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Quadratic Stabilization for Nonlinear Perturbed Discrete Time-Delay Systems

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Abstract: This paper discusses the quadratic stability and quadratic stabilization problem for a class of nonlinear perturbed discrete time-delay systems. Necessary and sufficient conditions for quadratic stability are presented via S-procedure technique and linear matrix inequality (LMI). Both static and dynamic output feedback controllers are constructed respectively. Furthermore, necessary and sufficient conditions for quadratic stabilization via static state feedback are constructed in the form of LMI. Finally, the effectiveness of new approach is demonstrated by numerical examples.

Keywords: *Discrete systems; time-delay; nonlinear perturbation; quadratic stabilization; linear matrix inequality.*

Mathematics Subject Classification (2000): 93C55, 93D15; 93C10.

1 Introduction

Quadratic stabilization theory for discrete-time systems has been receiving much attention in the last decade, see [5, 6, 8, 12, 15, 19 − 21]. Quadratic stability means that there exists a deterministic quadratic stable Lyapunov function for all admissible parameter perturbations. The objective of quadratic stabilization is to find a feedback controller such that the closed-loop systems are quadratically stable for all admissible parameter perturbations, where the associated Lyapunov function is quadratic and deterministic. By means of quasiconvex optimization approach, [6] constructs quadratic stabilizing controllers via linear static output feedback and state feedback for discrete-time linear systems with uncertainty. In [15], the robust stabilization for a class of single-input

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discrete-time nonlinear systems is formulated into a convex optimization problem in the form of LMI. Consequently, a static state feedback law is designed to stabilize the plant and to maximize the bound on the nonlinear perturbation terms.

Since time-delay usually results in unsatisfactory performances and is frequently a source of instability, many researchers have paid serious attention to those problems caused by time-delays. Recently several authors have used different approaches such as quadratic Lyapunov function, linear matrix inequalities to study stabilization problems for discrete-time (or continuous-time) linear systems with time-delays [3, 10, 17, 16]. [17] presents an interesting approach using state feedback control design for a class of discrete-time linear systems with time-delays and matched uncertainty, but their approach is based on *nonlinear matrix inequalities* (NLMIs). It is known that there has been no efficient way to construct the control law in terms of NLMIs so far.

The objective of this paper is to discuss quadratic stability and quadratic stabilization problem for a class of multi-input and multi-output (MIMO) discrete-time systems with nonlinear perturbation on both state and control-input perturbations. A necessary and sufficient condition for quadratic stability of unforced systems is presented by means of S-procedure technique and LMI. In addition, both static and dynamic output feedback are constructed if the corresponding LMI is feasible.

As compared with the existing results in the literature, this paper discusses more general class of systems than those in [5, 6, 8, 15, 17, 19, 21]. Both static and dynamic output feedback control designs are obtained in terms of LMI which is more computational efficient than the NLMI approach developed for linear uncertain systems by [17]. In addition, the single-input static state feedback design developed in [15] is a very special case of this paper, also time-delays and perturbation on control input are not considered in their work. Furthermore, a state feedback control design for linear uncertain systems based on the Riccati equation approach is developed by [5], which are also regarded as a special case of this paper.

2 Quadratic Stability for the Unforced Systems

Consider a class of unforced perturbed discrete time-delay systems as follows

$$\begin{aligned} z_{k+1} &= \tilde{A} z_k + \tilde{A}_1 z_{k-d} + g(k, z_k, z_{k-d}), \\ z_k &= \delta_k, \quad k = -d, -d+1, \cdots, 0, \end{aligned} \tag{1}$$

where $z_k \in \mathbf{R}^{\tilde{n}}$ is the system state; $\tilde{A}$, $\tilde{A}_1$ are constant matrices with appropriate dimensions; and positive integer d is maximal time-delay; δ_i $(i = -d, -d+1, \cdots, 0)$ are initial-value vectors for the delayed system; $g = g(k, z_k, z_{k-d})$ is a vector-valued nonlinear function which is regarded as a nonlinear perturbation and satisfies the following quadratic inequality for all (k, w, v)

$$\begin{aligned} g'(k, w, v) g(k, w, v) &\le w' G' G w + 2 w' G' G_1 v + v' G_1' G_1 v \\ &= \begin{pmatrix} w \\ v \end{pmatrix}' (G \quad G_1)' (G \quad G_1) \begin{pmatrix} w \\ v \end{pmatrix}, \end{aligned} \tag{2}$$

where G and G_1 are constant matrices with appropriate dimensions, w, v are vectors with the same dimension with z_k.

The following definition on quadratic stability is presented by [17].

Definition 2.1 *Systems* (1) *are* *quadratically stable* if there exist matrices $P > 0$, and $Q > 0$ such that for all admissible perturbation g, systems (1) satisfy

$$\Delta V_k = V_{k+1} - V_k < 0, \tag{3}$$

for all pair $(k, z_k, z_{k-d}) \in \mathbf{Z}_+ \times (\mathbf{R}^{\tilde{n}} \times \mathbf{R}^{\tilde{n}} - \{0\})$, $V_k = z_k' P z_k + \sum_{i=1}^{d} z_{k-i}' Q z_{k-i}$, $\mathbf{Z}_+ = \{0, 1, 2, \cdots\}$.

Lemma 2.1 (S-procedure lemma) [18] *Let $\Omega_0(x)$ and $\Omega_1(x)$ be two arbitrary quadratic forms over $\mathbf{R}^n$. Then $\Omega_0(x) < 0$ for all $x \in \mathbf{R}^n - \{0\}$ satisfying $\Omega_1(x) \le 0$ if there exists $\tau \ge 0$ such that*

$$\Omega_0(x) - \tau \Omega_1(x) < 0, \quad \forall\, x \in \mathbf{R}^n - \{0\}.$$

For convenience and compactness, the following notation of (4) will be used throughout this paper.

$$\mathcal{L}(X, X_1, \Gamma, \Gamma_1, \Psi, \Psi_1) := \begin{pmatrix} -X + X_1 & 0 & \Gamma' & \Psi' \\ 0 & -X_1 & \Gamma_1' & \Psi_1' \\ \Gamma & \Gamma_1 & I - X & 0 \\ \Psi & \Psi_1 & 0 & -I \end{pmatrix}, \tag{4}$$

where X, X_1, Γ, Γ_1, Ψ and Ψ_1 are matrices with appropriate dimensions, I is an identity matrix with appropriate dimension.

By means of S-procedure and LMI technique, the following theorem presents a necessary and sufficient condition for quadratic stability of unforced systems (1).

Theorem 2.1 *Unforced systems (1) are quadratically stable if and only if there exist positive definite matrices X and X_1 with appropriate dimension such that the following LMI is solvable*

$$\mathcal{L}(X, X_1, \tilde{A}X, \tilde{A}_1 X, GX, G_1 X) < 0. \tag{5}$$

Proof By means of the Schur Complement Lemma, LMI (5) is equivalent to the following matrix inequality

$$\begin{pmatrix} -X + X_1 + XG'GX & XG'G_1X & X\tilde{A}' \\ XG_1'GX & -X_1 + XG_1'G_1X & X\tilde{A}_1' \\ \tilde{A}X & \tilde{A}_1X & I - X \end{pmatrix} < 0. \tag{6}$$

Let $P = X^{-1}$, $Q = X^{-1}X_1X^{-1}$, and multiply both sides of the first inequality of (6) by $\operatorname{diag}\{X^{-1}, X^{-1}, I\}$, then (6) is equivalent to

$$\begin{pmatrix} -P + Q + G'G & G'G_1 & \tilde{A}' \\ G_1'G & -Q + G_1'G_1 & \tilde{A}_1' \\ \tilde{A} & \tilde{A}_1 & I - P^{-1} \end{pmatrix} < 0. \tag{7}$$

Similarly, (7) is equivalent to

$$\begin{gathered} \begin{pmatrix} -P + Q + G'G & G'G_1 \\ G_1'G & -Q + G_1'G_1 \end{pmatrix} + \begin{pmatrix} \tilde{A}' \\ \tilde{A}_1' \end{pmatrix} (P^{-1} - I)^{-1} (\tilde{A} \quad \tilde{A}_1) < 0, \\ P^{-1} - I > 0. \end{gathered} \tag{8}$$

Notice that

$$(P^{-1}-I)^{-1}=P+P(I-P)^{-1}P. \tag{9}$$

Then (8) is equivalent to

$$\begin{gathered}\begin{pmatrix} \tilde{A}'P\tilde{A}-P+Q+G'G & \tilde{A}'P\tilde{A}_1+G'G_1 \\ \tilde{A}_1'P\tilde{A}+G_1'G & \tilde{A}_1'P\tilde{A}_1-Q+G_1'G_1 \end{pmatrix}+\begin{pmatrix} \tilde{A}'P \\ \tilde{A}_1'P \end{pmatrix}(I-P)^{-1}\begin{pmatrix} P\tilde{A} & P\tilde{A}_1 \end{pmatrix}<0, \\ I-P>0. \end{gathered} \tag{10}$$

From the Schur Complement Lemma again, matrix inequalities (10) are equivalent to

$$\begin{pmatrix} \tilde{A}'P\tilde{A}-P+Q+G'G & \tilde{A}'P\tilde{A}_1+G'G_1 & \tilde{A}'P \\ \tilde{A}_1'P\tilde{A}+G_1'G & \tilde{A}_1'P\tilde{A}_1-Q+G_1'G_1 & \tilde{A}_1'P \\ P\tilde{A} & P\tilde{A}_1 & P-I \end{pmatrix}<0. \tag{11}$$

Sufficiency: If LMI (5) holds for X and X_1, then (11) holds for $P=X^{-1}$ and $Q=X^{-1}X_1X^{-1}$. In order to obtain the quadratic stability of systems (1), we construct the following quadratic Lyapunov functional candidate

$$V_k=z_k'Pz_k+\sum_{i=1}^{d}z_{k-i}'Qz_{k-i}. \tag{12}$$

Then along with systems (1), for $(z_k' \quad z_{k-d}' \neq 0$, from (11) we have

$$\begin{aligned} &V_{k+1}-V_k-\left[g'g-\begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}'(G \quad G_1)'(G \quad G_1)\begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}\right] \\ &=z_k'(\tilde{A}'P\tilde{A}-P+Q)z_k+z_{k-d}'(\tilde{A}_1'P\tilde{A}_1-Q)z_{k-d}+g'Pg+2z_k'\tilde{A}'P\tilde{A}_1z_{k-d} \\ &\quad+2z_k'Pg+2z_{k-d}'\tilde{A}_1'Pg-\left[g'g-\begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}'(G \quad G_1)'(G \quad G_1)\begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}\right] \\ &=\begin{pmatrix} z_k \\ z_{k-d} \\ g \end{pmatrix}'\begin{pmatrix} \tilde{A}'P\tilde{A}-P+Q+G'G & \tilde{A}'P\tilde{A}_1+G'G_1 & \tilde{A}'P \\ \tilde{A}_1'P\tilde{A}+G_1'G & \tilde{A}_1'P\tilde{A}_1-Q+G_1'G_1 & \tilde{A}_1'P \\ P\tilde{A} & P\tilde{A}_1 & P-I \end{pmatrix}\begin{pmatrix} z_k \\ z_{k-d} \\ g \end{pmatrix}<0. \end{aligned} \tag{13}$$

It follows from Lemma 2.1 that under constraint (2), $V_{k+1}-V_k<0$ for $(z_k' \quad z_{k-d}')\neq 0$, which implies that systems (1) are quadratically stable in the sense of Definition 2.1.

Necessity: If systems (1) are quadratically stable, that is, there exists a Lyapunov functional candidate as follows

$$V_k=z_k'\tilde{P}z_k+\sum_{i=1}^{d}z_{k-i}'\tilde{Q}z_{k-i}, \tag{14}$$

where $\tilde{P}$ and $\tilde{Q}$ are positive definite matrices with appropriate dimensions, and under constraint condition:

$$g'g-\begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}'(G \quad G_1)'(G \quad G_1)\begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}\leq 0, \tag{15}$$

we have

$$\Delta V_k = V_{k+1} - V_k < 0, \quad \forall\,(z_k' \quad z_{k-d}') \neq 0. \tag{16}$$

Notice that (15) and ΔV_k are quadratic on z_k, z_{k-d} and g, then it follows from Lemma 2.1 that there exists a constant $\tau \geq 0$ such that for all $(z_k' \quad z_{k-d}' \quad g') \neq 0$,

$$\Delta V_k - \tau \left[g'g - \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}' (G \quad G_1)'(G \quad G_1) \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix} \right] < 0. \tag{17}$$

However, if $\tau = 0$, then (17) implies that the original systems (1) can be quadratically stable for all g without constraint (2), which is impossible. Then (17) holds for some $\tau > 0$. In addition, (17) is equivalent to

$$\begin{pmatrix} \tilde{A}'\tilde{P}\tilde{A} - \tilde{P} + \tilde{Q} + \tau G'G & \tilde{A}'\tilde{P}\tilde{A}_1 + \tau G'G_1 & \tilde{A}'\tilde{P} \\ \tilde{A}_1'\tilde{P}\tilde{A} + \tau G_1'G & \tilde{A}_1'\tilde{P}\tilde{A}_1 - \tilde{Q} + \tau G_1'G_1 & \tilde{A}_1'\tilde{P} \\ \tilde{P}\tilde{A} & \tilde{P}\tilde{A}_1 & \tilde{P} - \tau I \end{pmatrix} < 0. \tag{18}$$

Let $P = \tau^{-1}\tilde{P}$, $Q = \tau^{-1}\tilde{Q}$, then (18) is the same as (11). This completes the proof.

Remark 2.1 Systems (1) with constraint (2) are more general than the systems discussed in [8, 19, 21]. Theorem 3.1 can be regarded as an extension of the results in literature above.

3 Static Output Feedback

Consider a class of MIMO discrete time-delay systems with nonlinear perturbation as follows

$$\begin{aligned} x_{k+1} &= Ax_k + A_1 x_{k-d} + Bu_k + f(k, x_k, x_{k-d}, u_k), \\ y_k &= Cx_k, \\ x_k &= \delta_k, \quad k = -d, -d+1, \cdots, 0, \end{aligned} \tag{19}$$

where $x_k \in \mathbf{R}^n$, $u_k \in \mathbf{R}^m$ and $y_k \in \mathbf{R}^p$ are the system state, control input and output, respectively; A, $A_1 \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{p\times n}$ are constant matrices with full-row rank; and positive integer d is maximal time-delay; $\delta_i \in \mathbf{R}^n$ $(i = -d, -d+1, \cdots, 0)$ are initial-value vectors for the delayed system; $f(k, w, v, u)$ is a vector-valued nonlinear function and satisfies the following quadratic inequality for all $(k, w, v, u) \in \mathbf{Z}_+ \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^m$;

$$f'(k, w, v, u) f(k, w, v, u) \leq (w' \quad v' \quad u') \begin{pmatrix} F' \\ F_1' \\ H' \end{pmatrix} (F \quad F_1 \quad H) \begin{pmatrix} w \\ v \\ u \end{pmatrix}, \tag{20}$$

where F, F_1 and H are constant matrices with appropriate dimensions.

Remark 3.1 If

$$f(k, x_k, x_{k-d}, u_k) = \Delta A(k)x_k + \Delta A_1(k)x_{k-d} + \Delta B(k)u_k + f_0(k, x_k, x_{k-d}, u_k),$$

where the norm of $\Delta A(k)$, $\Delta A_1(k)$ and $\Delta B(k)$ are uniformly bounded and $f(k, w, v, u)$ is global Lipschitz on $(w, v, u) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^m$ with $f(k, 0, 0, 0) = 0$ for any $k \in \mathbf{Z}_+$, then this class of Lipschitz systems with uncertainty can be included in systems (19). Therefore the discrete-time linear (or time-delay) systems with matched uncertainty considered by [5] and [17] are special cases of this paper.

In this section, we consider the following form of linear static output feedback controller

$$u_k = Ky_k + K_1 y_{k-d}, \tag{21}$$

where $K, K_1 \in \mathbf{R}^{m\times p}$ are constant matrices to be determined.

The purpose of this section is to find a controller in the form of (21) such that the closed-loop systems (19) and (21) are quadratically stable. In this case, the controller (21) is called a quadratic stabilisation controller.

The following theorem presents a way of constructing static output feedback controller law (21), in which sufficient condition is presented by means of LMI approach.

Theorem 3.1 *Systems (19) are quadratically stabilizable by means of static output feedback in the form of (21) if the following LMI (22) and matrix equation (23) on matrices $X, X_1 \in \mathbf{R}^{n\times n}$, $Y, Y_1 \in \mathbf{R}^{m\times p}$ and $Z \in \mathbf{R}^{p\times p}$ are solvable*

$$\mathcal{L}(X, X_1, \Gamma, \Gamma_1, \Psi, \Psi_1) < 0, \tag{22}$$

$$CX = ZC, \tag{23}$$

where

$$\begin{aligned} \Gamma &= AX + BYC, &\quad \Gamma_1 &= A_1X + BY_1C, \\ \Psi &= FX + HYC, &\quad \Psi_1 &= F_1X + HY_1C. \end{aligned} \tag{24}$$

Proof The conditions on full-row rank of C, $X > 0$ and matrix equation $CX = ZC$ imply that

$$p \geq \operatorname{rank}(Z) \geq \operatorname{rank}(ZC) = \operatorname{rank}(CX) \geq \operatorname{rank}[(CX)X^{-1}] = \operatorname{rank}(C) = p \tag{25}$$

that is, Z is non-singular. Then the gains of control law (21) can be chosen as follows:

$$K = YZ^{-1}, \quad K_1 = Y_1Z^{-1}. \tag{26}$$

In this case, the resulting closed-loop systems are systems (1) with

$$\tilde{A} = A + BKC, \quad \tilde{A}_1 = A_1 + BK_1C, \quad g = f(k, x_k, x_{k-d}, KCx_k + K_1Cx_{k-d}), \tag{27}$$

where

$$g'g \leq \begin{pmatrix} x_k \\ x_{k-d} \end{pmatrix}' \begin{pmatrix} G' \\ G_1' \end{pmatrix} (G \quad G_1) \begin{pmatrix} x_k \\ x_{k-d} \end{pmatrix}, \tag{28}$$

and

$$G = F + HKC, \quad G_1 = F_1 + HK_1C. \tag{29}$$

From (26) and matrix equality $CX = ZC$, we have

$$\begin{aligned} \tilde{A}X &= (A + BKC)X = AX + BKCX \\ &= AX + BKZC = AX + BYC = \Gamma, \\ \tilde{A}_1X &= (A_1 + BK_1C)X = A_1X + BK_1CX \\ &= A_1X + BK_1ZC = A_1X + BY_1C = \Gamma_1, \\ GX &= FX + HKCX = FX + HYC = \Psi, \\ G_1X &= F_1X + HK_1CX = F_1X + HY_1C = \Psi_1. \end{aligned} \tag{30}$$

Then it follows from Theorem 2.1 and (22) that the systems (1) with (27), that is, the closed-loop systems (19) and (21) with (26), are quadratically stable, which completes the proof.

As a direct application of Theorem 3.1, if $C = I$ is chosen in Theorem 3.1, we have the following result, which presents a necessary and sufficient condition under which systems can be quadratically stabilized via static state feedback law

$$u_k = Kx_k. \tag{31}$$

Corollary 3.1 *Systems (19) are quadratically stabilizable via static state feedback in the form of (31) if and only if the following LMI on matrices* $X, X_1 \in \mathbf{R}^{n\times n}$ *and* $Y \in \mathbf{R}^{m\times n}$ *is solvable*

$$\mathcal{L}(X, X_1, \Gamma, \Gamma_1, \Psi, \Psi_1) < 0, \tag{32}$$

where

$$\Gamma = AX + BY, \quad \Gamma_1 = A_1X, \quad \Psi = FX + HY, \quad \Psi_1 = F_1X. \tag{33}$$

In this case, a static state feedback law can be chosen as follows

$$u_k = YX^{-1}x_k. \tag{34}$$

The *Proof* for sufficiency of Corollary 3.1 follows directly from Theorem 3.1. The necessity can be obtained from Lemma 2.1.

Remark 3.2 [15] discusses a class of discrete-time systems with single-input and nonlinear perturbation (no control input perturbation is considered), and a static state feedback law is constructed by means of LMI. However a special structure matrix variable L is needed to guarantee the resulting matrix inequality to be an LMI such that a solution of K is obtained (see (19)–(21) in [15]). It is important to notice that Corollary 3.1 presents a more efficient approach to search for an explicit solution K. In addition, Theorem 2 by [15] is a special case of Theorems 3.1 and 3.2. Furthermore, the result in this section can be regarded as an extension of that by [6], where static output feedback is obtained by means of quasiconvex optimization approach.

Since the conditions (22)–(23) contain the constraint $CX = ZC$, MATLAB LMI Toolbox [4] can not be used to solve (22)–(23) directly. In order to convert the problem (22)–(23) into an LMI, we will show that this constraint on X and Z can be transformed into an equivalent constraint on X, then (22)–(23) will be equivalent to an LMI.

For convenience, we present the singular value decomposition of C as

$$C = U(C_0 \quad 0)V', \tag{35}$$

where $U \in \mathbf{R}^{p\times p}$ and $V \in \mathbf{R}^{n\times n}$ are unitary matrices and $C_0 \in \mathbf{R}^{p\times p}$ is a diagonal matrix with positive diagonal elements in decreasing order.

The following lemma presents an equivalent condition on matrix equation $CX = ZC$.

Lemma 3.1 *For a given* $C \in \mathbf{R}^{p\times n}$ *with* $\operatorname{rank}(C) = p$, *assume that* $X \in \mathbf{R}^{n\times n}$ *is a symmetric matrix, then there exists a matrix* $Z \in \mathbf{R}^{p\times p}$ *such that* $CX = ZC$ *if and only if* $X = V\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}V'$, *where* $X_1 \in \mathbf{R}^{p\times p}$, $X_2 \in \mathbf{R}^{(n-p)\times(n-p)}$.

Proof If $p = n$, from the proof of Theorem 3.1, C is non-singular, it is clear that the result is true. Without loss of generality, suppose $p < n$. From $CX = ZC$ and the singular value decomposition of C, that is, $C = U(C_0 \quad 0)V'$, we have that matrix equation $CX = ZC$ is equivalent to $U(C_0 \quad 0)V'X = ZU(C_0 \quad 0)V'$. That is,

$$(UC_0 \quad 0)V'XV = (ZUC_0 \quad 0). \tag{36}$$

Suppose $X = V \begin{pmatrix} X_1 & X_0' \\ X_0 & X_2 \end{pmatrix} V'$, where $X_1 \in \mathbf{R}^{p\times p}$, $X_2 \in \mathbf{R}^{(n-p)\times(n-p)}$ and $X_0 \in \mathbf{R}^{(n-p)\times p}$, then (36) is equivalent to

$$(UC_0X_1 \quad UC_0X_0) = (ZUC_0 \quad 0). \tag{37}$$

Matrix equation (37) is solvable on Z if and only if $UC_0X_0 = 0$, that is, $X_0 = 0$, which completes the proof.

Therefore we have the following result from Theorem 3.1 and Lemma 3.1.

Theorem 3.2 *Systems (19) are quadratically stabilizable by static output feedback law if the following LMI on matrices* $X_{11} \in \mathbf{R}^{p\times p}$, $X_{22} \in \mathbf{R}^{(n-p)\times(n-p)}$, $X_1 \in \mathbf{R}^{n\times n}$ *and* $Y, Y_1 \in \mathbf{R}^{m\times p}$ *is solvable*

$$\mathcal{L}(X, X_1, \Gamma, \Gamma_1, \Psi, \Psi_1) < 0, \tag{38}$$

where

$$\begin{gathered} X = V\,\mathrm{diag}\,\{X_{11}, X_{22}\}V', \quad \Gamma = AX + BYC, \quad \Gamma_1 = A_1X + BY_1C, \\ \Psi = FX + HYC, \quad \Psi_1 = F_1X + HY_1C. \end{gathered} \tag{39}$$

In this case, a static output feedback controller of form (21) can be chosen as follows

$$u_k = YUC_0X_{11}^{-1}C_0^{-1}U'y_k + Y_1UC_0X_{11}^{-1}C_0^{-1}U'y_{k-d}. \tag{40}$$

4 Dynamic Output Feedback

In this section, we consider stabilisation for systems (19) via the following Luenberger-like dynamic output feedback controller

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + A_1\hat{x}_{k-d} + Bu_k + L(y_k - C\hat{x}_k), \\ u_k &= K\hat{x}_k + K_1\hat{x}_{k-d}. \end{aligned} \tag{41}$$

Let the difference of x_k and $\hat{x}_k$ be e_k, that is, $e_k = x_k - \hat{x}_k$, then the closed-loop systems of (19) and (41) can be written as (1) with

$$\begin{gathered} z_k = \begin{pmatrix} \hat{x}_k \\ e_k \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} A + BK & LC \\ 0 & A - LC \end{pmatrix}, \quad \tilde{A}_1 = \begin{pmatrix} A_1 + BK_1 & 0 \\ 0 & A_1 \end{pmatrix}, \\ g(k, z_k, z_{k-d}) = \begin{pmatrix} 0 \\ f(k, \hat{x}_k + e_k, \hat{x}_{k-d} + e_k, K\hat{x}_k + K_1\hat{x}_{k-d}) \end{pmatrix}. \end{gathered} \tag{42}$$

From (20), after some algebraic manipulations, we have

$$\begin{aligned} &g'(k,z_k,z_{k-d})g(k,z_k,z_{k-d}) \\ &= [f(k,\hat{x}_k+e_k,\hat{x}_{k-d}+e_{k-d},K\hat{x}_k+K_1\hat{x}_{k-d})]'f(k,\hat{x}_k+e_k,\hat{x}_{k-d}+e_k,K\hat{x}_k+K_1\hat{x}_{k-d}) \\ &\le \begin{pmatrix} \hat{x}_k+e_k \\ \hat{x}_{k-d}+e_{k-d} \\ K\hat{x}_k+K_1\hat{x}_{k-d} \end{pmatrix}' \begin{pmatrix} F' \\ F_1' \\ H' \end{pmatrix} (F \quad F_1 \quad H) \begin{pmatrix} \hat{x}_k+e_k \\ \hat{x}_{k-d}+e_{k-d} \\ K\hat{x}_k+K_1\hat{x}_{k-d} \end{pmatrix} \\ &= \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}' \begin{pmatrix} G' \\ G_1' \end{pmatrix} (G \quad G_1) \begin{pmatrix} z_k \\ z_{k-d} \end{pmatrix}, \end{aligned} \tag{43}$$

where

$$G = (F+HK \quad F), \quad G_1 = (F_1+HK_1 \quad F_1). \tag{44}$$

Theorem 4.1 *Systems (19) are quadratically stable via dynamic output feedback in the form of (41) if there exist matrices* X_{11}, $X_{22} \in \mathbf{R}^{n\times n}$, $X_1 \in \mathbf{R}^{2n\times 2n}$, $Y_0 \in \mathbf{R}^{n\times p}$, $Y, Y_1 \in \mathbf{R}^{m\times n}$ *and* $Z \in \mathbf{R}^{p\times p}$ *such that the following LMI (45) and matrix equation (46) are solvable*

$$\mathcal{L}(X,X_1,\Gamma,\Gamma_1,\Psi,\Psi_1) < 0, \tag{45}$$

$$CX_{22} = ZC, \tag{46}$$

where

$$\begin{gathered} \Gamma = \begin{pmatrix} AX_{11}+BY & Y_0C \\ 0 & AX_{22}-Y_0C \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} A_1X_{11}+BY_1 & 0 \\ 0 & A_1X_{22} \end{pmatrix}, \\ X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}, \qquad \Psi = (FX_{11}+HY \quad FX_{22}), \\ \Psi_1 = (F_1X_{11}+HY_1 \quad F_1X_{22}). \end{gathered} \tag{47}$$

In this case, a dynamic output feedback controller can be given by (41) with

$$L = Y_0Z^{-1}, \qquad K = YX_{11}^{-1}, \qquad K_1 = Y_1X_{11}^{-1}. \tag{48}$$

Proof Similarly, we have that Z in non-singular from $CX_{22} = ZC$, then L, K, K_1 can be given by (48). In this case, the matrix parameters in the resulting closed-loop systems in the form of (1) satisfy the following conditions:

$$\begin{aligned} \tilde{A}X &= \begin{pmatrix} A+BK & LC \\ 0 & A-LC \end{pmatrix}\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} = \begin{pmatrix} AX_{11}+BKX_{11} & LCX_{22} \\ 0 & AX_{22}-LCX_{22} \end{pmatrix} \\ &= \begin{pmatrix} AX_{11}+BY & LZC \\ 0 & AX_{22}-LZC \end{pmatrix} = \begin{pmatrix} AX_{11}+BY & Y_0C \\ 0 & AX_{22}-Y_0C \end{pmatrix} = \Gamma. \end{aligned} \tag{49}$$

Similarly, we have

$$\tilde{A}_1X = \Gamma_1, \qquad GX = \Psi, \qquad G_1X = \Psi_1. \tag{50}$$

Then it follows from Lemma 2.1 that the systems (1) with (42) and (48), that is, the resulting closed-loop systems (19) and (41) with (48), are quadratically stable, which completes the proof of Theorem 4.1.

Similar to Theorem 3.2, the following result can be obtained from Theorem 4.1 and Lemma 3.1 directly.

Theorem 4.2 *Systems (19) are quadratically stabilizable by dynamic output feedback law (41) if there exist matrices* $X_{11} \in \mathbf{R}^{n\times n}$, $X_{221} \in \mathbf{R}^{p\times p}$, $X_{222} \in \mathbf{R}^{(n-p)\times(n-p)}$, $X_1 \in \mathbf{R}^{2n\times 2n}$, $Y_0 \in \mathbf{R}^{n\times p}$, *and* $Y, Y_1 \in \mathbf{R}^{m\times n}$ *such that the following LMI is solvable*

$$\mathcal{L}(X, X_1, \Gamma, \Gamma_1, \Psi, \Psi_1) < 0, \tag{51}$$

where Γ, Γ_1, Ψ, Ψ_1 *are defined in the same way as those in (47),*

$$X_{22} = V \operatorname{diag}\{X_{221}, X_{222}\}V', \qquad X = \operatorname{diag}\{X_{11}, X_{22}\}.$$

In this case, a dynamic output feedback controller can be given by (41) with

$$L = Y_0 U C_0 X_{221}^{-1} C_0 U', \quad K = Y X_{11}^{-1}, \quad K_1 = Y_1 X_{11}^{-1}. \tag{52}$$

Remark 4.1 This section can be regarded as an extension of the results in Section 3. In addition, this section presents a new approach to construction of dynamic output feedback controller for a class of discrete-time nonlinear time-delay systems.

5 Numerical Examples

All the numerical examples in this section are computed via the MATLAB LMI Toolbox [4].

The first example has been discussed by [17], where NLMIs are presented and no explicit algorithms are given. We shall present quadratic stability via LMI using the proposed explicit algorithms in this paper.

Example 5.1 [17] Consider the following unforced discrete-time systems:

$$z_{k+1} = \begin{pmatrix} -0.5 & -0.4 \\ 0.2 & -0.6 \end{pmatrix} z_k + \begin{pmatrix} 0.3 & 0.1 \\ -0.1 & 0.1 \end{pmatrix} z_{k-2} + g(k, z_k, z_{k-2}), \tag{53}$$

where $g(k, z_k, z_{k-2}) = MF(k)(N_A z_k + N_d z_{k-2})$, $M = \begin{pmatrix} 0.3 \\ 0.1 \end{pmatrix}$, $N_A = (0.15 \quad 0.1)$, $N_d = (0.2 \quad 0.1)$, $F(k)$ is an uncertain matrix with an appropriate dimension and satisfying $F'(k)F(k) \le I$ for all k.

Then we have

$$\begin{aligned} g'(k, z_k, z_{k-2}) g(k, z_k, z_{k-2}) &= (N_A z_k + N_d z_{k-2})' F'(k) M' M F(k)(N_A z_k + N_d z_{k-2}) \\ &\le 0.1 (z_k' \quad z_{k-2}') \begin{pmatrix} N_A' \\ N_d' \end{pmatrix} (N_A \quad N_d) \begin{pmatrix} z_k \\ z_{k-2} \end{pmatrix}. \end{aligned} \tag{54}$$

That is, $G = \sqrt{0.1} N_A = (0.0474 \quad 0.0316)$, $G_1 = \sqrt{0.1} N_d (0.0632 \quad 0.0316)$ in (2). We obtain a pair of solutions from LMI (5) with (4) as follows:

$$X = \begin{pmatrix} 9.8666 & -0.7210 \\ -0.7210 & 7.3597 \end{pmatrix}, \qquad X_1 = \begin{pmatrix} 4.2989 & 0.2233 \\ 0.2233 & 1.8045 \end{pmatrix}. \tag{55}$$

Therefore the systems (53) is quadratically stable.

The following example is a nonlinear system, we shall illustrate the construction of state feedback.

Example 5.2 Consider the following linear discrete-time systems with nonlinear perturbation:

$$\begin{aligned} x_{k+1} = &\begin{pmatrix} 1 & -0.6 \\ 0.4 & 0.5 \end{pmatrix} x_k + \begin{pmatrix} 0.5 & 0.2 \\ 0.6 & 0.4 \end{pmatrix} x_{k-2} + \begin{pmatrix} 0.1 & 0.2 \\ 0 & 0.1 \end{pmatrix} u_k \\ &+ M \sin(N_A x_k + N_d x_{k-2} + N_B u_k), \end{aligned} \tag{56}$$

where

$$M = (0.1 \quad 0.1)', \qquad N_A = (0.02 \quad 0.03), \qquad N_d = (0.02 \quad 0.01), \qquad N_B = (2 \quad 1.5). \tag{57}$$

Then it is easy to have $F = \mu N_A,\ F_1 = \mu N_d,\ H = \mu N_B,\ \mu = \sqrt{0.02}$. Now a static state feedback controller can be constructed by Corollary 3.1. A triple of matrix solutions X, X_1 and Y can be obtained from LMI (32) with (33) as follows:

$$X = \begin{pmatrix} 72.0747 & -23.0754 \\ -23.0754 & 93.6106 \end{pmatrix}, \qquad X_1 = \begin{pmatrix} 46.6402 & 12.4100 \\ 12.4100 & 21.4395 \end{pmatrix},$$

$$Y = \begin{pmatrix} 348.9366 & -154.6649 \\ -468.6426 & 208.1791 \end{pmatrix}.$$

Then the control gain of controller (34) can be given as follows:

$$K = YX^{-1} = \begin{pmatrix} 4.6818 & -0.4981 \\ -6.2863 & 0.6743 \end{pmatrix}. \tag{58}$$

The following example presents a very simple way of constructing both static output feedback control law and dynamic output feedback control law based on measurable output.

Example 5.3 Consider the systems (56) with the following output

$$y_k = Cx_k, \tag{59}$$

where $C = (1 \quad 0)$, and the constraint for nonlinear perturbation $f(k, x_k, x_{k-2}, u_k)$ is defined by (20) with F, F_1 and H presented by Example 5.2.

At first, we present a static output feedback control law in the form of (21) for Example 5.3.

Similarly, the following matrix solutions can be obtained from LMI (38) with (39)

$$X_{11} = 35.1887, \qquad X_{22} = 16.6820, \qquad X_1 = \begin{pmatrix} 17.5368 & 5.5927 \\ 5.5927 & 6.4090 \end{pmatrix},$$

$$Y = \begin{pmatrix} 129.7891 \\ -174.3170 \end{pmatrix}, \qquad Y_1 = \begin{pmatrix} 123.5208 \\ -165.2880 \end{pmatrix}.$$

In this case, it follows from Theorem 3.2 that a static output feedback control law can be given as follows:

$$u_k = Ky_k + K_1 y_{k-2} = \begin{pmatrix} 3.6884 \\ -4.9538 \end{pmatrix} y_k + \begin{pmatrix} 3.5102 \\ -4.6972 \end{pmatrix} y_{k-2}. \tag{60}$$

Then a dynamic output feedback controller can be constructed by means of Theorem 4.2. To this end, the following solutions can be computed from LMI (51) and (47)

$$X_{11} = 10^3 \begin{pmatrix} 1.4250 & 0.1199 \\ 0.1199 & 0.7585 \end{pmatrix}, \quad X_{221} = 216.5833, \quad X_{222} = 229.0485,$$

$$X_1 = \begin{pmatrix} 754.0081 & 265.3081 & 0.0611 & 0.1206 \\ 265.3081 & 240.4736 & -0.6242 & 0.0052 \\ 0.0611 & -0.6242 & 139.6342 & 70.6240 \\ 0.1206 & 0.0052 & 70.6240 & 64.9059 \end{pmatrix}, \qquad Y_0 = \begin{pmatrix} 155.7066 \\ 135.3971 \end{pmatrix},$$

$$Y = 10^3 \begin{pmatrix} 6.1647 & 0.9422 \\ -8.2406 & -1.2733 \end{pmatrix}, \qquad Y_1 = 10^3 \begin{pmatrix} 5.7601 & 2.1453 \\ -7.7000 & -2.8674 \end{pmatrix}.$$

Then it follows from Theorem 4.2 that a dynamic output feedback controller can be given in the form of (41) with the following gain matrices:

$$\begin{gathered} L = \begin{pmatrix} 0.7189 \\ 0.6252 \end{pmatrix}, \qquad K = \begin{pmatrix} 4.2786 & 0.5660 \\ -5.7178 & -0.7751 \end{pmatrix}, \\ K_1 = \begin{pmatrix} 3.8556 & 2.2190 \\ -5.1542 & -2.9658 \end{pmatrix}. \end{gathered} \tag{61}$$

6 Conclusion

This paper has studied the problems of quadratic stability and quadratic stabilisation problem for a class of discrete time-delay systems with nonlinear perturbations. It is shown that the problems can be reformulated as convex optimization problems in the form of LMI. The design technique in the existing literature has been improved and generalized in this paper. This paper presents a unified way of designing quadratic state feedback and output feedback laws for a class of perturbed discrete time-delay systems. It is easy to see that the approach in this paper can be fully extended to systems with multiple time-delays.

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