H V_{n+1} = F(t, W_n) + G(t, V_n), \quad V_{n+1}(0) = U_0, \tag{3.23}$$

$$D_H W_{n+1} = F(t, V_n) + G(t, W_n), \quad W_{n+1}(0) = U_0, \quad \text{on} \quad J. \tag{3.24}$$

Moreover, the monotone sequences $\{V_{2n}\}$, $\{V_{2n+1}\}$, $\{W_{2n}\}$, $\{W_{2n+1}\} \in K_c(R^n)$ *converge to* ρ, R, ρ^*, R^* *in* $K_c(R^n)$ *respectively and verify*

$$\begin{aligned}
&D_H R = F(t, R^*) + G(t, \rho), && R(0) = U_0,\\
&D_H \rho = F(t, \rho^*) + G(t, R), && \rho(0) = U_0,\\
&D_H R^* = F(t, R) + G(t, \rho^*), && R^*(0) = U_0,\\
&D_H \rho^* = F(t, \rho) + G(t, R^*), && \rho^*(0) = U_0, \quad \text{on} \quad J.
\end{aligned}$$

Proof We shall first show that coupled lower and upper solutions V_0, W_0 of type II of (3.1) exist on J satisfying $V_0 \le W_0$ on J. For this purpose, consider the IVP

$$D_H Z = F(t, \theta) + G(t, \theta), \quad Z(0) = U_0. \tag{3.25}$$

Let $Z(t)$ be the unique solution of (3.25) which exists on J. Define V_0, W_0 by

$$R_0 + V_0 = Z \quad \text{and} \quad W_0 = Z + R_0,$$

where the positive vector $R_0 = (R_{01}, R_{02}, \dots, R_{0n})$ is chosen sufficiently large so that we have $V_0 \le \theta \le W_0$ on J. Then using the monotone character of F and G, we arrive at

$$\begin{gathered}
D_H V_0 = D_H Z = F(t, \theta) + G(t, \theta) \le F(t, W_0) + G(t, V_0),\\
V_0(0) = Z(0) - R_0 \le Z(0) = U_0.
\end{gathered}$$

Similarly, $D_H W_0 \ge F(t, V_0) + G(t, W_0)$, $W_0(0) \ge U_0$. Thus V_0, W_0 are the coupled lower and upper solutions of type II of (3.1).

Let $U(t)$ be any solution of (3.1) such that $V_0 \le U \le W_0$ on J. We shall show that

$$\begin{gathered}
V_0 \le V_2 \le U \le V_3 \le V_1,\\
W_1 \le W_3 \le U \le W_2 \le W_0 \quad \text{on} \quad J.
\end{gathered} \tag{3.26}$$

Since U is a solution of (3.1), we have using the monotone character of F and G and the fact $V_0 \le U \le W_0$,

$$D_H U = F(t,U) + G(t,U) \le F(t,W_0) + G(t,V_0), \quad U(0) = U_0,$$

and V_1 satisfies

$$D_H V_1 = F(t,W_0) + G(t,V_0), \quad V_1(0) = U_0, \quad \text{on} \quad J. \tag{3.27}$$

Hence Corollary 2.1 yields $U \le V_1$ on J. Similarly, $W_1 \le U$ on J. Next we show that $V_2 \le U$ on J. Note that

$$D_H V_2 = F(t,W_1) + G(t,V_1), \quad V_2(0) = U_0,$$

and because of monotonicity of F and G, we get

$$D_H U = F(t,U) + G(t,U) \ge F(t,W_1) + G(t,V_1), \quad U(0) = U_0 \quad \text{on} \quad J.$$

Corollary 2.1 therefore gives $V_2 \le U$ on J. A similar argument shows that $U \le W_2$ on J. Next we find utilizing the assumption $V_0 \le V_2$, $W_2 \le W_0$ on J and monotonicity of F and G,

$$D_H V_3 = F(t,W_2) + G(t,V_2) \le F(t,W_0) + G(t,V_0), \quad V_3(0) = U_0 \quad \text{on} \quad J.$$

This together with (3.27) shows by Corollary 2.1 that $V_3 \le V_1$, on J. In the same way one can show that $W_1 \le W_3$ on J. Also, employing a similar reasoning, one can prove that $U \le V_3$ and $W_3 \le U$ on J, proving the relations (3.26).

Now assuming for some $n > 2$, the inequalities

$$\begin{gathered} V_{2n-4} \le V_{2n-2} \le U \le V_{2n-1} \le V_{2n-3}, \\ W_{2n-3} \le W_{2n-1} \le U \le W_{2n-2} \le W_{2n-4}, \quad \text{on} \quad J, \end{gathered}$$

to hold, it can be shown, employing similar arguments that

$$\begin{gathered} V_{2n-2} \le V_{2n} \le U \le V_{2n+1} \le V_{2n-1}, \\ W_{2n-1} \le W_{2n+1} \le U \le W_{2n} \le W_{2n-2}, \quad \text{on} \quad J. \end{gathered}$$

Thus by induction (3.21) and (3.22) are valid for all $n = 0, 1, 2, \ldots$.

Since V_n, $W_n \in K_c(R^n)$ for all n, employing a similar reasoning as in Theorem 3.1, we conclude that the limits

$$\begin{aligned} &\lim_{n\to\infty} V_{2n} = \rho, && \lim_{n\to\infty} V_{2n+1} = R, \\ &\lim_{n\to\infty} W_{n+1} = \rho^*, && \lim_{n\to\infty} W_{2n} = R^*, \end{aligned}$$

exist, in $K_c(R^n)$, uniformly on J. It therefore follows using the integral representations (3.23) and (3.24) suitably that ρ, ρ^*, R, R^* satisfy corresponding set differential equations given in Theorem 3.2 on J. Also, from (3.21)and (3.22), we arrive at

$$\rho \le U \le R, \quad \rho^* \le U \le R^* \quad \text{on} \quad J.$$

The proof is therefore complete.

Corollary 3.2 *Under the assumptions of Theorem 3.2 if F and G satisfy the assumptions of Corollary 3.1, then $\rho = \rho^* = R = R^* = U$ is the unique solution of (3.1).*

Proof Let $q_1 + \rho = R$, $q_2 + \rho^* = R^*$ so that q_1, $q_2 \geq 0$ on J, since $\rho \leq R$ and $\rho^* \leq R^*$ on J. It then follows using the assumptions, that

$$D_H(q_1 + q_2) \leq (N_1 + N_2)(q_1 + q_2), \quad q_1(0) + q_2(0) = 0 \quad \text{on} \quad J.$$

This implies that $q_1 + q_2 \leq 0$ on J and consequently, we get

$$U = \rho = R \quad \text{and} \quad \rho^* = R^* = U \quad \text{on} \quad J,$$

and this proves the claim of Corollary 3.2.

Theorem 3.2 also has several remarks which correspond to the remarks of Theorem 3.1. To avoid monotony we do not list them again. For similar results which unify monotone iterative technique refer to [5].

References

[1] Brandao Lopes Pinto, A.J., De Blasi, F.S. and Iervolino, F. Uniqueness and existence theorems for differential equations with compact convex-valued solutions. *Boll. Unione. Mat. Italy* **3** (1970) 47–54.

[2] Diamond, P. and Kloeden, P. *Metric Spaces of Fuzzy Sets.* World Scientific, Singapore, 1992.

[3] Hüllermeier, E. An approach to modeling and simulation of uncertain dynamical systems. *Int. J. Uncertainty, Fuzziness and Knowledge Based Systems* **5** (1997) 117–137.

[4] Ladde, G.S., Laksmikantham, V. and Vatsala, A.S. *Monotone Iterative Techniques in Nonlinear Differential Equations.* Pitman publishing, London, 1985.

[5] Lakshmikantham, V. and Koksal, S. *Monotone Flows and Rapid Convergence for Nonlinear Partial Differential Equations.* Taylor & Francis Ltd, U.K., (to appear).

[6] Lakshmikantham, V. and Mohapatra, R. *Theory of Fuzzy Differential Equations and Inclusions.* Taylor & Francis Ltd, U.K., (to appear).

[7] Lakshmikantham, V., Leela, S. and Vatsala, A.S. Setvalued hybrid differential equations and stability in terms of two measures. *Journal of Hybrid Systems* **2** (2002) 169–187.

[8] Lakshmikantham, V., Leela, S. and Vatsala, A.S. Interconnection between setvalued and fuzzy differential equations. *Nonlinear Analysis*, (to appear).

[9] Tolstonogov, A. *Differential Inclusions in a Banach Space.* Kluwer Academic Publishers, Dordrecht, 2000.

Nonlinear Dynamics and Systems Theory, **3**(2) (2003) 163–176

Nash-Optimisation Enhanced Distributed Model Predictive Control

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Received: May 28, 2003; Revised: September 15, 2003

Abstract: This study presents an efficient distributed model predictive control scheme based on Nash optimality, in which the on-line optimisation of the whole system is decomposed into that of several small co-operative agents in distributed structures, thus it can significantly reduce computational complexity in model predictive control of large-scale systems. The relevant nominal stability and the performance on single-step horizon under the communication disturbance are investigated. A three input and three output linear model is simulated to test the effectiveness of the proposed control algorithm.

Keywords: *Model predictive control (MPC); distributed control system; Nash optimality; multi-agents.*

Mathematics Subject Classification (2000): 93C55, 93C10, 49K99.

1 Introduction

Model predictive control (MPC) is a popular technique and has been successfully used in the control of various linear and nonlinear dynamic systems (see [1, 7, 17]). However, an obvious drawback of MPC involved in the formidable on-line computational effort limits its applicability to relatively fast and/or large processes with moderate number of inputs ([5]). Practically, there exists a great number of complex high dimensional systems, in which the number of variables and constraints is of ten several dozens or even several hundreds. Thus it has become very important to develop computationally efficient control architectures and algorithms with less computational burden. Unfortunately, with the possible exception of the studies by [9, 13, − 16]. Van Antwerp and Braatz [10] to reflect the large-scale nature of typical industrial plants, references for this topic are little in open literature. This probably is attributable to the inherent difficulties involved incomplex computation for large-scale processes. With the rapid development of communication network and the field-bus technology, centralised control has not been a sole structure in applications and has been gradually replaced by distributed control in large-scale systems. Distributed control structure brings new requirements to the traditional control field and allows the conceivability of new challenging control applications. For economic consideration and also no degrading performance, it is desirable to use several inexpensive microcomputers to replace a very high performance computer in control systems. The development of communication network and the field-bus technology has provided possibility for this distributed control. Xu, *et al.* [13] and Xi [12] proposed a decentralised predictive control algorithm. Zheng [14, 15], Zheng and Allgower [16], proposed a one-step approximation algorithm to reduce the on-line computation by decreasing the number of the decision variables. More recently Gurfil and Kasdin [3] developed an iterative ellipsoid algorithm to allow the quick computation of sub-optimal control moves. It should be pointed out that these approaches still take centralised computation and therefore need high cost computers. In this study, an efficient distributed optimisation scheme is developed based on Nash optimality for MPC of large-scale systems. Under this scheme, on-line optimisation of the whole system is decomposed into that of several small co-operative agents. These agents can co-operate and communicate each other in a distributed structure to achieve the objective of the whole system. Accordingly the computational complexity for such large-scale systems is significantly reduced. Since the protocol of mutual communication and information exchange is adequately taken into account, this approach can efficiently improve control performance and guarantee the Nash optimality ([6]). The second part of the study is to analyse the relevant performance of the developed method. The nominal stability and the convergent condition of this distributed control system are derived. The performance deviation on single-step horizon under the communication disturbance is also analysed with an assumption that the algorithm is convergent. The significance of this scheme is to reduce the computational burden in complex large-scale systems. Also it can be extended to the remote control and multi-agent systems. The main contents of the study is divided into five sections. In Section 2 distributed MPC algorithm based on Nash optimality is proposed. In Section 3 the convergent condition of the distributed predictive control algorithm for linear models is analysed. In Sections 4 and 5 the nominal stability and the performance deviation under disturbance are analysed respectively. In Section 6 a simulation example is presented to demonstrate the efficiency of the distributed MPC algorithm.

2 Distributed Model Predictive Control Algorithm Based on Nash Optimality

2.1 Model predictive control

Model predictive control (MPC) is formulated as resolving an on-lineopen-loop optimal control problem in moving horizon style. Using the current state, an input sequence is calculated to minimise a performance index while satisfying some specified constraints. Only the first element of the sequence is taken as controller output. At the next sampling time, the optimisation is resolved with new measurements from the plant. Thus both the control horizon and the prediction horizon move or recede ahead by one step at next sampling time. This is the reason why MPC is also sometimes referred to as receding horizon control (RHC) or moving horizon control (MHC). The purpose of taking new measurements at each sampling time is to compensate for unmeasured disturbances and model inaccuracy, both of which cause the system output to be different from its prediction. Suppose the prediction output model of the whole system is described as

$$Y(k+j \mid k) = f(Y(k), \Delta u(k|k)) \quad (j = 1, \dots, P), \tag{1}$$

where $\Delta u_M(k) = \left(\Delta u_{1,M}^T(k) \dots \Delta u_{m,M}^T(k)\right)^T$ is the increment of the manipulated (the controller output, also the input toplant) variables of the system, denotes the prediction horizon, denotes the control horizon, is the mapping function vector, where the element satisfied some smooth condition. The performance index of the whole system is

$$\min_{\Delta u_M(k|k)} J = \sum_{i=1}^{P} L[y(k+i \mid k), \Delta u_M(k \mid k)], \tag{2}$$

where L is the nonlinear function of input and output variables. The objective of the whole system is to regulate the system output to the expected values while keeping the performance minimal. For large-scale systems, because of the effect of control horizon M, the optimised variables $\Delta u_M(k)$ at each sampling time are highly dimensional, the computation is intensive, especially for nonlinear systems, which accordingly requires high performance computers or some advanced algorithms. To avoid the prohibitively high on-line computational demand, this study proposes a distributed scheme with inexpensive agent computers under network environment.

2.2 Distributed MPC strategy based on Nash optimality

The main idea of the distributed model predictive control algorithm is the on-line optimisation of MPC. Since an optimisation formulation in large-scale systems can be decomposed into a number of small-scale optimisations. These autonomous agents are connected via network with dynamic input coupling among them, share the common resources, communicate and co-ordinate each other in order to accomplish the whole objective. Suppose the behaviour of the whole system is described by m agents and the performance index (2) is separable for m agents. The local performance index for the i-th agent can be expressed as

$$\min_{\Delta u_i, M} J_i = \sum_{j=1}^{P} L_i[y_i(k+j \mid k), \Delta u_{i,M}(k \mid k)]. \tag{3}$$

This indicates the global performance index of the whole system is

$$\min J = \sum_{i=1}^{m} J_i. \tag{4}$$

At time instant k, the future predictive output of the i-th agent can be expressed as

$$y_i(k+j \mid k) = f_i[y_i(k), \Delta u_{1,M}(k \mid k), \cdots, \Delta u_{m,M}(k \mid k)], \quad (i = 1, \dots, P). \tag{5}$$

It can be seen that the global performance index can be decomposed into a number of local performance indices, but the output of each agent is still related to all the input variables due to the input coupling. Such distributed control problem with different goals can be resolved by means of Nash optimal concept ([6]). Concretely speaking, each agent optimises its objective (local performance index) only using its own input variables assuming that the other agent's optimal solutions are known, that is

$$\left.\frac{\partial J_i}{\partial \Delta u_{i,M}(k)}\right|_{\Delta u^*_{j,M}(k),\, j=1,\ldots,m,\, j\neq i} = 0 \quad (i = 1, \ldots, m). \tag{6}$$

Thus the resulted Nash optimal solution satisfies the Nash optimality condition

$$\begin{aligned} J_i(\Delta u^*_{1,M}(k), \cdots, \Delta u^*_{m,M}(k)) \le J_i(&\Delta u^*_{1,M}(k), \cdots, \Delta u^*_{i-1,M}(k), \\ &\Delta u_{i,M}(k), \Delta u^*_{i+1,M}(k), \cdots \Delta u^*_{m,M}(k)). \end{aligned} \tag{7}$$

Inspection of (5) to obtain the Nash optimal solution $\Delta u^*_{i,M}(k)$ of the i-th agent, it is necessary to know the other agent's Nash optimal solutions $\Delta u^*_{j,M}(k)$ $(j \neq i)$, so that the whole system could arrive at Nash optimal equilibrium in this coupling decision process. Here an iterative algorithm is proposed to seek the Nash optimal solution of the whole system at each sampling time. Each agent compares the newly computed optimal solution with that obtained in last iteration, and checks if its terminal condition is satisfied. If the algorithm is convergent, all the terminal conditions of the m agents will be satisfied, and the whole system will arrive at Nash equilibrium at this time. This Nash optimisation process will be repeated at next sampling time.

Algorithm:

Step 1: At sampling time k, each agent makes initial estimation of the input variables and announces it to the other agents, let the iterative index $l = 0$,

$$\Delta \bar{u}^l_{i,M}(k) = [\Delta \bar{u}^l_i(k), \Delta \bar{u}^l_i(k+1), \cdots, \Delta \bar{u}^l_i(k+M-1)]^T, \\ (i = 1, \cdots, m).$$

Step 2: Each agent resolves its optimal problem simultaneously to obtain its solution $\Delta u^*_{i,M}(k)$, $(i = 1, \cdots, m)$.

Step 3: Each agent checks if its terminal iteration condition is satisfied, that is, for the given error accuracy ε_i, $(i = 1, \cdots, m)$, if there exist

$$\|\Delta u^{l+1}_{i,M}(k) - \Delta \bar{u}^l_{i,M}(k)\| \le \varepsilon_i \quad (i = 1, \cdots, m).$$

If all the terminal conditions are satisfied, then end the iteration and go to step 4; otherwise, let $l = l + 1$, $\Delta \bar{u}^l_{i,M}(k) = \Delta u^*_{i,M}(k)$, $(i = 1, \cdots, m)$ all agents communicate to exchange this information, and take the latest solution to step 2.

Step 4: Compute the instant control law

$$\Delta u_i(k) = [I \ \cdots \ \mathbf{0}]\Delta u^*_{i,M}(k) \quad (i = 1, \cdots, m)$$

and take the first element as the controller output from each agent.

Step 5: Move horizon to the next sampling time, that is, $k + 1 \rightarrow k$, and go to step 1.

3 Computational Convergence for Linear Systems

Consider this distributed model predictive control of linear dynamic plants. At sampling time k, the output prediction model of the i-th agent can be described as

$$\tilde{y}_{i,PM}(k) = \tilde{y}_{i,P0}(k) + A_{ii}\Delta u_{i,M}(k) + \sum_{\substack{j=1 \\ j\neq i}}^{m} A_{ij}\Delta u_{j,M}, \qquad (i = 1, \cdots, m), \tag{8}$$

where A_{ii} and A_{ij} are the dynamic matrix of the i-th agent and the step response matrix of the i-th agent stimulated by the j-th agent respectively. They are expressed in terms of matrix

$$A_{ij} = \begin{bmatrix} a_{ij}(1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ a_{ij}(M) & \dots & a_{ij}(1) \\ \vdots & \vdots & \vdots \\ a_{ij}(P) & \cdots & a_{ij}(P-M+1) \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix},$$

where $a_{ij}(k)$, $(k = 1, 2, \dots,\ i,\ j = 1, \dots, m)$ is the step response matrix array. The local performance index for the i-th agent can be expressed as

$$\min J_i = \|\varpi_i(k) - \tilde{y}_{i,PM}(k)\|^2_{Q_i} + \|\Delta u_{i,M}(k)\|^2_{R_i} \quad (i = 1, \cdots, m), \tag{9}$$

where $\varpi_i(k) = [\varpi_i(k+1) \ \cdots \ \varpi_i(k+P)]^T$, $(i = 1, \cdots, m)$ is the expected output of the i-th agent, and

$$\begin{aligned} \tilde{y}_{i,PM}(k) &= [\tilde{y}_{i,M}(k+1 \mid k) \ \cdots \ \tilde{y}_{i,M}(k+P \mid k)]^T, \\ \tilde{y}_{i,P0}(k) &= [\tilde{y}_{i,0}(k+1 \mid k) \ \cdots \ \tilde{y}_{i,0}(k+P \mid k)]^T, \\ \Delta u_{i,M}(k) &= [\Delta u_i(k|k) \ \cdots \ \Delta u_i(k+M-1 \mid k)]^T. \end{aligned}$$

According to Nash optimality, at sampling time k, the Nash optimal solution of the i-th agent can be derived as

$$\Delta u^*_{i,M}(k) = D_{ii}[\varpi_i(k) - \tilde{y}_{i,P0}(k) - \sum_{\substack{j=1 \\ j\neq i}}^{m} A_{ij}\Delta u_{j,M}(k)] \quad (i = 1, \cdots, m), \tag{10}$$

where $D_{ii} = (A^T_{ii} Q A_{ii} + R_i)^{-1} A^T_{ii} Q_i$. If the algorithm is convergent, the Nash optimal solution of the whole system can be written as

$$\Delta u_M(k) = D_1[\varpi(k) - \tilde{y}_{P0}(k)] + D_0 \Delta u_M(k), \tag{11}$$

where

$$D_1 = \begin{bmatrix} D_{11} & & & \\ & D_{22} & & \\ & & \ddots & \\ & & & D_{mm} \end{bmatrix},$$

$$D_0 = \begin{bmatrix} 0 & -D_{11}A_{12} & \dots & -D_{11}A_{1m} \\ -D_{22}A_{21} & 0 & \dots & -D_{22}A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -D_{mm}A_{m1} & \dots & \dots & 0 \end{bmatrix}.$$

In the iteration procedure, equation (10) can be expressed as

$$\Delta u_M^{l+1}(k) = D_1[\varpi(k) - \tilde{y}_{P0}(k)] + D_0 \Delta u_M^l(k) \quad (l = 0, 1, \dots). \tag{12}$$

At time instant k, $\varpi(k)$ and $\tilde{y}_{P0}(k)$ are known in advance, hence $D_1[\varpi(k) - \tilde{y}_{P0}(k)]$ is the constant term irrelevant to the iteration. The convergence of expression (11) is then equivalent to that of the following

$$\Delta u_M^{l+1}(k) = D_0 \Delta u_M^l(k). \tag{13}$$

From the above analysis the convergence condition for the algorithm in application to distributed linear model predictive control is

$$|\rho(D_0)| < 1. \tag{14}$$

That is the spectrum radius must be less than 1 to guarantee a convergent computation.

4 Nominal Stability of Distributed Model Predictive Control System

In order to analyse the nominal stability, rewrite the prediction output model of (8) in terms of state space equation ([11]). The predictive state space model of the i-th agent at time instant can be written as

$$\begin{aligned} x_i(k+1) &= Sx_i(k) + a_{ii}\Delta u_i(k) + \sum_{\substack{j=1 \\ j\neq i}}^{m} a_{ij}\Delta u_j, \\ Y_i(k) &= GSx_i(k) + A_{ii}\Delta u_{i,M}(k) + \sum_{\substack{j=1 \\ j\neq i}}^{m} A_{ij}\Delta u_{j,M}, \\ &(i = 1, \cdots, m), \end{aligned} \tag{15}$$

where $\Delta u_i(k) = [1 \ \ \cdots \ \ 0]\Delta u_{i,M}(k)$

$$S = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 \end{bmatrix}_{(N*N)},$$

where N is the modelling horizon, and

$$a_{ij} = [a_{ij}(1) \quad \cdots \quad a_{ij}(N)]^T, \qquad x_i(k) = [x_{i1}(k) \quad \cdots \quad x_{iN}(k)]^T,$$
$$Y_i(k) = [y_i(k+1) \quad \cdots \quad y_i(k+P)]^T.$$

$G = [I_{P*P} \ \ \mathbf{0}_{P*(N-P)}]$ denotes the operation of taking out the first P vectors from the N dimensional vectors. The Nash optimal solution in state space expression of the i-th agent at time instant k is

$$\Delta u_{i,M}(k) = D_{ii}[\varpi_i(k) - GSx_i(k) - \sum_{\substack{j=1 \\ j\neq i}}^{m} A_{ij}\Delta u_{j,M}(k)]. \tag{16}$$

The integral Nash optimal solution of the whole system provided that the algorithm is convergent at each sampling time can be written as

$$\Delta U(k) = (I - D_0)^{-1}D_1[\varpi(k) - F_2X(k)]. \tag{17}$$

This is the state feedback control law. The instant control law of the whole system is $\Delta u(k) = L\Delta U(k)$, where

$$\begin{aligned}
&L = Block - diag(\underbrace{L_0 \quad \cdots \quad L_0}_{m}),\\
&L_0 = (1 \quad 0 \quad \cdots \quad 0)_{1*M},\\
&F_2 = Block - diag(\underbrace{GS, \cdots, GS}_{m}),\\
&\Delta U(k) = [\Delta u_{1,M}(k) \quad \cdots \quad \Delta u_{m,M}(k)]^T,\\
&\varpi(k) = [\varpi_1(k) \quad \cdots \quad \varpi_m(k)]^T,\\
&X(k) = [x_1(k) \quad \cdots \quad x_m(k)]^T.
\end{aligned}$$

Without loss of generality, let the expected output

$$\varpi_i(k+1) = 0, \quad (i = 1, \cdots, m).$$

Then the state space model of the whole system at time instant k can be expressed as

$$X(k+1) = F_1X(k) + BL\Delta U(k) = [F_1 - BL(I - D_0)^{-1}D_1F_2]X(k), \tag{18}$$

where

$$F_1 = Block - diag(\underbrace{S, \cdots, S}_{m}),$$

$$B = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mm} \end{bmatrix}.$$

The expression (17) shows the state mapping relationship of the distributed system between time instant k and time instant $k+1$. According to contraction mapping principle ([11]), the nominal stability of the whole distributed system can be guaranteed, if and only if

$$\lambda(F_1 - (I - D_0)D_1F_2) < 1. \tag{19}$$

That is, the eigen values of state mapping are less than 1.

5 Disturbance Analysis with Single-Step Horizon Control

In distributed control, each agent can work independently to achieve its local objective, but cannot accomplish the whole task on its own. These autonomous agents can communicate and co-ordinate each other, exchange information through network in order to accomplish the whole task or objective. If a distributed system is subjected to disturbance, does this strategy work well and what does the performance of the whole system change? In this section, the performance deviation on *single-step horizon* under the communication disturbance is discussed. Because MPC takes a receding-horizon control policy in which the optimisation is resolved on-line at each sampling time with updated measurements, it is reasonable to focus on single-step horizon.

In the following analysis, assume that the prediction horizon and the control horizon are equal and the communication disturbance is confined within stable region. To indicate the communication connection between agents, define a connection matrix E. All elements in the main diagonal of E are zeros and other elements in the non-main diagonal of E are 1 or 0. 1 denotes no communication disturbance, and 0 shows communication disturbance existed and the corresponding communication channel is shut up. The output prediction model and the Nash optimal solution of the i-th agent at time instant k can be respectively rewritten as

$$\begin{gathered}\tilde{y}_{i,PM}(k) = \tilde{y}_{i,P0}(k) + A_{ii}\Delta u_{i,M}(k) + \sum_{\substack{j=1\\ j\neq i}}^{m} e_{ij}A_{ij}\Delta u_{j,M} \\ (i = 1, \cdots, m),\end{gathered} \tag{20}$$

and

$$\begin{gathered}\Delta u^*_{i,M}(k) = (A_{ii}^TQ_iA_{ii} + R_i)^{-1}A_{ii}^TQ_i[\varpi_i - \tilde{y}_{i,P0}(k) - \sum_{\substack{j=1\\ j\neq i}}^{m} G_{ij}\Delta u^*_{j,M}(k)] \\ (i = 1, \cdots, m).\end{gathered} \tag{21}$$

Here $G = E \cdot A$, "$\cdot$" denotes the dot multiplication. The Nash optimal solution of the whole system under convergent computation is

$$\Delta u^*_M(k) = (I - D_E)^{-1}D_1[\varpi(k) - \tilde{y}_{P0}(k)], \tag{22}$$

where

$$D_E = \begin{bmatrix} 0 & -D_{11}e_{12}A_{12} & \dots & -D_{11}e_{1N}A_{1N} \\ -D_{22}e_{21}A_{21} & 0 & & -D_{22}e_{2N}A_{2N} \\ \vdots & & \ddots & \\ -D_{NN}e_{N1}A_{N1} & & & 0 \end{bmatrix}.$$

To analyse system performance deviation, define a disturbance matrix T. The disturbance matrix T is a diagonal matrix or block diagonal matrix. For diagonal matrix, define the elements of its main diagonal as 1 or 0. For block diagonal matrix, the elements of its main diagonal block are all 1s or all 0s. The value 0 corresponds to no disturbance, and 1 for the communication disturbance existed.

Remark 5.1 Here the communication disturbance is classified into three cases

(1) Row disturbance, that is, the disturbance happens on the receiving channels. In this case the agent cannot receive the information coming from other agents, equivalently the corresponding row of matrix G becomes 0 and G becomes G', or, $G' = G - G''$, $G'' = TG$ and the corresponding element of disturbance matrix T has changed from 0 to 1;
(2) Column disturbance, that is, the disturbance happens on the transmitting channels. In this case, the agent cannot send its information to other agents, equivalently the corresponding column of matrix G becomes 0 and G becomes G', or, $G' = G - G''$, $G'' = GT$;
(3) Mixed disturbance. In this case, both row and column disturbances exist and $G' = G - G'' = G - TGT$.

With these preliminaries a theorem is presented.

Theorem 5.1 *For a distributed system, assume that the prediction horizon and the control horizon are equal and the communication disturbance cannot affect the stability. Its performance at time instant k under the local communication disturbance is degrading. The degrading magnitude δ satisfies $0 \le \delta \le \delta_{\max}$, and the upper bound of this magnitude $\delta_{\max}$ is*

$$\delta_{\max} = \frac{t_W(W_{\max})}{\lambda_m(F)},$$

where $t_W(W_{\max})$ denotes the norm of $W_{\max}$ and $\lambda_m(F)$ is the minimal eigen value of F with

$$\begin{aligned} F &= [D_1^{-1}(I - D_E) - A]^T Q [D_1^{-1}(I - D_E) - A] + R, \\ W_{\max} &= (A - \overline{A} - G)^T Q (A - \overline{A} - G) + [D_1^{-1}(I - D_E) - A]^T Q (A - \overline{A} - G) \\ &\quad + (A - \overline{A} - G)^T Q [D_1^{-1}(I - D_E) - A], \end{aligned}$$

$$\overline{A} = \begin{bmatrix} A_{11} & & \\ & \ddots & \\ & & A_{NN} \end{bmatrix}.$$

Proof Without loss of generality, take the column disturbance as an example, it has

$$D_E'' = D_E T \quad D_E' = D_E - D_E'' = D_E - D_E T.$$

The Nash optimal solution of the whole system in this case is

$$\Delta u_M^{dis}(k) = (I - D_E + D_E T)^{-1} D_1 [\varpi(k) - \tilde{y}_{P0}(k)]. \tag{23}$$

Using the matrix decomposition technique, it gives

$$\begin{aligned}(I-D_E+D_ET)^{-1} &= [2(I-D_E)+(D_E+D_ET-I)]^{-1}\\ &= [2(I-D_E)]^{-1}-[2(I-D_E)]^{-1}\{[2(I-D_E)]^{-1}\\ &\quad +(D_E+D_ET-I)^{-1}\}^{-1}[2(I-D_E)]^{-1}.\end{aligned} \tag{24}$$

In general $(D_E+D_ET-I)^{-1}$ and $(I-D_E)^{-1}$ all exist, therefore the above equation holds. Substitute (24) into (23) to give

$$\begin{aligned}\Delta u_M^{dis}(k) &= \frac{1}{2}\,\Delta u_M^*(k)-\frac{1}{4}\,(I-D_E)^{-1}\,\frac{\Delta u_M^*(k)}{\frac{1}{2}\,(I-D_E)^{-1}+(D_E+D_ET-I)^{-1}}\\ &= S\Delta u_M^*(k)\end{aligned} \tag{25}$$

with

$$S=\frac{1}{2}\,I-\frac{1}{4}\,(I-D_E)^{-1}\left[\frac{1}{2}\,(I-D_E)^{-1}+(D_E+D_ET-I)^{-1}\right]^{-1}.$$

From $\Delta u_M^*(k)=(I-D_E)^{-1}D_1[\varpi(k)-\tilde{y}_{P0}(k)]$, it has

$$\varpi(k)-\tilde{y}_{P0}(k)=D_1^{-1}(I-D_E)\Delta u_M^*(k).$$

Then it gives

$$\begin{aligned}J^* &= \|\varpi(k)-\tilde{y}_{P0}(k)-A\Delta u_M^*(k)\|_Q^2+\|\Delta u_M^*(k)\|_R^2\\ &= \|D_1^{-1}(I-D_E)\Delta u_M^*(k)-A\Delta u_M^*(k)\|_Q^2+\|\Delta u_M^*(k)\|_R^2=\|\Delta u_M^*(k)\|_F^2\end{aligned} \tag{26}$$

with $F=[D_1^{-1}(I-D_E)-A]^TQ[D_1^{-1}(I-D_E)-A]+R$, let

$$\overline{A}=\begin{bmatrix}A_{11} & & \\ & \ddots & \\ & & A_{NN}\end{bmatrix}.$$

Then the prediction model of the whole distributed system under the column disturbance can be written as

$$\tilde{y}_M^{dis}(k)=\tilde{y}_{P0}(k)+(\overline{A}+G-GT)\Delta u_M^{dis}(k)=\tilde{y}_{P0}(k)+L\Delta u_M^{dis}(k), \tag{27}$$

where $L=\overline{A}+G-GT$.

Substitute (25) and (27) into (9), it can be derived

$$\begin{aligned}J^{dis} &= \|\varpi(k)-\tilde{y}_{P0}(k)-LS\Delta u_M^*(k)\|_Q^2+\|S\Delta u_M^*(k)\|_R^2\\ &= \|\varpi(k)-\tilde{y}_{P0}(k)-A\Delta u_M^*(k)+(A-LS)\Delta u_M^*(k)\|_Q^2\\ &\quad +\|\Delta u_M^*(k)+(S-I)\Delta u_M^*(k)\|_R^2\\ &= J^*+\|\Delta u_M^*(k)\|_W^2,\end{aligned} \tag{28}$$

where

$$W = (A - LS)^T Q(A - LS) + (S - I)^T R(S - I) + R(S - I) \\ + (S - I)^T R + (M - A)^T Q(A - LS) + (A - LS)^T Q(M - A).$$

Let $t_W(W)$ denotes the norm of W, it gives

$$\|\Delta u_M^*(k)\|_W^2 \le \Delta u_M^{*T}(k)\|W\|\Delta u_M^*(k) = t_W(W)\|\Delta u_M^*(k)\|^2 \\ \le \frac{t_W(W)}{\lambda_m(F)}\|\Delta u_M^*(k)\|_F^2 = \frac{t_W(W)}{\lambda_m(F)} J^*.$$

Here $\lambda_m(F)$ is the minimal eigen value of F. From the above derivations, the performance relationship between the free disturbance and disturbance can be expressed as

$$J^{dis} \le J^* + \frac{t_W(W)}{\lambda_m(F)} J^* = (1+\delta)J^*. \tag{29}$$

Inspection of (29), that $t_W(W)$ depends on G'' and D_E'', while G'' and D_E'' are affected by disturbance matrix T. So in case of free disturbance, $t_W(W)$ can arrive at the maximal value, at this time, $\|T\| = 0,\ G'' = 0,\ D_E'' = 0,\ L = \overline{A} + G,\ S = I$ and

$$W = W_{\max} = (A - \overline{A} - G)^T Q(A - \overline{A} - G) + [D_1^{-1}(I - D_E) - A]^T Q(A - \overline{A} - G) \\ + (A - \overline{A} - G)^T Q(D_1^{-1}(I - D_E) - A).$$

Therefore the upper bound of the performance deviation under the local communication disturbance is

$$\delta_{\max} = \frac{t_W(W)}{\lambda_m(F)}.$$

Theorem 5.2 *The convergence condition of the distributed linear model predictive control system under the communication disturbance is* $|\rho(D_E)| < 1$. D_E *is the same as defined before. This proof is similar to the analysis in Section 3.*

Remark 5.2 Under the communication disturbance, each agent cannot exchange information properly. In an extreme case the elements in matrix E are all 1s, D_E becomes null matrix, $|\rho(D_E)| < 1$ is always satisfied, which corresponds to the full decentralised architecture.

6 Simulation Study

Consider a linear continuous time dynamic plant model with three inputs and three outputs

$$G(s) = \begin{bmatrix} \frac{e^{-2s}}{100s+1} & \frac{e^{-6s}}{100s+1} & \frac{e^{-4s}}{200s+1} \\ \frac{-1.25e^{-2s}}{50s+1} & \frac{3.75e^{-6s}}{50s+1} & \frac{e^{-3s}}{50s+1} \\ \frac{-2e^{-2s}}{200s+1} & \frac{2e^{-4s}}{200s+1} & \frac{3.5e^{-2s}}{100s+1} \end{bmatrix}.$$

The expected output of this system is to follow a set of unit step reference signals. With the proposed distributed algorithm, first of all divide the whole system into three agents, they are

Agent 1: $G_1(x) = \frac{e^{-2s}}{100s+1}$.
Agent 2: $G_2(x) = \frac{3.75e^{-6s}}{50s+1}$.
Agent 3: $G_3(x) = \frac{3.5e^{-3s}}{100s+1}$.

The control parameters for each agent are set with $P = 8$, $M = 3$, $Q_i = I$, $R_i = 0.5I$, $(i = 1, 2, 3)$, sampling time of 20 sec, and $\varepsilon_i = 0.01$ $(i = 1, 2, 3)$. The Matlab based simulation results are shown in Figure 6.1. It can be observed that each agent in this distributed structure can properly arrive at the expected outputs while keeping the satisfactory performance to some extent. In addition, the design parameters for each agent such as prediction horizon, control horizon, weighting matrix and sample time etc. can all be designed and tuned separately, which is superior to the centralised control and significantly reduce the on-line computational burden. Notice that each agent is not necessary limited to SISO case and also it can be MIMO agent, whose dimension is still much lower than the whole system's.

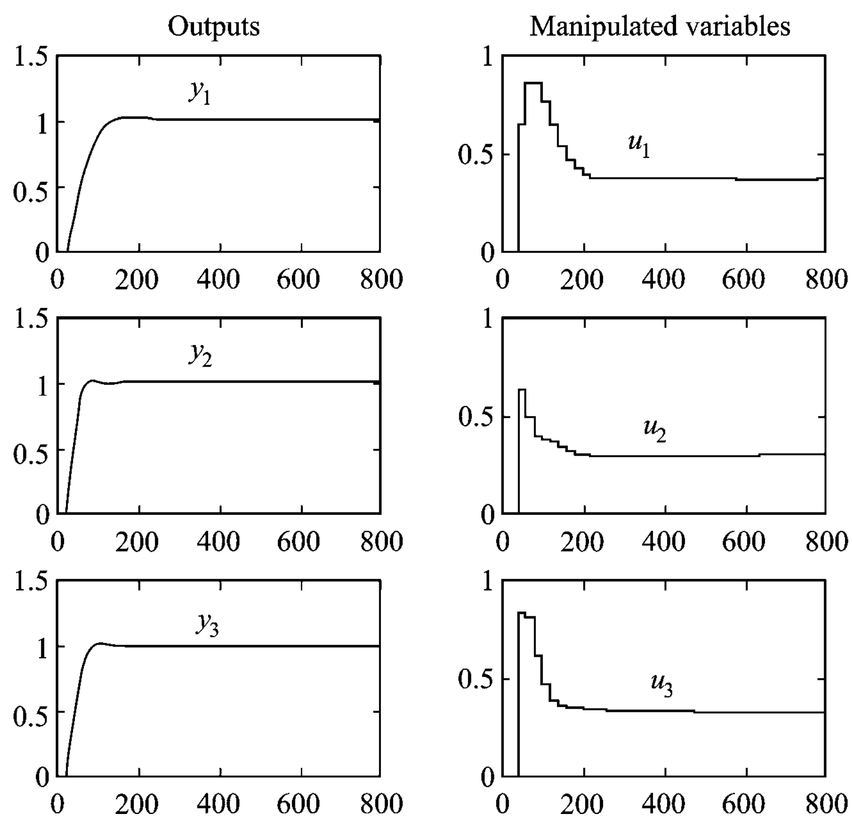

Figure 6.1. Output responses and manipulated/control signals on the experimental plant.

7 Conclusions

In this study a distributed model predictive control method based on Nash optimality is developed for large-scale linear systems. To avoid the prohibitively high on-line computational demand, the MPC is implemented in distributed scheme with the inexpensive agents within the network environment. These agents can co-operate and communicate each other to achieve the objective of the whole system. Coupling effects between the agents are fully taken into account in this scheme, which is superior to other traditional decentralised control methods. The main advantage of this scheme is that the on-line optimisation of a large-scale system can be converted to that of several small-scale systems, thus can significantly reduce the computational complexity while keeping satisfactory performance. In addition, the design parameters for each agent such as prediction horizon, control horizon, weighting matrix and sample time etc. can all be designed and tuned separately, which provides more flexibility for the analysis and applications. The second part of this study is to investigate the performance of the distributed control scheme. The nominal stability and the performance deviation on the single-step horizon under the communication disturbance are analysed. These will provide users better understanding of the developed algorithm and sensible guidance in applications. As the method is also expandable to complex large-scale nonlinear model predictive control with certain constraints, a further study to control nonlinear Hammertien models is under investigation.

Acknowledgement

This work was supported by the National Natural Science Foundation of China (under Grant: 60074004), and High Technology Research and Development Program of China (under Grant: 2002AA412130).

References

[1] Clark, D.W., Mohtadi, C. and Tuffs, P.S. General predictive control. *Automatica* **23**(3) (1987) 137–160.

[2] Du, X.N., Xi, Y.G. and Li, S.Y. Distributed model predictive control for large-scale systems. *Proceedings of American Control Conference, Arlington, Virginia*, 2001, 3142–3143.

[3] Gurfil, P. and Kasdin, N. J. Characterization and design of out-of-ecliptic trajectories using deterministic crowding genetic algorithms. *Computer Methods in Applied Mechanics and Engineering* **191**(19–20) (2002) 2141–2158.

[4] Katebi, M.R. and Johnson, M.A. Predictive controldesign for large-scale systems. *Automatica* **33**(3) (1997) 421–425.

[5] Mosca, E. *Optimal, Predictive and Adaptive Control.* Prentice Hall, Englewood Cliffs, New Jersey, 1995.

[6] Nash, S.G. A survey of truncated-Newton methods. *J. of Computational and Applied Mathematics* **124** (2000) 45–59.

[7] Qin, S.J. and Badgwell, T.A. An overview of industrial model predictive control technology. In: *Chemical process control-assessment and new directions for research*, (Eds.: Y.C. Kantor, C.E. Garcia and B.C. Garcia), AIChE Symposium series, AIChE and CACHE, **93**(316) (1997) 232-256.

[8] Richalet, J. Industrial applications of model based predictive control. *Automatica* **29**(5) (1993) 1251–1274.

[9] Ricker, N.L. and Lee, J.H. Nonlinear model predictive control of the Tennessee Eastman challenge process. *Computer and Chemical Engineering* **19**(9) (1995) 961–981.

[10] Van Antwerp, J.G. and Braatz, R.D. Model predictive control of large scale processes. *Journal of Process Control* **10**(1) (2000) 1–8.

[11] Xi, Y.G. New design method for discrete-time multi-variable predictive controllers. *Int. J. Control* **49**(1) (1989) 45–56.

[12] Xi, Y.G. *Predictive Control*. National defense and industry press, Beijing, 1993.

[13] Xu, X. M., Xi, Y.G. and Zhang, Z.J. Decentralised predictive control (DPC) of large-scale systems. *Information and Decision Technologies* **14**(3) (1988) 307–322.

[14] Zheng, A. A computationally efficient nonlinear MPC algorithm. *Proceedings of the American Control Conference, Albuquerque, New Mexico*, 1997, 1623–1627.

[15] Zheng, A. Reducing on-line computational demands in model predictive control by approximating QP constraints. *Journal of process control* **9**(4) (1999) 279–290.

[16] Zheng, A. and Allgower, F. Towards a practical nonlinear predictive control algorithm with guaranteed stability for large-scale systems. *Proceedings of the American Control Conference, Philadelphia, Pennsylvania*, 1998, 2534–2538.

[17] Zhu, Q.M., Warwick, K. and Douce, J.L. Adaptive general predictive controller for nonlinear systems. *IEE Proc. – Control Theory Appl.* **138**(1) (1991) 33–40.

Nonlinear Dynamics and Systems Theory, **3**(2) (2003) 177–189

Quadratic Stabilization for Nonlinear Perturbed Discrete Time-Delay Systems

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Received: March 12, 2003; Revised: June 26, 2003

Abstract: This paper discusses the quadratic stability and quadratic stabilization problem for a class of nonlinear perturbed discrete time-delay systems. Necessary and sufficient conditions for quadratic stability are presented via S-procedure technique and linear matrix inequality (LMI). Both static and dynamic output feedback controllers are constructed respectively. Furthermore, necessary and sufficient conditions for quadratic stabilization via static state feedback are constructed in the form of LMI. Finally, the effectiveness of new approach is demonstrated by numerical examples.

Keywords: *Discrete systems; time-delay; nonlinear perturbation; quadratic stabilization; linear matrix inequality.*

Mathematics Subject Classification (2000): 93C55, 93D15; 93C10.

1 Introduction

Quadratic stabilization theory for discrete-time systems has been receiving much attention in the last decade, see [5, 6, 8, 12, 15, 19 − 21]. Quadratic stability means that there exists a deterministic quadratic stable Lyapunov function for all admissible parameter perturbations. The objective of quadratic stabilization is to find a feedback controller such that the closed-loop systems are quadratically stable for all admissible parameter perturbations, where the associated Lyapunov function is quadratic and deterministic. By means of quasiconvex optimization approach, [6] constructs quadratic stabilizing controllers via linear static output feedback and state feedback for discrete-time linear systems with uncertainty. In [15], the robust stabilization for a class of single-input

⋆This work was supported by a grant from CityU (Project No. 7001425); the first author was also supported by a grant from Nantong Institute of Technology.

discrete-time nonlinear systems is formulated into a convex optimization problem in the form of LMI. Consequently, a static state feedback law is designed to stabilize the plant and to maximize the bound on the nonlinear perturbation terms.

Since time-delay usually results in unsatisfactory performances and is frequently a source of instability, many researchers have paid serious attention to those problems caused by time-delays. Recently several authors have used different approaches such as quadratic Lyapunov function, linear matrix inequalities to study stabilization problems for discrete-time (or continuous-time) linear systems with time-delays [3, 10, 17, 16]. [17] presents an interesting approach using state feedback control design for a class of discrete-time linear systems with time-delays and matched uncertainty, but their approach is based on *nonlinear matrix inequalities* (NLMIs). It is known that there has been no efficient way to construct the control law in terms of NLMIs so far.

The objective of this paper is to discuss quadratic stability and quadratic stabilization problem for a class of multi-input and multi-output (MIMO) discrete-time systems with nonlinear perturbation on both state and control-input perturbations. A necessary and sufficient condition for quadratic stability of unforced systems is presented by means of S-procedure technique and LMI. In addition, both static and dynamic output feedback are constructed if the corresponding LMI is feasible.

As compared with the existing results in the literature, this paper discusses more general class of systems than those in [5, 6, 8, 15, 17, 19, 21]. Both static and dynamic output feedback control designs are obtained in terms of LMI which is more computational efficient than the NLMI approach developed for linear uncertain systems by [17]. In addition, the single-input static state feedback design developed in [15] is a very special case of this paper, also time-delays and perturbation on control input are not considered in their work. Furthermore, a state feedback control design for linear uncertain systems based on the Riccati equation approach is developed by [5], which are also regarded as a special case of this paper.

2 Quadratic Stability for the Unforced Systems

Consider a class of unforced perturbed discrete time-delay systems as follows

$$\begin{aligned} z_{k+1} &= \tilde{A} z_k + \tilde{A}_1 z_{k-d} + g(k, z_k, z_{k-d}), \\ z_k &= \delta_k, \quad k = -d, -d+1, \cdots, 0, \end{aligned} \tag{1}$$

where $z_k \in \mathbf{R}^{\tilde{n}}$ is the system state; $\tilde{A}$, $\tilde{A}_1$ are constant matrices with appropriate dimensions; and positive integer d is maximal time-delay; δ_i $(i = -d, -d+1, \cdots, 0)$ are initial-value vectors for the delayed system; $g = g(k, z_k, z_{k-d})$ is a vector-valued nonlinear function which is regarded as a nonlinear perturbation and satisfies the following quadratic inequality for all (k, w, v)

$$\begin{aligned} g'(k, w, v) g(k, w, v) &\leq w' G' G w + 2 w' G' G_1 v + v' G_1' G_1 v \\ &= \begin{pmatrix} w \\ v \end{pmatrix}' (G \quad G_1)' (G \quad G_1) \begin{pmatrix} w \\ v \end{pmatrix}, \end{aligned} \tag{2}$$

where G and G_1 are constant matrices with appropriate dimensions, w, v are vectors with the same dimension with z_k.

The following definition on quadratic stability is presented by [17].

Definition 2.1 *Systems* (1) *are* *quadratically stable* if there exist matrices $P > 0$, and $Q > 0$ such that for all admissible perturbation g, systems (1) satisfy

$$\Delta V_k = V_{k+1} - V_k < 0, \tag{3}$$

for all pair $(k, z_k, z_{k-d}) \in \mathbf{Z}_+ \times (\mathbf{R}^{\tilde{n}} \times \mathbf{R}^{\tilde{n}} - \{0\})$, $V_k = z_k' P z_k + \sum_{i=1}^{d} z_{k-i}' Q z_{k-i}$, $\mathbf{Z}_+ = \{0, 1, 2, \cdots\}$.

Lemma 2.1 (S-procedure lemma) [18] *Let $\Omega_0(x)$ and $\Omega_1(x)$ be two arbitrary quadratic forms over $\mathbf{R}^n$. Then $\Omega_0(x) < 0$ for all $x \in \mathbf{R}^n - \{0\}$ satisfying $\Omega_1(x) \le 0$ if there exists $\tau \ge 0$ such that*

$$\Omega_0(x) - \tau\Omega_1(x) < 0, \quad \forall\, x \in \mathbf{R}^n - \{0\}.$$

For convenience and compactness, the following notation of (4) will be used throughout this paper.

$$\mathcal{L}(X, X_1, \Gamma, \Gamma_1, \Psi, \Psi_1) := \begin{pmatrix} -X + X_1 & 0 & \Gamma' & \Psi' \\ 0 & -X_1 & \Gamma_1' & \Psi_1' \\ \Gamma & \Gamma_1 & I - X & 0 \\ \Psi & \Psi_1 & 0 & -I \end{pmatrix}, \tag{4}$$

where X, X_1, Γ, Γ_1, Ψ and Ψ_1 are matrices with appropriate dimensions, I is an identity matrix with appropriate dimension.

By means of S-procedure and LMI technique, the following theorem presents a necessary and sufficient condition for quadratic stability of unforced systems (1).

Theorem 2.1 *Unforced systems (1) are quadratically stable if and only if there exist positive definite matrices X and X_1 with appropriate dimension such that the following LMI is solvable*

$$\mathcal{L}(X, X_1, \tilde{A}X, \tilde{A}_1 X, GX, G_1 X) < 0. \tag{5}$$

Proof By means of the Schur Complement Lemma, LMI (5) is equivalent to the following matrix inequality

$$\begin{pmatrix} -X + X_1 + XG'GX & XG'G_1X & X\tilde{A}' \\ XG_1'GX & -X_1 + XG_1'G_1X & X\tilde{A}_1' \\ \tilde{A}X & \tilde{A}_1X & I - X \end{pmatrix} < 0. \tag{6}$$

Let $P = X^{-1}$, $Q = X^{-1}X_1X^{-1}$, and multiply both sides of the first inequality of (6) by $\operatorname{diag}\{X^{-1}, X^{-1}, I\}$, then (6) is equivalent to

$$\begin{pmatrix} -P + Q + G'G & G'G_1 & \tilde{A}' \\ G_1'G & -Q + G_1'G_1 & \tilde{A}_1' \\ \tilde{A} & \tilde{A}_1 & I - P^{-1} \end{pmatrix} < 0. \tag{7}$$

Similarly, (7) is equivalent to

$$\begin{gathered} \begin{pmatrix} -P + Q + G'G & G'G_1 \\ G_1'G & -Q + G_1'G_1 \end{pmatrix} + \begin{pmatrix} \tilde{A}' \\ \tilde{A}_1' \end{pmatrix} (P^{-1} - I)^{-1} (\tilde{A} \quad \tilde{A}_1) < 0, \\ P^{-1} - I > 0. \end{gathered} \tag{8}$$

Notice that

$$(P^{-1} - I)^{-1} = P + P(I - P)^{-1}P. \tag{9}$$

Then (8) is equivalent to

$$\begin{gathered}\begin{pmatrix} \tilde{A}'P\tilde{A}-P+Q+G'G & \tilde{A}'P\tilde{A}_1+G'G_1 \\ \tilde{A}_1'P\tilde{A}+G_1'G & \tilde{A}_1'P\tilde{A}_1-Q+G_1'G_1 \end{pmatrix} + \begin{pmatrix} \tilde{A}'P \\ \tilde{A}_1'P \end{pmatrix} (I-P)^{-1} \begin{pmatrix} P\tilde{A} & P\tilde{A}_1 \end{pmatrix} < 0, \\ I - P > 0. \end{gathered} \tag{10}$$

From the Schur Complement Lemma again, matrix inequalities (10) are equivalent to

$$\begin{pmatrix} \tilde{A}'P\tilde{A} - P + Q + G'G & \tilde{A}'P\tilde{A}_1 + G'G_1 & \tilde{A}'P \\ \tilde{A}_1'P\tilde{A} + G_1'G & \tilde{A}_1'P\tilde{A}_1 - Q + G_1'G_1 & \tilde{A}_1'P \\ P\tilde{A} & P\tilde{A}_1 & P - I \end{pmatrix} < 0. \tag{11}$$

Sufficiency: If LMI (5) holds for X and X_1, then (11) holds for $P = X^{-1}$ and $Q = X^{-1}X_1X^{-1}$. In order to obtain the quadratic stability of systems (1), we construct the following quadratic Lyapunov functional candidate

$$V_k = z_k'Pz_k + \sum_{i=1}^{d} z_{k-i}'Qz_{k-i}. \tag{12}$$

Then along with systems (1), for $(z_k' \quad z_{k-d}' \neq 0$, from (11) we have

$$\begin{aligned} &V_{k+1} - V_k - \left[g'g - \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}' (G \quad G_1)'(G \quad G_1) \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix} \right] \\ &= z_k'(\tilde{A}'P\tilde{A} - P + Q)z_k + z_{k-d}'(\tilde{A}_1'P\tilde{A}_1 - Q)z_{k-d} + g'Pg + 2z_k'\tilde{A}'P\tilde{A}_1z_{k-d} \\ &\quad + 2z_k'Pg + 2z_{k-d}'\tilde{A}_1'Pg - \left[g'g - \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}' (G \quad G_1)'(G \quad G_1) \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix} \right] \\ &= \begin{pmatrix} z_k \\ z_{k-d} \\ g \end{pmatrix}' \begin{pmatrix} \tilde{A}'P\tilde{A} - P + Q + G'G & \tilde{A}'P\tilde{A}_1 + G'G_1 & \tilde{A}'P \\ \tilde{A}_1'P\tilde{A} + G_1'G & \tilde{A}_1'P\tilde{A}_1 - Q + G_1'G_1 & \tilde{A}_1'P \\ P\tilde{A} & P\tilde{A}_1 & P - I \end{pmatrix} \begin{pmatrix} z_k \\ z_{k-d} \\ g \end{pmatrix} < 0. \end{aligned} \tag{13}$$

It follows from Lemma 2.1 that under constraint (2), $V_{k+1} - V_k < 0$ for $(z_k' \quad z_{k-d}') \neq 0$, which implies that systems (1) are quadratically stable in the sense of Definition 2.1.

Necessity: If systems (1) are quadratically stable, that is, there exists a Lyapunov functional candidate as follows

$$V_k = z_k'\tilde{P}z_k + \sum_{i=1}^{d} z_{k-i}'\tilde{Q}z_{k-i}, \tag{14}$$

where $\tilde{P}$ and $\tilde{Q}$ are positive definite matrices with appropriate dimensions, and under constraint condition:

$$g'g - \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}' (G \quad G_1)'(G \quad G_1) \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix} \leq 0, \tag{15}$$

we have

$$\Delta V_k = V_{k+1} - V_k < 0, \quad \forall \, (z_k' \quad z_{k-d}') \neq 0. \tag{16}$$

Notice that (15) and ΔV_k are quadratic on z_k, z_{k-d} and g, then it follows from Lemma 2.1 that there exists a constant $\tau \geq 0$ such that for all $(z_k' \quad z_{k-d}' \quad g') \neq 0$,

$$\Delta V_k - \tau \left[g'g - \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}' (G \quad G_1)'(G \quad G_1) \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix} \right] < 0. \tag{17}$$

However, if $\tau = 0$, then (17) implies that the original systems (1) can be quadratically stable for all g without constraint (2), which is impossible. Then (17) holds for some $\tau > 0$. In addition, (17) is equivalent to

$$\begin{pmatrix} \tilde{A}'\tilde{P}\tilde{A} - \tilde{P} + \tilde{Q} + \tau G'G & \tilde{A}'\tilde{P}\tilde{A}_1 + \tau G'G_1 & \tilde{A}'\tilde{P} \\ \tilde{A}_1'\tilde{P}\tilde{A} + \tau G_1'G & \tilde{A}_1'\tilde{P}\tilde{A}_1 - \tilde{Q} + \tau G_1'G_1 & \tilde{A}_1'\tilde{P} \\ \tilde{P}\tilde{A} & \tilde{P}\tilde{A}_1 & \tilde{P} - \tau I \end{pmatrix} < 0. \tag{18}$$

Let $P = \tau^{-1}\tilde{P}$, $Q = \tau^{-1}\tilde{Q}$, then (18) is the same as (11). This completes the proof.

Remark 2.1 Systems (1) with constraint (2) are more general than the systems discussed in [8, 19, 21]. Theorem 3.1 can be regarded as an extension of the results in literature above.

3 Static Output Feedback

Consider a class of MIMO discrete time-delay systems with nonlinear perturbation as follows

$$\begin{aligned} x_{k+1} &= Ax_k + A_1 x_{k-d} + Bu_k + f(k, x_k, x_{k-d}, u_k), \\ y_k &= Cx_k, \\ x_k &= \delta_k, \quad k = -d, -d+1, \cdots, 0, \end{aligned} \tag{19}$$

where $x_k \in \mathbf{R}^n$, $u_k \in \mathbf{R}^m$ and $y_k \in \mathbf{R}^p$ are the system state, control input and output, respectively; A, $A_1 \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{p\times n}$ are constant matrices with full-row rank; and positive integer d is maximal time-delay; $\delta_i \in \mathbf{R}^n$ $(i = -d, -d+1, \cdots, 0)$ are initial-value vectors for the delayed system; $f(k, w, v, u)$ is a vector-valued nonlinear function and satisfies the following quadratic inequality for all $(k, w, v, u) \in \mathbf{Z}_+ \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^m$;

$$f'(k, w, v, u) f(k, w, v, u) \leq (w' \quad v' \quad u') \begin{pmatrix} F' \\ F_1' \\ H' \end{pmatrix} (F \quad F_1 \quad H) \begin{pmatrix} w \\ v \\ u \end{pmatrix}, \tag{20}$$

where F, F_1 and H are constant matrices with appropriate dimensions.

Remark 3.1 If

$$f(k, x_k, x_{k-d}, u_k) = \Delta A(k)x_k + \Delta A_1(k)x_{k-d} + \Delta B(k)u_k + f_0(k, x_k, x_{k-d}, u_k),$$

where the norm of $\Delta A(k)$, $\Delta A_1(k)$ and $\Delta B(k)$ are uniformly bounded and $f(k, w, v, u)$ is global Lipschitz on $(w, v, u) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^m$ with $f(k, 0, 0, 0) = 0$ for any $k \in \mathbf{Z}_+$, then this class of Lipschitz systems with uncertainty can be included in systems (19). Therefore the discrete-time linear (or time-delay) systems with matched uncertainty considered by [5] and [17] are special cases of this paper.

In this section, we consider the following form of linear static output feedback controller

$$u_k = Ky_k + K_1 y_{k-d}, \tag{21}$$

where $K, K_1 \in \mathbf{R}^{m\times p}$ are constant matrices to be determined.

The purpose of this section is to find a controller in the form of (21) such that the closed-loop systems (19) and (21) are quadratically stable. In this case, the controller (21) is called a quadratic stabilisation controller.

The following theorem presents a way of constructing static output feedback controller law (21), in which sufficient condition is presented by means of LMI approach.

Theorem 3.1 *Systems (19) are quadratically stabilizable by means of static output feedback in the form of (21) if the following LMI (22) and matrix equation (23) on matrices* $X, X_1 \in \mathbf{R}^{n\times n}$*,* $Y, Y_1 \in \mathbf{R}^{m\times p}$ *and* $Z \in \mathbf{R}^{p\times p}$ *are solvable*

$$\mathcal{L}(X, X_1, \Gamma, \Gamma_1, \Psi, \Psi_1) < 0, \tag{22}$$

$$CX = ZC, \tag{23}$$

where

$$\begin{aligned} \Gamma &= AX + BYC, \qquad & \Gamma_1 &= A_1X + BY_1C, \\ \Psi &= FX + HYC, \qquad & \Psi_1 &= F_1X + HY_1C. \end{aligned} \tag{24}$$

Proof The conditions on full-row rank of C, $X > 0$ and matrix equation $CX = ZC$ imply that

$$p \geq \operatorname{rank}(Z) \geq \operatorname{rank}(ZC) = \operatorname{rank}(CX) \geq \operatorname{rank}[(CX)X^{-1}] = \operatorname{rank}(C) = p \tag{25}$$

that is, Z is non-singular. Then the gains of control law (21) can be chosen as follows:

$$K = YZ^{-1}, \quad K_1 = Y_1 Z^{-1}. \tag{26}$$

In this case, the resulting closed-loop systems are systems (1) with

$$\tilde{A} = A + BKC, \quad \tilde{A}_1 = A_1 + BK_1C, \quad g = f(k, x_k, x_{k-d}, KCx_k + K_1Cx_{k-d}), \tag{27}$$

where

$$g'g \leq \begin{pmatrix} x_k \\ x_{k-d} \end{pmatrix}' \begin{pmatrix} G' \\ G_1' \end{pmatrix} (G \quad G_1) \begin{pmatrix} x_k \\ x_{k-d} \end{pmatrix}, \tag{28}$$

and

$$G = F + HKC, \quad G_1 = F_1 + HK_1C. \tag{29}$$

From (26) and matrix equality $CX = ZC$, we have

$$\begin{aligned} \tilde{A}X &= (A + BKC)X = AX + BKCX \\ &= AX + BKZC = AX + BYC = \Gamma, \\ \tilde{A}_1X &= (A_1 + BK_1C)X = A_1X + BK_1CX \\ &= A_1X + BK_1ZC = A_1X + BY_1C = \Gamma_1, \\ GX &= FX + HKCX = FX + HYC = \Psi, \\ G_1X &= F_1X + HK_1CX = F_1X + HY_1C = \Psi_1. \end{aligned} \tag{30}$$

Then it follows from Theorem 2.1 and (22) that the systems (1) with (27), that is, the closed-loop systems (19) and (21) with (26), are quadratically stable, which completes the proof.

As a direct application of Theorem 3.1, if $C = I$ is chosen in Theorem 3.1, we have the following result, which presents a necessary and sufficient condition under which systems can be quadratically stabilized via static state feedback law

$$u_k = Kx_k. \tag{31}$$

Corollary 3.1 *Systems (19) are quadratically stabilizable via static state feedback in the form of (31) if and only if the following LMI on matrices X, $X_1 \in \mathbf{R}^{n\times n}$ and $Y \in \mathbf{R}^{m\times n}$ is solvable*

$$\mathcal{L}(X, X_1, \Gamma, \Gamma_1, \Psi, \Psi_1) < 0, \tag{32}$$

where

$$\Gamma = AX + BY, \quad \Gamma_1 = A_1X, \quad \Psi = FX + HY, \quad \Psi_1 = F_1X. \tag{33}$$

In this case, a static state feedback law can be chosen as follows

$$u_k = YX^{-1}x_k. \tag{34}$$

The *Proof* for sufficiency of Corollary 3.1 follows directly from Theorem 3.1. The necessity can be obtained from Lemma 2.1.

Remark 3.2 [15] discusses a class of discrete-time systems with single-input and nonlinear perturbation (no control input perturbation is considered), and a static state feedback law is constructed by means of LMI. However a special structure matrix variable L is needed to guarantee the resulting matrix inequality to be an LMI such that a solution of K is obtained (see (19)–(21) in [15]). It is important to notice that Corollary 3.1 presents a more efficient approach to search for an explicit solution K. In addition, Theorem 2 by [15] is a special case of Theorems 3.1 and 3.2. Furthermore, the result in this section can be regarded as an extension of that by [6], where static output feedback is obtained by means of quasiconvex optimization approach.

Since the conditions (22)–(23) contain the constraint $CX = ZC$, MATLAB LMI Toolbox [4] can not be used to solve (22)–(23) directly. In order to convert the problem (22)–(23) into an LMI, we will show that this constraint on X and Z can be transformed into an equivalent constraint on X, then (22)–(23) will be equivalent to an LMI.

For convenience, we present the singular value decomposition of C as

$$C = U(C_0 \quad 0)V', \tag{35}$$

where $U \in \mathbf{R}^{p\times p}$ and $V \in \mathbf{R}^{n\times n}$ are unitary matrices and $C_0 \in \mathbf{R}^{p\times p}$ is a diagonal matrix with positive diagonal elements in decreasing order.

The following lemma presents an equivalent condition on matrix equation $CX = ZC$.

Lemma 3.1 *For a given $C \in \mathbf{R}^{p\times n}$ with $\operatorname{rank}(C) = p$, assume that $X \in \mathbf{R}^{n\times n}$ is a symmetric matrix, then there exists a matrix $Z \in \mathbf{R}^{p\times p}$ such that $CX = ZC$ if and only if $X = V\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}V'$, where $X_1 \in \mathbf{R}^{p\times p}$, $X_2 \in \mathbf{R}^{(n-p)\times(n-p)}$.*

Proof If $p = n$, from the proof of Theorem 3.1, C is non-singular, it is clear that the result is true. Without loss of generality, suppose $p < n$. From $CX = ZC$ and the singular value decomposition of C, that is, $C = U(C_0 \quad 0)V'$, we have that matrix equation $CX = ZC$ is equivalent to $U(C_0 \quad 0)V'X = ZU(C_0 \quad 0)V'$. That is,

$$(UC_0 \quad 0)V'XV = (ZUC_0 \quad 0). \tag{36}$$

Suppose $X = V \begin{pmatrix} X_1 & X_0' \\ X_0 & X_2 \end{pmatrix} V'$, where $X_1 \in \mathbf{R}^{p\times p}$, $X_2 \in \mathbf{R}^{(n-p)\times(n-p)}$ and $X_0 \in \mathbf{R}^{(n-p)\times p}$, then (36) is equivalent to

$$(UC_0X_1 \quad UC_0X_0) = (ZUC_0 \quad 0). \tag{37}$$

Matrix equation (37) is solvable on Z if and only if $UC_0X_0 = 0$, that is, $X_0 = 0$, which completes the proof.

Therefore we have the following result from Theorem 3.1 and Lemma 3.1.

Theorem 3.2 *Systems (19) are quadratically stabilizable by static output feedback law if the following LMI on matrices $X_{11} \in \mathbf{R}^{p\times p}$, $X_{22} \in \mathbf{R}^{(n-p)\times(n-p)}$, $X_1 \in \mathbf{R}^{n\times n}$ and $Y, Y_1 \in \mathbf{R}^{m\times p}$ is solvable*

$$\mathcal{L}\left(X, X_1, \Gamma, \Gamma_1, \Psi, \Psi_1\right) < 0, \tag{38}$$

where

$$\begin{gathered} X = V \operatorname{diag}\{X_{11}, X_{22}\}V', \quad \Gamma = AX + BYC, \quad \Gamma_1 = A_1X + BY_1C, \\ \Psi = FX + HYC, \quad \Psi_1 = F_1X + HY_1C. \end{gathered} \tag{39}$$

In this case, a static output feedback controller of form (21) can be chosen as follows

$$u_k = YUC_0X_{11}^{-1}C_0^{-1}U'y_k + Y_1UC_0X_{11}^{-1}C_0^{-1}U'y_{k-d}. \tag{40}$$

4 Dynamic Output Feedback

In this section, we consider stabilisation for systems (19) via the following Luenberger-like dynamic output feedback controller

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + A_1\hat{x}_{k-d} + Bu_k + L(y_k - C\hat{x}_k), \\ u_k &= K\hat{x}_k + K_1\hat{x}_{k-d}. \end{aligned} \tag{41}$$

Let the difference of x_k and $\hat{x}_k$ be e_k, that is, $e_k = x_k - \hat{x}_k$, then the closed-loop systems of (19) and (41) can be written as (1) with

$$\begin{gathered} z_k = \begin{pmatrix} \hat{x}_k \\ e_k \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} A + BK & LC \\ 0 & A - LC \end{pmatrix}, \quad \tilde{A}_1 = \begin{pmatrix} A_1 + BK_1 & 0 \\ 0 & A_1 \end{pmatrix}, \\ g(k, z_k, z_{k-d}) = \begin{pmatrix} 0 \\ f(k, \hat{x}_k + e_k, \hat{x}_{k-d} + e_k, K\hat{x}_k + K_1\hat{x}_{k-d}) \end{pmatrix}. \end{gathered} \tag{42}$$

From (20), after some algebraic manipulations, we have

$$
\begin{aligned}
&g'(k,z_k,z_{k-d})g(k,z_k,z_{k-d})\\
&= [f(k,\hat{x}_k+e_k,\hat{x}_{k-d}+e_{k-d},K\hat{x}_k+K_1\hat{x}_{k-d})]'f(k,\hat{x}_k+e_k,\hat{x}_{k-d}+e_k,K\hat{x}_k+K_1\hat{x}_{k-d})\\
&\le \begin{pmatrix} \hat{x}_k+e_k \\ \hat{x}_{k-d}+e_{k-d} \\ K\hat{x}_k+K_1\hat{x}_{k-d}\end{pmatrix}' \begin{pmatrix} F' \\ F_1' \\ H'\end{pmatrix} (F \quad F_1 \quad H) \begin{pmatrix} \hat{x}_k+e_k \\ \hat{x}_{k-d}+e_{k-d} \\ K\hat{x}_k+K_1\hat{x}_{k-d}\end{pmatrix}\\
&= \begin{pmatrix} z_k \\ z_{k-d}\end{pmatrix}' \begin{pmatrix} G' \\ G_1'\end{pmatrix} (G \quad G_1) \begin{pmatrix} z_k \\ z_{k-d}\end{pmatrix},
\end{aligned}
\tag{43}
$$

where

$$
G = (F+HK \quad F), \quad G_1 = (F_1+HK_1 \quad F_1). \tag{44}
$$

Theorem 4.1 *Systems (19) are quadratically stable via dynamic output feedback in the form of (41) if there exist matrices* X_{11}, $X_{22} \in \mathbf{R}^{n\times n}$, $X_1 \in \mathbf{R}^{2n\times 2n}$, $Y_0 \in \mathbf{R}^{n\times p}$, Y, $Y_1 \in \mathbf{R}^{m\times n}$ *and* $Z \in \mathbf{R}^{p\times p}$ *such that the following LMI (45) and matrix equation (46) are solvable*

$$
\mathcal{L}(X, X_1, \Gamma, \Gamma_1, \Psi, \Psi_1) < 0, \tag{45}
$$

$$
CX_{22} = ZC, \tag{46}
$$

where

$$
\begin{gathered}
\Gamma = \begin{pmatrix} AX_{11}+BY & Y_0C \\ 0 & AX_{22}-Y_0C\end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} A_1X_{11}+BY_1 & 0 \\ 0 & A_1X_{22}\end{pmatrix},\\
X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22}\end{pmatrix}, \qquad \Psi = (FX_{11}+HY \quad FX_{22}),\\
\Psi_1 = (F_1X_{11}+HY_1 \quad F_1X_{22}).
\end{gathered}
\tag{47}
$$

In this case, a dynamic output feedback controller can be given by (41) with

$$
L = Y_0Z^{-1}, \quad K = YX_{11}^{-1}, \quad K_1 = Y_1X_{11}^{-1}. \tag{48}
$$

Proof Similarly, we have that Z in non-singular from $CX_{22} = ZC$, then L, K, K_1 can be given by (48). In this case, the matrix parameters in the resulting closed-loop systems in the form of (1) satisfy the following conditions:

$$
\begin{aligned}
\tilde{A}X &= \begin{pmatrix} A+BK & LC \\ 0 & A-LC\end{pmatrix}\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22}\end{pmatrix} = \begin{pmatrix} AX_{11}+BKX_{11} & LCX_{22} \\ 0 & AX_{22}-LCX_{22}\end{pmatrix}\\
&= \begin{pmatrix} AX_{11}+BY & LZC \\ 0 & AX_{22}-LZC\end{pmatrix} = \begin{pmatrix} AX_{11}+BY & Y_0C \\ 0 & AX_{22}-Y_0C\end{pmatrix} = \Gamma.
\end{aligned}
\tag{49}
$$

Similarly, we have

$$
\tilde{A}_1X = \Gamma_1, \quad GX = \Psi, \quad G_1X = \Psi_1. \tag{50}
$$

Then it follows from Lemma 2.1 that the systems (1) with (42) and (48), that is, the resulting closed-loop systems (19) and (41) with (48), are quadratically stable, which completes the proof of Theorem 4.1.

Similar to Theorem 3.2, the following result can be obtained from Theorem 4.1 and Lemma 3.1 directly.

Theorem 4.2 *Systems (19) are quadratically stabilizable by dynamic output feedback law (41) if there exist matrices* $X_{11} \in \mathbf{R}^{n\times n}$, $X_{221} \in \mathbf{R}^{p\times p}$, $X_{222} \in \mathbf{R}^{(n-p)\times(n-p)}$, $X_1 \in \mathbf{R}^{2n\times 2n}$, $Y_0 \in \mathbf{R}^{n\times p}$, *and* Y, $Y_1 \in \mathbf{R}^{m\times n}$ *such that the following LMI is solvable*

$$\mathcal{L}(X, X_1, \Gamma, \Gamma_1, \Psi, \Psi_1) < 0, \tag{51}$$

where Γ, Γ_1, Ψ, Ψ_1 *are defined in the same way as those in (47),*

$$X_{22} = V \operatorname{diag}\{X_{221}, X_{222}\}V', \qquad X = \operatorname{diag}\{X_{11}, X_{22}\}.$$

In this case, a dynamic output feedback controller can be given by (41) with

$$L = Y_0 U C_0 X_{221}^{-1} C_0 U', \quad K = Y X_{11}^{-1}, \quad K_1 = Y_1 X_{11}^{-1}. \tag{52}$$

Remark 4.1 This section can be regarded as an extension of the results in Section 3. In addition, this section presents a new approach to construction of dynamic output feedback controller for a class of discrete-time nonlinear time-delay systems.

5 Numerical Examples

All the numerical examples in this section are computed via the MATLAB LMI Toolbox [4].

The first example has been discussed by [17], where NLMIs are presented and no explicit algorithms are given. We shall present quadratic stability via LMI using the proposed explicit algorithms in this paper.

Example 5.1 [17] Consider the following unforced discrete-time systems:

$$z_{k+1} = \begin{pmatrix} -0.5 & -0.4 \\ 0.2 & -0.6 \end{pmatrix} z_k + \begin{pmatrix} 0.3 & 0.1 \\ -0.1 & 0.1 \end{pmatrix} z_{k-2} + g(k, z_k, z_{k-2}), \tag{53}$$

where $g(k, z_k, z_{k-2}) = MF(k)(N_A z_k + N_d z_{k-2})$, $M = \begin{pmatrix} 0.3 \\ 0.1 \end{pmatrix}$, $N_A = (0.15 \quad 0.1)$, $N_d = (0.2 \quad 0.1)$, $F(k)$ is an uncertain matrix with an appropriate dimension and satisfying $F'(k)F(k) \le I$ for all k.

Then we have

$$\begin{aligned} g'(k, z_k, z_{k-2})g(k, z_k, z_{k-2}) &= (N_A z_k + N_d z_{k-2})' F'(k) M' M F(k)(N_A z_k + N_d z_{k-2}) \\ &\le 0.1(z_k' \quad z_{k-2}') \begin{pmatrix} N_A' \\ N_d' \end{pmatrix} (N_A \quad N_d) \begin{pmatrix} z_k \\ z_{k-2} \end{pmatrix}. \end{aligned} \tag{54}$$

That is, $G = \sqrt{0.1} N_A = (0.0474 \quad 0.0316)$, $G_1 = \sqrt{0.1} N_d (0.0632 \quad 0.0316)$ in (2). We obtain a pair of solutions from LMI (5) with (4) as follows:

$$X = \begin{pmatrix} 9.8666 & -0.7210 \\ -0.7210 & 7.3597 \end{pmatrix}, \qquad X_1 = \begin{pmatrix} 4.2989 & 0.2233 \\ 0.2233 & 1.8045 \end{pmatrix}. \tag{55}$$

Therefore the systems (53) is quadratically stable.

The following example is a nonlinear system, we shall illustrate the construction of state feedback.

Example 5.2 Consider the following linear discrete-time systems with nonlinear perturbation:

$$\begin{aligned} x_{k+1} = & \begin{pmatrix} 1 & -0.6 \\ 0.4 & 0.5 \end{pmatrix} x_k + \begin{pmatrix} 0.5 & 0.2 \\ 0.6 & 0.4 \end{pmatrix} x_{k-2} + \begin{pmatrix} 0.1 & 0.2 \\ 0 & 0.1 \end{pmatrix} u_k \\ & + M \sin(N_A x_k + N_d x_{k-2} + N_B u_k), \end{aligned} \tag{56}$$

where

$$M = (0.1 \quad 0.1)', \qquad N_A = (0.02 \quad 0.03), \qquad N_d = (0.02 \quad 0.01), \qquad N_B = (2 \quad 1.5). \tag{57}$$

Then it is easy to have $F = \mu N_A,\ F_1 = \mu N_d,\ H = \mu N_B,\ \mu = \sqrt{0.02}$. Now a static state feedback controller can be constructed by Corollary 3.1. A triple of matrix solutions X, X_1 and Y can be obtained from LMI (32) with (33) as follows:

$$X = \begin{pmatrix} 72.0747 & -23.0754 \\ -23.0754 & 93.6106 \end{pmatrix}, \qquad X_1 = \begin{pmatrix} 46.6402 & 12.4100 \\ 12.4100 & 21.4395 \end{pmatrix},$$
$$Y = \begin{pmatrix} 348.9366 & -154.6649 \\ -468.6426 & 208.1791 \end{pmatrix}.$$

Then the control gain of controller (34) can be given as follows:

$$K = YX^{-1} = \begin{pmatrix} 4.6818 & -0.4981 \\ -6.2863 & 0.6743 \end{pmatrix}. \tag{58}$$

The following example presents a very simple way of constructing both static output feedback control law and dynamic output feedback control law based on measurable output.

Example 5.3 Consider the systems (56) with the following output

$$y_k = Cx_k, \tag{59}$$

where $C = (1 \quad 0)$, and the constraint for nonlinear perturbation $f(k, x_k, x_{k-2}, u_k)$ is defined by (20) with F, F_1 and H presented by Example 5.2.

At first, we present a static output feedback control law in the form of (21) for Example 5.3.

Similarly, the following matrix solutions can be obtained from LMI (38) with (39)

$$X_{11} = 35.1887, \qquad X_{22} = 16.6820, \qquad X_1 = \begin{pmatrix} 17.5368 & 5.5927 \\ 5.5927 & 6.4090 \end{pmatrix},$$
$$Y = \begin{pmatrix} 129.7891 \\ -174.3170 \end{pmatrix}, \qquad Y_1 = \begin{pmatrix} 123.5208 \\ -165.2880 \end{pmatrix}.$$

In this case, it follows from Theorem 3.2 that a static output feedback control law can be given as follows:

$$u_k = Ky_k + K_1 y_{k-2} = \begin{pmatrix} 3.6884 \\ -4.9538 \end{pmatrix} y_k + \begin{pmatrix} 3.5102 \\ -4.6972 \end{pmatrix} y_{k-2}. \tag{60}$$

Then a dynamic output feedback controller can be constructed by means of Theorem 4.2. To this end, the following solutions can be computed from LMI (51) and (47)

$$X_{11} = 10^3 \begin{pmatrix} 1.4250 & 0.1199 \\ 0.1199 & 0.7585 \end{pmatrix}, \quad X_{221} = 216.5833, \quad X_{222} = 229.0485,$$

$$X_1 = \begin{pmatrix} 754.0081 & 265.3081 & 0.0611 & 0.1206 \\ 265.3081 & 240.4736 & -0.6242 & 0.0052 \\ 0.0611 & -0.6242 & 139.6342 & 70.6240 \\ 0.1206 & 0.0052 & 70.6240 & 64.9059 \end{pmatrix}, \qquad Y_0 = \begin{pmatrix} 155.7066 \\ 135.3971 \end{pmatrix},$$

$$Y = 10^3 \begin{pmatrix} 6.1647 & 0.9422 \\ -8.2406 & -1.2733 \end{pmatrix}, \qquad Y_1 = 10^3 \begin{pmatrix} 5.7601 & 2.1453 \\ -7.7000 & -2.8674 \end{pmatrix}.$$

Then it follows from Theorem 4.2 that a dynamic output feedback controller can be given in the form of (41) with the following gain matrices:

$$\begin{gathered} L = \begin{pmatrix} 0.7189 \\ 0.6252 \end{pmatrix}, \qquad K = \begin{pmatrix} 4.2786 & 0.5660 \\ -5.7178 & -0.7751 \end{pmatrix}, \\ K_1 = \begin{pmatrix} 3.8556 & 2.2190 \\ -5.1542 & -2.9658 \end{pmatrix}. \end{gathered} \tag{61}$$

6 Conclusion

This paper has studied the problems of quadratic stability and quadratic stabilisation problem for a class of discrete time-delay systems with nonlinear perturbations. It is shown that the problems can be reformulated as convex optimization problems in the form of LMI. The design technique in the existing literature has been improved and generalized in this paper. This paper presents a unified way of designing quadratic state feedback and output feedback laws for a class of perturbed discrete time-delay systems. It is easy to see that the approach in this paper can be fully extended to systems with multiple time-delays.

References

[1] Barmish, B.R. Necessary and sufficient conditions for quadratic stabilizability of an uncertain system. *J. Opt. Theory Appl.* **46** (1985) 399–408.
[2] Boyd, S.P., Ghaoui, L.E., Feron, E. and Balakrishnam, V. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, 1994.
[3] Dugard, L. and Verriest, E.I. *Stability and Control of Time-Delay Systems.* Springer-Verlag, London, 1997.
[4] Gahinet, P., Nemirovski, A., Laub, A.J. and Chilali, M. *LMI Control Toolbox.* The Math Works Inc., 1995.
[5] Garcia, G., Bernussou, J. and Arzelier, D. Robust stabilization of discrete-time linear systems with norm-bounded time-varying uncertainty. *Systems & Control Letters* **22** (1994) 327–339.

[6] Gu, K.Q. Designing stabilizing control of uncertain systems by quasiconvex optimization. *IEEE Trans. Automat. Contr.* **39** (1994) 127–131.

[7] Iwasaki, T. and Skelton, R.E. All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas. *Automatica* **30** (1994) 1307–1317.

[8] Halicka, M. and Rosinova, D. Stability robustness bound estimates of discrete systems: Analysis and comparison. *International Journal of Control* **60** (1994) 297–314.

[9] LaSalle, J.P. *The Stability and Control of Discrete Process.* Springer, Berlin, 1986.

[10] Mahmoud, M.S. *Robust Control and Filtering for Time-Delay Systems.* Marcel Dekker, Inc., New York, 2000.

[11] Nesterov, Y. and Nemirovsky, A. 'Interior point polynomial methods in convex programming', **13** (Philadelphia, PA: SIAM., 1994)

[12] Petersen, I.R. and Hollot, C.V. A Riccati equation approach to the stabilization of uncertain linear systems. *Automatica* **22** (1986) 397–411.

[13] Šiljak, D.D. *Decentralized Control of Complex Systems.* Academic Press, Boston, 1991.

[14] Šiljak, D.D. and Stipanovic, D.M. Robust stabilization of nonlinear systems: The LMI approach. *Mathematical Problems in Engineering* **6** (2000) 461–493.

[15] Stipanovic, D.M. and Šiljak, D.D. Robust stability and stabilization of discrete-time nonlinear systems: the LMI approach. *Int. J. Control* **74** (2001) 873–879.

[16] Wu, H. and Mizukami, K. Linear and nonlinear stabilizing continuous controllers of uncertain dynamical systems including state delay. *IEEE Trans. Automat. Contr.* **41** (1996) 116–121.

[17] Xu, S.Y., Lam, J. and Yang, C.W. Quadratic stability and stabilization of uncertain linear discrete-time systems with state delay. *Systems & Control Letters* **43** (2001) 77–84.

[18] Yakubovich, V.A. The S-procedure in nonlinear control theory. *Vestnik Leningrad Univ. Math.* **4** (1977) 73–93.

[19] Yaz, E. and Niu, X. Stability robustness of linear discrete-time systems in the presence of uncertainty. *International Journal of Control* **50** (1989) 173–182.

[20] Yuan, L., Achenie L.E.K. and Jiang, W.S. Robust H_∞ control for linear discrete-time systems with norm-bounded time-varying uncertainty. *Systems & Control Letters* **27** (1996) 199–208.

[21] Zeng, X.J. Robust stability for linear discrete-time systems with structure perturbations. *International Journal of Control* **61** (1995) 739–748.

Nonlinear Dynamics and Systems Theory, **3**(2) (2003) 191–202

A New Generalization of Direct Lyapunov Method for Uncertain Dynamical Systems

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Received: October 10, 2002; Revised: June 18, 2003

Abstract: In this paper we study a class of uncertain dynamical systems and sufficient conditions, in terms of matrix-valued Liapunov functions are provided for the new concept of stability and uniform asymptotic stability.

Keywords: *Uncertain dynamical systems; matrix-valued Liapunov functions; stability; uniform asymptotic stability.*

Mathematics Subject Classification (2000): 26E50, 34A19, 93C42, 93D05.

1 Introduction

In many cases the motion of uncertain systems is successfully analysed in terms of the development of ideas of direct Lyapunov method [1]. Recent surveys by Corless [2] and Leitmann [3] of papers in this direction provide a comprehensive idea of what has been done in the field of uncertain system investigations for the last decades. The aim of this paper is to give an account of results of qualitative investigation of solutions to uncertain systems with respect to the moving invariant set. To this end the method of matrix Lyapunov functions is applied.

It should be noted that the investigation of uncertain system dynamics in terms of matrix-valued functions allows the extension of the set of the direct Lyapunov method.

2 Statement of the Problem

2.1 Description of the system

We consider a mechanical system whose motion is modelled by the differential equations

$$\frac{dx}{dt} = f(t, x, \alpha), \qquad x(t_0) = x_0, \tag{2.1}$$

where $x(t) \in R^n$, $t \in \mathcal{T}_0 = [t_0, +\infty)$, $t_0 \in \mathcal{T}_i$, $\mathcal{T}_i \subseteq R$ and $f \in C(\mathcal{T}_0 \times R^n \times R^d,\, R^n)$. The parameter $\alpha \in R^d$, $d \geq 1$, represents the "uncertainties" of the system under consideration. Here and on it is assumed that the motion of system (2.1) described by the solution $x(t; t_0, x_0, \alpha) \triangleq x(t, \alpha)$ possesses the following properties.

(A_1) for an open neighborhood D of the state $x = 0$, $D \subseteq R^n$
   (a) system (2.1) has unique solution $x(t; t_0, x_0, \alpha)$ taking the value x_0 for $t = t_0$ for any $(t_0, x_0, \alpha) \in \mathcal{T}_i \times D \times \mathcal{S}$, $\mathcal{S} \subset R^d$, $\mathcal{S}$ is a compact set;
   (b) the motion $x(t; t_0, x_0, \alpha)$ of system (2.1) is defined and continuously (differentiable) in $(t, t_0, x_0) \in \mathcal{T}_0 \times \mathcal{T}_i \times D$ for any $\alpha \in \mathcal{S}$.

Note that the initial values x_0 and the uncertainty parameters α may be related by the correlations which ensure the presence of properties (A_1) for the motions of system (2.1).

Model (2.1) embraces many systems whose dynamics is modelled by ordinary differential equations with uncertain values of parameters.

According to Leitmann [4], Chen [5], etc., the parameter α:

(a) can represent an uncertain value of some parameter of the system or the outer perturbation;
(b) can be a function mapping R into R^d and representing some parameter value which is uncertainly time-varying or input effects;
(c) can be a function mapping $\mathcal{T}_0 \times R^n$ into R^d and representing nonlinear elements of the mechanical system in question whose exact description is difficult;
(d) can be just an index showing the existence of some uncertainties in the system;
(e) can be a combination of (a)−(c).

Let a function $r = r(\alpha) > 0$ be given such that $r(\alpha) \to r_0$ $(r_0 = \text{const} > 0)$ as $\|\alpha\| \to 0$ and $r(\alpha) \to +\infty$ as $\|\alpha\| \to +\infty$. In the Euclidean space $(R^n, \|\cdot\|)$ the moving set

$$A(r) = \{x \in R^n \colon \|x\| = r(\alpha)\} \tag{2.2}$$

is determined and the set $A(r)$ is assumed non-empty for any $(\alpha \neq 0) \in \mathcal{S} \subset R^d$.

Definition 2.0 The solution $x(t, \alpha)$ of system (2.1) is called *non-continuable*, if for any $x(t, \alpha)$ there is not a continuation, which would be different from $x(t, \alpha)$ on the interval of definition $J \subset \mathcal{T}_0$ for all $\alpha \in \mathcal{S} \subseteq R^d$.

Definition 2.1 The set $A(r)$ is called *moving invariant set of system (2.1)*, if for every $x_0 \in A(r)$ and all solutions $x(t, \alpha) = x(t; t_0, x_0, \alpha)$ of system (2.1) determined on some interval $J \subset \mathcal{T}_0$ and such that $x(t_0; t_0, x_0, \alpha) = x_0$ for all $(\alpha \neq 0) \in \mathcal{S} \subset R^d$ the inclusion $x(t, \alpha) \in A(r)$ is satisfied for every $t \in J$.

2.2 Definitions

Taking into account the results of paper [6] and monograph [7] we shall formulate definitions necessary for the subsequent presentation.

Definition 2.2 The solutions of system (2.1) are

(a) *stable with respect to the sets* $A(r)$ *and* $\mathcal{T}_i \subset R$, iff given $r(\alpha) > 0$, $\varepsilon > 0$ and $t_0 \in \mathcal{T}_i$, given $\delta = \delta(t_0, \varepsilon) > 0$ such that under the initial conditions

$$r(\alpha) - \delta < \|x_0\| < r(\alpha) + \delta,$$

the solution of system (2.1) satisfies the estimate

$$r(\alpha) - \varepsilon < \|x(t,\alpha)\| < r(\alpha) + \varepsilon,$$

for all $t \in \mathcal{T}_0$ and all $\alpha \in \mathcal{S} \subseteq R^d$;

(b) *uniformly stable with respect to the sets* $A(r)$ *and* $\mathcal{T}_i$, iff the conditions of Definition 2.2(a) are satisfied and for any $\varepsilon > 0$ the corresponding maximal value δ_M satisfying the conditions of Definition 2.2() is such that

$$\inf\,[\delta_M(t,\varepsilon)\colon t \in \mathcal{T}_i] > 0;$$

(c) *stable in the whole with respect to* $\mathcal{T}_i$, iff the condition of Remark 2.2 are satisfied as well as the conditions of Definition 2.2() with the function

$$\delta_M(t,\varepsilon) \to +\infty \quad \text{as} \quad \varepsilon \to +\infty, \quad \forall\, t \in \mathcal{T}_i;$$

(d) *uniformly stable in the whole with respect to* $\mathcal{T}_i$, iff the conditions of Definitions 2.2(b) and 2.2(c) are satisfied.

Definition 2.3 For the solutions of system (2.1) the moving set $A(r)$ is called

(a) *attractive with respect to* $\mathcal{T}_i$, iff given function $r(\alpha) > 0$ and $t_0 \in \mathcal{T}_i$, there exists a $\delta(t_0) > 0$ and for any $\zeta > 0$ a $\tau(t_0,x_0,\zeta) \in [0,\infty)$ exists such that the condition

$$r(\alpha) - \delta < \|x_0\| < r(\alpha) + \delta$$

implies the estimate

$$r(\alpha) - \zeta < \|x(t,\alpha)\| < r(\alpha) + \zeta$$

for all $t \in (t_0 + \tau(t_0,x_0,\zeta),\, +\infty)$ and all $\alpha \in \mathcal{S} \subseteq R^d$;

(b) x_0-*uniformly attractive with respect to* $\mathcal{T}_i$, iff the conditions of Definition 2.3(a) are satisfied and for any $t_0 \in \mathcal{T}_i$ there exists a $\delta(t_0) > 0$ and for any $\zeta \in (0,+\infty)$ a $\tau_u(t_0,\Delta(t_0),\zeta) \in [0,\infty)$ exists such that

$$\sup\,\{\tau_m(t_0,x_0,\zeta)\colon r(\alpha) - \Delta \le \|x_0\| < r(\alpha) + \Delta\} = \tau_u(t_0,\Delta(t_0),\zeta);$$

(c) t_0-*uniformly attractive with respect to* $\mathcal{T}_i$, iff the conditions of Definition 1.3(a) are satisfied, there exists a $\Delta^* > 0$ and for any

$$(x_0,\zeta) \in \{r(\alpha) - \Delta^* \le \|x_0\| < r(\alpha) + \Delta^*\} \times (0,+\infty)$$

there exists a $\tau_u(\mathcal{T}_i,x_0,\zeta) \in [0,+\infty)$ such that

$$\sup\,\{\tau_m(t_0,x_0,\zeta)\colon t_0 \in \mathcal{T}_i\} = \tau_u(\mathcal{T}_i,x_0,\zeta);$$

(d) *uniformly attractive with respect to* $\mathcal{T}_i$, if conditions of Definitions 2.3(b) and 2.3(c) are satisfied or, what is the same, the conditions of Definition 2.3() are satisfied and there exists a $\delta > 0$ and for any $\zeta \in (0,+\infty)$ a $\tau_u(\mathcal{T}_i,\Delta,\zeta) \in [0,\infty)$ exists such that

$$\sup\,[\tau_m(t_0,x_0,\zeta)\colon (t_0,x_0) \in \mathcal{T}_i \times \{r(\alpha) - \Delta < \|x_0\| < r(\alpha) + \Delta\}] = \\ = \tau_u(\mathcal{T}_i,\Delta,\zeta);$$

(e) the attraction properties 2.3(a)–2.3(d) *take place in the whole*, if conditions of Definition 2.3(a) are satisfied for any $\Delta(t_0) \in (0,+\infty)$ and any $t_0 \in \mathcal{T}_i$, if $r(\alpha) \to +\infty$ as $\|\alpha\| \to +\infty$.

The expression "with respect to $\mathcal{T}_i$" in Definitions 2.3 is omitted, iff $\mathcal{T}_i = R$.

Definition 2.4 For system (2.1) the moving set $A(r)$ is called

(a) *asymptotically stable with respect to* $\mathcal{T}_i$, iff it is stable with respect to $\mathcal{T}_i$ and attractive with respect to $\mathcal{T}_i$;

(b) *equi-asymptotically stable with respect to* $\mathcal{T}_i$, if it is stable with respect to $\mathcal{T}_i$ and x_0-uniformly attractive with respect to $\mathcal{T}_i$;

(c) *quasi-uniformly asymptotically stable with respect to* $\mathcal{T}_i$, if it is uniformly stable with respect to $\mathcal{T}_i$ and t_0-uniformly attractive with respect to $\mathcal{T}_i$;

(d) *uniformly asymptotically stable with respect to the sets* $A(r)$ *and* $\mathcal{T}_i$, if it is uniformly stable with respect to the sets $A(r)$ and $\mathcal{T}_i$ and uniformly attractive with respect to the sets $A(r)$ and $\mathcal{T}_i$;

(e) *uniformly exponentially stable with respect to* $\mathcal{T}_i$, if given function $r(\alpha)$ and constants β_1, β_2 and λ, there exists a $\delta > 0$ such that the condition

$$r(\alpha) - \delta < \|x_0\| < r(\alpha) + \delta$$

implies the estimate

$$\begin{aligned} r(\alpha) - \beta_1\|x_0\|\exp[-\lambda(t-t_0)] \le \|x(t,\alpha)\| \le \\ \le r(\alpha) + \beta_2\|x_0\|\exp[-\lambda(t-t_0)] \qquad \forall\, t \in \mathcal{T}_0, \quad \forall\, t_0 \in \mathcal{T}_i; \end{aligned}$$

(f) *exponentially stable in the whole with respect to* $\mathcal{T}_i$, if the conditions of Definition 2.4(e) are satisfied for $r(\alpha) \to \infty$, $\|\alpha\| \to +\infty$ and $\delta \to +\infty$.

The expression "with respect to $\mathcal{T}_i$" in Definitions 1.4 is omitted, iff $\mathcal{T}_i = R$.

3 Properties of Matrix-Valued Functions on the Moving Set

Under some assumptions it is possible to construct for system (2.1) a two-index system of functions (see [8, 10])

$$U(t,x) = [u_{ij}(t,x)], \qquad i,\, j = 1, 2, \dots, s. \tag{3.1}$$

Here the elements $u_{ij} \in C(\mathcal{T}_0 \times R^n,\, R)$, for all $i,\, j = 1, 2, \dots, s$.

We construct by means of vector $y \in R^s$ $(y \ne 0)$ the scalar function

$$V(t,x,y) = y^T U(t,x,)y, \qquad (y \ne 0) \in R^s. \tag{3.2}$$

The total upper right Dini derivative of function (3.2) along solutions of system (2.1) is defined by the formula

$$D^+V(t,x,y) = y^T D^+U(t,x,)y, \qquad (y \ne 0) \in R^s. \tag{3.3}$$

Here the upper right Dini derivative of the matrix $U(t,x)$

$$D^+U(t,x) = \limsup\left\{[U(t+\theta,\, x+\theta f(t,x,\alpha)) - U(t,x)]\theta^{-1} \colon \theta \to 0^+\right\}$$

is computed element-wise.

Further for the set $A(r)$ moving in R^n we shall consider its moving σ-neighborhood and the internal and external parts $\operatorname{int} A(r)$ and $\operatorname{ext} A(r)$, i.e. the sets

$$N(A,\sigma) = \{x \in R^n \colon 0 < \rho(x,A) < \sigma\},$$

where $\rho(x,A) = \inf\limits_{q \in A(r)} \rho(x,q)$ and σ is some number,

$$\operatorname{int} A(r) = \{x \in R^n \colon \|x\| < r(\alpha)\} \quad \text{and} \quad \operatorname{ext} A(r) = \{x \in R^n \colon \|x\| > r(\alpha)\},$$

respectively.

In view of results of the monograph [10] we shall cite the following definitions.

Definition 3.1 The matrix-valued function $U\colon R\times R^n\to R^{s\times s}$ is called *positive semi-definite on $\mathcal{T}_\tau=[\tau,+\infty)$, $\tau\in R$ with respect to the moving set $A(r)$*, if

(i) U is continuous in $(t,x)\in\mathcal{T}_\tau\times N(A,\sigma)$,

$$U\in C\left(\mathcal{T}_\tau\times N(A,\sigma),\, R^{s\times s}\right);$$

(ii) U is nonnegative on $N(A,\sigma)$:

$$y^TU(t,x)y\ge 0 \quad \forall\,(t,y)\in\mathcal{T}_\tau\times R^s \quad\text{and}\quad \forall\,x\notin A(r);$$

(iii) U vanishes when $x\in A(r)$.

Definition 3.2 (see [11]) The continuous function $\varphi\colon[0,\beta]\to R_+$ belongs to the class K, i.e. $\varphi\in K$, if $\varphi(0)=0$ and $\varphi(u)$ is strictly increasing on $[0,\beta]$.

Definition 3.3 The matrix-valued function $U\colon R\times R^n\to R^{s\times s}$ is called *positive definite on $\mathcal{T}_\tau$, $\tau\in R$ with respect to the moving set $A(r)$*, if conditions (i) − (iii) of Definition 3.1 are satisfied and there exists a function a of class K satisfying the inequality

$$a(\|x\|)\le y^TU(t,x)y, \quad \forall\,(t,y)\in\mathcal{T}_\tau\times R^s \quad\text{and}\quad \forall\,x\notin A(r).$$

The expression "on $\mathcal{T}_\tau$" in Definition 3.3(a) is omitted, iff all conditions of these definitions are satisfied for every $\tau\in R$.

Definition 3.4 The matrix-valued function $U\colon R\times R^n\to R^{s\times s}$ is called *decreasing on $\mathcal{T}_\tau$ with respect to the moving set $A(r)$*,

(i) U is continuous in $(t,x)\in\mathcal{T}_\tau\times N(A,\sigma)$,

$$U\in C\left(\mathcal{T}_\tau\times N(A,\sigma),\, R^{s\times s}\right);$$

(ii) there exists a function b of class K satisfying the inequality

$$y^TU(t,x)y\le b(\|x\|), \quad \forall\,(t,y)\in\mathcal{T}_\tau\times R^s \quad\text{and}\quad \forall\,x\notin A(r);$$

(iii) U vanishes when $x\in A(r)$.

The expression "on $\mathcal{T}_\tau$" in Definition 3.4 is omitted, iff the conditions of Definition 3.4 are satisfied for every $\tau\in R$.

4 On Stability and Uniform Asymptotic Stability of Uncertain Systems

Theorem 4.1 *Assume that in system (2.1) $f(t,x,\alpha)$ is continuous on $\mathcal{T}_0\times R^n\times R^d$ and the following conditions are satisfied*

(1) *for every $\alpha\in\mathcal{S}\subseteq R^d$ there exists a function $r=r(\alpha)>0$ such that the set $A(r)$ is nonempty for all $\alpha\in\mathcal{S}\subseteq R^d$;*

(2) *there exists a matrix-valued function $U\in C(\mathcal{T}_0\times R^n,R^{s\times s})$, $U(t,x)$ is locally Lipschitzian in x, the vector $y\in R^s$, $s\times s$-matrices $\theta_1(r)$ and $\theta_2(r)$ are such that*

(a) $\quad a(\|x\|)\le V(t,x,y)$ *for* $\|x\|>r(\alpha)$,

and

(b) $V(t,x,y) \le b(\|x\|)$ *for* $\|x\| \le r(\alpha)$,

where a *and* b *are of class* K;

(c) $D^+V(t,x,y)|_{(2.1)} \le \varphi^T(\|x\|)\theta_1(r)\varphi(\|x\|)$

if $\|x\| > r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

and

(d) $D^+V(t,x,y)|_{(2.1)} = 0$ *iff* $\|x\| = r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(e) $D^+V(t,x,y)|_{(2.1)} > \psi^T(\|x\|)\theta_2(r)\psi(\|x\|)$

if $\|x\| < r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

where $\varphi^T(\|x\|) = \Big(\varphi_1^{1/2}(\|x_1\|),\dots,\varphi_s^{1/2}(\|x_s\|)\Big)$, $\varphi_i \in K$,

$\psi^T(\|x\|) = \Big(\psi_1^{1/2}(\|x_1\|),\dots,\psi_s^{1/2}(\|x_s\|)\Big)$, $\psi_i \in K$;

(3) *there exist constant* $s \times s$ *-matrices* $\overline{\theta}_1$ *and* $\overline{\theta}_2$ *such that*

(a) $\dfrac{1}{2}\big(\theta_1(r) + \theta_1^T(r)\big) \le \overline{\theta}_1$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$

and

(b) $\dfrac{1}{2}\big(\theta_2(r) + \theta_2^T(r)\big) \ge \overline{\theta}_2$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

and moreover, $\overline{\theta}_1$ *is negative semi-definite and* $\overline{\theta}_2$ *is positive semi-definite;*

(4) *for any* $r(\alpha) > 0$ *and functions* $a(r)$ *and* $b(r)$

$$a(r) = b(r).$$

Then the set $A(r)$ *is invariant with respect to the solutions of system (2.1) and the solutions of system (2.1) are stable with respect to the set* $A(r)$.

For the *proof* see [12].

4.1 Corollary

In cases when it is possible to construct scalar Lyapunov function for system (2.1) the stability of solutions can be studied in terms of the following assertion.

Theorem 4.2 *The set* $A(r)$ *is invariant with respect to the solutions of system (2.1) and the solutions of system (2.1) are stable with respect to the set* $A(r)$ *if*

(1) *for every* $\alpha \in \mathcal{S} \subseteq R^d$ *there exists a function* $r = r(\alpha) > 0$ *such that* $r(\alpha) \to r_0$ $(r_0 = \mathrm{const} > 0)$ *as* $\|\alpha\| \to 0$ *and* $r(\alpha) \to +\infty$ *as* $\|\alpha\| \to +\infty$;

(2) *there exist scalar functions* $V \in C^1(\mathcal{T}_0 \times R^n, R_+)$, $W_1\colon R^n \times R^d \to R$ *and* $W_2\colon R^n \times R^d \to R$ *such that*

(a) $a(\|x\|) \le V(t,x)$ *for* $\|x\| > r(\alpha)$,

(b) $V(t,x) \le b(\|x\|)$ *for* $\|x\| \le r(\alpha)$,

where a *and* b *are of class* K;

(c) $DV(t,x)|_{(2.1)} \le W_1(x,\alpha)$ *for* $\|x\| > r(\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$,

(d) $DV(t,x)|_{(2.1)} = 0$ *iff* $\|x\| = r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(e) $DV(t,x)|_{(2.1)} \ge W_1(x,\alpha)$ *for* $\|x\| < r(\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$;

(3) *there exist functions* $\overline{W}_1(x)$ *and* $\underline{W}_2(x)$ *such that*

(a) $W_1(x,\alpha) \le \overline{W}_1(x) < 0$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(b) $W_2(x,\alpha) \ge \underline{W}_2(x) > 0$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$;

(4) *for any* $r(\alpha) > 0$ *and the functions* $a(r)$ *and* $b(r)$

$$a(r) = b(r).$$

The assertion of Theorem 4.2 follows from Theorem 4.1.

4.2 Example

Considered is the uncertain equation

$$\frac{dx}{dt} = x - f^2(\alpha)x^3, \qquad x(0) \neq 0, \tag{4.1}$$

where $f(\alpha)$ is the function of the uncertainty parameter $\alpha \in \mathcal{S} \subseteq R^d$, $f(\alpha) \to f_0$ ($f_0 = \text{const} > 0$) as $\|\alpha\| \to 0$ and $f(\alpha) \to 0$ as $\|\alpha\| \to \infty$.

Zero solution $x = 0$ of this equation is unstable by Lyapunov, since its first approximation

$$\frac{dx}{dt} = x, \qquad x(0) \neq 0$$

has the eigenvalue $\lambda = 1 > 0$.

Let $r(\alpha) = (f(\alpha))^{-1} > 0$. It is clear that $r(\alpha) \to r_0$ as $\alpha \to 0$ and $r(\alpha) \to \infty$ as $\alpha \to \infty$.

The set $A(r)$ is

$$A(r) = \left\{ x \colon |x| = \frac{1}{f(\alpha)} \right\}. \tag{4.2}$$

We take $V = x^2$ and compute

$$\frac{dV}{dt} = 2x\,\frac{dx}{dt} = 2x^2(1 - f^2(\alpha)x^2).$$

Hence, it is clear

$$\begin{aligned} \frac{dV}{dt} &< 0 \quad \text{for} \quad |x| > \frac{1}{f(\alpha)}, \quad t \geq 0, \\ \frac{dV}{dt} &= 0 \quad \text{for} \quad |x| = \frac{1}{f(\alpha)}, \quad t \geq 0, \\ \frac{dV}{dt} &> 0 \quad \text{for} \quad |x| < \frac{1}{f(\alpha)}, \quad t \geq 0. \end{aligned}$$

Therefore, if $f(\alpha)$ satisfies the conditions $\lim\limits_{\alpha\to 0} f(\alpha) = f_0$ and $\lim\limits_{\alpha\to\infty} f(\alpha) = 0$, then by Theorem 4.2 the set $A(r)$ is invariant with respect to the equation (4.1) and all solutions of this equation are stable with respect to the set (4.2) in the sense of Definition 2.2(a).

Note that the equation (4.1) was considered in [13] for $f^2(\alpha) = \beta^2$, β is a control parameter.

Theorem 4.3 *Assume that in system (2.1) $f(t, x, \alpha)$ is continuous on $\mathcal{T}_0 \times R^n \times R^d$ and*

(1) *for any $\alpha \in \mathcal{S} \subseteq R^d$ there exists a function $r = r(\alpha) > 0$ such that the set $A(r)$ is nonempty for all $\alpha \in \mathcal{S} \subseteq R^d$;*

(2) *there exists a matrix-valued function* $U \in C(\mathcal{T}_0 \times R^n, R^{s\times s})$, $U(t,x)$ *is locally Lipschitzian in* x, *the vector* $y \in R^s$, $s \times s$*-matrices* $\Phi_1(r)$ *and* $\Phi_2(r)$ *are such that*

(a) $a(\|x\|) \le V(t,x,y) \quad \text{for} \quad \|x\| > r(\alpha),$

and

(b) $0 < V(t,x,y) \le b(\|x\|) \quad \text{for} \quad \|x\| \le r(\alpha),$

where a *and* b *are of class* K*;*

(c) $D^+V(t,x,y)|_{(2.1)} < \varphi^T(\|x\|)\Phi_1(r)\varphi(\|x\|) \quad \text{for} \quad \|x\| > r(\alpha),$
$\alpha \in \mathcal{S} \subseteq R^d,$

and

(d) $D^+V(t,x,y)|_{(2.1)} = 0$ *iff* $\|x\| = r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(e) $D^+V(t,x,y)|_{(2.1)} > \psi^T(\|x\|)\Phi_2(r)\psi(\|x\|) \quad \text{for} \quad \|x\| < r(\alpha),$
$\alpha \in \mathcal{S} \subseteq R^d,$

where

$$\varphi^T(\|x\|) = \left(\varphi_1^{1/2}(\|x_1\|), \dots, \varphi_s^{1/2}(\|x_s\|)\right), \qquad \varphi_i \in K,$$

$$\psi^T(\|x\|) = \left(\psi_1^{1/2}(\|x_1\|), \dots, \psi_s^{1/2}(\|x_s\|)\right), \qquad \psi_i \in K,$$

$$x_s \in R^{n_s}, \qquad n_1 + n_2 + \dots n_s = n;$$

(3) *there exist constant* $s \times s$*-matrices* $\overline{\Phi}_1$ *and* $\overline{\Phi}_2$ *such that*

(a) $\frac{1}{2}\left(\Phi_1(r) + \Phi_1^T(r)\right) \le \overline{\Phi}_1 \quad \text{for all} \quad \alpha \in \mathcal{S} \subseteq R^d,$

(b) $\frac{1}{2}\left(\Phi_2(r) + \Phi_2^T(r)\right) \ge \overline{\Phi}_2 \quad \text{for all} \quad \alpha \in \mathcal{S} \subseteq R^d,$

and moreover $\overline{\Phi}_1$ *is negative definite and* $\overline{\Phi}_2$ *is positive definite;*

(4) *for any* $r(\alpha) > 0$ *and the functions* $a(r)$ *and* $b(r)$

$$a(r) = b(r).$$

Then the set $A(r)$ *is invariant with respect to the solutions of system (2.1) and the solutions of system (2.1) are uniformly asymptotically stable with respect to the set* $A(r)$.

For the *proof* see [14].

4.3 Corollary

Theorem 4.4 *The set* $A(r)$ *is invariant with respect to the solutions of system (2.1) and the solutions of system (2.1) are uniformly asymptotically stable with respect to the set* $A(r)$ *if*

(1) *for every* $\alpha \in \mathcal{S} \subseteq R^d$ *there exists a function* $r = r(\alpha)$ *such that* $r(\alpha) \to r_0$ *if* $\|\alpha\| \to 0$ *and* $r(\alpha) \to +\infty$ *if* $\|\alpha\| \to +\infty$*;*

(2) *there exist scalar functions* $V \in C^1(\mathcal{T}_0 \times R^n, R_+)$, $W_1 \colon R^n \times R^d \to R$ *and* $W_2 \colon R^n \times R^d \to R$ *such that*

(a) $a(\|x\|) \le V(t,x) \quad \text{for} \quad \|x\| > r(\alpha),$

(b) $V(t,x) \le b(\|x\|) \quad \text{for} \quad \|x\| \le r(\alpha),$

where a *and* b *are of class* K*;*

(c) $DV(t,x)|_{(2.1)} \le W_1(x,\alpha)$ *for* $\|x\| > r(\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$,

(d) $DV(t,x)|_{(2.1)} = 0$ *iff* $\|x\| = r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(e) $DV(t,x)|_{(2.1)} \ge W_1(x,\alpha)$ *for* $\|x\| < r(\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$;

(3) *there exist functions* $\overline{W}_1(x)$ *and* $\underline{W}_2(x)$ *of definite sign in the sense of Lyapunov such that*

(a) $W_1(x,\alpha) \le \overline{W}_1(x) < 0$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(b) $W_2(x,\alpha) \ge \underline{W}_2(x) > 0$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$;

(4) *for any* $r(\alpha) > 0$ *and functions* $a(r)$ *and* $b(r)$

$$a(r) = b(r).$$

4.4 Examples

Example 4.4.1 Let the equations

$$\begin{aligned} \frac{dx}{dt} &= n(t)y + \left(1 - \frac{1}{a^2}\, m^2(\alpha)(x^2+y^2)\right) x(x^2+y^2), \\ \frac{dy}{dt} &= -n(t)x + \left(1 - \frac{1}{a^2}\, m^2(\alpha)(x^2+y^2)\right) y(x^2+y^2) \end{aligned} \tag{4.3}$$

be given, where $n(t) \in C(R,R)$, $m(\alpha)$ is the uncertainty function in system (4.3) with the same properties that the function $f(\alpha)$ in Example 4.2.

Let $r(\alpha) = \frac{a}{m(\alpha)}$, $\alpha \in \mathcal{S} \subseteq R$. The set $A(r)$ is determined as

$$A(r) = \left\{x,\, y\colon (x^2+y^2)^{1/2} = r(\alpha)\right\}. \tag{4.4}$$

We take the function V in the form

$$V = x^2 + y^2.$$

Its derivative by virtue of equations (4.3) is

$$\frac{dV}{dt} = 2\left(1 - \frac{1}{a^2}\, m^2(\alpha)(x^2+y^2)\right)\left(x^2+y^2\right)^2.$$

Hence, it follows

$$\begin{aligned} \frac{dV}{dt} &< 0 \quad \text{for} \quad \left(x^2+y^2\right)^{1/2} > r(\alpha), \quad t \ge t_0, \\ \frac{dV}{dt} &= 0 \quad \text{for} \quad \left(x^2+y^2\right)^{1/2} = r(\alpha), \quad t \ge t_0, \\ \frac{dV}{dt} &> 0 \quad \text{for} \quad \left(x^2+y^2\right)^{1/2} < r(\alpha), \quad t \ge t_0. \end{aligned}$$

It is easy to see that if the function $m(\alpha)$ satisfies the conditions $\lim\limits_{\|\alpha\|\to 0} m(\alpha) = m_0$ and $\lim\limits_{\|\alpha\|\to\infty} m(\alpha) = \infty$, all conditions of Theorem 4.4 are satisfied and the set $A(r)$ is invariant for system (4.3) and all solutions of the system are uniformly asymptotically stable with respect to the set $A(r)$.

Example 4.4.2 We consider the systems

$$\begin{aligned}\frac{dx}{dt} &= \mu x + y - g(x,y,\alpha)x(x^2+y^2),\\ \frac{dy}{dt} &= \mu y - x - g(x,y,\alpha)y(x^2+y^2), \qquad \alpha \in \mathcal{S} \subseteq R^d,\end{aligned} \tag{4.5}$$

where $\mu = \text{const} > 0$, $g(x,y,\alpha) > 0$ is a function characteristics of "uncertainties" of system (4.5) (cf. [15]).

In system (4.5) we substitute the variables

$$x = -r\cos\theta, \qquad y = r\sin\theta$$

and reduce the system to the form

$$\frac{dr}{dt} = \mu r - g(r,\theta,\alpha)r^3, \qquad \frac{d\theta}{dt} = 1, \tag{4.6}$$

where

$$g^m \le g(r,\theta,\alpha) \le g^M \tag{4.7}$$

for all $(r,\theta,\alpha) \in R_+ \times [0,2\pi] \times \mathcal{S}$, $g^m < g^M$ are given constants.

Note that the solution $r = 0$ of the first approximation equations (4.6) is unstable in the sense of Liapunov, since the linear approximation $\dfrac{dr}{dt} = \mu r$ has its eigen value $\lambda = \mu > 0$.

Together with system (4.6) consider function $V = r^2$.

For derivative dV/dt by virtue of system (4.6) we get

$$\left.\frac{dV}{dt}\right|_{(2.79)} = 2r\,\frac{dr}{dt} = 2r^2[\mu - g(r,\theta,\alpha)r^2], \qquad \alpha \in \mathcal{S} \subseteq R^d. \tag{4.8}$$

Hence it follows that if for any function $g(r,\theta,\alpha)$ satisfying condition (4.7) the following inequalities hold

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(4.6)} &< 0 \qquad \text{for} \quad r^2 > \frac{\mu}{g(r,\theta,\alpha)}, \quad t \ge 0,\\ \left.\frac{dV}{dt}\right|_{(4.6)} &= 0 \qquad \text{for} \quad r^2 = \frac{\mu}{g(r,\theta,\alpha)}, \quad t \ge 0,\\ \left.\frac{dV}{dt}\right|_{(4.6)} &> 0 \qquad \text{for} \quad r^2 < \frac{\mu}{g(r,\theta,\alpha)}, \quad t \ge 0,\end{aligned}$$

then the moving set

$$\mathcal{S}^*(\varkappa) = \left\{ r\colon r^2 = \frac{\mu}{g(r,\theta,\alpha)} \right\}, \qquad \alpha \in \mathcal{S} \subseteq R^d$$

is uniformly asymptotically stable.

Further consider the motion of system (4.6) with respect to the domains

$$S_1 = \{r\colon r^2 < H\}, \quad 0 < H < \infty,$$
$$S_2 = \{r\colon r^2 \le \delta\}, \quad \delta = \left(\frac{\mu}{g^M}\right)^{1/2},$$
$$S_3 = \{r\colon r^2 \ge \eta\}, \quad \eta = \left(\frac{\mu}{g^m}\right)^{1/2}$$

under restrictions (4.7).

Let the motion of system (4.6) begin outside the ring with radius $r_0 + \sigma$, where $r_0 = \left(\frac{\mu}{g^m}\right)^{1/2}$ and σ is an arbitrary small constant value. Since

$$\left.\frac{dV}{dt}\right|_{(4.6)} = 2\mu V - 4g(r,\theta,\alpha)V^2,$$

by Theorem 1 from [16] the interval of time for which the solutions of system (4.6) will get to the moving surface

$$r^2 = \frac{\mu}{g(r,\theta,\alpha)}$$

is estimated by the inequality

$$\tau \le \int\limits_{\varkappa_1}^{\varkappa} \frac{dc}{2\mu c - 4g^m c^2}, \tag{4.9}$$

where $\varkappa_1 < \varkappa$, $\varkappa_1 = \frac{1}{2}\,r^2$, $\varkappa = \frac{1}{2}\,(r_0+\sigma)^2$. Estimate (4.9) implies

$$\tau \le \frac{1}{2\mu}\,\ln\left|\frac{(r_0+\sigma)^2}{r^2}\,\frac{(r^2-r_0^2)}{2r_0\sigma+\sigma^2}\right|.$$

Similarly we estimate the interval of time sufficient for solutions starting in the domain $r^* - \sigma \ge 0$, where $r^* = \left(\frac{\mu}{g^M}\right)^{1/2}$, to get to the moving surface $r^2 = \frac{\mu}{g(r,\theta,\alpha)}$.

Note that the function $g(r,\theta,\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$, is not assumed continuously differentiable, therefore equation (4.6) is efficiently studied by qualitative technique whereas its immediate integration is difficult.

5 Concluding Remarks

This investigation of uncertain system dynamics contributes to the well-known results for this class of equations in several directions. First, it is shown that under certain conditions the problem of qualitative analysis of solutions to the uncertain system is reduced to the investigation of the property of having a fixed sign of special matrices estimating the matrix-valued function and its total derivative along solutions of the system under consideration. Second, non-smooth and non-differentiable functions may be used as the elements of the matrix-valued function. Note also that our results possess a considerable potential for their extension to new classes of equations modelling the dynamics of uncertain systems and in particular uncertain controlled systems.

Acknowledgements

The author expresses his gratitude to Professors P. Borne and E.A. Galperin for their helpful comments on this paper.

References

[1] Lyapunov, A.M. *General Problem on Motion Stability.* Izdat. Akad. Nauk SSSR, Moscow, 1956. [Russian].

[2] Corless, M. Control of uncertain nonlinear systems. *ASME Journal of Dynamic Systems, Measurement, and Control* **115** (1993) 362–372.

[3] Leitmann, G. One approach to the control of uncertain dynamical systems. *Applied Mathematics and Computation* **70** (1995) 261–272.

[4] Leitmann, G. One approach to the control of uncertain dynamical systems. *Sixth Workshop of Dynamics and Control*, Vienna, Austria, 1993.

[5] Chen, Y.-H. Performance analysis of controlled uncertain systems. *Dynamics and Control* **6** (1996) 131–142.

[6] Lakshmikantham, V. and Vatsala, A.S. Stability of moving invariant sets. In: *Advances in Nonlinear Dynamics.* (Eds.: Sivasundaram, S. and Martynyuk, A.A.), Gordon and Breach Scientific Publishers, Amsterdam, **Vol.5**, 1997, 79–83.

[7] Grujić, Lj.T., Martynyuk, A.A. and Ribbens-Pavella, M. *Large Scale Systems Stability under Structural and Singular Perturbation.* Springer-Verlag, Berlin, 1987.

[8] Djordjević, M.Z. Stability analysis of interconnected systems with possibly unstable subsystems. *Systems and Control Letters* **3** (1983) 165–169.

[9] Martynyuk, A.A. On matrix Lyapunov functions and stability of motion. *Dokl. Akad. Nauk USSR* **249**(5) (1985) 59–63. [Russian].

[10] Martynyuk, A.A. *Stability by Liapunov's Matrix Functions Method with Applications*, Marcel Dekker, Inc., New York, 1998.

[11] Hahn, W. *Stability of Motion.* Springer-Verlag, Berlin, 1967.

[12] Martynyuk-Chernienko, Yu.A. To the theory of motion stability of uncertain systems. *Dokl. Natsional. Akad. Nauk Ukrainy* **1** (1998) 28–31. [Russian].

[13] Galperin, E.A. and Skowronski, J.M. Geometry of V-functions and the Lyapunov stability theory. *Proceedings of 24th Conference on Decision and Control Ft.*, Lauderdate, FL, 1985, 302–303.

[14] Martynyuk-Chernienko, Yu.A. On uniform asymptotic stability of solutions to uncertain systems with respect to invariant set. *Dokl. Akad. Nauk Russia* **364**(2) (1999) 163–166. [Russian].

[15] Kamenkov, G.V. *Stability and Oscillations of Nonlinear Systems.* Nauka, Moscow, 1972. [Russian].

[16] Mel'nikov, G.I. Some questions of direct Lyapunov's method. *Dokl. Akad. Nauk SSSR* **110**(3) (1956) 326–329. [Russian].

Nonlinear Dynamics and Systems Theory, **3**(2) (2003) 203–226

Control of Chaos in a Convective Loop System

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Received: March 25, 2003; Revised: July 4, 2003

Abstract: A convective loop is a system in which a fluid circulates freely inside a closed circular pipe. The circulating fluid works as a transport media of heat from a source to a sink. First order lumped parameter modelling of this system leads to a set of nonlinear ordinary differential equations. Depending on heating rate this system can show chaotic behavior. In this paper, the performance of nonlinear model predictive control is compared with other conventional nonlinear control law and it is found that although a simple linear or, nonlinear controller may stabilize the system, nonlinear model predictive controller outperforms other controllers.

Keywords: *Chaos; Lyapunov stability; nonlinear model predictive control.*

Mathematics Subject Classification (2000): 34D20, 34C23, 49J15, 70K55, 76E99.

1 Introduction

Natural convection loops showing chaotic behavior are used in solar energy heating and cooling systems, reactor, turbine, engine cooling systems, greenhouses, geothermal power production and in process industries. Chaos in such convective loop systems in general can be beneficial or detrimental depending on the process and the objective. Since it is associated with vigorous change in states under nominal operating condition without any change in input energy, it is beneficial for processes where mixing, heat transport and chemical reactions are important. However due to the oscillation, chaos may lead to vibrations and fatigue failure to the physical equipment, irregular and oscillation of process operating conditions and increased drag of fluid flow systems. Ehrhard and Müller [9] in their paper investigated natural convection in a closed loop. They first developed a first principal model of the loop based on heat transfer law. They also accounted for the nonsymmetric arrangement of heat sources and sinks. Finally the model is reduced to a set of nonlinear ordinary differential equations. Then through experimental and analytical data it is shown that this loop is characterized by nonlinear effects and can show stable, unstable or, chaotic regimes based on the heating rate. The model development and its analysis is further discussed in Section 2.3.

Abed and Fu [1, 2] in their papers have shown ways for local stabilization of nonlinear systems with Hopf and Stationary bifurcation. Sufficient conditions are also obtained for the local stabilization of nonlinear systems whose linearization has a pair of simple, nonzero imaginary eigenvalues. The greatest contribution in this area lies perhaps upon Ott, *et al.* [21] who have shown that small time dependent perturbations can be effectively used to convert a chaotic attractor to any of a large number of possible attracting time periodic motions. The method utilizes delay coordinate embedding and can be used on experimental situations where knowledge of the system dynamics are not available.

Like Ott, *et al.* [21], Singer, *et al.* [26] in their paper through experimental and simulation results have also shown how a simple low energy feedback controller like on-off controller can stabilize a chaotic system. The developed control action is based on the deviation of the vertical temperature difference from the equilibrium point which stabilizes the states to their equilibrium points. Wang and Abed [30] have also suggested a feedback control synthesis technique for relocating and ensuring stability of bifurcated limit cycles to a convective loop problem. They showed that stability can be ensured in several different ways, one of which is replacing the chaotic behavior by its equilibrium or, replacing the limit cycle with a relatively small amplitude limit cycle. For this purpose they have used a small washout filter to delay and to extinguish chaos in the model and developed linear and nonlinear feedback control law. Recently Bošković and Krstić [5] have investigated a thermal convective loop and developed a nonlinear feedback control law to achieve global stability using boundary control of velocity and temperature. The nonlinear control law is developed based on the discretized model of nonlinear PDE in space using the finite difference method and resultant high order system of coupled nonlinear ODE's.

In this paper, we will apply linear and nonlinear control law and investigate their performance among each other. For this case it is found that proportional state feedback control law with setpoint tracking $(u = -k(x_3 - x_{3e}))$ gives the best result where the proportional constant can be found out by stability analysis of linearized model or LQR. A nonlinear control law similar to the previous structure $(u = -(x_1 + x_2)(x_3 - x_{3e}))$ gives better result in terms of quick stabilization of the states to the desired setpoints (here, the desired setpoints are the equilibrium points). This controller is equivalent to taking $-k(x) = x_1 + x_2$ and depends a lot on the initial values of the states at the time when the control law is applied. Nonlinear control law based on backstepping method is also developed here which stabilizes the system but can not bring the states to the desired equilibrium points. Other advanced control law like Nonlinear Model Predictive Control (NMPC) stabilizes the system very efficiently compared to Linear MPC. Results from these simulations are also included for comparision.

2 Process Description

The presence of chaos is very common in physical systems. It is desirable to reduce the chaos so that system performance can be improved. We can do it in two ways (Ott, *et al.* [21]). First make some large costly alteration to the system which completely changes its dynamics to the desired dynamic behavior. Second improve performance by making small time dependant perturbations in an accessible parameters. In this case chaotic system holds advantage over other systems in that it can be made stable to any existing orbit without much effort or alteration of the system.

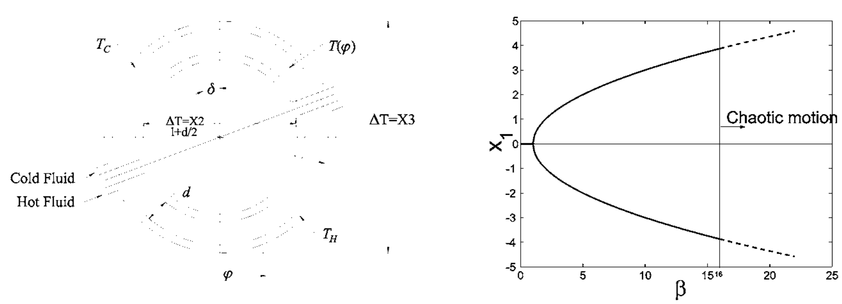
2.1 Definition of chaos

There is no universally agreed definition of chaos. Wang and Abed [30] defined chaos as "an irregular, seemingly random, dynamic behavior of a deterministic system displaying extreme sensitivity to initial condition" which most people accept as working definition. It has two main parts: 1) the system is deterministic meaning that the system has no irregular input; the chaotic behavior solely comes from the highly nonlinear nature of the system, and 2) the system is extremely sensitive to the initial conditions. Usually this kind of system has different stable region and can show periodic jump among these states depending on the external condition.

2.2 Description of thermal convection loop model

Natural convection in a closed loop system consists of a heat source and several sinks positioned above the source. The source and sink are connected by pipe forming at least one closed loop system. The heat is transported from the source to the sink by circulating fluid inside the loop. Unlike the forced convection (as in refrigerator), the heat is transported by natural convection only. Solar heating system and nuclear reactors are example of such system. For a detailed review of closed loop natural convection system, (see [9, 11, 19, 32]).

Figure 2.1(a) shows a schematic diagram of the system. The sink and the source are connected by a circular loop filled with an incompressible fluid which works as a transporting media of heat from source to sink. The cross section, A of this loop is circular and constant. The lower semicircle of the loop is heated by a hot fluid at a temperature T_H and the upper semicircle is cooled by a coolant at a temperature T_C. The cooling and heating zones are tilted by an angle δ from the symmetric position. If the temperature difference $\Delta T = T_H - T_C$ is increased, the fluid is at first at no motion state. During this stage, heat is transported by conduction only. As the heating rate is increased, a steady state convection arises either in clockwise or counter-clockwise direction. If heating rate is further increased, the steady state convection becomes unstable and shows oscillatory and chaotic motion.



(a) Schematic description of natural convection loop

(b) Bifurcation diagram of the state x_2 vs. β

Figure 2.1. a) System, b) Bifurcation of the system depends on the heating rate, β.

2.3 First order model development

Assuming $d << l$, material and energy balance (see [9] for detail derivation) leads to the following equations,

$$\begin{aligned} &\frac{\partial u}{l\partial\varphi} = 0, \\ &\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{l\partial\varphi} - \rho(T) g \sin(\varphi) - f_w, \\ &\rho_0 c_p \left\{ \frac{\partial T}{\partial t} + u \frac{\partial T}{l\partial\varphi} \right\} - \lambda \frac{\partial^2 T}{l^2 \partial\varphi^2} = h_w [T_w(\varphi(T) - T] + q_w(\varphi), \end{aligned} \tag{2.3.1}$$

where

$$f_w = \frac{1}{2}\rho_0 f_{w0} u, \tag{2.3.2a}$$

$$T(\varphi, t) = T_0(t) + \sum_{n=1}^{\infty} \{S_n(t)\sin(n\varphi) + C_n(t)\cos(n\varphi)\}, \tag{2.3.2b}$$

$$\begin{aligned} Q(\varphi) &= Q_0 + \sum_{n=1}^{\infty} \{Q_n \sin(n\varphi) + R_n \cos(n\varphi)\}, \\ &= \frac{1}{\rho_0 c_p} l \{h_w T_w(\varphi) + q_w(\varphi)\}. \end{aligned} \tag{2.3.2c}$$

Introducing the dimensionless variables as follows,

$$\textit{Time},\ t' = \frac{h_{w0}}{\rho_0 c_p} t, \tag{2.3.3a}$$

$$x_1 = \frac{\rho_0 c_p}{l h_{w0}} u, \tag{2.3.3b}$$

$$x_2 = \frac{\rho_0 c_p}{h_{w0}} \frac{\gamma g}{f_{w0} l} S_1, \tag{2.3.3c}$$

$$x_3 = \frac{\rho_0 c_p}{h_{w0}} \frac{\gamma g}{f_{w0} l} \left\{ \frac{\rho_0 c_p}{h_{w0}} R_1 - C1 \right\}, \tag{2.3.3d}$$

where

$$\begin{aligned} \gamma &= \text{coefficient of thermal expansion}, \\ c_p &= \text{specific heat}, \\ \rho_0 &= \text{reference density}, \\ \lambda &= \text{heat conductivity}, \\ g &= \text{acceleration due to gravity}, \\ h_w &= \text{heat transfer coefficient} = h_{w0} \left\{ 1 + K|x_1|^{1/3} \right\}. \end{aligned} \tag{2.3.4}$$

Neglecting higher order terms in equation (2.3.2),

$$T(\varphi,t) = T_0(t) + S_1(t)\sin(\varphi) + C_1(t)\cos(\varphi),$$
$$Q(\varphi) = Q_0 + Q_1\sin(\varphi) + R_1\cos(\varphi) = \frac{1}{\rho_0 c_p}\, l\{h_w T_w(\varphi) + q_w(\varphi)\}$$

and assuming that the heat transfer coefficient h_w is constant *i.e.*, $K = 0$, the parameters S_1, C_1, R_1 are found to be

$$C_1(t) = \frac{T(0^\circ,t) - T(180^\circ,t)}{2}, \tag{2.3.5a}$$

$$S_1(t) = \frac{T(90^\circ,t) - T(270^\circ,t)}{2}, \tag{2.3.5b}$$

$$R_1 = \frac{h_{w0}}{\rho_0 c_p}\frac{T_H - T_C}{2}, \tag{2.3.5c}$$

where T_H and T_C are the temperature of the heating and cooling zone respectively. Further assuming that there is no tilting between the heating and cooling zone *i.e.*, $\delta = 0$ and there is negligible heat transfer in the direction of the tube axis, the system can be described by the following set of ordinary differential equations:

$$\begin{aligned} \dot{x}_1 &= \alpha\left(-x_1 + x_2\right), \\ \dot{x}_2 &= -x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2 - x_3 - \beta, \end{aligned} \tag{2.3.6}$$

where,

$$\alpha = \frac{\rho_0 c_p}{h_{w0}}\frac{f_{w0}}{2}, \tag{2.3.7}$$

$$\beta = \frac{\gamma g}{f_{w0} l}\left(\frac{\rho_0 c_p}{h_{w0}}\right)^2 R_1 = \frac{\gamma g}{f_{w0} l}\frac{\rho_0 c_p}{h_{w0}}\frac{T_H - T_C}{2}. \tag{2.3.8}$$

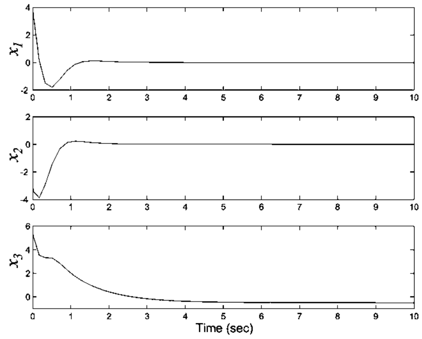
Here, α is comparable to the Prandtl number and β is the heating rate which is directly proportional to the temperature difference ΔT and is equivalent to the Rayleigh number. The states x_1, x_2 and x_3 are proportional to the average cross sectional velocity inside the loop, temperature difference along the horizontal direction and temperature difference along the vertical direction. All of the states are measurable and hence available for computation.

2.4 Open loop response

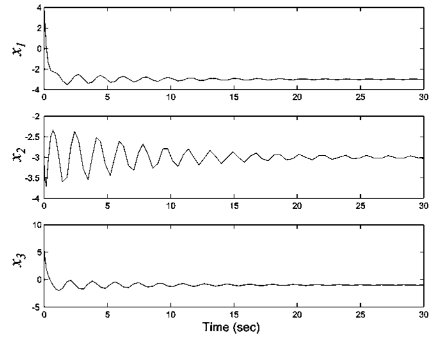
In the equation (2.3.6), α stands for Prandtl number and can be assumed constant. The other parameter β stands for Rayleigh number which is proportional to the heating rate. At equilibrium, $\dot{x}_i$'s are zero. Putting these values in equation (2.3.6) and solving them the following two cases arise:

Case a: $\beta \le 1$, $x_{1e} = x_{2e} = 0$ and $x_{3e} = -\beta$

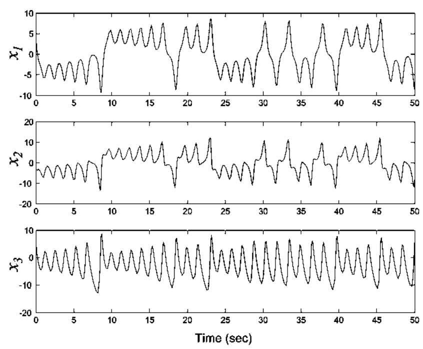
In this case, the states are globally stable and converge to the equilibrium points



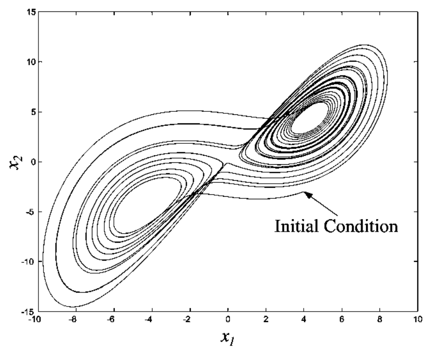
(a) $\beta = 0.5$: system is globally stable



(b) $\beta = 10$: system has two sets of stable region



(c) $\beta = 20$: system is chaotic



(d) x_2 vs. x_1 for $\beta = 20$

Figure 2.2. Open loop response of the system for different β's; Initial conditions of the state variables are $x_{10} = 4.0$, $x_{20} = -3.0$ and $x_{30} = 5.5$, taken arbitrarily.

irrespective of the initial conditions. The state x_1 i.e., average cross-sectional velocity of the fluid is zero at equilibrium which means that the fluid is at no motion state in this case and heat is transported from the source to the sink by conduction only.

Case b: $\beta > 1$, $x_{1e} = x_{2e} = \pm\sqrt{\beta - 1}$ and $x_{3e} = -1$

In this case, the states have two equilibrium points. The fluid average velocity may be clockwise or counter-clockwise. Heat is transported at this stage by convection. Depending on the value of the parameter β the system may show stable or unstable and chaotic behavior. This is because as heating rate is increased fluid velocity is also increased and at higher value of β it becomes locally unstable and jumps from one equilibrium point to another from time to time making the system chaotic.

The different cases are depicted in Figure 2.1(b). The open loop response for different β are given in Figures 2.2(a−c). These figures show how the system responses to the same initial condition with different β.

From Figure 2.2(d), it is obvious that at chaos the system has two different orbits. Solution of x_1 and x_2 remains in this orbit but never becomes stable to any single equilibrium point (see, Figure 2.2(c)). From the bifurcation diagram it is obvious that at some critical value of β the system starts showing chaotic behavior. To find out this critical value we need to do stability analysis of the open loop system:

2.4.1 Stability analysis of the linearized open loop system for $\boldsymbol{\beta > 1}$

If there is a nonlinear equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

then linearization of the above equation around the equilibrium point leads to the following equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

is evaluated at equilibrium points. For the system to be stable all the eigenvalues of the matrix $\mathbf{A}$ must have negative real parts. For the convective loop described by equation (2.3.6), the linearized equation becomes

$$\dot{x} = Ax,$$

where

$$A = \begin{bmatrix} -\alpha & \alpha & 0 \\ -x_3 & -1 & -x_1 \\ x_2 & x_1 & -1 \end{bmatrix}_{evaluated\ at\ equilibrium} = \begin{bmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & -\sqrt{\beta-1} \\ \sqrt{\beta-1} & \sqrt{\beta-1} & -1 \end{bmatrix}.$$

Here, the positive equilibrium values of x_1 and x_2 are taken for analysis with $\beta > 1$. Making the real parts of the eigenvalues of the A matrix equal to zero leads to the following relation[1]:

$$\beta_{crit} = \frac{\alpha(\alpha+4)}{\alpha-2}.$$

So, if β is greater than this critical value then the system will show chaotic behavior. For example for $\alpha = 4$, the critical value of β is 16 over which the system is chaotic. Notice that the critical value is found by linearization of the nonlinear system. So, in practice the transition from stable to chaotic behavior will not happen exactly at this critical value of β. In fact, there is a transition region where the system actually is semi-chaotic meaning that it shows chaotic response initially and after some period the oscillation decays resulting into settling down of the response to one of its stable equilibrium points.

3 Controlling Chaos

Unlike linear systems, control of nonlinear and chaotic system is difficult due to the heavy computational duty which makes nonlinear control not feasible. Also, when the main

[1] All the eigenvalue analysis is done by using Maple V.

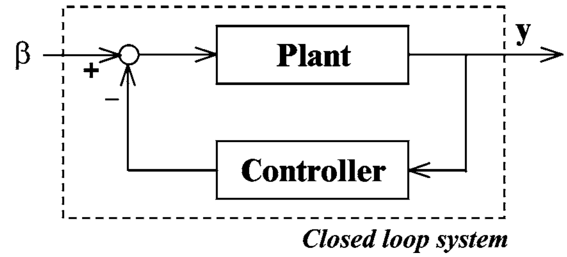


Figure 3.1. Schematic diagram of closed loop system.

target is to keep the operating point steady, it often suffices to linearize the nonlinear system around the operating point and apply linear control law.

Whenever any feedback control action is taken, the open loop system is changed to a desired closed loop stable system (Figure 3.1). In the following sections several methods for controlling chaos in the convective loop is discussed. For a review of different control strategy of chaotic system and bifurcation control see [21, 26, 30, 3, 16, 1, 2, 14]. There are several works on linear feedback control of chaotic system (see [30]). For different well established nonlinear controller design technique see [13, 17, 25]. The main theme is to set the control action to be a function of some observable state so that it can be calculated and implemented. In case of convective loop, the parameter β (heating rate) is proportional to the temperature difference in the vertical direction which is the state x_3. So the control action , u in the convective loop system is taken as the deviation of heating rate from its nominal value

$$\begin{aligned} \dot{x}_1 &= \alpha(-x_1 + x_2), \\ \dot{x}_2 &= -x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2 - x_3 \qquad \underbrace{-\beta + u}_{\text{Total heating rate, } U} \qquad . \end{aligned} \tag{3.0.1}$$

3.1 Proportional controller

For the convective loop system the control action, u in equation (3.0.1) is taken as proportional to the state, x_3 i.e.,

$$u = -kx_3.$$

Stability analysis of the closed loop system leads to the following relationship for the linear system

$$\beta = \frac{\alpha(4 + \alpha + 5k + \alpha k + k^2)}{\alpha - k - 2}.$$

This means that if the system were linear for $\alpha = 4$, $k = 2$ would be sufficient for stabilizing the system for any value of β. Since the system is highly nonlinear, feedback gain $k = 2$ may not suffice for higher values of β. However for small β, small negative feedback gain suffices to make the system steady [see, Figure 3.2]. However in this case the system equilibrium point is not the same as the open loop system. The equilibrium point of the average cross-sectional velocity is determined by $\pm\sqrt{\beta - k - 1}$ and the final fluid velocity stabilizes at this new equilibrium point instead of open loop equilibrium point $x_{2e} = \pm\sqrt{\beta - 1}$. The heating rate does not remain the same as β instead it

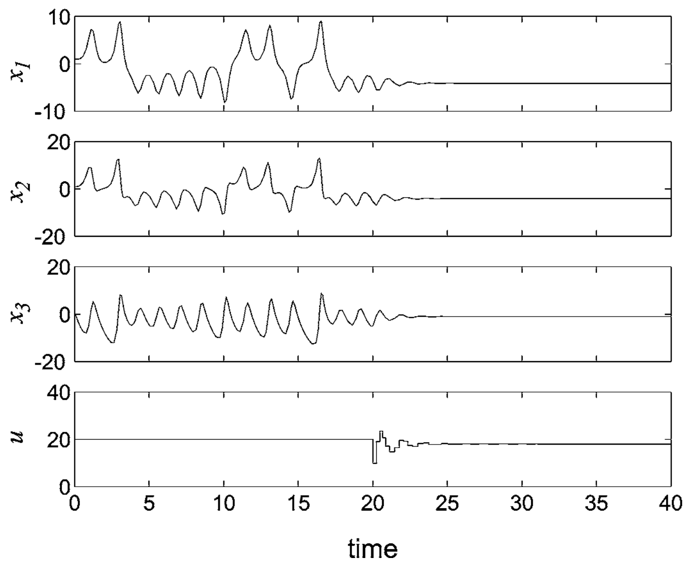


Figure 3.2. Closed loop response with proportional controller for $k = 2$ for system with $\beta = 20$. The control is applied at time, $t = 20$.

becomes $\beta - u$ where u is a constant value at steady state. This actually changes the heating rate to some extent.

3.2 Setpoint tracking

This is same as the proportional controller but the control law is defined by

$$u = -k(x_3 - x_{3e}), \tag{3.2.1}$$

where x_{3e} is the open loop equilibrium point of the state, x_3. The closed loop equilibrium point is same as the open loop equilibria and the steady state value of the control action, u is zero. This is given in the Figure 3.3.

3.3 Nonlinear control law: Lyapunov stability criterion

The main difficulties with designing a controller based on Lyapunov stability criterion is in choosing the energy function. For this case the best candidate for the energy function should be of the form:

$$V(x) = mx_3^2 + nx_1^2, \quad m, n > 0, \tag{3.3.1}$$

because of the fact that heating rate is proportional to x_3 (vertical temperature difference) and energy loss due to friction is proportional to x_1^2. Here m and n are two proportional constants which depends on the parameters used during conversion from PDE to ODE of the system model. But this energy function is positive semi-definite. Nevertheless using this “wrong” energy function, and Taylor series approximation to approximate $\sqrt{\beta - u + 1} = f(u) \approx a + bu$, where a and b are linearization constants and truncating constant terms in the final control law which accounts for lowering the heating rate

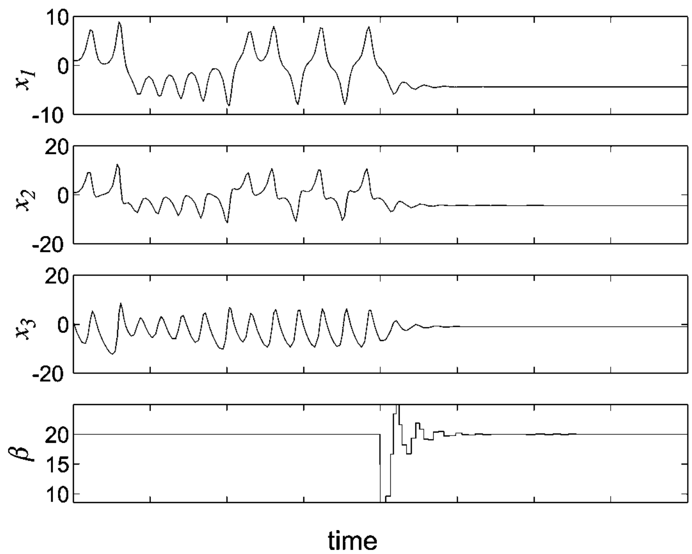


Figure 3.3. Feed back control with reference point tracking; here, $k = 2$ for system with $\beta = 20$.

(similar things are discussed in Section 3.4), we can finally come up with the following control law[2]:

$$u = (x_1 + x_2)(x_3 + 1). \tag{3.3.2}$$

Surprisingly this control gives better stabilizing effect than that developed by backstepping method as will be discussed next. But it depends greatly on the initial condition. Simulation result is given in Figure 3.4.

As we said earlier that this control law is based on the "wrong" energy function $V(x)$. So why does it work then? The answer is that with so many assumption during the development of the control law, the control law u is not associated with the positive semi-definite energy function any more. Rather it belongs to some other unknown energy function. If we take an energy function of the form $V(z) = \frac{1}{2}(z_1^2 + z_2^2 + z_3^2)$, where z_i's are the transformed states for $\dot{z} = f(z)$ with equilibrium points at the origin, it can be shown that $\dot{V}$ is negative provided that the open loop system is bounded (which is true for this case without any external excitation even in unstable chaotic region).

3.4 Nonlinear control law: Back stepping method

The system

$$\dot{x}_1 = \alpha(-x_1 + x_2), \tag{3.4.1}$$

$$\dot{x}_2 = -x_2 - x_1 x_3, \tag{3.4.2}$$

$$\dot{x}_3 = x_1 x_2 - x_3 - \beta + u, \tag{3.4.2}$$

[2]Detailed derivation is omitted here due to page constraints.

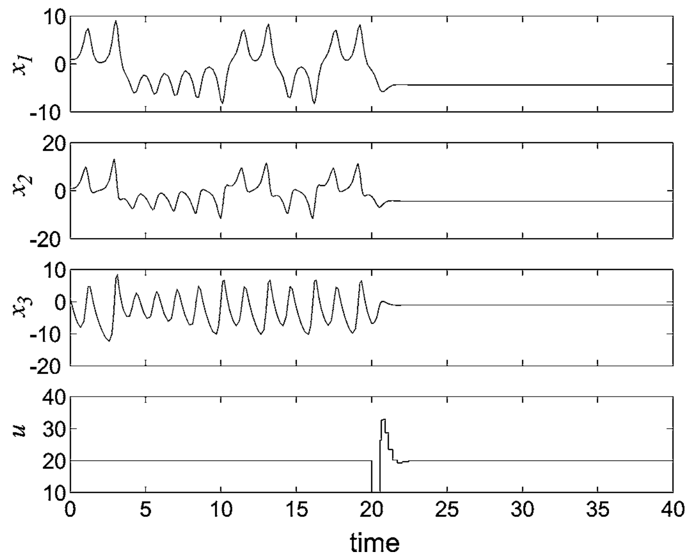


Figure 3.4. Feed back control for system with $\beta = 20$: Lyapunov stability criterion.

can be written in the following strict feedback system form:

$$\dot{x}_1 = f(x_1) + g(x_1)\xi_1, \tag{3.4.4}$$

$$\dot{\xi}_1 = f_1(x_1,\xi_1) + g_1(x_1,\xi_1)\xi_2, \tag{3.4.5}$$

$$\dot{\xi}_2 = f_2(x_1,\xi_1,\xi_2) + g_2(x_1,\xi_1,\xi_2)u, \tag{3.4.6}$$

where

$$\begin{aligned} &f(x_1) = -\alpha x_1, && g(x_1) = \alpha, && \xi_1 = x_2,\\ &f_1(x_1,\xi_1) = -x_2, && g_1(x_1,\xi_1) = -x_1, && \xi_2 = x_3,\\ &f_2(x_1,\xi_1,\xi_2) = x_1x_2 - x_3 - \beta, && g_2(x_1,\xi_1,\xi_2) = 1. && \end{aligned}$$

The first target is to stabilize the x_1 sub-system defined by equation (3.4.4). Let the Lyapunov function be $V_1 = \frac{1}{2}x_1^2$. Then

$$\dot{v}_1 = \frac{\partial V_1}{\partial x_1}\dot{x}_1 = x_1(-\alpha x_1 + \alpha x_2) = -\alpha x_1^2 + \alpha x_1 x_2.$$

Let us take the control law to be

$$x_2 = \phi(x_1) = -a x_1, \qquad a \in \Re^+. \tag{3.4.7}$$

We have included an unknown parameter a in the control law $\phi(x_1)$ which we will see in the later section increases degree of freedom and will help removing singularity in the final control law. For better flexibility and more degree of freedom we could also take the following control law instead:

$$x_2 = \phi(x_1) = -a x_1^{2b+1}, \quad a > 0, \quad b \geq 0. \tag{3.4.8}$$

But the addition of parameter b increases complexity in the final control law and so we assumed $b = 0$ for now. If necessary we can always come back and assume it to be nonzero.

With this control law [equation (3.4.7)] the sub-system equation (3.4.4) becomes:

$$\dot{x}_1 = -(a+1)\alpha x_1 \tag{3.4.9}$$

and the derivative of the energy function V becomes:

$$\dot{V} = -(a+1)\alpha x_1^2, \qquad a,\, \alpha > 0 \tag{3.4.10}$$

which is negative definite. Hence the sub-system is globally asymptotically stable. The energy function for the next sub-system equation (3.4.5) can be written as:

$$V_2 = V_1 + \frac{1}{2}[\xi_1 - \phi]^2 = \frac{1}{2}x_1^2 + \frac{1}{2}[x_2 + ax_1]^2. \tag{3.4.11}$$

Then the control law that makes the derivative of V_2 negative definite can be expressed as

$$\begin{aligned} x_3 = \phi_1 &= \frac{1}{g_1}\left[\frac{\partial\phi}{\partial x_1}(f + g\xi_1) - \frac{\partial V_1}{\partial x_1}g - k_1(\xi_1 - \phi) - f_1\right] \\ &= -\frac{1}{x_1}\left[-a(-\alpha x_1 + \alpha x_2) - x_1\alpha - k_1(x_2 + ax_1) + x_2\right] \\ &= -(a\alpha - \alpha - k_1 a) + (a\alpha + k_1 - 1)\frac{x_2}{x_1}, \qquad k_1 > 0. \end{aligned} \tag{3.4.12}$$

Similarly the final control law can be written as:

$$\begin{aligned} u &= \frac{1}{g_2}\left[\frac{\partial\phi_1}{\partial x_1}(f + g\xi_1) + \frac{\partial\phi_1}{\partial\xi_1}(f_1 + g_1\xi_2) - \frac{\partial V_2}{\partial\xi_1}g_1 - k_2(\xi_2 - \phi_1) - f_2\right] \\ &= -(a\alpha + k_1 - 1)\frac{x_2}{x_1^2}(-\alpha x_1 + \alpha x_2) + \frac{a\alpha + k_1 - 1}{x_1}(-x_2 - x_1x_3) \\ &\quad - (x_2 + ax_1)(-x_1) - k_2\left(x_3 + (a\alpha - \alpha - k_1 a) - (a\alpha + k_1 - 1)\frac{x_2}{x_1}\right) \\ &\quad - (x_1x_2 - x_3 - \beta) \\ &= \underbrace{(a\alpha + k_1 - 1)\left(\alpha + k_2 - 1 - \alpha\frac{x_2}{x_1}\right)\frac{x_2}{x_1}}_{\text{Singularity}} \\ &\quad + (ax_1^2 - (k_2 - 1)x_3 - k_2(a\alpha - \alpha - k_1) + \beta). \end{aligned} \tag{3.4.13}$$

The above control law is not feasible in terms of implementation due to the first term which has x_1 in the denominator. So, whenever x_1 goes near zero the control action becomes very large. For example, with $\alpha = 4$, $a = 1$, $k_1 = 1$ and $k_2 = 2$, the control action rises to infinity making the system unstable. To evade this problem we have two options in hand:

1. Switching to an alternative control law [e.g., $u = -k(x_3 + 1)$] that can stabilize the system to the desired setpoint whenever control action calculated from the control law [equation (3.4.13)] exceeds a predefined boundary.
2. Choose the parameters a and k_1 in such a way so that the term containing singularity vanishes.

Of the two options, the first option will always work as long as the alternative control law works. For the second case we need to set the parameter values a and k_1 so that the terms containing x_1 in the denominator vanishes away. For this purpose set

$$a\alpha - k_1 + 1 = 0 \Rightarrow k_1 = 1 - a\alpha. \tag{3.4.14}$$

Since by assumption k_1 should be a positive number, choose

$$a = \frac{1}{n\alpha}, \qquad n > 1 \tag{3.4.15}$$

which gives the final control law to be:

$$\begin{aligned} u &= ax_1^2 - (k_2 - 1)x_3 - k_2(a\alpha - \alpha - k_1) + \beta \\ \Rightarrow u &= \underbrace{\frac{1}{n\alpha}\,x_1^2}_{\text{Nonlinear Part}} \underbrace{-(k_2 - 1)x_3}_{\text{Linear Part}} \underbrace{-k_2\left(\frac{2}{n} - \alpha - 1\right) + \beta}_{\text{Constant Part}}. \end{aligned} \tag{3.4.16}$$

The final control law defined by equation (3.4.16) has three parts: Nonlinear, Linear and Constant terms. If we take $k_2 = 1$, the linear term vanishes away. From the simulation result it is found that presence of this linear term enhances quick stability of the system to the desired equilibrium points. So, it is better to choose

$$k_2 > 1. \tag{3.4.17}$$

The constant term however stabilizes the system in a slightly different manner. What it does is that it reduces the heating rate β to the region where the overall open loop system is stable. Since we want to keep the system in the region where the open loop system is unstable and want to diminish the chaos, the constant term in the control law does not serve our purpose. So, removing the constant part we have the following control law which is actually perturbation around the nominal heating rate:

$$u = \frac{1}{n\alpha}\,x_1^2 - (k_2 - 1)x_3, \qquad n,\, k_2 > 1. \tag{3.4.18}$$

Notice that heating rate is proportional to x_3. Also x_1 denotes fluid velocity inside the convective loop and hence energy loss due to the fluid flow is proportional to x_1^2 $\left[h_L = f\frac{LV^2}{2gD}\right]$. So, the control law is actually an energy term which makes it physically understandable. But with this truncated control law the question that immediately comes into the mind is that "Does this truncated control law still makes the system stable?". To answer this question we have to analyze the stability of the closed loop system with the truncated control law defined by equation (3.4.18). The energy function for the closed loop system with the full control law [equation(3.4.16)] is given by:

$$
\begin{aligned}
V_3 &= V_2 + \frac{1}{2}[\xi_2 - \phi_1]^2 \\
&= \frac{1}{2}x_1^2 + \frac{1}{2}[x_2 + ax_1]^2 + \frac{1}{2}[x_3 + a\alpha - \alpha - k_1 a]^2 \\
\Rightarrow \dot{V}_3 &= \begin{bmatrix}(1+a^2)x_1 + ax_2 & ax_1 + x_2 & x_3 + a\alpha - \alpha - k_1 a\end{bmatrix} \\
&\quad \times \begin{pmatrix} -\alpha x_1 + \alpha x_2 \\ -x_2 - x_1 x_3 \\ x_1 x_2 - x_3 - \beta + u \end{pmatrix} \\
&= \underbrace{-(\alpha + a\alpha + k_1 a^2)x_1^2 - k_1 x_2^2 - k_2 x_3^2}_{\text{negative}} \\
&\quad + \underbrace{(k_1 k_2 + k_2\alpha + k_1 k_2 a - k_2 - \beta)x_3 - 2k_1 a x_1 x_2}_{\text{depends on the sign of } x_1 \text{ and } x_2} \\
&\quad + \underbrace{(1 - k_1 - \alpha + k_1 a)}_{\text{negative}}\beta.
\end{aligned}
\tag{3.4.19}
$$

Here $\dot{V}_3$ has three terms as shown in equation (3.4.19): a negative quadratic term consisting of x_1^2, x_2^2 and x_3^2, a term containing x_1x_2 and x_3 which depends on the sign of the variable and a constant term. In the constant term $1 - k_1 = \frac{1}{n} \in (0,1)$ and $k_1 a = \Big(1 - \frac{1}{n}\big)\frac{1}{n\alpha} \in (0,1)$. Usually the parameter α has value 4, which makes the term $(1 - k_1 - \alpha + k_1 a)$ negative. Nothing can be said about the other two terms containing x_1x_2 and x_3. But if we take a look at the simulation result it is found that except near zero x_1 and x_2 have the same sign making $-2k_1 a x_1 x_2$ negative and even in the extreme conditions when x_3 is negative making $(k_1k_2 + k_2\alpha + k_1k_2a - k_2 - \beta)x_3$ positive but smaller than the other negative terms. This is due to the fact that though the system shows chaotic behavior the states are always confined in a boundary. Hence the equilibrium points of the system are locally stable with this control action defined by equation (3.4.18). For $\alpha = 4$, $k_2 = 3$ and $n = 2$ $[k_1 = 1 - 1/n = 0.5,\ a = 1/n\alpha = 1/8]$, the control law becomes:

$$u = \frac{1}{8}\,x_1^2 - 2x_3. \tag{3.4.20}$$

With the same initial condition as before the response of the controlled system is given in Figure 3.5.

3.5 Model predictive control (MPC)

In model predictive control[3], a set of future control action including the current control action is calculated based on the model of the system. That is why it is sometimes called the model based predictive control. The model can be linear or non-linear. The main purpose is to minimize an objective function (which is often a quadratic function of the states and inputs) subject to the model equation and some physical constraints. For

[3]For a review of different model predictive control technique see [4, 6, 12, 24, 31, 23, 20, 7, 8, 22, 28, 18].

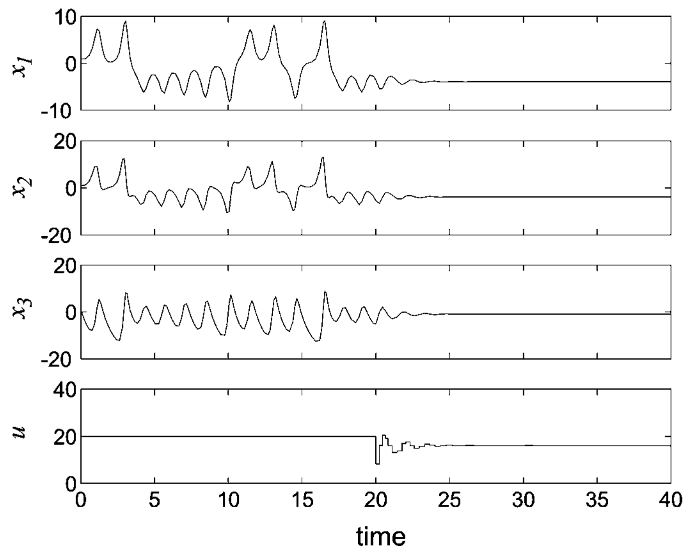


Figure 3.5. Controlled system for $\beta = 20$: Back Stepping method.

linear time invariant model this problem can be solved to give a control law as a function of current output and past input. For nonlinear case there is usually no explicit solution of the minimization problem and one is forced to solve it numerically.

3.5.1 Nonlinear model predictive control (NMPC)

The objective of all control problem is to minimize the difference of output, y with the desired value[4], y_{ref}. One such objective function is

$$\begin{aligned}\min_{u,x_1,\dots,x_n} J = \sum_{i=1}^{n} \gamma_i [\mathbf{x}_i(t) - \mathbf{x}_{i,ref}]^T [\mathbf{x}_i(t) - \mathbf{x}_{i,ref}] \\ + \gamma_u [u(t) - u_{ref}]^T [u(t) - u_{ref}] + \gamma_{\Delta u} \Delta\mathbf{u}^T \Delta\mathbf{u}\end{aligned} \tag{3.5.1}$$

subject to,

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), u, t), \tag{3.5.2}$$

where γ's are penalty functions on x_i's and for the convective loop system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} -px_1 + px_2 \\ -x_1x_3 - x_2 \\ x_1x_2 - x_3 - R + u \end{pmatrix}. \tag{3.5.3}$$

[4]In convective loop problem, the desired reference points are the equilibrium points.

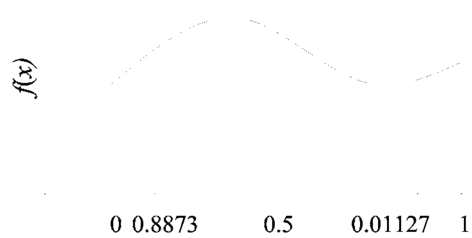


Figure 3.6. Approximation of a function by three point collocation on one step ahead prediction.

This minimization problem (3.5.1) has continuous nonlinear model constraint (3.5.2). To solve this problem the continuous model constraint needs to be discretized. Any finite element method can be used for this purpose:

1. using the conventional numerical method to predict future values e.g., Runge-Kutta 23 method etc.
2. by converting dynamic constraints to algebraic constraints using
   - Orthogonal Collocation Method;
   - Galerkin method;
   - Flatness based technique etc.

Of these methods only orthogonal collocation method will be applied on the convective loop model to control chaos.

3.5.2 Orthogonal collocation method, prediction horizon 1

In the orthogonal collocation method, any function can be approximated by an interpolating polynomials with nodes located at the roots of a set of orthogonal polynomials (see [10, 29, 15, 6, 27] for detail), i.e.,

$$y(x) = \sum_{i=1}^{N+2} b_i P_{i-1}(x), \tag{3.5.4}$$

where

$$P_m(x) = \sum_{j=0}^{m} c_j x^j \tag{3.5.5}$$

is the m-th polynomial such that

$$\int_a^b W(x) P_k(x) P_m(x)\, dx = 0, \qquad k = 0, 1, 2, \dots, m-1.$$

Here, the polynomial m has m-roots in the interval $[a, b]$ and thus users do not need to pick the collocation points arbitrarily. This has advantage over the conventional collocation method where there is a good chance of poor choice of these nodes by inexperienced users and thus bad approximation of the function. Typically, the integration range is taken as 0 to 1 to generalize the problem. Equations (3.5.4) and (3.5.5) can be combined to give

$$y(x_j) = \sum_{i=1}^{N+2} d_i x_j^{i-1}. \tag{3.5.6}$$

Derivatives can also be approximated by orthogonal polynomials and finally we get the following forms

$$\frac{dy}{dx}(x_j) = \sum_{i=1}^{N+2} d_i (i-1) x_j^{i-2}, \tag{3.5.7}$$

$$\frac{d^2y}{dx^2}(x_j) = \sum_{i=1}^{N+2} d_i (i-1)(i-2) x_j^{i-3}. \tag{3.5.8}$$

In matrix notation,

$$y = \mathbf{Q}d, \qquad \frac{dy}{dx} = \mathbf{C}d, \qquad \frac{d^2y}{dx^2} = \mathbf{D}d,$$

where

$$\begin{aligned} Q_{ji} &= x_j^{i-1}, \\ C_{ji} &= (i-1)x_j^{i-2}, \\ D_{ji} &= (i-1)(i-2)x_j^{i-3}. \end{aligned} \tag{3.5.9}$$

Therefore,

$$\frac{dy}{dx} = \mathbf{C}Q^{-1}y \equiv \mathbf{A}y, \tag{3.5.10}$$

$$\frac{d^2y}{dx^2} = \mathbf{D}Q^{-1}y \equiv \mathbf{B}y. \tag{3.5.11}$$

For our case, the three point collocation method is used. The collocation points and the A-matrices are given in the Table 3.1 and Table 3.2.

Table 3.1. Polynomial roots and the weighting functions.

N	x_j	W_j
1	0.50000 00000	0.66666 66667
2	0.21332 48654 0.78867 51346	0.50000 00000 0.50000 00000
3	0.11270 16654 0.50000 00000 0.88729 83346	0.27777 77778 0.44444 44444 0.27777 77778

Table 3.2. Matrices for orthogonal collocation found from equation (3.5.9).

N	A
1	$\begin{pmatrix} -3 & 4 & -1 \\ -1 & 0 & 1 \\ 1 & -4 & 3 \end{pmatrix}$
2	$\begin{pmatrix} -7 & 8.196 & -2.196 & 1 \\ -2.732 & 1.732 & 1.732 & -07321 \\ 0.7321 & -1.732 & -1.732 & 2.732 \\ -1 & 2.196 & -8.196 & 7 \end{pmatrix}$
3	$\begin{pmatrix} -13 & 14.79 & -2.67 & 1.88 & -1 \\ -5.32 & 3.87 & 2.07 & -1.29 & 0.68 \\ 1.5 & -3.23 & 0 & 3.23 & -1.5 \\ -0.68 & 1.29 & -2.07 & -3.87 & 5.32 \\ 1 & -1.88 & 2.67 & -14.79 & 13 \end{pmatrix}$

The matrices given in Table 3.2 for different collocation points are for interval $[0, 1]$. But the constraint equation (3.5.2) has the interval $[0, \Delta t]$, where Δt is the sampling interval. To account for it the following changes are made to convert the dynamic constraint into algebraic constraint:

$$\begin{aligned} \frac{d\mathbf{x}}{dt'} &= \mathbf{f}(\mathbf{x}, u), \qquad t' \in [0, \Delta t'], \\ \frac{d\mathbf{x}}{dt} &= \mathbf{A}x, \qquad t \in [0, 1], \\ \Rightarrow \mathbf{A}x &= \Delta t' \mathbf{f}(\mathbf{x}, u). \end{aligned} \tag{3.5.12}$$

To take into account the initial condition (i.e., previous control effects) the first row of A-matrix needs to change so that it becomes,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -5.32 & 3.87 & 2.07 & -1.29 & 0.68 \\ 1.5 & -3.23 & 0 & 3.23 & -1.5 \\ -0.68 & 1.29 & -2.07 & -3.87 & 5.32 \\ 1 & -1.88 & 2.67 & -14.79 & 13 \end{bmatrix}. \tag{3.5.13}$$

3.5.3 Orthogonal collocation method, prediction horizon > 1

Similar to the conversion of dynamic constraint to algebraic constraint for prediction horizon one, when prediction horizon is greater than one, same equation (3.5.12) is used

$$\tilde{\mathbf{A}}\mathbf{x} = \Delta t' \tilde{\mathbf{F}}(\tilde{\mathbf{x}}, u), \tag{3.5.14}$$

where

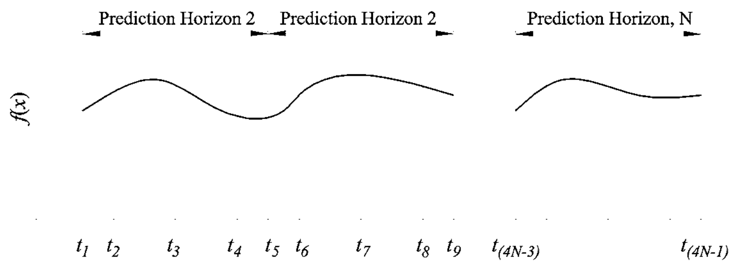


Figure 3.7. Polynomial approximation of a function using three point collocation method with prediction horizon > 1.

$$\tilde{A} = \begin{pmatrix} \boxed{A_0} & & & & & \\ & \boxed{A} & & & & \\ & & \boxed{A} & & & \\ & & & \ddots & & \\ & & & & \boxed{A} & \\ & & & & & \boxed{A} \end{pmatrix}, \tag{3.5.15}$$

$$\tilde{\mathbf{F}}(\tilde{\mathbf{x}}, u) = \begin{pmatrix} x_{init}^T \\ f^T(x_{2*}^T, u_0, t_2) \\ f^T(x_{3*}^T, u_0, t_3) \\ f^T(x_{4*}^T, u_0, t_4) \\ f^T(x_{5*}^T, u_0, t_5) \\ f^T(x_{6*}^T, u_1, t_6) \\ f^T(x_{7*}^T, u_1, t_7) \\ \vdots \\ f^T(x_{(4N)*}^T, u_{N-1}, t_{4N}) \\ f^T(x_{(4N+1)*}^T, u_{N-1}, t_{4N+1}) \end{pmatrix} \tag{3.5.16}$$

and

$$\tilde{x} = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \\ x_{4,1} & x_{4,2} & x_{4,3} \\ \vdots & \vdots & \vdots \\ x_{4N,1} & x_{4N,2} & x_{4N,3} \\ x_{4N+1,1} & x_{4N+1,2} & x_{4N+1,3} \end{pmatrix}. \tag{3.5.17}$$

Here, the first subscript denotes the collocation points in the time interval and the second means state. Using the formulations stated in the equations (3.5.15)–(3.5.17) (see [12] for detail) simulation was run for different prediction and control horizons.

3.5.4 Simulation result for model predictive control

The performance of the controller based on linear or, nonlinear MPC depends on the sampling time, ΔT, and the penalty of the state and input variables in the objective function, γ_i's. The system is highly nonlinear and shows the peculiarity of chaos and bifurcation as is described in the Section 2.4. The fast dynamic system with highly nonlinear behavior makes it difficult to laminarize (or, stabilize) the system using linear model predictive controller. Surprisingly linear MPC with prediction horizon one gives better control than with prediction horizon greater than one for the same sampling time. It is evident from the fact that for this fast chaotic dynamic system a linear model with smaller prediction horizon (Figure 3.8) can track the system better than that of a linear model with large prediction horizon (Figure 3.9).

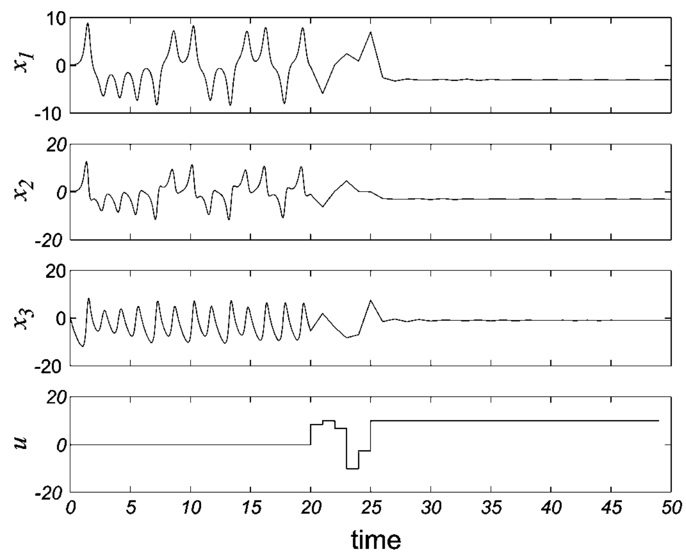


Figure 3.8. Linear MPC; Prediction Horizon = 1, Control Horizon = 1, $\Delta T = 1$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_u = 1$, $\gamma_{\Delta u} = 0$.

In every case however the control action never comes to zero as in the nonlinear model predictive control. The control action takes the higher limit and stays there which in fact in most cases drags the system from the chaotic region to nonchaotic one and thus making the system stable. However nonlinear MPC can stabilize the chaotic system very well. The control action decays rapidly to zero (see Figures 3.10 − 3.12). The time for stabilization depends greatly on the penalty functions on the states and input in the objective function of the optimization problem (3.5.1) as well as the sampling rate[5]. The input limit and its change depend on the constraint used in the minimization problem. Thus in Figure 3.10 due to the input rate constraint limited to 5 control action does not change instantly as in Figure 3.11 or, Figure 3.12 but it takes more time to stabilize the system. So, less stabilization time comes at the cost of larger control energy.

[5]The time interval for implementing control action is also equal to the sampling rate.

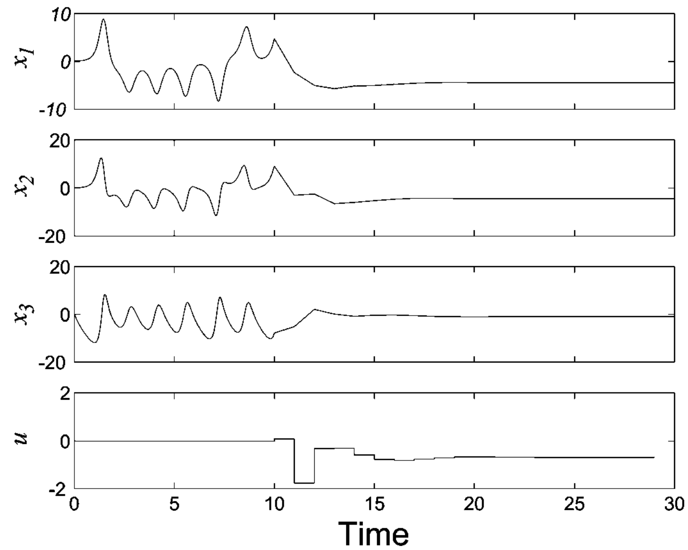


Figure 3.9. Linear MPC; Prediction Horizon = 5, Control Horizon = 2, $\gamma_1 = \gamma_2 = \gamma_3 = 1$, $\gamma_u = \gamma_{\Delta u} = 0$, $\Delta T = 1$.

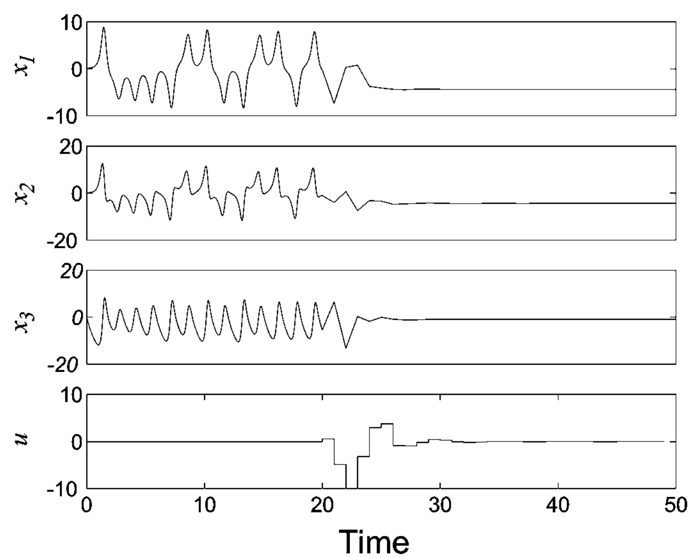


Figure 3.10. Nonlinear MPC; Prediction Horizon = 5, Control Horizon = 2, $\Delta T = 1$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_u = 1$, $\gamma_{\Delta u} = 0$.

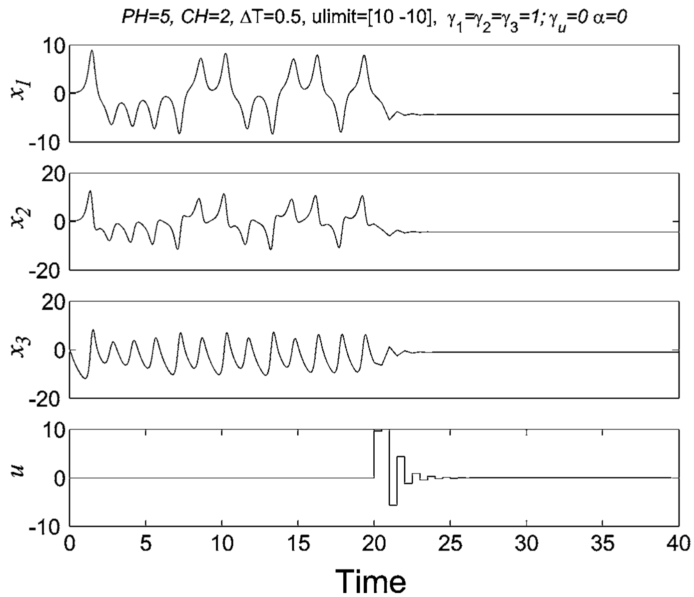


Figure 3.11. Nonlinear MPC; Prediction Horizon = 5, Control Horizon = 2, $\Delta T = 0.5$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_u = 1$, $\gamma_{\Delta u} = 0$.

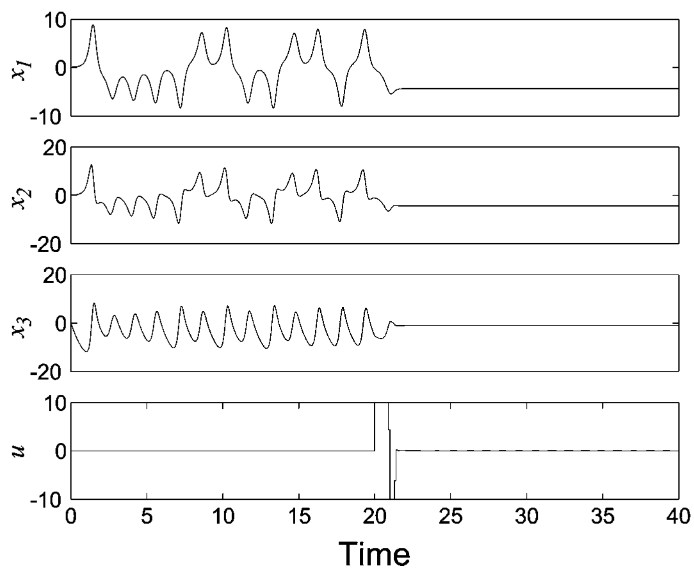


Figure 3.12. Nonlinear MPC; Prediction Horizon = 5, Control Horizon = 2, $\Delta T = 0.1$, $\gamma_1 = \gamma_2 = \gamma_3 = 1$, $\gamma_u = \gamma_{\Delta u} = 0$.

4 Conclusion

For this system Nonlinear Model Predictive Control (NMPC) outperforms other controller in terms of stabilizing time and control action. One of its main disadvantage is high computational time. With the advent of high performance computer however this is not a major problem anymore. Another disadvantage of NMPC is that tuning of the parameters in the objective functions has to be tried through a lot of simulations. Also the model parameters (α and β) need to be correctly identified for the implementation of the controller. Although computational time for Linear MPC is much smaller than Nonlinear MPC, it cannot regulate the system to its desired setpoint unless the sampling time is very small. Among others, linear state feedback controller with setpoint tracking [equation (3.2.1)] and nonlinear controller based on Lyapunov Stability Criterion [equation (3.3.2)] also give better result than others in terms of stabilizing time and movement rate of controller. Of these two, the linear controller is less sensitive to the initial condition *i.e.*, the time when controller is implemented and gives less fluctuation in the control action when measurement noise is present. The nonlinear controller stabilizes the system very quickly but gives a lot of spikes in the control action if noise is present. In this case we assumed that the states are measurable and available for calculation. If any state is not measurable, then a nonlinear observer can be used to estimate the unknown states and calculate the control law. In this case, the performance of the controller will depend on the performance of the observer as well.

References

[1] Abed, E.H. and Fu, J.H. Local feedback stabilization and bifurcation control; I. Hopf bifurcation. *Systems and Control Letters* **7**(1) (1986) 11–17.

[2] Abed, E.H. and Fu, J.H. Local feedback stabilization and bifurcation control; II. Stationary bifurcation. *Systems and Control Letters* **8**(5) (1987) 467–473.

[3] Bau, H.H. and Singer, J. Controlling a chaotic system. *Physical Review Letter* **66** (1991) 1123–1125.

[4] Bewley, T.R. Flow control: new challenges for a new renaissance. *Progress in Aerospace Sciences* **37** (2001) 21–58.

[5] Bošković, D.M. and Krstić, M. Nonlinear stabilization of a thermal convective loop by state feedback. *Automatica* **37**(12) (2001) 2033–2040.

[6] Burns, J.A. Optimal sensor location for robust control of distributed parameter systems. In: *Proceedings of 33rd IEEE conference on Decision and Control.* Lake Buena Vista, Florida, USA, 1992, pp. 3967–3972.

[7] Burns, J.A., King, B.B. and Rudio, D. Feedback control of a thermal fluid using state estimation. *Int. J. Comput. Fluid D.* **11**(1–2) (1998) 93–112.

[8] Chen, Wen-Hua and Ballance, D.J. Model predictive control of nonlinear systems: Computational burden and stability. *Csc report: Csc-99007.* Center for Systems and Control and Department of Mechanical Engineering. University of Glasgow, UK, 1999.

[9] Ehrhard, P. and Müller, M. Dynamical behaviour of natural convection in a single-phase loop. *J. Fluid Mech.* **217** (1990) 487–518.

[10] Finlayson, B.A. *Nonlinear Analysis in Chemical Engineering.* McGraw-Hill International Book Co., New York, 1980.

[11] Greif, R. Natural circulation loops. *Trans. ASME C: J. Heat Transfer* **110** (1988) 1243–1258.

[12] Meadows, E.S. and Rawlings, J.B. Model predictive control. In: *Nonlinear Process Controls*, (Eds.: Henson, M.A. and Seborg, D.E.), Prentice Hall, Englewood Cliffs, NJ, 1997, pp. 233–310.

[13] Khalil, H.K. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, NJ, 1996.
[14] Kim, T. and Abed, E.H. Stationary bifurcation control of systems with uncontrollable linearization. *Int. J. Control* **74**(5) (2001) 445–452.
[15] Lapidus, L. and Seinfeld, J.H. *Numerical Solution of Ordinary Differential Equations*. Vol. 74 of "Mathematics in Science and Engineering". Academic Press, Boston, 1971.
[16] Lee, H.C. and Abed, E.H. Washout filters in the bifurcation control of high alpha flight dynamics. *Proc. American Control Conf.* (1991) 206–211.
[17] Marquez, H.J. *Nonlinear Control Systems - Analysis and Design.* Wiley-Interscience, Hoboken, NJ, 2003.
[18] Maune, D.Q. Nonlinear model predictive control: An assessment. *Fifth International Conference on Chemical Process Control-V* **93**(316) (1997) 217–231.
[19] Metrol, A. and Greif, R. A review of natural circulation loops. *NATO Advanced Study Inst. of Natural Convection: Fundam. and Applic.* (1984) 1033–1081.
[20] Nicolao, G. De, Magni, L. and Scattolini, R. Stability and robustness of nonlinear receding horizon control. *Progress in Systems and Control Theory* **26** (2000) 3–22.
[21] Ott, E., Grebogi, C. and Yorke, J.A. Controlling chaos. *Physical Review Letters* **64**(11) (1990) 1196–1199.
[22] Qin, S.J. and Badgwell, T.A. An overview of industrial model predictive control technology. *Fifth International Conference on Chemical Process Control-V* **93**(316) (1997) 232–256.
[23] Qin, S.J. and Badgwell, T.A. Review of nonlinear model predictive control applications. In: *Nonlinear Predictive Control: Theory and Practice*. (Eds.: Kouvaritakis, B., *et. al.*). IEE, Stevenage, Control Engineering Series, **Vol. 61**, 2001, pp. 3–32.
[24] Rao, C.V. and Rawlings, J.B. Nonlinear moving horizon state estimation. *Progress in Systems and Control Theory* **26** (2000) 45–69.
[25] Sastry, S. *Nonlinear Systems: Analysis, Stability and Control*. Springer-Verlag New York, Inc., New York, 1999.
[26] Singer, J., Wang, Y.-Z. and Bau, H.H. Controlling a chaotic system. *Physical Review Letters* **66**(9) (1991) 1123–1125.
[27] Stephens, A.B. A finite difference galerkin formulation for the incompressible Navier-Stokes equation. *Journal of Computational Physics* **53** (1984) 152–172.
[28] Trierweiler, J.O. and Secchi, A.R. Exploring the potentiality of using multiple model approach in nonlinear model predictive control. *Progress in Systems and Control Theory* **26** (2000) 191–203.
[29] Villadsen, J. and Michelsen, M.L. *Solution of Differential Equation Models by Polynomial Approximation*. Prentice Hall Inc., Englewood Cliffs, NJ, 1978.
[30] Wang, H.O. and Abed, E.A. Bifurcation control of a chaotic system. *Automatica* **31**(9) (1995) 1213–1226.
[31] Zheng, A. Some practical issues and possible solutions for nonlinear model predictive control. *Progress in Systems and Control Theory* **26** (2000) 129–143.
[32] Zvirin, Y. A review of natural circulation loops in pressurized water ractors and other systems. *Nuclear Engineering Design* **67** (1981) 203–225.

Nonlinear Dynamics and Systems Theory, **3**(2) (2003) 227–238

Optimal Control of Nonlinear Uncertain Systems over an Infinite Horizon via Finite-Horizon Approximations

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Received: May 3, 2003; Revised: August 9, 2003

Abstract: It is well-known that the Hamilton-Jacobi-Isaacs (HJI) equation associated with a nonlinear H^∞-optimal control problem on an infinite-time horizon generally admits nonunique, and in fact infinitely many, viscosity solutions. This makes it difficult to pick the relevant viscosity solution for the problem at hand, particularly when it is computed numerically. For the finite-horizon version of the problem, however, there is generally a unique viscosity solution (under appropriate conditions), which brings up the question of obtaining the viscosity solution relevant to the infinite-horizon problem as the limit of the unique solution of the finite-horizon one. This paper addresses this question for nonlinear systems affine in the control and the disturbance, and with a cost function quadratic in the control, where the control is not restricted to lie in a compact set. It establishes the existence of a well-defined limit, and also obtains a result on global asymptotic stability of closed-loop system under the H^∞ controller and the corresponding worst-case disturbance.

Keywords: *Nonlinear H^∞ control; Isaacs equation; viscosity solutions; global stability.*

Mathematics Subject Classification (2000): 49L25, 49J20, 49L20.

1 Introduction

An approach toward solving the nonlinear H^∞-optimal control problem is to treat it as a zero-sum differential game (e.g. [1]), for which a sufficient condition for the existence of a solution is expressed in terms of a Hamilton-Jacobi-Isaacs (HJI) equation. Such equations do not generally admit classical solutions, because the *values* of the corresponding differential games are not smooth enough to satisfy the HJI equations in the classical sense. Evans and Souganidis [5], and Bardi and Soravia [2] were among the first to show that the values of certain classes of differential games are viscosity solutions of the corresponding HJI equations. In the context of nonlinear H^∞ control problems, several authors have studied before the existence of a value function, and when the value function is a viscosity solution of the corresponding HJI equation, the uniqueness of such viscosity solutions (see [3, 4, 9 − 12]). But most of these studies have pertained to the *a priori* assumption that control sets are bounded.

The system considered in this paper has the input-affine form, leading (along with a quadratic-in-control cost function) to a Hamiltonian that is quadratic in both the control and the disturbance. This structure allows us to establish a comparison theorem which yields the uniqueness of the viscosity solution of the corresponding finite-horizon HJI equation, and this solution in turn can be used to approximate the desired viscosity solution of the corresponding infinite-horizon HJI equation. Thus one objective of this paper is to establish the connection between two HJI equations, one of which has multiple solutions and the other one has a unique solution which can be used to approach the desired solution of the former. A second objective is to show connections between such viscosity solutions and stabilizing feedback controller design. As indicated above, most current work which relate to nonlinear H^∞ control problems requires the control set to be compact in order to prove the uniqueness of the viscosity solution of the corresponding HJI equations (e.g. see [7, 9]). Clearly the boundedness assumption on the control space could be overly restrictive, and is not convenient for technical approaches. In this paper, such a restriction is relaxed and the uniqueness of HJI equations holds under standard assumptions.

The paper is organized as follows. In Section 2, we present the problem formulation and describe some necessary assumptions for the systems. In Section 3, we show that the HJI equation in the finite-horizon case admits a viscosity solution. Section 4 proves that the viscosity solution discussed in Section 3 is unique. In Section 5, we study how to obtain the viscosity solution of the HJI equation of infinite-horizon case from the unique solution of the finite-horizon one. An example is given in Section 6 to illustrate the main result of the paper. Some final remarks in Section 7 conclude the paper.

2 Preliminaries and Assumptions

Consider a system of the input-affine form

$$\frac{dx}{ds} = a(x) + B(x)u + D(x)w, \qquad x(t) = x^0, \tag{2.1}$$

where $x(s)$ is the state vector with values in $\mathcal{R}^n$, and $u(s)$ is the control vector with values in $\mathcal{R}^p$. The other input, $w(s) \in \mathcal{R}^m$, is the driving noise, which is an unknown $L^2[0,\infty)$ function; it represents modeling errors in a and other possible errors or inaccuracies in the dynamics. Thus the system model (2.1) accommodates uncertainties.

We will assume that there exist positive constants K_a, K_B, K_D such that

$$\begin{aligned} |a(x) - a(y)| &\le K_a|x-y| \qquad && \forall\, x,\, y \in \mathcal{R}^n, \\ |a(x)| &\le K_a(1+|x|) && \forall\, x \in \mathcal{R}^n, \\ |B(x) - B(y)| &\le K_B|x-y| && \forall\, x,\, y \in \mathcal{R}^n, \\ |D(x) - D(y)| &\le K_D|x-y| && \forall\, x,\, y \in \mathcal{R}^n, \end{aligned} \tag{2.2}$$

where the symbol $|\cdot|$ denotes the Euclidean norm. Since the state of the system probably operates over some compact subset of $\mathcal{R}^n$, we may only need (2.2) to hold on this compact set as a, B, D can be extended to all of $\mathcal{R}^n$.

Let the running cost be $q(x) + u^T R(x)u$, where $q(x) \geq 0$, $\forall\, x \in \mathcal{R}^n$, and satisfies the following bounds and growth conditions:

$$|q(x) - q(y)| \leq C_q(1 + |x| + |y|)|x - y|, \qquad C_q \geq 0.$$

For $R(x)$, on the other hand, there exist positive constants k_1, k_2 such that for all $x \in \mathcal{R}^n$

$$k_1 I^{p\times p} \leq R(x) \leq k_2 I^{p\times p},$$

which in particular implies that $R(x)$ is invertible for all x.

Further let the terminal state cost function be $g(x)$, satisfying the bound

$$|g(x) - g(y)| \leq K_g(r)|x - y|, \quad \forall\, |x| \leq r, \quad |y| \leq r.$$

For a given $t_f > 0$, we consider the lower-value function

$$V(t; x, t_f) = \sup_w \inf_u J_\gamma^{t_f}(t, x, u, w), \tag{2.3}$$

where $J_\gamma^{t_f}(t, x, u, w) = g(x(t_f)) + L_\gamma^{t_f}(t, x, u, w)$ and

$$L_\gamma^\tau(t, x, u, w) = \int_t^\tau \left(q(x(s)) + u^T R(x(s))u - \gamma^2 |w(s)|^2\right) ds.$$

The corresponding Hamilton-Jacobi-Isaacs (HJI) equation is

$$-V_t(t; x, t_f) + H(x, V(t; x, t_f)) = 0 \quad \text{and} \quad V(t_f; x, t_f) = g(x), \tag{2.4}$$

where the Hamiltonian for this case is given by

$$\begin{aligned} H(x,p) :&= -\sup_w \inf_u \{q + u^T Ru - \gamma^2 w^T w + p^T[a + Bu + Dw]\} \\ &= -q - p^T a + \frac{1}{4} p^T \left(BR^{-1}B^T - \frac{1}{\gamma^2} DD^T\right) p \end{aligned}$$

with the assumption that for fixed x

$$|p| \to \infty \quad \text{implies} \quad |H(x,p)| \to \infty. \tag{2.5}$$

Remark 2.1 Let $\mathcal{M}$ be the space of all state-feedback controllers, i.e. measurable mappings from $\mathcal{R}^n$ into $\mathcal{R}^p$. Then, the quantity we are really interested in (the one that is relevant to nonlinear H^∞ control) is in fact the upper value of the game: $\inf_{\mathcal{M}} \sup_w J_\gamma^{t_f}(t, x, u, w)$. Note, however, that since the Isaacs' condition is satisfied, the Hamiltonian admits a saddle-point solution, which makes the upper and lower values equal. In view of this, we are allowed to work with the lower value of the game and thus avoid some technical issues that arise in a direct study of the upper value of the game.

3 Lower Value of the Differential Game and the Viscosity Solution

Lemma 3.1 *Let V be defined as in (2.3). Then, for any $0 \le t \le \tau \le t_f$, and with $x(t) = x$,*

$$V(t; x, t_f) \ge \sup_w \inf_u \Big\{ L_\gamma^\tau(t; x, u, w) + V(\tau; x(\tau), t_f) \Big\}.$$

If the upper value of the game is finite, then the above inequality becomes an equality.

Proof This involves a standard dynamic programming type argument in the context of differential games.

Theorem 3.1 *If in (2.3) $V(\cdot; \cdot, t_f) \in C([0, t_f] \times \Omega)$, then V is a viscosity supersolution of (2.4). Furthermore, if the upper value of the game, $\inf_{\mathcal{M}} \sup_w J_\gamma^{t_f}(t; x, \mu, w)$, is finite, then V is a viscosity solution of (2.4).*

Proof Suppose that to the contrary V is not a viscosity supersolution of (2.4). Then there would exist an $\varepsilon > 0$, and a pair $(t_0, x_0) \in [0, t_f] \times \Omega$ and a function Φ: $[0, t_f] \times \Omega \to \mathcal{R}$ such that $V(\cdot; \cdot, t_f) - \Phi$ has a local minimum at (t_0, x_0), and

$$-\Phi_t(t_0, x_0) + H(x_0, \Phi_x(t_0, x_0)) \le -\varepsilon.$$

By making use of (2.1), as $t \downarrow t_0$, we have

$$\Phi(t_0, x_0) - \Phi(t, x(t)) \le -\varepsilon(t - t_0) + \inf_u L_\gamma^t(t_0; x_0, u, w).$$

Since (t_0, x_0) is a local minimizer of $V(\cdot; \cdot, t_f) - \Phi$, in a small neighborhood of (t_0, x_0),

$$V(t_0; x_0, t_f) - V(t; x, t_f) \le \Phi(t_0, x_0) - \Phi(t, x(t)).$$

Therefore we arrive at

$$V(t_0; x_0, t_f) \le \sup_w \inf_u \big\{ L_\gamma^t(t_0; x_0, u, w) + V(t; x, t_f) \big\} - \varepsilon(t - t_0),$$

which contradicts the statement of Lemma 3.1. For the case of viscosity subsolution, let $(t_0, x_0) \in [0, t_f] \times \Omega$ and $\Psi \in C^1([0, t_f] \times \Omega)$ be such that (t_0, x_0) is a local maximizer of $V - \Psi$ with $V(t_0; x_0, t_f) = \Psi(t_0, x_0)$. By Lemma 3.1, for any $t \in (t_0, t_f]$,

$$\Psi(t_0, x_0) = V(t_0; x_0, t_f) \le \sup_w \big\{ L_\gamma^t(t_0, x_0, u^*, w) + V(t; x(t), t_f) \big\}, \tag{3.1}$$

where $u^* = -R^{-1} B^T \Psi_x$. Observing that $\{x(t, x_0, u^*, w)\}$ is continuous in t, when $t > t_0$ is sufficiently close to t_0, we have

$$V(t_0; x_0, t_f) - V(t; x, t_f) \ge \Psi(t_0, x_0) - \Psi(t, x(t, x_0, u^*, w)).$$

Divide (3.1) by $t - t_0$, and let $t \downarrow t_0$, to obtain

$$\sup_w \left\{ \frac{1}{2}[q + (u^*)^T R u^* - \gamma^2 |w|^2] + \Psi_t + \Psi_x^T (a + B u^* + D w) \right\} \ge 0$$

which yields at $(t,x) = (t_0, x_0)$

$$\Psi_t + q + \Psi_x^T a - \frac{1}{4}\,\Psi_x^T\left(BR^{-1}B^T - \frac{1}{\gamma^2}\,DD^T\right)\Psi_x \geq 0$$

that is to say, V is a viscosity solution of (2.4).

4 Uniqueness of the Viscosity Solution of (2.4)

In this section, we show that the viscosity solution of (2.4) is unique. Suppose that V, W are a viscosity supersolution and a viscosity subsolution, respectively, of (2.4) on $Q_{t_f}^{\Omega} = [0, t_f] \times \Omega$. Furthermore, assume that

$$W \leq V \quad \text{on } (\{t = t_f\} \times \Omega) \cup ([0, t_f] \times \partial\Omega).$$

Lemma 4.1 *Let $R < \infty$, and a function $\Lambda \in C^1(Q_{t_f}^{\Omega})$ be such that $\Lambda \geq 0$ if $|x| \geq R$, and*

$$\Lambda_t < 0 \quad on (\text{supp } \Lambda)^o \cap \left(Q_{t_f}^{\Omega}\right)^o, \tag{4.1}$$

where the superscript "o" indicates interior. Then $W \leq V$ on $(\text{supp}\,\Lambda)^o \cap (Q_{t_f}^{\Omega})^o$.

Proof Suppose that $(t_0, x_0) \in (\text{supp}\,\Lambda)^o \cap (Q_{t_f}^{\Omega})^o$ such that

$$M_0 = \Lambda(t_0, x_0)[W(t_0, x_0) - V(t_0, x_0)] = \max_{Q_{t_f}^{\Omega}} \Lambda(t,x)[W(t,x) - V(t,x)] > 0 \tag{4.2}$$

since otherwise the result has already been established. Introduce a function $\Phi^{\varepsilon,\delta} \colon Q_{t_f}^{\Omega} \times Q_{t_f}^{\Omega} \to \mathcal{R}^n$ by

$$\Phi^{\varepsilon,\delta} = \Lambda(s,y)W(t,x) - \Lambda(t,x)V(s,y) - \frac{1}{2\varepsilon}\,|x-y|^2 - \frac{1}{2\delta}\,|t-s|^2. \tag{4.3}$$

Since $\Phi^{\varepsilon\,\delta}$ is upper semicontinuous and Λ has a compact support, there exists $(t_\delta, x_\varepsilon, s_\delta, y_\varepsilon) \in Q_{t_f}^{\Omega} \times Q_{t_f}^{\Omega}$ such that

$$\Phi^{\varepsilon,\delta}(t_\delta, x_\varepsilon, s_\delta, y_\varepsilon) = \max_{Q_{t_f}^{\Omega} \times Q_{t_f}^{\Omega}} \Phi^{\varepsilon,\delta}(t,x,s,y). \tag{4.4}$$

Let $M^{\varepsilon,\delta} = \Phi^{\varepsilon,\delta}(t_\delta, x_\varepsilon, s_\delta, y_\varepsilon)$, and consider $0 < \varepsilon_2 \leq \varepsilon_1$ and $0 < \delta_2 \leq \delta_1$. Then

$$\begin{aligned}
&M^{\varepsilon_1,\delta_1} - \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\frac{|x_{\varepsilon_2} - y_{\varepsilon_2}|^2}{2} - \left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\frac{|t_{\delta_2} - s_{\delta_2}|^2}{2} \\
&\geq \Phi^{\varepsilon_1,\delta_1}(t_{\delta_2}, x_{\varepsilon_2}, s_{\delta_2}, y_{\varepsilon_2}) - \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\frac{|x_{\varepsilon_2} - y_{\varepsilon_2}|^2}{2} - \left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\frac{|t_{\delta_2} - s_{\delta_2}|^2}{2} \\
&= \Lambda(s_{\delta_2}, y_{\varepsilon_2})W(t_{\delta_2}, x_{\varepsilon_2}) - \Lambda(t_{\delta_2}, x_{\varepsilon_2})V(s_{\delta_2}, y_{\varepsilon_2}) - \frac{1}{2\varepsilon_1}\,|x_{\varepsilon_2} - y_{\varepsilon_2}|^2
\end{aligned}$$

$$- \frac{1}{2\delta_2}\,|t_{\delta_2} - s_{\delta_2}|^2 - \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\frac{|x_{\varepsilon_2} - y_{\varepsilon_2}|^2}{2} - \left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\frac{|t_{\delta_2} - s_{\delta_2}|^2}{2}$$
$$= \Phi^{\varepsilon_2,\delta_2}(t_{\delta_2}, x_{\varepsilon_2}, s_{\delta_2}, y_{\varepsilon_2}) = M^{\varepsilon_2,\delta_2}.$$

Hence, we can see that $(\varepsilon, \delta) \mapsto M^{\varepsilon,\delta}$ is nondecreasing. Let $\varepsilon_1 = 2\varepsilon, \varepsilon_2 = \varepsilon$ and $\delta_1 = \delta_2 = \delta$; then

$$M^{2\varepsilon,\delta} - M^{\varepsilon,\delta} \geq \frac{1}{2\varepsilon}\frac{|x_\varepsilon - y_\varepsilon|^2}{2}. \tag{4.5}$$

Note that $M^{2\varepsilon,\delta} - M^{\varepsilon,\delta} \to 0$ as $\varepsilon \downarrow 0$. Hence, $(\varepsilon, \delta) \to M^{\varepsilon,\delta}$ is nondecreasing. Thus we have

$$\frac{1}{2\varepsilon}\,|x_\varepsilon - y_\varepsilon|^2 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \tag{4.6}$$

Similarly, we have

$$\frac{1}{2\delta}\,|t_\delta - s_\delta|^2 \to 0 \quad \text{as} \quad \delta \downarrow 0. \tag{4.7}$$

Since Λ has compact support, and (4.6), (4.7) hold, there exist sequences $\{\varepsilon_n\}$ and $\{\delta_m\}$ which converge to *zero* such that

$$x_{\varepsilon_n} \to \hat{x}, \quad y_{\varepsilon_n} \to \hat{x}, \quad \text{as} \quad n \to \infty \tag{4.8}$$

and

$$t_{\delta_m} \to \hat{t}, \quad s_{\delta_m} \to \hat{t} \quad \text{as} \quad m \to \infty, \tag{4.9}$$

where $(\hat{t}, \hat{x}) \in Q^{\Omega}_{t_f}$. In fact it is easy to see that $\hat{x} = x_0$, $\hat{t} = t_0$. Note that under the initial hypotheses, $(t_0, x_0) \in (\operatorname{supp}\Lambda)^o \cap (Q^{\Omega}_{t_f})^o$. Therefore, for sufficiently large n and m, we have that $(t_{\delta_m}, x_{\varepsilon_n}), (s_{\delta_m}, y_{\varepsilon_n}) \in (\operatorname{supp}\Lambda)^o \cap (Q^{\Omega}_{t_f})^o$. Since the function

$$W(t,x) - \frac{1}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})}\left[\Lambda(t,x)V(t_{\delta_m}, x_{\varepsilon_n}) + \frac{1}{2\varepsilon_n}|x - y_{\varepsilon_n}|^2 + \frac{1}{2\delta_m}|t - s_{\delta_m}|^2\right] \tag{4.10}$$

attains its maximum at $(t,x) = (t_{\delta_m}, x_{\varepsilon_n})$, by the definition of viscosity subsolution, we have

$$\begin{aligned} &\frac{\Lambda_t(t_{\varepsilon_n}, x_{\varepsilon_n})V(t_{\delta_m}, x_{\varepsilon_n}) + (t_{\delta_m} - s_{\delta_m})/\delta_m}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})} \\ &\quad + H\left(x_{\varepsilon_n}, \frac{\Lambda_x(t_{\delta_m}, x_{\varepsilon_n})V(t_{\delta_m}, x_{\varepsilon_n}) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})}\right) \leq 0. \end{aligned} \tag{4.11}$$

Similarly, the function

$$V(s,y) - \frac{1}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\left[\Lambda(s,y)W(t_{\delta_m}, x_{\varepsilon_n}) - \frac{1}{2\varepsilon_n}|x_{\varepsilon_n} - y|^2 - \frac{1}{2\delta_m}|t_{\delta_m} - s|^2\right] \tag{4.12}$$

has a minimum at $(s_{\delta_m}, y_{\varepsilon_n})$. Note that $W(\cdot,\cdot)$ is a supersolution, which results in

$$\begin{aligned} &\frac{\Lambda_s(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}) + (t_{\delta_m} - s_{\delta_m})/\delta_m}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})} \\ &\quad + H\left(y_{\varepsilon_n}, \frac{\Lambda_y(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\right) \geq 0. \end{aligned} \tag{4.13}$$

Fix ε_n and let $m \to \infty$; (4.11) and (4.13) then imply that the sequence $\{(t_{\delta_m} - s_{\delta_m})/\delta_m\}$ is bounded. Thus there exists a converging subsequence, which we still denote by $\{(t_{\delta_m} - s_{\delta_m})/\delta_m\}$. By (4.11) and assumption (2.5), we have that $(x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n$ is also bounded. Note that by (4.8) and (4.9), the difference between

$$H\left(x_{\varepsilon_n}, \frac{\Lambda_x(t_{\delta_m}, x_{\varepsilon_n})V(t_{\delta_m}, x_{\varepsilon_n}) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})}\right)$$

and

$$H\left(y_{\varepsilon_n}, \frac{\Lambda_y(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\right)$$

approaches zero as $m, n \to \infty$. Hence letting $m \to \infty$ in both (4.11) and (4.13), subtracting (4.13) from (4.11), and letting $n \to \infty$, leads to

$$-\frac{\Lambda_t(t_0, x_0)[W(t_0, x_0) - V(t_0, x_0)]}{\Lambda(t_0, x_0)} \leq 0. \tag{4.14}$$

This, in turn, implies that $\Lambda_t(t_0, x_0) \geq 0$, which contradicts the assumption of the lemma. Therefore

$$\max_{Q^{\Omega}_{t_f} \times Q^{\Omega}_{t_f}} \Lambda(t, x)[W(t, x) - V(t, x)] \leq 0 \tag{4.15}$$

and this completes the proof of Lemma 4.1.

Now we are ready to state the following comparison theorem:

Theorem 4.2 *If condition (4.1) holds, then we have*

$$W \leq V \quad on \quad Q^{\Omega}_{t_f}. \tag{4.16}$$

Proof We are interested in finding a function Λ such that the conditions of Lemma 4.1 are satisfied. A natural choice for this function is:

$$\Lambda(t, x) = \begin{cases} \exp\left\{\dfrac{R^2}{|x|^2 - R^2} - t\right\}, & |x| < R, \\ 0, & |x| \geq R \end{cases}. \tag{4.17}$$

Suppose that there were $(t_0, x_0, i_0) \in Q^{\Omega}_{t_f}$ such that

$$W(t_0, x_0) > V(t_0, x_0). \tag{4.18}$$

Let $R > |x_0|$, and Λ be as above. Clearly, (3.2) is satisfied under this specific choice of Λ. Applying Lemma 4.1, we know that (4.18) could not hold. Therefore (4.16) must be true.

Under our assumptions, the comparison theorem leads to uniqueness of the viscosity solution of (2.4):

Corollary 4.1 *Let V, W be two viscosity solutions of (2.4) with boundary and terminal conditions*

$$V(t,x) = W(t,x) = \varphi(t,x) \quad on \quad [0,t_f] \times \partial\Omega,$$
$$V(t_f,x) = W(t_f,x) = g(x) \quad on \quad \Omega.$$

Under assumptions (2.2) – (2.3) and (2.5), we have

$$V = W \quad on \quad [0,t_f] \times \Omega. \tag{4.19}$$

5 Feedback Optimal Control

Letting $V(t;x,t_f)$ denote the unique viscosity solution of (2.4), we consider in this section the limit $\lim\limits_{t_f\to\infty} V(t;x,t_f)$ provided that such a limit exits. Toward this end, we introduce another HJI equation which corresponds to the infinite-horizon case:

$$H(x, V(x)) = 0, \tag{5.1}$$

where H is as given in (2.5). Henceforth we denote the viscosity solution of (5.1) by $\hat{V}$.

Lemma 5.1 *$\hat{V}$ is a viscosity solution of (5.1) with $x \in \Omega$ if and only if*

$$H(x, p(x)) = 0, \tag{5.2}$$

for $p \in D^-\hat{V}(x)$, where

$$D^-\hat{V}(x) := \left\{ p \in \mathcal{R}^n,\ \liminf_{y\to x} \frac{\hat{V}(y) - \hat{V}(x) - p(y-x)}{|y-x|} \geq 0 \right\}.$$

Proof See page 80 of [7].

Theorem 5.1 *Let $g \equiv 0$. Assume that $\hat{V}$ is the smallest nonnegative viscosity solution on any open bounded subset $\Omega \subset \mathcal{R}^n$ with properties*

(1) *The state feedback controller*

$$\mu(x) = -\frac{1}{2} R^{-1}(x) B^T(x) p(x), \qquad p \in D^-\hat{V}(x), \tag{5.3}$$

is an admissible state feedback controller, that is, under it, the state equation admits at least one solution in $L^2_{loc}(0,\infty;\mathcal{R}^n)$.

(2) *There exists a nonnegative function $\varphi\colon \Omega \to \mathcal{R}$, with $\nabla_x\varphi$ existing a.e. on Ω, such that*

$$\nabla_x \varphi = p, \quad a.e.\ x \in \Omega. \tag{5.4}$$

(3) *$q(x(\cdot)) \in L^1(\mathcal{R}^+;\mathcal{R})$ implies $x \in L^2(\mathcal{R}^+;\mathcal{R}^n)$.*

Then, under the feedback controller (5.3), the worst-case system trajectory generated by

$$\dot{x}^* = a(x^*) - \frac{1}{2}\, B(x^*)R^{-1}(x)B(x^*)^T p(x^*) + \frac{1}{2\gamma^2}\, D(x^*)D(x^*)^T p(x^*)$$

is globally asymptotically stable, i.e.

$$x^* \in C(\mathcal{R}^+;\mathcal{R}^n) \cap L^2(\mathcal{R}^+;\mathcal{R}^n);\ \lim_{t\to\infty} x^*(t) = 0,$$

and for any $w \in L_2([0,\infty);\mathcal{R}^m)$ *we have*

$$\int_{t_1}^{t_2} \left\{ q(x) + \mu(x)^T R(x)\mu(x) - \gamma^2|w|^2 \right\} dt + \varphi(x(t_2)) \le \varphi(x(t_1)), \tag{5.5}$$

where x *satisfies*

$$\dot{x} = a(x) + B(x)\mu(x) + D(x)w. \tag{5.6}$$

Proof By hypothesis (2) of the theorem, and Lemma 5.1, we have

$$q + \nabla_x\varphi^T a - \frac{1}{4}\left(\nabla_x\varphi B R^{-1} B^T \nabla_x\varphi - \frac{1}{\gamma^2}\nabla_x\varphi^T D^T D \nabla_x\varphi\right) = 0 \quad a.e. \quad x \in \Omega. \tag{5.7}$$

Note that

$$\frac{d\varphi(x(t))}{dt} = \nabla_x\varphi^T \left[a(x) + B(x)\mu(x) + D(x)w\right] \tag{5.8}$$

and integrating (5.8) on (t_1,t_2), and making use of (5.7), we get

$$\varphi(x(t_2)) + \int_{t_1}^{t_2} \left\{ q(x) + \mu(x)^T R(x)\mu(x) - \gamma^2|w|^2 \right\} ds$$

$$= \varphi(x(t_1)) - \int_{t_1}^{t_2} \left| \gamma w(s) + \frac{1}{2\gamma} D(x(s))p(x(s)) \right|^2 ds \le \varphi(x(t_1)).$$

Let $t_1 = 0$, $t_2 = T$, and note that by above inequality we have, for some constant C,

$$\int_0^t q(x^*(s))\, ds \le C, \qquad \forall\, T > 0.$$

In view of this, and hypothesis (3), we have $x^* \in L^2([0,\infty);\mathcal{R}^n)$. Hence $x^*(t) \to 0$ as $t \to \infty$.

Theorem 5.2 *Let* $\hat{V}$ *be the smallest nonnegative viscosity solution of (5.1) and* V *be the viscosity solution of (2.4) with* $g \equiv 0$. *Under the three hypotheses of Theorem 5.2, we have:*

(1) *There exists a function* $\psi\colon \Omega \to \mathcal{R} \cup \{\infty\}$ *such that* $\forall\, x \in \Omega$

$$V(0;x,t_f) \uparrow \psi(x) \quad as \quad t_f \to \infty. \tag{5.9}$$

(2) *If the above convergence is uniform on compact subsets of* Ω, *then* $\psi = \hat{V}$.

Proof (1) According to Theorem 3.1, we know that

$$V(0;x,t_f) = \sup_w \inf_u J^{t_f}(0;x,u,w). \tag{5.10}$$

Note that $V(0;\cdot,t_f)$ is monotonically nondecreasing with increasing t_f, since the lower value of the game $J^{t_f}(0;x,\cdot,\cdot)$ defined on $[0,t_f]$ cannot be larger than that of the one defined on a longer interval, $[0,t_{f'}]$, $t_{f'} > t_f$, as the maximizing player can always play zero control on the subinterval $[t_f,t_{f'}]$. According to the proof of Theorem 5.2, we have $V(0;x,t_f) = \sup_w \inf_u J^{t_f}(0;x,u,w) \le \varphi(x_0)$. Hence there exists a function ψ such that

$$V(0;x,t_f) \uparrow \psi(x) \quad \forall x \in \Omega. \tag{5.11}$$

(2) For all $x \in \Omega$ and $t \ge -t_f$, introduce

$$V^{t_f}(t,x) = V(0;x,t+t_f) \tag{5.12}$$

and note that V^{t_f} is a viscosity solution of

$$-V_t^{t_f}(t,x) + H(x, V^{t_f}(t,x)) = 0. \tag{5.13}$$

Thus ψ is a continuous viscosity solution of (5.13) according to the uniform convergence theorem of [7]. Since ψ is t-invariant, it is a continuous viscosity solution of

$$H(x,V(x)) = 0. \tag{5.14}$$

The proof of Theorem 5.2 is thus complete.

6 Example

Revisiting the example in Chapter 4 of [1] (p. 170), consider the bilinear system

$$\dot{x}(t) = (u(t)+w(t))x(t), \quad x(0) = x_0, \tag{6.1}$$

and the cost function

$$J_\gamma(x;u,w) = \int_0^\infty \{x^2(t) + u^2(t) - \gamma^2 w^2(t)\}\, dt.$$

The associated HJI equation is

$$-x^2 + \frac{1}{4}\left(1 - \frac{1}{\gamma^2}\right) V_x^2 x^2 = 0 \tag{6.2}$$

whose smallest nonnegative viscosity solution is

$$\hat{V}(x) = \frac{2\gamma}{\sqrt{\gamma^2-1}}\,|x| \tag{6.3}$$

provided that $\gamma > 1$. It can be shown that $\hat{V}$ is in fact the lower value of the game $J_\gamma(x,\cdot,\cdot)$ (e.g. see [11]), that is,

$$\hat{V}(x) = \sup_w \inf_u J_\gamma(x; u, w). \tag{6.4}$$

The subdifferential $D^-\hat{V}(x)$ of $\hat{V}$ is

$$D^-\hat{V}(x) = \frac{2\gamma}{\sqrt{\gamma^2-1}}\,\frac{x}{|x|}, \qquad x \neq 0. \tag{6.5}$$

In this case, the function φ introduced in (5.4) is $\varphi(x) = \hat{V}(x)$. According to Theorem 5.1, the H^∞ optimal state feedback controller is

$$\mu(x) = -\frac{\gamma}{\sqrt{\gamma^2-1}}\,|x|. \tag{6.6}$$

For any positive $t_f > 0$, let

$$J^{t_f}(t,x;u,w) = \int_t^{t_f} \left\{x^2(s) + u^2(s) - \gamma^2 w^2(s)\right\} ds.$$

The associated HJI equation

$$-V_t - x^2 + \frac{1}{4}\left(1 - \frac{1}{\gamma^2}\right) V_x^2 x^2 = 0 \tag{6.7}$$

with terminal condition $V_t(t_f; x, t_f) = 0$ has a unique viscosity solution according to Corollary 4.1, and such a viscosity solution is also the lower value of the game with cost function $J^{t_f}(t,x;\cdot,\cdot)$. Theorem 5.2 assures $V(0;x,t_f) \to \hat{V}(x)$ as $t_f \to \infty$. This conclusion, however, can also be verified directly by the Arzelá-Ascoli Theorem for this particular example. Consider the system $\dot{y} = (\mu(y) + \nu(y))y$, $y(0) = y$, where $\mu(\cdot)$ is as given by (6.6), and $\nu(x) = \frac{1}{\gamma\sqrt{\gamma^2-1}}\,|x|$. Note that by the proof of Theorem 5.1, we have

$$\begin{aligned} |V(0,x;t_f) - V(0,y;t_f)| &= |\varphi(x) - \varphi(y) + \varphi(y(t_f)) - \varphi(x(t_f))| \\ &\le \frac{\gamma}{\sqrt{\gamma^2-1}}\,\left(|x-y| + |y(t_f) - x(t_f)|\right) \\ &\le \frac{\gamma}{\sqrt{\gamma^2-1}}\,C|x-y|, \quad \text{for some} \quad C > 0. \end{aligned}$$

By the Arzelá-Ascoli Theorem, we have that V converges to $\hat{V}$ uniformly on compact sets.

7 Concluding Remarks

In this paper we have shown that for input-affine nonlinear systems the relevant viscosity solution of the HJI equation associated with the infinite-horizon nonlinear H^∞-optimal control problem can be obtained as the limit of the unique viscosity solution of the HJI equation associated with a particular finite-horizon version, as the length of the time interval goes to infinity. This result has been obtained without necessarily restricting the control to a bounded set. Once such a viscosity solution is obtained, the resulting unique H^∞ controller makes the closed-loop system asymptotically stable under worst-case disturbances. The result also extends to more general nonlinear systems, as long as the underlying differential game admits a saddle-point solution.

References

[1] Başar, T. and Bernhard, P. *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach.* Birkhäuser, Boston, 1995.

[2] Bardi, M. and Soravia, P. Hamilton-Jocobi equations with singular boundary conditions on a free boundary and applications to differential games. *Trans. Amer. Math. Soc.* **325** (1991) 205–229.

[3] Crandall, M. and Lions, P. On existence and uniqueness of solutions of Hamilton-Jacobi equations. *Nonlinear Analysis, Theory, Methods and Applications* **10** (1986) 353–370.

[4] Crandall, M. and Lions, P. Viscosity solutions of Hamilton-Jacobi equations. *Tran. Amer. Math. Soc.* **277** (1983) 1–42.

[5] Evans, L. and Souganidis, P. Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. *Indiana University Math. J.* **33** (1984) 773–797.

[6] Frankowska, H. and Quincampoix, M. Dissipative control systems and disturbance attenuation for nonlinear H^∞-problem. *Preprints of 7th International Symposium on Differential Games and Applications*, Kanagawa, Japan, December 16-18, 1996, 199–212.

[7] Fleming, W. and Soner, H. *Controlled Markov Processes and Viscosity Solutions.* Springer-Verlag, Berlin, 1992.

[8] Lions, P.L. *Generalized Solution of Hamilton-Jacobi Equations.* Pitman Publishing Inc., Boston, 1982.

[9] McEneaney, W. Uniqueness for viscosity solutions of nonstationary Hamilton-Jacobi-Bellman equations under some a priori conditions (with applications). *SIAM J. Control and Optimization* **33** (1995) 1560–1576.

[10] McEneaney, W. Robust control and differential games on a finite time horizon. *Math. Control Signals Systems* **8** (1995) 138–166.

[11] Soravia, P. H_∞ control of nonlinear systems: differential games and viscosity solutions. *SIAM J. Control and Optimization* **34** (1996) 1071–1097.

[12] Xiao, M. and Başar, T. Viscosity supersolutions of a class of Hamilton-Jacobi-Isaccs equations arising in nonlinear H^∞ control. *Proc. of 36th IEEE Conference on Decision and Control*, 1997, 1761–1766.

[13] Xiao, M. and Başar, T. Nonlinear H^∞ control with unbounded controls: viscosity solutions and feedback design. *Proc. of IFAC Nonlinear Control Symposium*, Enschede, Netherlands, July 1-3, 1998, 169–174.

[14] Xiao, M. and Başar, T. Viscosity solution of a class of coupled HJB equations. *J. of Inequalities and Applications* **6** (2001) 519–545.

[15] Xiao, M. and Başar, T. H^∞ controller design via viscosity supersolutions of the Isaacs equations. In: *Stochastic Analysis, Control, Optimization and Applications, Systems Control Found. Appl.*. (Eds.: McEneaney, W.M., Yin, G. and Zhang, Q.), Birkhäuser, Boston, 1999, pp.151–170.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

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Volume 3 **Number 1** **2003**

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