Nonlinear Dynamics and Systems Theory, 3(2) (2003) 227–238

Optimal Control of Nonlinear Uncertain Systems over an Infinite Horizon via Finite-Horizon Approximations

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Received: May 3, 2003; Revised: August 9, 2003

Abstract: It is well-known that the Hamilton-Jacobi-Isaacs (HJI) equation associated with a nonlinear H^∞-optimal control problem on an infinite-time horizon generally admits nonunique, and in fact infinitely many, viscosity solutions. This makes it difficult to pick the relevant viscosity solution for the problem at hand, particularly when it is computed numerically. For the finite-horizon version of the problem, however, there is generally a unique viscosity solution (under appropriate conditions), which brings up the question of obtaining the viscosity solution relevant to the infinite-horizon problem as the limit of the unique solution of the finite-horizon one. This paper addresses this question for nonlinear systems affine in the control and the disturbance, and with a cost function quadratic in the control, where the control is not restricted to lie in a compact set. It establishes the existence of a well-defined limit, and also obtains a result on global asymptotic stability of closed-loop system under the H^∞ controller and the corresponding worst-case disturbance.

Keywords: *Nonlinear H^∞ control; Isaacs equation; viscosity solutions; global stability.*

Mathematics Subject Classification (2000): 49L25, 49J20, 49L20.

1 Introduction

An approach toward solving the nonlinear H^∞-optimal control problem is to treat it as a zero-sum differential game (e.g. [1]), for which a sufficient condition for the existence of a solution is expressed in terms of a Hamilton-Jacobi-Isaacs (HJI) equation. Such equations do not generally admit classical solutions, because the *values* of the corresponding differential games are not smooth enough to satisfy the HJI equations in the classical sense. Evans and Souganidis [5], and Bardi and Soravia [2] were among the first to show that the values of certain classes of differential games are viscosity solutions of the corresponding HJI equations. In the context of nonlinear H^∞ control problems, several authors have studied before the existence of a value function, and when the value function is a viscosity solution of the corresponding HJI equation, the uniqueness of such viscosity solutions (see [3, 4, 9 − 12]). But most of these studies have pertained to the *a priori* assumption that control sets are bounded.

The system considered in this paper has the input-affine form, leading (along with a quadratic-in-control cost function) to a Hamiltonian that is quadratic in both the control and the disturbance. This structure allows us to establish a comparison theorem which yields the uniqueness of the viscosity solution of the corresponding finite-horizon HJI equation, and this solution in turn can be used to approximate the desired viscosity solution of the corresponding infinite-horizon HJI equation. Thus one objective of this paper is to establish the connection between two HJI equations, one of which has multiple solutions and the other one has a unique solution which can be used to approach the desired solution of the former. A second objective is to show connections between such viscosity solutions and stabilizing feedback controller design. As indicated above, most current work which relate to nonlinear H^∞ control problems requires the control set to be compact in order to prove the uniqueness of the viscosity solution of the corresponding HJI equations (e.g. see [7, 9]). Clearly the boundedness assumption on the control space could be overly restrictive, and is not convenient for technical approaches. In this paper, such a restriction is relaxed and the uniqueness of HJI equations holds under standard assumptions.

The paper is organized as follows. In Section 2, we present the problem formulation and describe some necessary assumptions for the systems. In Section 3, we show that the HJI equation in the finite-horizon case admits a viscosity solution. Section 4 proves that the viscosity solution discussed in Section 3 is unique. In Section 5, we study how to obtain the viscosity solution of the HJI equation of infinite-horizon case from the unique solution of the finite-horizon one. An example is given in Section 6 to illustrate the main result of the paper. Some final remarks in Section 7 conclude the paper.

2 Preliminaries and Assumptions

Consider a system of the input-affine form

$$\frac{dx}{ds} = a(x) + B(x)u + D(x)w, \qquad x(t) = x^0, \tag{2.1}$$

where $x(s)$ is the state vector with values in $\mathcal{R}^n$, and $u(s)$ is the control vector with values in $\mathcal{R}^p$. The other input, $w(s) \in \mathcal{R}^m$, is the driving noise, which is an unknown $L^2[0,\infty)$ function; it represents modeling errors in a and other possible errors or inaccuracies in the dynamics. Thus the system model (2.1) accommodates uncertainties.

We will assume that there exist positive constants K_a, K_B, K_D such that

$$\begin{aligned} |a(x) - a(y)| &\le K_a|x-y| \qquad && \forall\, x,\, y \in \mathcal{R}^n, \\ |a(x)| &\le K_a(1+|x|) && \forall\, x \in \mathcal{R}^n, \\ |B(x) - B(y)| &\le K_B|x-y| && \forall\, x,\, y \in \mathcal{R}^n, \\ |D(x) - D(y)| &\le K_D|x-y| && \forall\, x,\, y \in \mathcal{R}^n, \end{aligned} \tag{2.2}$$

where the symbol $|\cdot|$ denotes the Euclidean norm. Since the state of the system probably operates over some compact subset of $\mathcal{R}^n$, we may only need (2.2) to hold on this compact set as a, B, D can be extended to all of $\mathcal{R}^n$.

Let the running cost be $q(x) + u^T R(x) u$, where $q(x) \geq 0$, $\forall\, x \in \mathcal{R}^n$, and satisfies the following bounds and growth conditions:

$$|q(x) - q(y)| \leq C_q(1 + |x| + |y|)|x - y|, \qquad C_q \geq 0.$$

For $R(x)$, on the other hand, there exist positive constants k_1, k_2 such that for all $x \in \mathcal{R}^n$

$$k_1 I^{p\times p} \leq R(x) \leq k_2 I^{p\times p},$$

which in particular implies that $R(x)$ is invertible for all x.

Further let the terminal state cost function be $g(x)$, satisfying the bound

$$|g(x) - g(y)| \leq K_g(r)|x - y|, \quad \forall\, |x| \leq r, \quad |y| \leq r.$$

For a given $t_f > 0$, we consider the lower-value function

$$V(t; x, t_f) = \sup_w \inf_u J_\gamma^{t_f}(t, x, u, w), \tag{2.3}$$

where $J_\gamma^{t_f}(t, x, u, w) = g(x(t_f)) + L_\gamma^{t_f}(t, x, u, w)$ and

$$L_\gamma^\tau(t, x, u, w) = \int_t^\tau \left(q(x(s)) + u^T R(x(s))u - \gamma^2 |w(s)|^2\right) ds.$$

The corresponding Hamilton-Jacobi-Isaacs (HJI) equation is

$$-V_t(t; x, t_f) + H(x, V(t; x, t_f)) = 0 \quad \text{and} \quad V(t_f; x, t_f) = g(x), \tag{2.4}$$

where the Hamiltonian for this case is given by

$$\begin{aligned} H(x,p) : &= -\sup_w \inf_u \{q + u^T R u - \gamma^2 w^T w + p^T[a + Bu + Dw]\} \\ &= -q - p^T a + \frac{1}{4} p^T \left(BR^{-1}B^T - \frac{1}{\gamma^2} DD^T\right) p \end{aligned}$$

with the assumption that for fixed x

$$|p| \to \infty \quad \text{implies} \quad |H(x,p)| \to \infty. \tag{2.5}$$

Remark 2.1 Let $\mathcal{M}$ be the space of all state-feedback controllers, i.e. measurable mappings from $\mathcal{R}^n$ into $\mathcal{R}^p$. Then, the quantity we are really interested in (the one that is relevant to nonlinear H^∞ control) is in fact the upper value of the game: $\inf\limits_{\mathcal{M}} \sup\limits_w J_\gamma^{t_f}(t, x, u, w)$. Note, however, that since the Isaacs' condition is satisfied, the Hamiltonian admits a saddle-point solution, which makes the upper and lower values equal. In view of this, we are allowed to work with the lower value of the game and thus avoid some technical issues that arise in a direct study of the upper value of the game.

3 Lower Value of the Differential Game and the Viscosity Solution

Lemma 3.1 *Let V be defined as in (2.3). Then, for any $0 \le t \le \tau \le t_f$, and with $x(t) = x$,*

$$V(t; x, t_f) \ge \sup_w \inf_u \Big\{ L_\gamma^\tau(t; x, u, w) + V(\tau; x(\tau), t_f) \Big\}.$$

If the upper value of the game is finite, then the above inequality becomes an equality.

Proof This involves a standard dynamic programming type argument in the context of differential games.

Theorem 3.1 *If in (2.3) $V(\cdot;\cdot,t_f) \in C([0,t_f] \times \Omega)$, then V is a viscosity supersolution of (2.4). Furthermore, if the upper value of the game, $\inf_{\mathcal{M}} \sup_w J_\gamma^{t_f}(t; x, \mu, w)$, is finite, then V is a viscosity solution of (2.4).*

Proof Suppose that to the contrary V is not a viscosity supersolution of (2.4). Then there would exist an $\varepsilon > 0$, and a pair $(t_0, x_0) \in [0, t_f] \times \Omega$ and a function Φ: $[0, t_f] \times \Omega \to \mathcal{R}$ such that $V(\cdot;\cdot,t_f) - \Phi$ has a local minimum at (t_0, x_0), and

$$-\Phi_t(t_0, x_0) + H(x_0, \Phi_x(t_0, x_0)) \le -\varepsilon.$$

By making use of (2.1), as $t \downarrow t_0$, we have

$$\Phi(t_0, x_0) - \Phi(t, x(t)) \le -\varepsilon(t - t_0) + \inf_u L_\gamma^t(t_0; x_0, u, w).$$

Since (t_0, x_0) is a local minimizer of $V(\cdot;\cdot,t_f) - \Phi$, in a small neighborhood of (t_0, x_0),

$$V(t_0; x_0, t_f) - V(t; x, t_f) \le \Phi(t_0, x_0) - \Phi(t, x(t)).$$

Therefore we arrive at

$$V(t_0; x_0, t_f) \le \sup_w \inf_u \big\{ L_\gamma^t(t_0; x_0, u, w) + V(t; x, t_f) \big\} - \varepsilon(t - t_0),$$

which contradicts the statement of Lemma 3.1. For the case of viscosity subsolution, let $(t_0, x_0) \in [0, t_f] \times \Omega$ and $\Psi \in C^1([0, t_f] \times \Omega)$ be such that (t_0, x_0) is a local maximizer of $V - \Psi$ with $V(t_0; x_0, t_f) = \Psi(t_0, x_0)$. By Lemma 3.1, for any $t \in (t_0, t_f]$,

$$\Psi(t_0, x_0) = V(t_0; x_0, t_f) \le \sup_w \big\{ L_\gamma^t(t_0, x_0, u^*, w) + V(t; x(t), t_f) \big\}, \tag{3.1}$$

where $u^* = -R^{-1} B^T \Psi_x$. Observing that $\{x(t, x_0, u^*, w)\}$ is continuous in t, when $t > t_0$ is sufficiently close to t_0, we have

$$V(t_0; x_0, t_f) - V(t; x, t_f) \ge \Psi(t_0, x_0) - \Psi(t, x(t, x_0, u^*, w)).$$

Divide (3.1) by $t - t_0$, and let $t \downarrow t_0$, to obtain

$$\sup_w \left\{ \frac{1}{2}[q + (u^*)^T R u^* - \gamma^2 |w|^2] + \Psi_t + \Psi_x^T (a + Bu^* + Dw) \right\} \ge 0$$

which yields at $(t,x)=(t_0,x_0)$

$$\Psi_t + q + \Psi_x^T a - \frac{1}{4}\,\Psi_x^T\left(BR^{-1}B^T - \frac{1}{\gamma^2}\,DD^T\right)\Psi_x \geq 0$$

that is to say, V is a viscosity solution of (2.4).

4 Uniqueness of the Viscosity Solution of (2.4)

In this section, we show that the viscosity solution of (2.4) is unique. Suppose that V, W are a viscosity supersolution and a viscosity subsolution, respectively, of (2.4) on $Q_{t_f}^{\Omega} = [0,t_f]\times\Omega$. Furthermore, assume that

$$W \leq V \quad \text{on } (\{t=t_f\}\times\Omega)\cup([0,t_f]\times\partial\Omega).$$

Lemma 4.1 *Let $R<\infty$, and a function $\Lambda \in C^1(Q_{t_f}^{\Omega})$ be such that $\Lambda \geq 0$ if $|x|\geq R$, and*

$$\Lambda_t < 0 \quad on (\text{supp }\Lambda)^o \cap \left(Q_{t_f}^{\Omega}\right)^o, \tag{4.1}$$

where the superscript "o" indicates interior. Then $W\leq V$ on $(\text{supp}\,\Lambda)^o\cap(Q_{t_f}^{\Omega})^o$.

Proof Suppose that $(t_0,x_0)\in(\text{supp}\,\Lambda)^o\cap(Q_{t_f}^{\Omega})^o$ such that

$$M_0 = \Lambda(t_0,x_0)[W(t_0,x_0)-V(t_0,x_0)] = \max_{Q_{t_f}^{\Omega}} \Lambda(t,x)[W(t,x)-V(t,x)] > 0 \tag{4.2}$$

since otherwise the result has already been established. Introduce a function $\Phi^{\varepsilon,\delta}\colon Q_{t_f}^{\Omega}\times Q_{t_f}^{\Omega}\to\mathcal{R}^n$ by

$$\Phi^{\varepsilon,\delta} = \Lambda(s,y)W(t,x) - \Lambda(t,x)V(s,y) - \frac{1}{2\varepsilon}\,|x-y|^2 - \frac{1}{2\delta}\,|t-s|^2. \tag{4.3}$$

Since $\Phi^{\varepsilon\,\delta}$ is upper semicontinuous and Λ has a compact support, there exists $(t_\delta,x_\varepsilon,s_\delta,y_\varepsilon)\in Q_{t_f}^{\Omega}\times Q_{t_f}^{\Omega}$ such that

$$\Phi^{\varepsilon,\delta}(t_\delta,x_\varepsilon,s_\delta,y_\varepsilon) = \max_{Q_{t_f}^{\Omega}\times Q_{t_f}^{\Omega}} \Phi^{\varepsilon,\delta}(t,x,s,y). \tag{4.4}$$

Let $M^{\varepsilon,\delta}=\Phi^{\varepsilon,\delta}(t_\delta,x_\varepsilon,s_\delta,y_\varepsilon)$, and consider $0<\varepsilon_2\leq\varepsilon_1$ and $0<\delta_2\leq\delta_1$. Then

$$\begin{aligned}
M^{\varepsilon_1,\delta_1} &- \left(\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_1}\right)\frac{|x_{\varepsilon_2}-y_{\varepsilon_2}|^2}{2} - \left(\frac{1}{\delta_2}-\frac{1}{\delta_1}\right)\frac{|t_{\delta_2}-s_{\delta_2}|^2}{2} \\
&\geq \Phi^{\varepsilon_1,\delta_1}(t_{\delta_2},x_{\varepsilon_2},s_{\delta_2},y_{\varepsilon_2}) - \left(\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_1}\right)\frac{|x_{\varepsilon_2}-y_{\varepsilon_2}|^2}{2} - \left(\frac{1}{\delta_2}-\frac{1}{\delta_1}\right)\frac{|t_{\delta_2}-s_{\delta_2}|^2}{2} \\
&= \Lambda(s_{\delta_2},y_{\varepsilon_2})W(t_{\delta_2},x_{\varepsilon_2}) - \Lambda(t_{\delta_2},x_{\varepsilon_2})V(s_{\delta_2},y_{\varepsilon_2}) - \frac{1}{2\varepsilon_1}\,|x_{\varepsilon_2}-y_{\varepsilon_2}|^2
\end{aligned}$$

$$- \frac{1}{2\delta_2} |t_{\delta_2} - s_{\delta_2}|^2 - \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\frac{|x_{\varepsilon_2} - y_{\varepsilon_2}|^2}{2} - \left(\frac{1}{\delta_2} - \frac{1}{\delta_1}\right)\frac{|t_{\delta_2} - s_{\delta_2}|^2}{2}$$
$$= \Phi^{\varepsilon_2,\delta_2}(t_{\delta_2}, x_{\varepsilon_2}, s_{\delta_2}, y_{\varepsilon_2}) = M^{\varepsilon_2,\delta_2}.$$

Hence, we can see that $(\varepsilon, \delta) \mapsto M^{\varepsilon,\delta}$ is nondecreasing. Let $\varepsilon_1 = 2\varepsilon, \varepsilon_2 = \varepsilon$ and $\delta_1 = \delta_2 = \delta$; then

$$M^{2\varepsilon,\delta} - M^{\varepsilon,\delta} \geq \frac{1}{2\varepsilon}\frac{|x_\varepsilon - y_\varepsilon|^2}{2}. \tag{4.5}$$

Note that $M^{2\varepsilon,\delta} - M^{\varepsilon,\delta} \to 0$ as $\varepsilon \downarrow 0$. Hence, $(\varepsilon, \delta) \to M^{\varepsilon,\delta}$ is nondecreasing. Thus we have

$$\frac{1}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \tag{4.6}$$

Similarly, we have

$$\frac{1}{2\delta} |t_\delta - s_\delta|^2 \to 0 \quad \text{as} \quad \delta \downarrow 0. \tag{4.7}$$

Since Λ has compact support, and (4.6), (4.7) hold, there exist sequences $\{\varepsilon_n\}$ and $\{\delta_m\}$ which converge to *zero* such that

$$x_{\varepsilon_n} \to \hat{x}, \quad y_{\varepsilon_n} \to \hat{x}, \quad \text{as} \quad n \to \infty \tag{4.8}$$

and

$$t_{\delta_m} \to \hat{t}, \quad s_{\delta_m} \to \hat{t} \quad \text{as} \quad m \to \infty, \tag{4.9}$$

where $(\hat{t}, \hat{x}) \in Q^\Omega_{t_f}$. In fact it is easy to see that $\hat{x} = x_0$, $\hat{t} = t_0$. Note that under the initial hypotheses, $(t_0, x_0) \in (\operatorname{supp}\Lambda)^o \cap (Q^\Omega_{t_f})^o$. Therefore, for sufficiently large n and m, we have that $(t_{\delta_m}, x_{\varepsilon_n}), (s_{\delta_m}, y_{\varepsilon_n}) \in (\operatorname{supp}\Lambda)^o \cap (Q^\Omega_{t_f})^o$. Since the function

$$W(t,x) - \frac{1}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})}\left[\Lambda(t,x)V(t_{\delta_m}, x_{\varepsilon_n}) + \frac{1}{2\varepsilon_n}|x - y_{\varepsilon_n}|^2 + \frac{1}{2\delta_m}|t - s_{\delta_m}|^2\right] \tag{4.10}$$

attains its maximum at $(t,x) = (t_{\delta_m}, x_{\varepsilon_n})$, by the definition of viscosity subsolution, we have

$$\begin{aligned}&\frac{\Lambda_t(t_{\varepsilon_n}, x_{\varepsilon_n})V(t_{\delta_m}, x_{\varepsilon_n}) + (t_{\delta_m} - s_{\delta_m})/\delta_m}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})} \\ &+ H\left(x_{\varepsilon_n}, \frac{\Lambda_x(t_{\delta_m}, x_{\varepsilon_n})V(t_{\delta_m}, x_{\varepsilon_n}) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})}\right) \leq 0.\end{aligned} \tag{4.11}$$

Similarly, the function

$$V(s,y) - \frac{1}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\left[\Lambda(s,y)W(t_{\delta_m}, x_{\varepsilon_n}) - \frac{1}{2\varepsilon_n}|x_{\varepsilon_n} - y|^2 - \frac{1}{2\delta_m}|t_{\delta_m} - s|^2\right] \tag{4.12}$$

has a minimum at $(s_{\delta_m}, y_{\varepsilon_n})$. Note that $W(\cdot,\cdot)$ is a supersolution, which results in

$$\begin{aligned}&\frac{\Lambda_s(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}) + (t_{\delta_m} - s_{\delta_m})/\delta_m}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})} \\ &+ H\left(y_{\varepsilon_n}, \frac{\Lambda_y(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\right) \geq 0.\end{aligned} \tag{4.13}$$

Fix ε_n and let $m \to \infty$; (4.11) and (4.13) then imply that the sequence $\{(t_{\delta_m} - s_{\delta_m})/\delta_m\}$ is bounded. Thus there exists a converging subsequence, which we still denote by $\{(t_{\delta_m} - s_{\delta_m})/\delta_m\}$. By (4.11) and assumption (2.5), we have that $(x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n$ is also bounded. Note that by (4.8) and (4.9), the difference between

$$H\left(x_{\varepsilon_n}, \frac{\Lambda_x(t_{\delta_m}, x_{\varepsilon_n})V(t_{\delta_m}, x_{\varepsilon_n}) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})}\right)$$

and

$$H\left(y_{\varepsilon_n}, \frac{\Lambda_y(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\right)$$

approaches zero as $m, n \to \infty$. Hence letting $m \to \infty$ in both (4.11) and (4.13), subtracting (4.13) from (4.11), and letting $n \to \infty$, leads to

$$-\frac{\Lambda_t(t_0, x_0)[W(t_0, x_0) - V(t_0, x_0)]}{\Lambda(t_0, x_0)} \le 0. \tag{4.14}$$

This, in turn, implies that $\Lambda_t(t_0, x_0) \ge 0$, which contradicts the assumption of the lemma. Therefore

$$\max_{Q^{\Omega}_{t_f} \times Q^{\Omega}_{t_f}} \Lambda(t, x)[W(t, x) - V(t, x)] \le 0 \tag{4.15}$$

and this completes the proof of Lemma 4.1.

Now we are ready to state the following comparison theorem:

Theorem 4.2 *If condition (4.1) holds, then we have*

$$W \le V \quad \text{on} \quad Q^{\Omega}_{t_f}. \tag{4.16}$$

Proof We are interested in finding a function Λ such that the conditions of Lemma 4.1 are satisfied. A natural choice for this function is:

$$\Lambda(t, x) = \begin{cases} \exp\left\{\dfrac{R^2}{|x|^2 - R^2} - t\right\}, & |x| < R, \\ 0, & |x| \ge R \end{cases}. \tag{4.17}$$

Suppose that there were $(t_0, x_0, i_0) \in Q^{\Omega}_{t_f}$ such that

$$W(t_0, x_0) > V(t_0, x_0). \tag{4.18}$$

Let $R > |x_0|$, and Λ be as above. Clearly, (3.2) is satisfied under this specific choice of Λ. Applying Lemma 4.1, we know that (4.18) could not hold. Therefore (4.16) must be true.

Under our assumptions, the comparison theorem leads to uniqueness of the viscosity solution of (2.4):

Corollary 4.1 *Let V, W be two viscosity solutions of (2.4) with boundary and terminal conditions*

$$V(t,x) = W(t,x) = \varphi(t,x) \quad \text{on} \quad [0,t_f]\times\partial\Omega,$$
$$V(t_f,x) = W(t_f,x) = g(x) \quad \text{on} \quad \Omega.$$

Under assumptions (2.2)–(2.3) and (2.5), we have

$$V = W \quad \text{on} \quad [0,t_f]\times\Omega. \tag{4.19}$$

5 Feedback Optimal Control

Letting $V(t;x,t_f)$ denote the unique viscosity solution of (2.4), we consider in this section the limit $\lim_{t_f\to\infty} V(t;x,t_f)$ provided that such a limit exits. Toward this end, we introduce another HJI equation which corresponds to the infinite-horizon case:

$$H(x, V(x)) = 0, \tag{5.1}$$

where H is as given in (2.5). Henceforth we denote the viscosity solution of (5.1) by $\hat{V}$.

Lemma 5.1 *$\hat{V}$ is a viscosity solution of (5.1) with $x\in\Omega$ if and only if*

$$H(x, p(x)) = 0, \tag{5.2}$$

for $p\in D^-\hat{V}(x)$, where

$$D^-\hat{V}(x) := \left\{ p\in\mathcal{R}^n,\ \liminf_{y\to x} \frac{\hat{V}(y)-\hat{V}(x)-p(y-x)}{|y-x|} \geq 0 \right\}.$$

Proof See page 80 of [7].

Theorem 5.1 *Let $g\equiv 0$. Assume that $\hat{V}$ is the smallest nonnegative viscosity solution on any open bounded subset $\Omega\subset\mathcal{R}^n$ with properties*

(1) *The state feedback controller*

$$\mu(x) = -\frac{1}{2}R^{-1}(x)B^T(x)p(x), \qquad p\in D^-\hat{V}(x), \tag{5.3}$$

is an admissible state feedback controller, that is, under it, the state equation admits at least one solution in $L^2_{loc}(0,\infty;\mathcal{R}^n)$.

(2) *There exists a nonnegative function $\varphi\colon\Omega\to\mathcal{R}$, with $\nabla_x\varphi$ existing a.e. on Ω, such that*

$$\nabla_x\varphi = p, \quad \text{a.e. } x\in\Omega. \tag{5.4}$$

(3) *$q(x(\cdot))\in L^1(\mathcal{R}^+;\mathcal{R})$ implies $x\in L^2(\mathcal{R}^+;\mathcal{R}^n)$.*

Then, under the feedback controller (5.3), the worst-case system trajectory generated by

$$\dot{x}^* = a(x^*) - \frac{1}{2}\, B(x^*)R^{-1}(x)B(x^*)^T p(x^*) + \frac{1}{2\gamma^2}\, D(x^*)D(x^*)^T p(x^*)$$

is globally asymptotically stable, i.e.

$$x^* \in C(\mathcal{R}^+;\mathcal{R}^n) \cap L^2(\mathcal{R}^+;\mathcal{R}^n);\ \lim_{t\to\infty} x^*(t) = 0,$$

and for any $w \in L_2([0,\infty);\mathcal{R}^m)$ *we have*

$$\int_{t_1}^{t_2} \big\{q(x) + \mu(x)^T R(x)\mu(x) - \gamma^2|w|^2\big\}\, dt + \varphi(x(t_2)) \le \varphi(x(t_1)), \tag{5.5}$$

where x *satisfies*

$$\dot{x} = a(x) + B(x)\mu(x) + D(x)w. \tag{5.6}$$

Proof By hypothesis (2) of the theorem, and Lemma 5.1, we have

$$q + \nabla_x\varphi^T a - \frac{1}{4}\bigg(\nabla_x\varphi B R^{-1} B^T \nabla_x\varphi - \frac{1}{\gamma^2}\nabla_x\varphi^T D^T D\nabla_x\varphi\bigg) = 0 \quad a.e. \quad x \in \Omega. \tag{5.7}$$

Note that

$$\frac{d\varphi(x(t))}{dt} = \nabla_x\varphi^T\big[a(x) + B(x)\mu(x) + D(x)w\big] \tag{5.8}$$

and integrating (5.8) on (t_1,t_2), and making use of (5.7), we get

$$\varphi(x(t_2)) + \int_{t_1}^{t_2} \big\{q(x) + \mu(x)^T R(x)\mu(x) - \gamma^2|w|^2\big\}\, ds$$

$$= \varphi(x(t_1)) - \int_{t_1}^{t_2} \bigg|\gamma w(s) + \frac{1}{2\gamma} D(x(s))p(x(s))\bigg|^2 ds \le \varphi(x(t_1)).$$

Let $t_1 = 0$, $t_2 = T$, and note that by above inequality we have, for some constant C,

$$\int_0^t q(x^*(s))\, ds \le C, \qquad \forall\, T > 0.$$

In view of this, and hypothesis (3), we have $x^* \in L^2([0,\infty);\mathcal{R}^n)$. Hence $x^*(t) \to 0$ as $t \to \infty$.

Theorem 5.2 *Let* $\hat{V}$ *be the smallest nonnegative viscosity solution of (5.1) and* V *be the viscosity solution of (2.4) with* $g \equiv 0$. *Under the three hypotheses of Theorem 5.2, we have:*

(1) *There exists a function* $\psi\colon \Omega \to \mathcal{R} \cup \{\infty\}$ *such that* $\forall\, x \in \Omega$

$$V(0;x,t_f) \uparrow \psi(x) \quad as \quad t_f \to \infty. \tag{5.9}$$

(2) *If the above convergence is uniform on compact subsets of* Ω, *then* $\psi = \hat{V}$.

Proof (1) According to Theorem 3.1, we know that

$$V(0;x,t_f) = \sup_w \inf_u J^{t_f}(0;x,u,w). \tag{5.10}$$

Note that $V(0;\cdot,t_f)$ is monotonically nondecreasing with increasing t_f, since the lower value of the game $J^{t_f}(0;x,\cdot,\cdot)$ defined on $[0,t_f]$ cannot be larger than that of the one defined on a longer interval, $[0,t_{f'}]$, $t_{f'} > t_f$, as the maximizing player can always play zero control on the subinterval $[t_f,t_{f'}]$. According to the proof of Theorem 5.2, we have $V(0;x,t_f) = \sup_w \inf_u J^{t_f}(0;x,u,w) \le \varphi(x_0)$. Hence there exists a function ψ such that

$$V(0;x,t_f) \uparrow \psi(x) \quad \forall\, x \in \Omega. \tag{5.11}$$

(2) For all $x \in \Omega$ and $t \ge -t_f$, introduce

$$V^{t_f}(t,x) = V(0;x,t+t_f) \tag{5.12}$$

and note that V^{t_f} is a viscosity solution of

$$-V_t^{t_f}(t,x) + H(x, V^{t_f}(t,x)) = 0. \tag{5.13}$$

Thus ψ is a continuous viscosity solution of (5.13) according to the uniform convergence theorem of [7]. Since ψ is t-invariant, it is a continuous viscosity solution of

$$H(x,V(x)) = 0. \tag{5.14}$$

The proof of Theorem 5.2 is thus complete.

6 Example

Revisiting the example in Chapter 4 of [1] (p. 170), consider the bilinear system

$$\dot{x}(t) = (u(t)+w(t))x(t), \quad x(0) = x_0, \tag{6.1}$$

and the cost function

$$J_\gamma(x;u,w) = \int_0^\infty \{x^2(t) + u^2(t) - \gamma^2 w^2(t)\}\, dt.$$

The associated HJI equation is

$$-x^2 + \frac{1}{4}\left(1 - \frac{1}{\gamma^2}\right) V_x^2 x^2 = 0 \tag{6.2}$$

whose smallest nonnegative viscosity solution is

$$\hat{V}(x) = \frac{2\gamma}{\sqrt{\gamma^2-1}}\,|x| \tag{6.3}$$

provided that $\gamma > 1$. It can be shown that $\hat{V}$ is in fact the lower value of the game $J_\gamma(x,\cdot,\cdot)$ (e.g. see [11]), that is,

$$\hat{V}(x) = \sup_w \inf_u J_\gamma(x;u,w). \tag{6.4}$$

The subdifferential $D^-\hat{V}(x)$ of $\hat{V}$ is

$$D^-\hat{V}(x) = \frac{2\gamma}{\sqrt{\gamma^2-1}}\,\frac{x}{|x|}, \qquad x \neq 0. \tag{6.5}$$

In this case, the function φ introduced in (5.4) is $\varphi(x) = \hat{V}(x)$. According to Theorem 5.1, the H^∞ optimal state feedback controller is

$$\mu(x) = -\frac{\gamma}{\sqrt{\gamma^2-1}}\,|x|. \tag{6.6}$$

For any positive $t_f > 0$, let

$$J^{t_f}(t,x;u,w) = \int_t^{t_f} \left\{x^2(s) + u^2(s) - \gamma^2 w^2(s)\right\} ds.$$

The associated HJI equation

$$-V_t - x^2 + \frac{1}{4}\left(1 - \frac{1}{\gamma^2}\right) V_x^2 x^2 = 0 \tag{6.7}$$

with terminal condition $V_t(t_f;x,t_f) = 0$ has a unique viscosity solution according to Corollary 4.1, and such a viscosity solution is also the lower value of the game with cost function $J^{t_f}(t,x;\cdot,\cdot)$. Theorem 5.2 assures $V(0;x,t_f) \to \hat{V}(x)$ as $t_f \to \infty$. This conclusion, however, can also be verified directly by the Arzelá-Ascoli Theorem for this particular example. Consider the system $\dot{y} = (\mu(y) + \nu(y))y$, $y(0) = y$, where $\mu(\cdot)$ is as given by (6.6), and $\nu(x) = \frac{1}{\gamma\sqrt{\gamma^2-1}}\,|x|$. Note that by the proof of Theorem 5.1, we have

$$\begin{aligned}
|V(0,x;t_f) - V(0,y;t_f)| &= |\varphi(x) - \varphi(y) + \varphi(y(t_f)) - \varphi(x(t_f))| \\
&\leq \frac{\gamma}{\sqrt{\gamma^2-1}}\,(|x-y| + |y(t_f) - x(t_f)|) \\
&\leq \frac{\gamma}{\sqrt{\gamma^2-1}}\,C|x-y|, \quad \text{for some} \quad C > 0.
\end{aligned}$$

By the Arzelá-Ascoli Theorem, we have that V converges to $\hat{V}$ uniformly on compact sets.

7 Concluding Remarks

In this paper we have shown that for input-affine nonlinear systems the relevant viscosity solution of the HJI equation associated with the infinite-horizon nonlinear H^∞-optimal control problem can be obtained as the limit of the unique viscosity solution of the HJI equation associated with a particular finite-horizon version, as the length of the time interval goes to infinity. This result has been obtained without necessarily restricting the control to a bounded set. Once such a viscosity solution is obtained, the resulting unique H^∞ controller makes the closed-loop system asymptotically stable under worst-case disturbances. The result also extends to more general nonlinear systems, as long as the underlying differential game admits a saddle-point solution.

References

[1] Başar, T. and Bernhard, P. *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach.* Birkhäuser, Boston, 1995.

[2] Bardi, M. and Soravia, P. Hamilton-Jocobi equations with singular boundary conditions on a free boundary and applications to differential games. *Trans. Amer. Math. Soc.* **325** (1991) 205–229.

[3] Crandall, M. and Lions, P. On existence and uniqueness of solutions of Hamilton-Jacobi equations. *Nonlinear Analysis, Theory, Methods and Applications* **10** (1986) 353–370.

[4] Crandall, M. and Lions, P. Viscosity solutions of Hamilton-Jacobi equations. *Tran. Amer. Math. Soc.* **277** (1983) 1–42.

[5] Evans, L. and Souganidis, P. Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. *Indiana University Math. J.* **33** (1984) 773–797.

[6] Frankowska, H. and Quincampoix, M. Dissipative control systems and disturbance attenuation for nonlinear H^∞-problem. *Preprints of 7th International Symposium on Differential Games and Applications*, Kanagawa, Japan, December 16-18, 1996, 199–212.

[7] Fleming, W. and Soner, H. *Controlled Markov Processes and Viscosity Solutions.* Springer-Verlag, Berlin, 1992.

[8] Lions, P.L. *Generalized Solution of Hamilton-Jacobi Equations.* Pitman Publishing Inc., Boston, 1982.

[9] McEneaney, W. Uniqueness for viscosity solutions of nonstationary Hamilton-Jacobi-Bellman equations under some a priori conditions (with applications). *SIAM J. Control and Optimization* **33** (1995) 1560–1576.

[10] McEneaney, W. Robust control and differential games on a finite time horizon. *Math. Control Signals Systems* **8** (1995) 138–166.

[11] Soravia, P. H_∞ control of nonlinear systems: differential games and viscosity solutions. *SIAM J. Control and Optimization* **34** (1996) 1071–1097.

[12] Xiao, M. and Başar, T. Viscosity supersolutions of a class of Hamilton-Jacobi-Isaccs equations arising in nonlinear H^∞ control. *Proc. of 36th IEEE Conference on Decision and Control*, 1997, 1761–1766.

[13] Xiao, M. and Başar, T. Nonlinear H^∞ control with unbounded controls: viscosity solutions and feedback design. *Proc. of IFAC Nonlinear Control Symposium*, Enschede, Netherlands, July 1-3, 1998, 169–174.

[14] Xiao, M. and Başar, T. Viscosity solution of a class of coupled HJB equations. *J. of Inequalities and Applications* **6** (2001) 519–545.

[15] Xiao, M. and Başar, T. H^∞ controller design via viscosity supersolutions of the Isaacs equations. In: *Stochastic Analysis, Control, Optimization and Applications, Systems Control Found. Appl.*. (Eds.: McEneaney, W.M., Yin, G. and Zhang, Q.), Birkhäuser, Boston, 1999, pp.151–170.