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Control of Chaos in a Convective Loop System

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Abstract: A convective loop is a system in which a fluid circulates freely inside a closed circular pipe. The circulating fluid works as a transport media of heat from a source to a sink. First order lumped parameter modelling of this system leads to a set of nonlinear ordinary differential equations. Depending on heating rate this system can show chaotic behavior. In this paper, the performance of nonlinear model predictive control is compared with other conventional nonlinear control law and it is found that although a simple linear or, nonlinear controller may stabilize the system, nonlinear model predictive controller outperforms other controllers.

Keywords: *Chaos; Lyapunov stability; nonlinear model predictive control.*

Mathematics Subject Classification (2000): 34D20, 34C23, 49J15, 70K55, 76E99.

1 Introduction

Natural convection loops showing chaotic behavior are used in solar energy heating and cooling systems, reactor, turbine, engine cooling systems, greenhouses, geothermal power production and in process industries. Chaos in such convective loop systems in general can be beneficial or detrimental depending on the process and the objective. Since it is associated with vigorous change in states under nominal operating condition without any change in input energy, it is beneficial for processes where mixing, heat transport and chemical reactions are important. However due to the oscillation, chaos may lead to vibrations and fatigue failure to the physical equipment, irregular and oscillation of process operating conditions and increased drag of fluid flow systems. Ehrhard and Müller [9] in their paper investigated natural convection in a closed loop. They first developed a first principal model of the loop based on heat transfer law. They also accounted for the nonsymmetric arrangement of heat sources and sinks. Finally the model is reduced to a set of nonlinear ordinary differential equations. Then through experimental and analytical data it is shown that this loop is characterized by nonlinear effects and can show stable, unstable or, chaotic regimes based on the heating rate. The model development and its analysis is further discussed in Section 2.3.

Abed and Fu [1, 2] in their papers have shown ways for local stabilization of nonlinear systems with Hopf and Stationary bifurcation. Sufficient conditions are also obtained for the local stabilization of nonlinear systems whose linearization has a pair of simple, nonzero imaginary eigenvalues. The greatest contribution in this area lies perhaps upon Ott, *et al.* [21] who have shown that small time dependent perturbations can be effectively used to convert a chaotic attractor to any of a large number of possible attracting time periodic motions. The method utilizes delay coordinate embedding and can be used on experimental situations where knowledge of the system dynamics are not available.

Like Ott, *et al.* [21], Singer, *et al.* [26] in their paper through experimental and simulation results have also shown how a simple low energy feedback controller like on-off controller can stabilize a chaotic system. The developed control action is based on the deviation of the vertical temperature difference from the equilibrium point which stabilizes the states to their equilibrium points. Wang and Abed [30] have also suggested a feedback control synthesis technique for relocating and ensuring stability of bifurcated limit cycles to a convective loop problem. They showed that stability can be ensured in several different ways, one of which is replacing the chaotic behavior by its equilibrium or, replacing the limit cycle with a relatively small amplitude limit cycle. For this purpose they have used a small washout filter to delay and to extinguish chaos in the model and developed linear and nonlinear feedback control law. Recently Bošković and Krstić [5] have investigated a thermal convective loop and developed a nonlinear feedback control law to achieve global stability using boundary control of velocity and temperature. The nonlinear control law is developed based on the discretized model of nonlinear PDE in space using the finite difference method and resultant high order system of coupled nonlinear ODE's.

In this paper, we will apply linear and nonlinear control law and investigate their performance among each other. For this case it is found that proportional state feedback control law with setpoint tracking $(u = -k(x_3 - x_{3e}))$ gives the best result where the proportional constant can be found out by stability analysis of linearized model or LQR. A nonlinear control law similar to the previous structure $(u = -(x_1 + x_2)(x_3 - x_{3e}))$ gives better result in terms of quick stabilization of the states to the desired setpoints (here, the desired setpoints are the equilibrium points). This controller is equivalent to taking $-k(x) = x_1 + x_2$ and depends a lot on the initial values of the states at the time when the control law is applied. Nonlinear control law based on backstepping method is also developed here which stabilizes the system but can not bring the states to the desired equilibrium points. Other advanced control law like Nonlinear Model Predictive Control (NMPC) stabilizes the system very efficiently compared to Linear MPC. Results from these simulations are also included for comparision.

2 Process Description

The presence of chaos is very common in physical systems. It is desirable to reduce the chaos so that system performance can be improved. We can do it in two ways (Ott, *et al.* [21]). First make some large costly alteration to the system which completely changes its dynamics to the desired dynamic behavior. Second improve performance by making small time dependant perturbations in an accessible parameters. In this case chaotic system holds advantage over other systems in that it can be made stable to any existing orbit without much effort or alteration of the system.

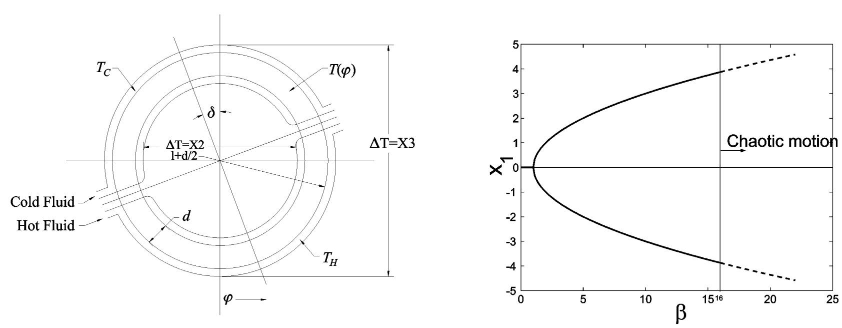
2.1 Definition of chaos

There is no universally agreed definition of chaos. Wang and Abed [30] defined chaos as "an irregular, seemingly random, dynamic behavior of a deterministic system displaying extreme sensitivity to initial condition" which most people accept as working definition. It has two main parts: 1) the system is deterministic meaning that the system has no irregular input; the chaotic behavior solely comes from the highly nonlinear nature of the system, and 2) the system is extremely sensitive to the initial conditions. Usually this kind of system has different stable region and can show periodic jump among these states depending on the external condition.

2.2 Description of thermal convection loop model

Natural convection in a closed loop system consists of a heat source and several sinks positioned above the source. The source and sink are connected by pipe forming at least one closed loop system. The heat is transported from the source to the sink by circulating fluid inside the loop. Unlike the forced convection (as in refrigerator), the heat is transported by natural convection only. Solar heating system and nuclear reactors are example of such system. For a detailed review of closed loop natural convection system, (see [9, 11, 19, 32]).

Figure 2.1(a) shows a schematic diagram of the system. The sink and the source are connected by a circular loop filled with an incompressible fluid which works as a transporting media of heat from source to sink. The cross section, A of this loop is circular and constant. The lower semicircle of the loop is heated by a hot fluid at a temperature T_H and the upper semicircle is cooled by a coolant at a temperature T_C. The cooling and heating zones are tilted by an angle δ from the symmetric position. If the temperature difference $\Delta T = T_H - T_C$ is increased, the fluid is at first at no motion state. During this stage, heat is transported by conduction only. As the heating rate is increased, a steady state convection arises either in clockwise or counter-clockwise direction. If heating rate is further increased, the steady state convection becomes unstable and shows oscillatory and chaotic motion.



(a) Schematic description of natural convection loop

(b) Bifurcation diagram of the state x_2 vs. β

Figure 2.1. a) System, b) Bifurcation of the system depends on the heating rate, β.

2.3 First order model development

Assuming $d << l$, material and energy balance (see [9] for detail derivation) leads to the following equations,

$$\begin{aligned}
&\frac{\partial u}{l\partial\varphi} = 0,\\
&\rho_0\frac{\partial u}{\partial t} = -\frac{\partial p}{l\partial\varphi} - \rho(T)g\sin(\varphi) - f_w, \qquad (2.3.1)\\
&\rho_0 c_p\left\{\frac{\partial T}{\partial t} + u\frac{\partial T}{l\partial\varphi}\right\} - \lambda\frac{\partial^2 T}{l^2\partial\varphi^2} = h_w[T_w(\varphi(T) - T] + q_w(\varphi),
\end{aligned}$$

where

$$f_w = \frac{1}{2}\rho_0 f_{w0} u, \qquad (2.3.2a)$$

$$T(\varphi,t) = T_0(t) + \sum_{n=1}^{\infty}\{S_n(t)\sin(n\varphi) + C_n(t)\cos(n\varphi)\}, \qquad (2.3.2b)$$

$$\begin{aligned}
Q(\varphi) &= Q_0 + \sum_{n=1}^{\infty}\{Q_n\sin(n\varphi) + R_n\cos(n\varphi)\},\\
&= \frac{1}{\rho_0 c_p}\, l\{h_w T_w(\varphi) + q_w(\varphi)\}. \qquad (2.3.2c)
\end{aligned}$$

Introducing the dimensionless variables as follows,

$$\mathit{Time},\ t' = \frac{h_{w0}}{\rho_0 c_p}\, t, \qquad (2.3.3a)$$

$$x_1 = \frac{\rho_0 c_p}{l h_{w0}}\, u, \qquad (2.3.3b)$$

$$x_2 = \frac{\rho_0 c_p}{h_{w0}}\frac{\gamma g}{f_{w0} l}\, S_1, \qquad (2.3.3c)$$

$$x_3 = \frac{\rho_0 c_p}{h_{w0}}\frac{\gamma g}{f_{w0} l}\left\{\frac{\rho_0 c_p}{h_{w0}}\, R_1 - C1\right\}, \qquad (2.3.3d)$$

where

$$\begin{aligned}
\gamma &= \text{coefficient of thermal expansion},\\
c_p &= \text{specific heat},\\
\rho_0 &= \text{reference density},\\
\lambda &= \text{heat conductivity}, \qquad (2.3.4)\\
g &= \text{acceleration due to gravity},\\
h_w &= \text{heat transfer coefficient} = h_{w0}\left\{1 + K|x_1|^{1/3}\right\}.
\end{aligned}$$

Neglecting higher order terms in equation (2.3.2),

$$T(\varphi,t) = T_0(t) + S_1(t)\sin(\varphi) + C_1(t)\cos(\varphi),$$

$$Q(\varphi) = Q_0 + Q_1\sin(\varphi) + R_1\cos(\varphi) = \frac{1}{\rho_0 c_p}\, l\{h_w T_w(\varphi) + q_w(\varphi)\}$$

and assuming that the heat transfer coefficient h_w is constant *i.e.*, $K = 0$, the parameters S_1, C_1, R_1 are found to be

$$C_1(t) = \frac{T(0^\circ,t) - T(180^\circ,t)}{2}, \tag{2.3.5a}$$

$$S_1(t) = \frac{T(90^\circ,t) - T(270^\circ,t)}{2}, \tag{2.3.5b}$$

$$R_1 = \frac{h_{w0}}{\rho_0 c_p}\frac{T_H - T_C}{2}, \tag{2.3.5c}$$

where T_H and T_C are the temperature of the heating and cooling zone respectively. Further assuming that there is no tilting between the heating and cooling zone *i.e.*, $\delta = 0$ and there is negligible heat transfer in the direction of the tube axis, the system can be described by the following set of ordinary differential equations:

$$\begin{aligned} \dot{x}_1 &= \alpha\left(-x_1 + x_2\right), \\ \dot{x}_2 &= -x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - x_3 - \beta, \end{aligned} \tag{2.3.6}$$

where,

$$\alpha = \frac{\rho_0 c_p}{h_{w0}}\frac{f_{w0}}{2}, \tag{2.3.7}$$

$$\beta = \frac{\gamma g}{f_{w0} l}\left(\frac{\rho_0 c_p}{h_{w0}}\right)^2 R_1 = \frac{\gamma g}{f_{w0} l}\frac{\rho_0 c_p}{h_{w0}}\frac{T_H - T_C}{2}. \tag{2.3.8}$$

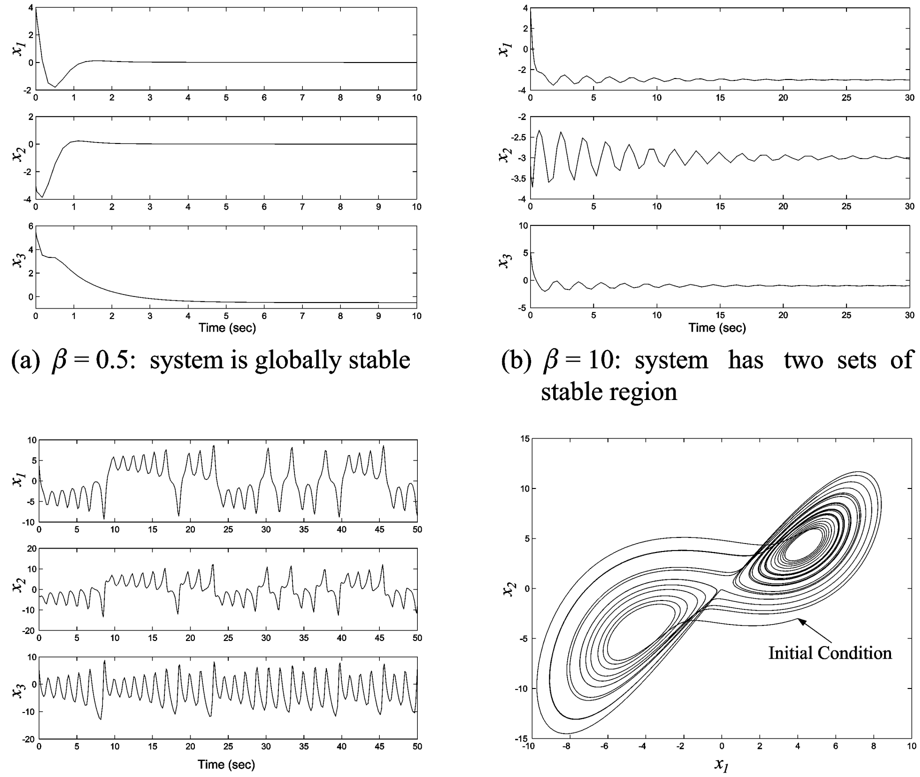
Here, α is comparable to the Prandtl number and β is the heating rate which is directly proportional to the temperature difference ΔT and is equivalent to the Rayleigh number. The states x_1, x_2 and x_3 are proportional to the average cross sectional velocity inside the loop, temperature difference along the horizontal direction and temperature difference along the vertical direction. All of the states are measurable and hence available for computation.

2.4 Open loop response

In the equation (2.3.6), α stands for Prandtl number and can be assumed constant. The other parameter β stands for Rayleigh number which is proportional to the heating rate. At equilibrium, $\dot{x}_i$'s are zero. Putting these values in equation (2.3.6) and solving them the following two cases arise:

Case a: $\beta \le 1$, $x_{1e} = x_{2e} = 0$ and $x_{3e} = -\beta$

In this case, the states are globally stable and converge to the equilibrium points



(a) $\beta = 0.5$: system is globally stable

(b) $\beta = 10$: system has two sets of stable region

(c) $\beta = 20$: system is chaotic

(d) x_2 vs. x_1 for $\beta = 20$

Figure 2.2. Open loop response of the system for different β's; Initial conditions of the state variables are $x_{10} = 4.0$, $x_{20} = -3.0$ and $x_{30} = 5.5$, taken arbitrarily.

irrespective of the initial conditions. The state x_1 i.e., average cross-sectional velocity of the fluid is zero at equilibrium which means that the fluid is at no motion state in this case and heat is transported from the source to the sink by conduction only.

Case b: $\beta > 1$, $x_{1e} = x_{2e} = \pm\sqrt{\beta - 1}$ and $x_{3e} = -1$

In this case, the states have two equilibrium points. The fluid average velocity may be clockwise or counter-clockwise. Heat is transported at this stage by convection. Depending on the value of the parameter β the system may show stable or unstable and chaotic behavior. This is because as heating rate is increased fluid velocity is also increased and at higher value of β it becomes locally unstable and jumps from one equilibrium point to another from time to time making the system chaotic.

The different cases are depicted in Figure 2.1(b). The open loop response for different β are given in Figures 2.2(a–c). These figures show how the system responses to the same initial condition with different β.

From Figure 2.2(d), it is obvious that at chaos the system has two different orbits. Solution of x_1 and x_2 remains in this orbit but never becomes stable to any single equilibrium point (see, Figure 2.2(c)). From the bifurcation diagram it is obvious that at some critical value of β the system starts showing chaotic behavior. To find out this critical value we need to do stability analysis of the open loop system:

2.4.1 Stability analysis of the linearized open loop system for $\boldsymbol{\beta > 1}$

If there is a nonlinear equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

then linearization of the above equation around the equilibrium point leads to the following equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

is evaluated at equilibrium points. For the system to be stable all the eigenvalues of the matrix $\mathbf{A}$ must have negative real parts. For the convective loop described by equation (2.3.6), the linearized equation becomes

$$\dot{x} = Ax,$$

where

$$A = \begin{bmatrix} -\alpha & \alpha & 0 \\ -x_3 & -1 & -x_1 \\ x_2 & x_1 & -1 \end{bmatrix}_{evaluated\ at\ equilibrium} = \begin{bmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & -\sqrt{\beta-1} \\ \sqrt{\beta-1} & \sqrt{\beta-1} & -1 \end{bmatrix}.$$

Here, the positive equilibrium values of x_1 and x_2 are taken for analysis with $\beta > 1$. Making the real parts of the eigenvalues of the A matrix equal to zero leads to the following relation[1]:

$$\beta_{crit} = \frac{\alpha(\alpha+4)}{\alpha-2}.$$

So, if β is greater than this critical value then the system will show chaotic behavior. For example for $\alpha = 4$, the critical value of β is 16 over which the system is chaotic. Notice that the critical value is found by linearization of the nonlinear system. So, in practice the transition from stable to chaotic behavior will not happen exactly at this critical value of β. In fact, there is a transition region where the system actually is semi-chaotic meaning that it shows chaotic response initially and after some period the oscillation decays resulting into settling down of the response to one of its stable equilibrium points.

3 Controlling Chaos

Unlike linear systems, control of nonlinear and chaotic system is difficult due to the heavy computational duty which makes nonlinear control not feasible. Also, when the main

[1] All the eigenvalue analysis is done by using Maple V.

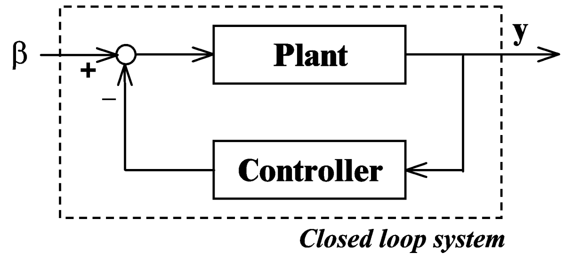


Figure 3.1. Schematic diagram of closed loop system.

target is to keep the operating point steady, it often suffices to linearize the nonlinear system around the operating point and apply linear control law.

Whenever any feedback control action is taken, the open loop system is changed to a desired closed loop stable system (Figure 3.1). In the following sections several methods for controlling chaos in the convective loop is discussed. For a review of different control strategy of chaotic system and bifurcation control see [21, 26, 30, 3, 16, 1, 2, 14]. There are several works on linear feedback control of chaotic system (see [30]). For different well established nonlinear controller design technique see [13, 17, 25]. The main theme is to set the control action to be a function of some observable state so that it can be calculated and implemented. In case of convective loop, the parameter β (heating rate) is proportional to the temperature difference in the vertical direction which is the state x_3. So the control action , u in the convective loop system is taken as the deviation of heating rate from its nominal value

$$\begin{aligned}\dot{x}_1 &= \alpha(-x_1 + x_2),\\ \dot{x}_2 &= -x_2 - x_1 x_3,\\ \dot{x}_3 &= x_1 x_2 - x_3 \qquad \underbrace{-\beta + u}_{\text{Total heating rate, } U} \qquad .\end{aligned} \tag{3.0.1}$$

3.1 Proportional controller

For the convective loop system the control action, u in equation (3.0.1) is taken as proportional to the state, x_3 i.e.,

$$u = -kx_3.$$

Stability analysis of the closed loop system leads to the following relationship for the linear system

$$\beta = \frac{\alpha(4 + \alpha + 5k + \alpha k + k^2)}{\alpha - k - 2}.$$

This means that if the system were linear for $\alpha = 4$, $k = 2$ would be sufficient for stabilizing the system for any value of β. Since the system is highly nonlinear, feedback gain $k = 2$ may not suffice for higher values of β. However for small β, small negative feedback gain suffices to make the system steady [see, Figure 3.2]. However in this case the system equilibrium point is not the same as the open loop system. The equilibrium point of the average cross-sectional velocity is determined by $\pm\sqrt{\beta - k - 1}$ and the final fluid velocity stabilizes at this new equilibrium point instead of open loop equilibrium point $x_{2e} = \pm\sqrt{\beta - 1}$. The heating rate does not remain the same as β instead it

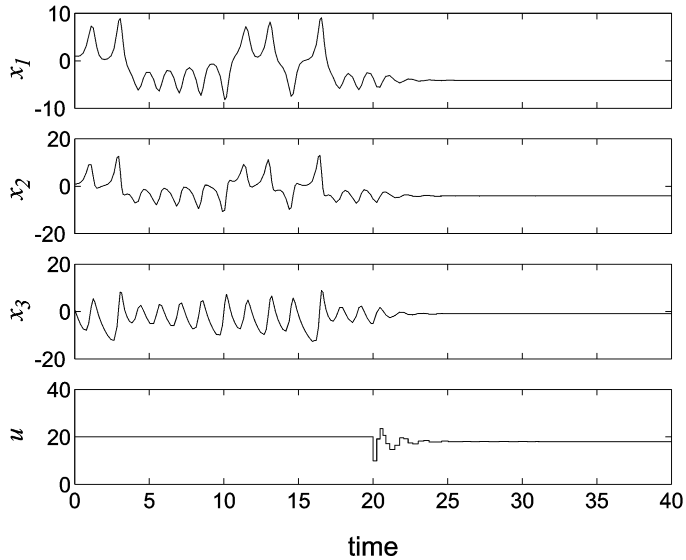


Figure 3.2. Closed loop response with proportional controller for $k = 2$ for system with $\beta = 20$. The control is applied at time, $t = 20$.

becomes $\beta - u$ where u is a constant value at steady state. This actually changes the heating rate to some extent.

3.2 Setpoint tracking

This is same as the proportional controller but the control law is defined by

$$u = -k(x_3 - x_{3e}), \tag{3.2.1}$$

where x_{3e} is the open loop equilibrium point of the state, x_3. The closed loop equilibrium point is same as the open loop equilibria and the steady state value of the control action, u is zero. This is given in the Figure 3.3.

3.3 Nonlinear control law: Lyapunov stability criterion

The main difficulties with designing a controller based on Lyapunov stability criterion is in choosing the energy function. For this case the best candidate for the energy function should be of the form:

$$V(x) = mx_3^2 + nx_1^2, \quad m, n > 0, \tag{3.3.1}$$

because of the fact that heating rate is proportional to x_3 (vertical temperature difference) and energy loss due to friction is proportional to x_1^2. Here m and n are two proportional constants which depends on the parameters used during conversion from PDE to ODE of the system model. But this energy function is positive semi-definite. Nevertheless using this "wrong" energy function, and Taylor series approximation to approximate $\sqrt{\beta - u + 1} = f(u) \approx a + bu$, where a and b are linearization constants and truncating constant terms in the final control law which accounts for lowering the heating rate

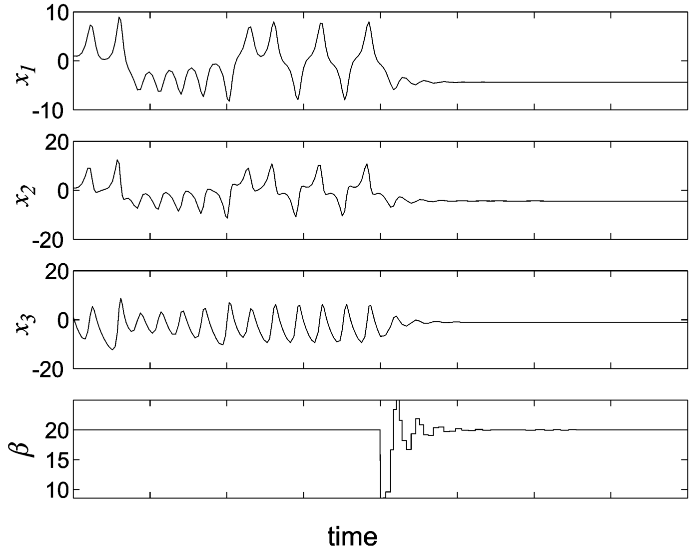


Figure 3.3. Feed back control with reference point tracking; here, $k = 2$ for system with $\beta = 20$.

(similar things are discussed in Section 3.4), we can finally come up with the following control law[2]:

$$u = (x_1 + x_2)(x_3 + 1). \tag{3.3.2}$$

Surprisingly this control gives better stabilizing effect than that developed by backstepping method as will be discussed next. But it depends greatly on the initial condition. Simulation result is given in Figure 3.4.

As we said earlier that this control law is based on the "wrong" energy function $V(x)$. So why does it work then? The answer is that with so many assumption during the development of the control law, the control law u is not associated with the positive semi-definite energy function any more. Rather it belongs to some other unknown energy function. If we take an energy function of the form $V(z) = \frac{1}{2}(z_1^2 + z_2^2 + z_3^2)$, where z_i's are the transformed states for $\dot{z} = f(z)$ with equilibrium points at the origin, it can be shown that $\dot{V}$ is negative provided that the open loop system is bounded (which is true for this case without any external excitation even in unstable chaotic region).

3.4 Nonlinear control law: Back stepping method

The system

$$\dot{x}_1 = \alpha(-x_1 + x_2), \tag{3.4.1}$$

$$\dot{x}_2 = -x_2 - x_1 x_3, \tag{3.4.2}$$

$$\dot{x}_3 = x_1 x_2 - x_3 - \beta + u, \tag{3.4.2}$$

[2]Detailed derivation is omitted here due to page constraints.

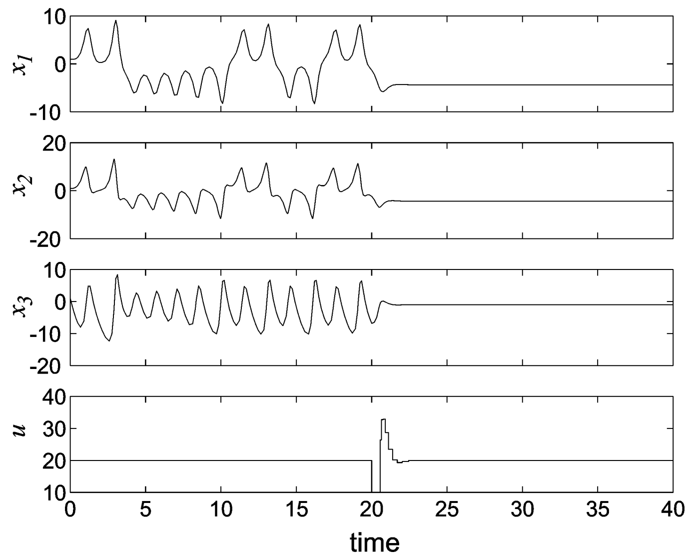


Figure 3.4. Feed back control for system with $\beta = 20$: Lyapunov stability criterion.

can be written in the following strict feedback system form:

$$\dot{x}_1 = f(x_1) + g(x_1)\xi_1, \tag{3.4.4}$$

$$\dot{\xi}_1 = f_1(x_1, \xi_1) + g_1(x_1, \xi_1)\xi_2, \tag{3.4.5}$$

$$\dot{\xi}_2 = f_2(x_1, \xi_1, \xi_2) + g_2(x_1, \xi_1, \xi_2)u, \tag{3.4.6}$$

where

$$\begin{aligned} &f(x_1) = -\alpha x_1, && g(x_1) = \alpha, && \xi_1 = x_2, \\ &f_1(x_1, \xi_1) = -x_2, && g_1(x_1, \xi_1) = -x_1, && \xi_2 = x_3, \\ &f_2(x_1, \xi_1, \xi_2) = x_1 x_2 - x_3 - \beta, && g_2(x_1, \xi_1, \xi_2) = 1. && \end{aligned}$$

The first target is to stabilize the x_1 sub-system defined by equation (3.4.4). Let the Lyapunov function be $V_1 = \frac{1}{2}x_1^2$. Then

$$\dot{v}_1 = \frac{\partial V_1}{\partial x_1}\dot{x}_1 = x_1(-\alpha x_1 + \alpha x_2) = -\alpha x_1^2 + \alpha x_1 x_2.$$

Let us take the control law to be

$$x_2 = \phi(x_1) = -a x_1, \qquad a \in \Re^+. \tag{3.4.7}$$

We have included an unknown parameter a in the control law $\phi(x_1)$ which we will see in the later section increases degree of freedom and will help removing singularity in the final control law. For better flexibility and more degree of freedom we could also take the following control law instead:

$$x_2 = \phi(x_1) = -a x_1^{2b+1}, \quad a > 0, \quad b \geq 0. \tag{3.4.8}$$

But the addition of parameter b increases complexity in the final control law and so we assumed $b = 0$ for now. If necessary we can always come back and assume it to be nonzero.

With this control law [equation (3.4.7)] the sub-system equation (3.4.4) becomes:

$$\dot{x}_1 = -(a+1)\alpha x_1 \tag{3.4.9}$$

and the derivative of the energy function V becomes:

$$\dot{V} = -(a+1)\alpha x_1^2, \qquad a,\, \alpha > 0 \tag{3.4.10}$$

which is negative definite. Hence the sub-system is globally asymptotically stable. The energy function for the next sub-system equation (3.4.5) can be written as:

$$V_2 = V_1 + \frac{1}{2}[\xi_1 - \phi]^2 = \frac{1}{2}x_1^2 + \frac{1}{2}[x_2 + ax_1]^2. \tag{3.4.11}$$

Then the control law that makes the derivative of V_2 negative definite can be expressed as

$$\begin{aligned} x_3 = \phi_1 &= \frac{1}{g_1}\left[\frac{\partial\phi}{\partial x_1}(f + g\xi_1) - \frac{\partial V_1}{\partial x_1}g - k_1(\xi_1 - \phi) - f_1\right] \\ &= -\frac{1}{x_1}\left[-a(-\alpha x_1 + \alpha x_2) - x_1\alpha - k_1(x_2 + ax_1) + x_2\right] \\ &= -(a\alpha - \alpha - k_1 a) + (a\alpha + k_1 - 1)\frac{x_2}{x_1}, \qquad k_1 > 0. \end{aligned} \tag{3.4.12}$$

Similarly the final control law can be written as:

$$\begin{aligned} u &= \frac{1}{g_2}\left[\frac{\partial\phi_1}{\partial x_1}(f + g\xi_1) + \frac{\partial\phi_1}{\partial\xi_1}(f_1 + g_1\xi_2) - \frac{\partial V_2}{\partial\xi_1}g_1 - k_2(\xi_2 - \phi_1) - f_2\right] \\ &= -(a\alpha + k_1 - 1)\frac{x_2}{x_1^2}(-\alpha x_1 + \alpha x_2) + \frac{a\alpha + k_1 - 1}{x_1}(-x_2 - x_1x_3) \\ &\quad - (x_2 + ax_1)(-x_1) - k_2\left(x_3 + (a\alpha - \alpha - k_1 a) - (a\alpha + k_1 - 1)\frac{x_2}{x_1}\right) \\ &\quad - (x_1x_2 - x_3 - \beta) \\ &= \underbrace{(a\alpha + k_1 - 1)\left(\alpha + k_2 - 1 - \alpha\frac{x_2}{x_1}\right)\frac{x_2}{x_1}}_{\text{Singularity}} \\ &\quad + (ax_1^2 - (k_2 - 1)x_3 - k_2(a\alpha - \alpha - k_1) + \beta). \end{aligned} \tag{3.4.13}$$

The above control law is not feasible in terms of implementation due to the first term which has x_1 in the denominator. So, whenever x_1 goes near zero the control action becomes very large. For example, with $\alpha = 4$, $a = 1$, $k_1 = 1$ and $k_2 = 2$, the control action rises to infinity making the system unstable. To evade this problem we have two options in hand:

1. Switching to an alternative control law [e.g., $u = -k(x_3 + 1)$] that can stabilize the system to the desired setpoint whenever control action calculated from the control law [equation (3.4.13)] exceeds a predefined boundary.
2. Choose the parameters a and k_1 in such a way so that the term containing singularity vanishes.

Of the two options, the first option will always work as long as the alternative control law works. For the second case we need to set the parameter values a and k_1 so that the terms containing x_1 in the denominator vanishes away. For this purpose set

$$a\alpha - k_1 + 1 = 0 \Rightarrow k_1 = 1 - a\alpha. \tag{3.4.14}$$

Since by assumption k_1 should be a positive number, choose

$$a = \frac{1}{n\alpha}, \qquad n > 1 \tag{3.4.15}$$

which gives the final control law to be:

$$\begin{aligned} u &= ax_1^2 - (k_2 - 1)x_3 - k_2(a\alpha - \alpha - k_1) + \beta \\ \Rightarrow u &= \underbrace{\frac{1}{n\alpha}\,x_1^2}_{\text{Nonlinear Part}} \underbrace{-(k_2-1)x_3}_{\text{Linear Part}} \underbrace{-k_2\left(\frac{2}{n} - \alpha - 1\right) + \beta}_{\text{Constant Part}}. \end{aligned} \tag{3.4.16}$$

The final control law defined by equation (3.4.16) has three parts: Nonlinear, Linear and Constant terms. If we take $k_2 = 1$, the linear term vanishes away. From the simulation result it is found that presence of this linear term enhances quick stability of the system to the desired equilibrium points. So, it is better to choose

$$k_2 > 1. \tag{3.4.17}$$

The constant term however stabilizes the system in a slightly different manner. What it does is that it reduces the heating rate β to the region where the overall open loop system is stable. Since we want to keep the system in the region where the open loop system is unstable and want to diminish the chaos, the constant term in the control law does not serve our purpose. So, removing the constant part we have the following control law which is actually perturbation around the nominal heating rate:

$$u = \frac{1}{n\alpha}\,x_1^2 - (k_2 - 1)x_3, \qquad n,\, k_2 > 1. \tag{3.4.18}$$

Notice that heating rate is proportional to x_3. Also x_1 denotes fluid velocity inside the convective loop and hence energy loss due to the fluid flow is proportional to x_1^2 $\left[h_L = f\frac{LV^2}{2gD}\right]$. So, the control law is actually an energy term which makes it physically understandable. But with this truncated control law the question that immediately comes into the mind is that "Does this truncated control law still makes the system stable?". To answer this question we have to analyze the stability of the closed loop system with the truncated control law defined by equation (3.4.18). The energy function for the closed loop system with the full control law [equation(3.4.16)] is given by:

$$
\begin{aligned}
V_3 &= V_2 + \frac{1}{2}[\xi_2 - \phi_1]^2 \\
&= \frac{1}{2}x_1^2 + \frac{1}{2}[x_2 + ax_1]^2 + \frac{1}{2}[x_3 + a\alpha - \alpha - k_1 a]^2 \\
\Rightarrow \dot{V}_3 &= \big[(1+a^2)x_1 + ax_2 \quad ax_1 + x_2 \quad x_3 + a\alpha - \alpha - k_1 a\big] \\
&\qquad \times \begin{pmatrix} -\alpha x_1 + \alpha x_2 \\ -x_2 - x_1 x_3 \\ x_1 x_2 - x_3 - \beta + u \end{pmatrix} \\
&= \underbrace{-(\alpha + a\alpha + k_1 a^2)x_1^2 - k_1 x_2^2 - k_2 x_3^2}_{\text{negative}} \\
&\qquad + \underbrace{(k_1 k_2 + k_2\alpha + k_1 k_2 a - k_2 - \beta)x_3 - 2k_1 a x_1 x_2}_{\text{depends on the sign of } x_1 \text{ and } x_2} \\
&\qquad + \underbrace{(1 - k_1 - \alpha + k_1 a)}_{\text{negative}}\beta.
\end{aligned}
\tag{3.4.19}
$$

Here $\dot{V}_3$ has three terms as shown in equation (3.4.19): a negative quadratic term consisting of x_1^2, x_2^2 and x_3^2, a term containing x_1x_2 and x_3 which depends on the sign of the variable and a constant term. In the constant term $1 - k_1 = \frac{1}{n} \in (0,1)$ and $k_1 a = \Big(1 - \frac{1}{n}\Big)\frac{1}{n\alpha} \in (0,1)$. Usually the parameter α has value 4, which makes the term $(1 - k_1 - \alpha + k_1 a)$ negative. Nothing can be said about the other two terms containing x_1x_2 and x_3. But if we take a look at the simulation result it is found that except near zero x_1 and x_2 have the same sign making $-2k_1 a x_1 x_2$ negative and even in the extreme conditions when x_3 is negative making $(k_1k_2 + k_2\alpha + k_1k_2a - k_2 - \beta)x_3$ positive but smaller than the other negative terms. This is due to the fact that though the system shows chaotic behavior the states are always confined in a boundary. Hence the equilibrium points of the system are locally stable with this control action defined by equation (3.4.18). For $\alpha = 4$, $k_2 = 3$ and $n = 2$ $[k_1 = 1 - 1/n = 0.5,\ a = 1/n\alpha = 1/8]$, the control law becomes:

$$u = \frac{1}{8}\,x_1^2 - 2x_3. \tag{3.4.20}$$

With the same initial condition as before the response of the controlled system is given in Figure 3.5.

3.5 Model predictive control (MPC)

In model predictive control[3], a set of future control action including the current control action is calculated based on the model of the system. That is why it is sometimes called the model based predictive control. The model can be linear or non-linear. The main purpose is to minimize an objective function (which is often a quadratic function of the states and inputs) subject to the model equation and some physical constraints. For

[3]For a review of different model predictive control technique see [4, 6, 12, 24, 31, 23, 20, 7, 8, 22, 28, 18].

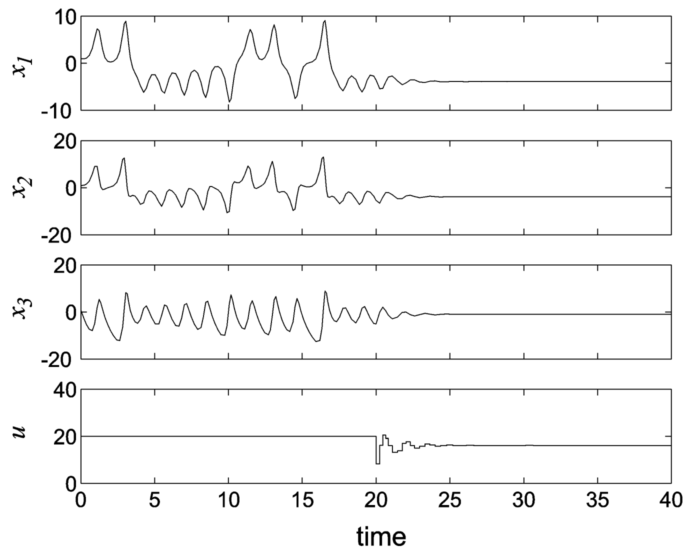


Figure 3.5. Controlled system for $\beta = 20$: Back Stepping method.

linear time invariant model this problem can be solved to give a control law as a function of current output and past input. For nonlinear case there is usually no explicit solution of the minimization problem and one is forced to solve it numerically.

3.5.1 Nonlinear model predictive control (NMPC)

The objective of all control problem is to minimize the difference of output, y with the desired value[4], y_{ref}. One such objective function is

$$\begin{aligned}\min_{u,x_1,\dots,x_n} J = \sum_{i=1}^{n} \gamma_i [\mathbf{x}_i(t) - \mathbf{x}_{i,ref}]^T [\mathbf{x}_i(t) - \mathbf{x}_{i,ref}] \\ + \gamma_u [u(t) - u_{ref}]^T [u(t) - u_{ref}] + \gamma_{\Delta u} \Delta\mathbf{u}^T \Delta\mathbf{u}\end{aligned} \tag{3.5.1}$$

subject to,

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), u, t), \tag{3.5.2}$$

where γ's are penalty functions on x_i's and for the convective loop system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} -px_1 + px_2 \\ -x_1x_3 - x_2 \\ x_1x_2 - x_3 - R + u \end{pmatrix}. \tag{3.5.3}$$

[4]In convective loop problem, the desired reference points are the equilibrium points.

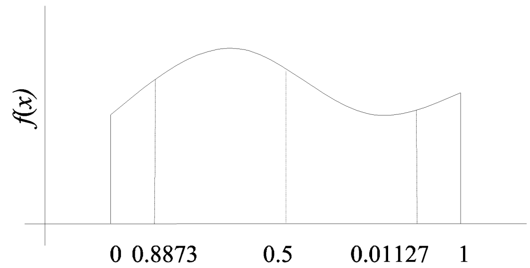


Figure 3.6. Approximation of a function by three point collocation on one step ahead prediction.

This minimization problem (3.5.1) has continuous nonlinear model constraint (3.5.2). To solve this problem the continuous model constraint needs to be discretized. Any finite element method can be used for this purpose:

1. using the conventional numerical method to predict future values e.g., Runge-Kutta 23 method etc.
2. by converting dynamic constraints to algebraic constraints using
   - Orthogonal Collocation Method;
   - Galerkin method;
   - Flatness based technique etc.

Of these methods only orthogonal collocation method will be applied on the convective loop model to control chaos.

3.5.2 Orthogonal collocation method, prediction horizon 1

In the orthogonal collocation method, any function can be approximated by an interpolating polynomials with nodes located at the roots of a set of orthogonal polynomials (see [10, 29, 15, 6, 27] for detail), i.e.,

$$y(x) = \sum_{i=1}^{N+2} b_i P_{i-1}(x), \tag{3.5.4}$$

where

$$P_m(x) = \sum_{j=0}^{m} c_j x^j \tag{3.5.5}$$

is the m-th polynomial such that

$$\int_a^b W(x) P_k(x) P_m(x)\, dx = 0, \qquad k = 0, 1, 2, \dots, m-1.$$

Here, the polynomial m has m-roots in the interval $[a, b]$ and thus users do not need to pick the collocation points arbitrarily. This has advantage over the conventional collocation method where there is a good chance of poor choice of these nodes by inexperienced users and thus bad approximation of the function. Typically, the integration range is taken as 0 to 1 to generalize the problem. Equations (3.5.4) and (3.5.5) can be combined to give

$$y(x_j) = \sum_{i=1}^{N+2} d_i x_j^{i-1}. \tag{3.5.6}$$

Derivatives can also be approximated by orthogonal polynomials and finally we get the following forms

$$\frac{dy}{dx}(x_j) = \sum_{i=1}^{N+2} d_i (i-1) x_j^{i-2}, \tag{3.5.7}$$

$$\frac{d^2y}{dx^2}(x_j) = \sum_{i=1}^{N+2} d_i (i-1)(i-2) x_j^{i-3}. \tag{3.5.8}$$

In matrix notation,

$$y = \mathbf{Q}d, \qquad \frac{dy}{dx} = \mathbf{C}d, \qquad \frac{d^2y}{dx^2} = \mathbf{D}d,$$

where

$$\begin{aligned} Q_{ji} &= x_j^{i-1}, \\ C_{ji} &= (i-1)x_j^{i-2}, \\ D_{ji} &= (i-1)(i-2)x_j^{i-3}. \end{aligned} \tag{3.5.9}$$

Therefore,

$$\frac{dy}{dx} = \mathbf{C}Q^{-1}y \equiv \mathbf{A}y, \tag{3.5.10}$$

$$\frac{d^2y}{dx^2} = \mathbf{D}Q^{-1}y \equiv \mathbf{B}y. \tag{3.5.11}$$

For our case, the three point collocation method is used. The collocation points and the A-matrices are given in the Table 3.1 and Table 3.2.

Table 3.1. Polynomial roots and the weighting functions.

N	x_j	W_j
1	0.50000 00000	0.66666 66667
2	0.21332 48654 0.78867 51346	0.50000 00000 0.50000 00000
3	0.11270 16654 0.50000 00000 0.88729 83346	0.27777 77778 0.44444 44444 0.27777 77778

Table 3.2. Matrices for orthogonal collocation found from equation (3.5.9).

N	A
1	$\begin{pmatrix} -3 & 4 & -1 \\ -1 & 0 & 1 \\ 1 & -4 & 3 \end{pmatrix}$
2	$\begin{pmatrix} -7 & 8.196 & -2.196 & 1 \\ -2.732 & 1.732 & 1.732 & -07321 \\ 0.7321 & -1.732 & -1.732 & 2.732 \\ -1 & 2.196 & -8.196 & 7 \end{pmatrix}$
3	$\begin{pmatrix} -13 & 14.79 & -2.67 & 1.88 & -1 \\ -5.32 & 3.87 & 2.07 & -1.29 & 0.68 \\ 1.5 & -3.23 & 0 & 3.23 & -1.5 \\ -0.68 & 1.29 & -2.07 & -3.87 & 5.32 \\ 1 & -1.88 & 2.67 & -14.79 & 13 \end{pmatrix}$

The matrices given in Table 3.2 for different collocation points are for interval $[0, 1]$. But the constraint equation (3.5.2) has the interval $[0, \Delta t]$, where Δt is the sampling interval. To account for it the following changes are made to convert the dynamic constraint into algebraic constraint:

$$\begin{aligned} \frac{d\mathbf{x}}{dt'} &= \mathbf{f}(\mathbf{x}, u), \qquad t' \in [0, \Delta t'], \\ \frac{d\mathbf{x}}{dt} &= \mathbf{A}x, \qquad t \in [0, 1], \\ \Rightarrow \mathbf{A}x &= \Delta t' \mathbf{f}(\mathbf{x}, u). \end{aligned} \tag{3.5.12}$$

To take into account the initial condition (i.e., previous control effects) the first row of A-matrix needs to change so that it becomes,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -5.32 & 3.87 & 2.07 & -1.29 & 0.68 \\ 1.5 & -3.23 & 0 & 3.23 & -1.5 \\ -0.68 & 1.29 & -2.07 & -3.87 & 5.32 \\ 1 & -1.88 & 2.67 & -14.79 & 13 \end{bmatrix}. \tag{3.5.13}$$

3.5.3 Orthogonal collocation method, prediction horizon > 1

Similar to the conversion of dynamic constraint to algebraic constraint for prediction horizon one, when prediction horizon is greater than one, same equation (3.5.12) is used

$$\tilde{\mathbf{A}}\mathbf{x} = \Delta t' \tilde{\mathbf{F}}(\tilde{\mathbf{x}}, u), \tag{3.5.14}$$

where

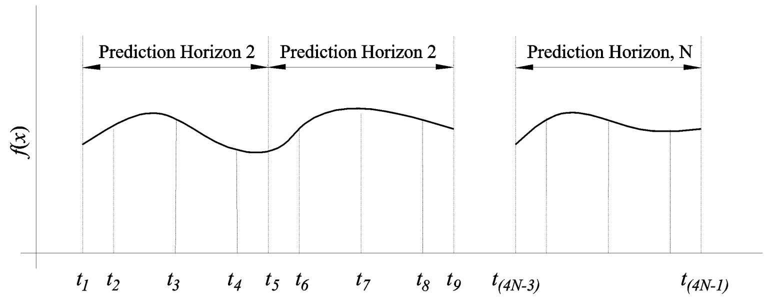


Figure 3.7. Polynomial approximation of a function using three point collocation method with prediction horizon > 1.

$$\tilde{A} = \begin{pmatrix} \boxed{A_0} & & & & & \\ & \boxed{A} & & & & \\ & & \boxed{A} & & & \\ & & & \ddots & & \\ & & & & \boxed{A} & \\ & & & & & \boxed{A} \end{pmatrix}, \tag{3.5.15}$$

$$\tilde{\mathbf{F}}(\tilde{\mathbf{x}}, u) = \begin{pmatrix} x_{init}^T \\ f^T(x_{2*}^T, u_0, t_2) \\ f^T(x_{3*}^T, u_0, t_3) \\ f^T(x_{4*}^T, u_0, t_4) \\ f^T(x_{5*}^T, u_0, t_5) \\ f^T(x_{6*}^T, u_1, t_6) \\ f^T(x_{7*}^T, u_1, t_7) \\ \vdots \\ f^T(x_{(4N)*}^T, u_{N-1}, t_{4N}) \\ f^T(x_{(4N+1)*}^T, u_{N-1}, t_{4N+1}) \end{pmatrix} \tag{3.5.16}$$

and

$$\tilde{x} = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \\ x_{4,1} & x_{4,2} & x_{4,3} \\ \vdots & \vdots & \vdots \\ x_{4N,1} & x_{4N,2} & x_{4N,3} \\ x_{4N+1,1} & x_{4N+1,2} & x_{4N+1,3} \end{pmatrix}. \tag{3.5.17}$$

Here, the first subscript denotes the collocation points in the time interval and the second means state. Using the formulations stated in the equations (3.5.15)–(3.5.17) (see [12] for detail) simulation was run for different prediction and control horizons.

3.5.4 Simulation result for model predictive control

The performance of the controller based on linear or, nonlinear MPC depends on the sampling time, ΔT, and the penalty of the state and input variables in the objective function, γ_i's. The system is highly nonlinear and shows the peculiarity of chaos and bifurcation as is described in the Section 2.4. The fast dynamic system with highly nonlinear behavior makes it difficult to laminarize (or, stabilize) the system using linear model predictive controller. Surprisingly linear MPC with prediction horizon one gives better control than with prediction horizon greater than one for the same sampling time. It is evident from the fact that for this fast chaotic dynamic system a linear model with smaller prediction horizon (Figure 3.8) can track the system better than that of a linear model with large prediction horizon (Figure 3.9).

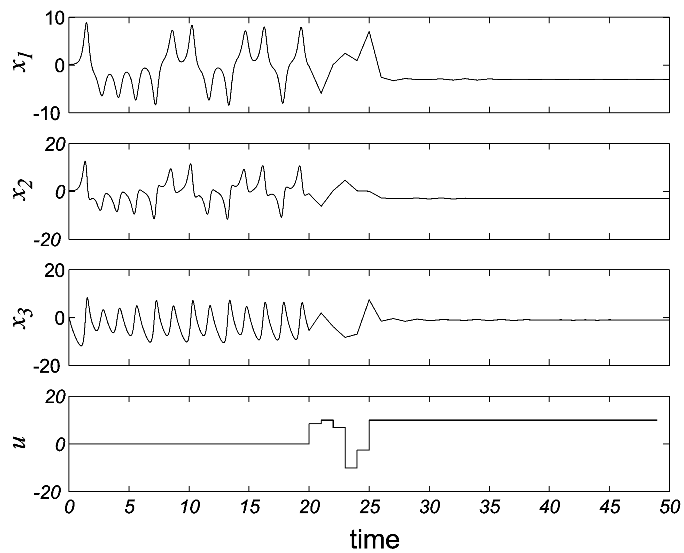


Figure 3.8. Linear MPC; Prediction Horizon = 1, Control Horizon = 1, $\Delta T = 1$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_u = 1$, $\gamma_{\Delta u} = 0$.

In every case however the control action never comes to zero as in the nonlinear model predictive control. The control action takes the higher limit and stays there which in fact in most cases drags the system from the chaotic region to nonchaotic one and thus making the system stable. However nonlinear MPC can stabilize the chaotic system very well. The control action decays rapidly to zero (see Figures 3.10 − 3.12). The time for stabilization depends greatly on the penalty functions on the states and input in the objective function of the optimization problem (3.5.1) as well as the sampling rate[5]. The input limit and its change depend on the constraint used in the minimization problem. Thus in Figure 3.10 due to the input rate constraint limited to 5 control action does not change instantly as in Figure 3.11 or, Figure 3.12 but it takes more time to stabilize the system. So, less stabilization time comes at the cost of larger control energy.

[5]The time interval for implementing control action is also equal to the sampling rate.

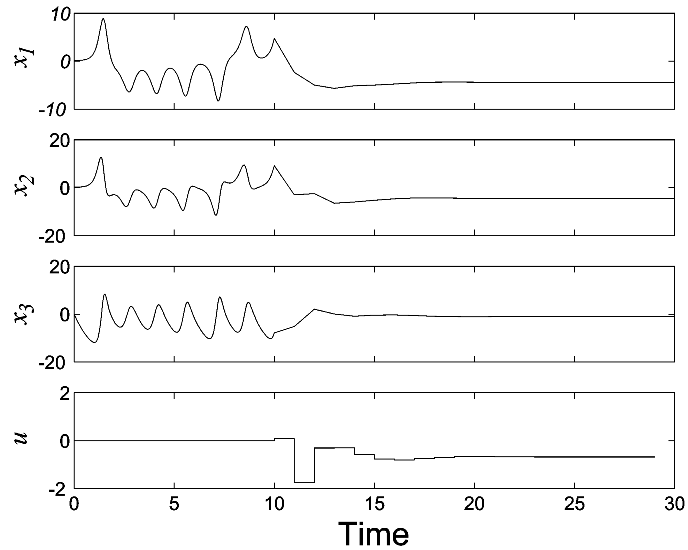


Figure 3.9. Linear MPC; Prediction Horizon = 5, Control Horizon = 2, $\gamma_1 = \gamma_2 = \gamma_3 = 1$, $\gamma_u = \gamma_{\Delta u} = 0$, $\Delta T = 1$.

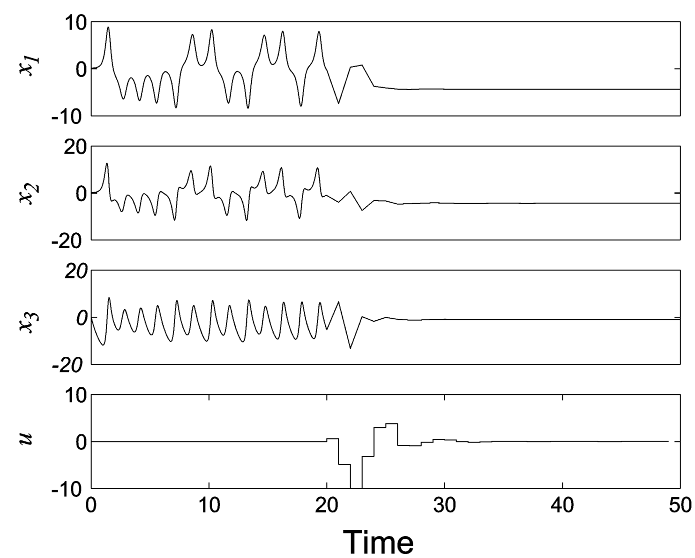


Figure 3.10. Nonlinear MPC; Prediction Horizon = 5, Control Horizon = 2, $\Delta T = 1$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_u = 1$, $\gamma_{\Delta u} = 0$.

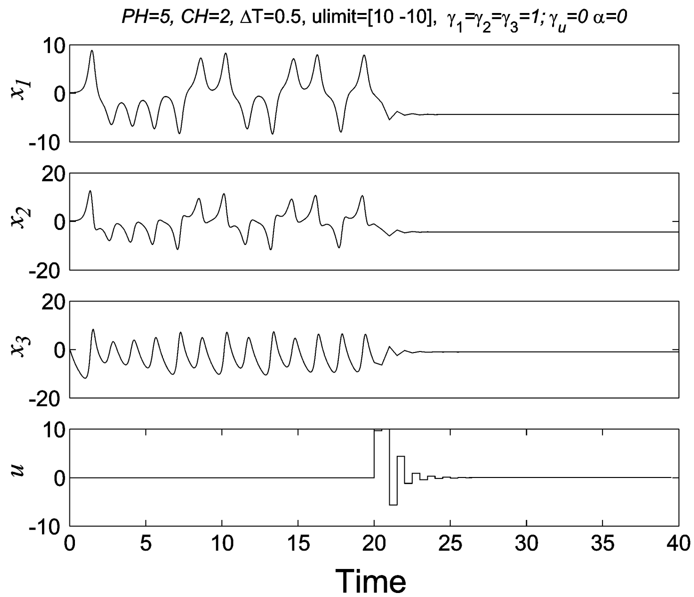


Figure 3.11. Nonlinear MPC; Prediction Horizon = 5, Control Horizon = 2, $\Delta T = 0.5$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_u = 1$, $\gamma_{\Delta u} = 0$.

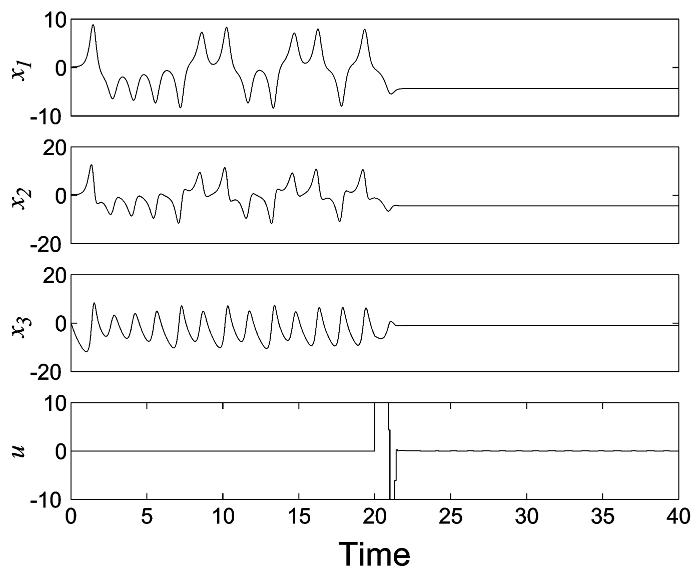


Figure 3.12. Nonlinear MPC; Prediction Horizon = 5, Control Horizon = 2, $\Delta T = 0.1$, $\gamma_1 = \gamma_2 = \gamma_3 = 1$, $\gamma_u = \gamma_{\Delta u} = 0$.

4 Conclusion

For this system Nonlinear Model Predictive Control (NMPC) outperforms other controller in terms of stabilizing time and control action. One of its main disadvantage is high computational time. With the advent of high performance computer however this is not a major problem anymore. Another disadvantage of NMPC is that tuning of the parameters in the objective functions has to be tried through a lot of simulations. Also the model parameters (α and β) need to be correctly identified for the implementation of the controller. Although computational time for Linear MPC is much smaller than Nonlinear MPC, it cannot regulate the system to its desired setpoint unless the sampling time is very small. Among others, linear state feedback controller with setpoint tracking [equation (3.2.1)] and nonlinear controller based on Lyapunov Stability Criterion [equation (3.3.2)] also give better result than others in terms of stabilizing time and movement rate of controller. Of these two, the linear controller is less sensitive to the initial condition *i.e.*, the time when controller is implemented and gives less fluctuation in the control action when measurement noise is present. The nonlinear controller stabilizes the system very quickly but gives a lot of spikes in the control action if noise is present. In this case we assumed that the states are measurable and available for calculation. If any state is not measurable, then a nonlinear observer can be used to estimate the unknown states and calculate the control law. In this case, the performance of the controller will depend on the performance of the observer as well.

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