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Set Differential Equations and Monotone Flows

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Abstract: Monotone iterative technique is extended to set differential equations. The nonlinear function involved is allowed to be difference of two monotone functions, which takes care of several results known and new.

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1 Preliminaries

Let $K(R^n)(K_c(R^n))$ denote the collection of all nonempty, compact (compact, convex) subsets of R^n. Define the Hausdorff metric

$$D[A,B] = \max\big[\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\big], \tag{1.1}$$

where $d[x,A] = \inf[d(x,y)\colon y \in A]$, A, B are bounded sets in R^n. We note that $K(R^n)$, $(K_c(R^n))$, with the metric is a complete metric space.

It is known that if the space $K_c(R^n)$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, then $K_c(R^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space [1, 9].

The Hausdorff metric (1.1) satisfies the following properties.

$$D[A+C, B+C] = D[A,B] \quad \text{and} \quad D[A,B] = D[B,A], \tag{1.2}$$

$$D[\lambda A, \lambda B] = \lambda D[A,B], \tag{1.3}$$

$$D[A,B] \le D[A,C] + D[C,B], \tag{1.4}$$

for all A, B, $C \in K_c(R^n)$ and $\lambda \in R^+$.

Let $A, B \in K_c(R^n)$. The set $C \in K_c(R^n)$ satisfying $A = B + C$ is known as the geometric difference of the sets A and B and is denoted by the symbol $A - B$. We say that the mapping $F\colon I \to K_c(R^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if there exists an element $D_H F(t_0) \in K_c(R^n)$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h}, \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist in the topology of $K_c(R^n)$ and are equal to $D_H F(t_0)$. Here I is any interval in R.

By embedding $K_c(R^n)$ as a complete cone in a corresponding Banach space and taking into account the result on differentiation of Bochner integral, we find that if

$$F(t) = X_0 + \int_0^t \Phi(s)\, ds, \qquad X_0 \in K_c(R^n), \tag{1.5}$$

where $\Phi\colon I \to K_c(R^n)$ is integrable in the sense of Bochner, then $D_H F(t)$ exists and the equality

$$D_H F(t) = \Phi(t), \quad \text{a.e on} \quad I, \tag{1.6}$$

holds. Also, the Hukuhara integral

$$\int_I F(s)\, ds = \left[\int_I f(s)\, ds\colon\ f \text{ is a continuous selector of } F \right],$$

for any compact set $I \subset R_+$. With these preliminaries, we consider the set differential equation

$$D_H U = F(t, U), \quad U(t_0) = U_0 \in K_c(R^n), \quad t_0 \geq 0, \tag{1.7}$$

where $F \in C[R_+ \times K_c(R^n),\, K_c(R^n)]$.

The mapping $U \in C^1[J, K_c(R^n)]$, $J = [t_0, t_0 + a]$ is said to be a solution of (1.7) on J if it satisfies (1.7) on J. Since $U(t)$ is continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^t D_H U(s)\, ds, \qquad t \in J. \tag{1.8}$$

Thus we associate with the initial value problem (IVP) (1.7) the following

$$U(t) = U_0 + \int_{t_0}^t F(s, U(s))\, ds, \qquad t \in J, \tag{1.9}$$

where the integral is the Hukuhara integral. Observe also that $U(t)$ is a solution of (1.7) iff it satisfies (1.9) on J. The investigation of set differential equation (1.7) as an independent subject has some advantages. For example, when $U(t)$ is a singlevalued mapping, it is easy to see that Hukuhara derivative and the integral reduce to the ordinary vector derivative and the integral, and therefore, the results obtained in this new framework of (1.7) become the corresponding results of ordinary differential systems. Also, we have only semilinear complete metric space to work with, in the present setup, compared to the complete normed linear space one employs in the study of ordinary differential systems. Furthermore, set differential equations, that are generated by multivalued differential inclusions, when the multivalued functions involved do not possess convex values, can be used as a tool for studying multivalued differential inclusions. See Tolstonogov [9]. Moreover one can utilize set differential equations of the type (1.7) to investigate profitably fuzzy differential equations, since the original formulation of which suffers from grave disadvantages and does not reflect the rich behavior of corresponding differential equations without fuzziness [2, 3, 6]. This is due to the fact that the diameter of any solution of fuzzy differential equation increases as time increases because of the necessity of the fuzzification of the derivative involved.

It is well known that the ideas embedded in the interesting and fruitful method of monotone iterative technique have proved to be of immense value and have played a crucial role in unifying a wide variety of nonlinear problems [4]. In this paper, we shall develop this technique to set differential equations (1.7) in a unified way following the work in [5]. In [7], we initiated the study of set differential equations of the type (1.7) as an independent subject and in [8] an interconnection between fuzzy differential equations and set differential equation is investigated.

2 Comparison Results

Let us introduce a partial ordering in the metric space $(K_c(R^n), D)$ which is needed in order to prove a basic comparison result that is required for our discussion.

We denote by $K(K^0)$ the subfamily of $K_c(R^n)$ consisting of sets $X \in K_c(R^n)$ such that any $x \in X$ is a nonnegative (positive) vector of n-components satisfying $x_i \geq 0$ $(x_i > 0)$ for $i = 1, 2, ..., n$. Thus K is a cone in $K_c(R^n)$ and K^0 is the nonempty interior of K. We can therefore induce a partial ordering in $K_c(R^n)$. See [1] for this approach.

Definition 2.1 For any X and $Y \in K_c(R^n)$, if there exists a $Z \in K_c(R^n)$ such that $Z \in K(K^0)$ and

$$X = Y + Z,$$

then we write $X \geq Y (X > Y)$ respectively. Similarly, one can define $X \leq Y (X < Y)$.

We can now prove the following basic result on set differential inequalities.

Theorem 2.1 *Assume that*

(i) $V, W \in C^1[R_+, K_c(R^n)]$, $F \in C[R_+ \times K_c(R^n), K_c(R^n)]$, $F(t, X)$ *is monotone nondecreasing in* X *for each* $t \in R_+$ *and*

$$D_H V \leq F(t, V), \quad D_H W \geq F(t, W), \quad t \in R_+;$$

(ii) *for any* X, $Y \in K_c(R^n)$ *such that* $X \geq Y$, $t \in R_+$,

$$F(t, X) \leq F(t, Y) + L(X - Y)$$

for some $L > 0$.

Then $V(t_0) \leq W(t_0)$ *implies*

$$V(t) \leq W(t), \quad t \geq t_0. \tag{2.1}$$

Proof Let $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) > 0$ and define $\tilde{W} = W + \epsilon e^{2Lt}$. Since $V(t_0) \le W(t_0) < \tilde{W}(t_0)$, it is enough to prove that

$$V(t) < \tilde{W}(t), \quad t \ge t_0, \tag{2.2}$$

to prove the conclusion (2.1) in view of the fact $\epsilon > 0$ is arbitrary.

Let $t_1 > 0$ be the supremum of all positive numbers $\delta > 0$ such that $V(t_0) < \tilde{W}(t_0)$ implies $V(t) < \tilde{W}(t)$ on $[t_0, \delta]$. It is clear that $t_1 > t_0$ and $V(t_1) \le \tilde{W}(t_1)$. From this follows, using the nondecreasing nature of F and condition (ii), that

$$\begin{aligned} D_H V(t_1) &\le F(t_1, V(t_1)) \le F(t_1, \tilde{W}(t_1)) \le F(t_1, W(t_1)) + L(\tilde{W} - W) \\ &\le D_H W(t_1) + L \in e^{2Lt_1} < D_H W(t_1) + 2L \in e^{2Lt_1} = D_H \tilde{W}(t_1). \end{aligned}$$

Consequently, it follows that there exists an $\eta > 0$ satisfying

$$V(t) - \tilde{W}(t) > V(t_1) - \tilde{W}(t_1), \quad t_1 - \eta < t < t_1.$$

This implies that $t_1 > t_0$ cannot be the supremum in view of the continuity of the functions involved and therefore the relation (2.2) is true, which, in turn, leads to the conclusion (2.1). The proof is complete.

The following corollary is useful.

Corollary 2.1 *Let V, $W \in C^1[R_+, K_c(R^n)]$, $\sigma \in C[R_+, K_c(R^n)]$. Suppose that*

$$D_H V \le \sigma, \quad D_H W \ge \sigma \quad \text{for} \quad t \ge t_0.$$

Then $V(t) \le W(t)$, $t \ge t_0$, provided $V(t_0) \le W(t_0)$.

3 Monotone Flows

In this section, we shall consider the following set differential equation

$$D_H U = F(t, U) + G(t, U), \quad U(0) = U_0 \in K_c(R^n), \tag{3.1}$$

where F, $G \in C[J \times K_c(R^n), K_c(R^n)]$ and $J = [0, T]$. We need the following definition which gives various possible notions of lower and upper solutions relative to (3.1).

Definition 3.1 Let V, $W \in C^1[J, K_c(R^n)]$. Then V, W are said to be

(a) *natural lower and upper solutions* of (3.1) if

$$D_H V \le F(t, V) + G(t, V), \quad D_H W \ge F(t, W) + G(t, W), \quad t \in J; \tag{3.2}$$

(b) *coupled lower and upper solutions* of type I of (3.1) if

$$D_H V \le F(t, V) + G(t, W), \quad D_H W \ge F(t, W) + G(t, V), \quad t \in J; \tag{3.3}$$

(c) *coupled lower and upper solutions* of type II of (3.1) if

$$D_H V \le F(t, W) + G(t, V), \quad D_H W \ge F(t, V) + G(t, W), \quad t \in J; \tag{3.4}$$

(d) *coupled lower and upper solutions* of type III of (3.1) if

$$D_H V \le F(t, W) + G(t, W), \quad D_H W \ge F(t, V) + G(t, V), \quad t \in J. \tag{3.5}$$

We observe that whenever we have $V(t) \le W(t)$, $t \in J$, if $F(t, X)$ is nondecreasing in X for each $t \in J$ and $G(t, Y)$ is nonincreasing in Y for each $t \in J$, the lower and upper solutions defined by (3.2) and (3.5) reduce to (3.4) and consequently, it is sufficient to investigate the cases (3.3) and (3.4).

We are now in a position to prove the following result.

Theorem 3.1 *Assume that*

(A_1) *V, $W \in C^1[J, K_c(R^n)]$ are coupled lower and upper solutions of type I relative to (3.1) with $V(t) \le W(t)$, $t \in J$;*

(A_2) *F, $G \in C[J \times K_c(R^n), K_c(R^n)]$, $F(t, X)$ is nondecreasing in X and $G(t, Y)$ is nonincreasing in Y, for each $t \in J$.*

Then there exist monotone sequences $\{V_n(t)\}$, $\{W_n(t)\} \in K_c(R^n)$ such that $V_n(t) \to \rho(t)$, $W_n(t) \to R(t)$ in $K_c(R^n)$ and (ρ, R) are the coupled minimal and maximal solutions of (3.1) respectively, that is, they satisfy

$$D_H \rho = F(t, \rho) + G(t, R), \quad \rho(0) = U_0,$$
$$D_H R = F(t, R) + G(t, \rho), \quad R(0) = U_0, \ \text{on } J.$$

Proof For each $n \ge 0$, define the unique solutions $V_{n+1}(t)$, $W_{n+1}(t)$ by

$$D_H V_{n+1} = F(t, V_n) + G(t, W_n), \quad V_{n+1}(0) = U_0, \tag{3.6}$$
$$D_H W_{n+1} = F(t, W_n) + G(t, V_n), \quad W_{n+1}(0) = U_0, \quad t \in J, \tag{3.7}$$

where $V(0) \le U_0 \le W(0)$. We set $V_0 = V$, $W_0 = W$. Our aim is to prove

$$V_0 \le V_1 \le V_2 \le \dots \le V_n \le W_n \le \dots \le W_2 \le W_1 \le W_0, \quad t \in J. \tag{3.8}$$

Since V_0 is the coupled lower solutions of type I of (3.1), we have using the fact $V_0 \le W_0$ and the nondecreasing character of F,

$$D_H V_0 \le F(t, V_0) + G(t, W_0).$$

Also from (3.6), we get for $n = 0$,

$$D_H V_1 = F(t, V_0) + G(t, W_0).$$

Consequently, we arrive at $V_0 \le V_1$ on J. A similar argument shows that $W_1 \le W_0$ on J. We next prove $V_1 \le W_1$ on J. For this purpose consider

$$D_H V_1 = F(t, V_0) + G(t, W_0) \quad \text{and}$$
$$D_H W_1 = F(t, W_0) + G(t, V_0), \quad V_1(0) = W_1(0) = U_0.$$

Then, the monotone nature of F and G respectively yield

$$D_H V_1 \le F(t, W_0) + G(t, W_0), \quad D_H W_1 \ge F(t, W_0) + G(t, W_0), \quad t \in J.$$

We therefore have, by Corollary 2.1, $V_1 \le W_1$ on J. As a result, we obtain

$$V_0 \le V_1 \le W_1 \le W_0 \quad \text{on} \quad J. \tag{3.9}$$

Assume that for some $j > 1$, we have

$$V_{j-1} \le V_j \le W_j \le W_{j-1} \quad \text{on} \quad J. \tag{3.10}$$

Then we show that

$$V_j \leq V_{j+1} \leq W_{j+1} \leq W_j \quad \text{on} \quad J. \tag{3.11}$$

To do this, consider

$$\begin{aligned} &D_H V_j = F(t, V_{j-1}) + G(t, W_{j-1}), \quad V_j(0) = U_0, \\ &D_H V_{j+1} = F(t, V_j) + G(t, W_j) \geq F(t, V_{j-1}) + G(t, W_{j-1}), \quad t \in J. \end{aligned}$$

Here we have employed (3.10) and the monotone nature of F and G. Corollary 2.1 now gives $V_j \leq V_{j+1}$ on J. Similarly, we can get $W_{j+1} \leq W_j$ on J. Next we show that $V_{j+1} \leq W_{j+1}$, $t \in J$. We have from (3.6) and (3.7)

$$\begin{aligned} &D_H V_{j+1} = F(t, V_j) + G(t, W_j), \quad V_{j+1}(0) = U_0, \\ &D_H W_{j+1} = F(t, W_j) + G(t, V_j), \quad W_{j+1}(0) = U_0, \quad t \in J. \end{aligned}$$

Using (3.10) and the monotone character of F and G, we arrive at

$$\begin{aligned} &D_H V_{j+1} \leq F(t, W_j) + G(t, W_j), \\ &D_H W_{j+1} \geq F(t, W_j) + G(t, W_j), \quad t \in J, \end{aligned}$$

and therefore Corollary 2.1 implies that $V_{j+1} \leq W_{j+1}$, $t \in J$. Hence (3.11) follows and consequently, by induction (3.8) is valid for all n. Clearly the sequences $\{V_n\}$, $\{W_n\}$ are uniformly bounded on J. To show that they are equicontinuous, consider for any $s < t$, where t, $s \in J$,

$$\begin{aligned} D[V_n(t), V_n(s)] = D\Bigg[&U_0 + \int_0^t \{F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi))\}\, d\xi, \\ &U_0 + \int_0^s \{F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi))\}\, d\xi\Bigg] \end{aligned}$$

$$= D\left[\int_0^t \{F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi)\}\, d\xi,\ \int_0^s \{F(\xi, V_{n-1}(\xi) + G(\xi, W_{n-1}(\xi))\}\, d\xi\right]$$

$$\leq \int_s^t D[F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi)), \theta]\, d\xi \leq M|t - s|.$$

Here we have utilized the properties of integral and the metric D, together with the fact $F + G$ are bounded since $\{V_n\}$, $\{W_n\}$ are uniformly bounded. Hence $\{V_n(t)\}$ is equicontinuous on J. The corresponding Ascoli's Theorem [9] now gives a subsequence $\{V_{n_k}(t)\}$ which converges uniformly to $\rho(t) \in K_c(R^n)$, $t \in J$, and since $\{V_n(t)\}$ is monotone nondecreasing sequence, the entire sequence $\{V_n(t)\}$ converges uniformly to $\rho(t)$ on J. Similar arguments apply to the sequence $\{W_n(t)\}$ and $W_n(t) \to R(t)$ uniformly on J. It therefore follows, using the integral representations of (3.6) and (3.7) that $\rho(t)$, $R(t)$ satisfy

$$\left[\begin{aligned} &D_H \rho(t) = F(t, \rho(t)) + G(t, R(t)), \quad \rho(0) = U_0, \\ &D_H R(t) = F(t, R(t)) + G(t, \rho(t)), \quad R(0) = U_0, \quad t \in J, \end{aligned}\right. \tag{3.12}$$

and that

$$V_0 \le \rho \le R \le W_0, \quad t \in J. \tag{3.13}$$

Next we claim that (ρ, R) are coupled minimal and maximal solution of (3.1), that is, if $U(t)$ is any solution of (3.1) such that

$$V_0 \le U \le W_0, \quad t \in J, \tag{3.14}$$

then

$$V_0 \le \rho \le U \le R \le V_0, \quad t \in J. \tag{3.15}$$

Suppose that for some n,

$$V_n \le U \le W_n \quad \text{on} \quad J. \tag{3.16}$$

Then we have using monotone nature of F, G and (3.16),

$$\begin{aligned}
&D_H U = F(t,U) + G(t,U) \ge F(t,V_n) + G(t,W_n), \quad U(0) = U_0,\\
&D_H V_{n+1} = F(t,V_n) + G(t,W_n), \quad V_{n+1}(0) = U_0.
\end{aligned}$$

Corollary 2.1 yields $V_{n+1} \le U$ on J. Similarly $W_{n+1} \ge U$ on J. Hence by induction (3.16) is true for all $n \ge 1$. Now taking limit as $n \to \infty$, we get (3.15) proving the claim. The proof is therefore complete.

Corollary 3.1 *If, in addition to the assumptions of Theorem 3.1, F and G satisfy whenever $X \ge Y$, $X, Y \in K_c(R^n)$,*

$$F(t,X) \le F(t,Y) + N_1(X - Y)$$

and

$$G(t,X) + N_2(X - Y) \ge G(t,Y),$$

where N_1, $N_2 > 0$. Then $\rho = R = U$ is the unique solution of (3.1).

Proof Since $\rho \le R$ on J, it is enough to prove that $R \le \rho$ on J. We know that

$$\begin{aligned}
&D_H \rho = F(t,\rho) + G(t,R), \quad \rho(0) = U_0,\\
&D_H R = F(t,R) + G(t,\rho), \quad R(0) = U_0, \quad t \in J.
\end{aligned}$$

Using the assumptions, we then get

$$D_H(R - \rho) \le (N_1 + N_2)(R - \rho),$$

which leads to by Theorem 2.1, $R \le \rho$ on J, proving the claimed uniqueness of $\rho = R = U$, completing the proof.

Several remarks are now in order.

Remark 3.1

(1) In Theorem 3.1, if $G(t,Y) \equiv 0$, then we get a result when F is nondecreasing.

(2) In (1) above, suppose that F is not nondecreasing but $\tilde{F}(t,X) = F(t,X) + MX$ is nondecreasing in X for each $t \in J$, for some $M > 0$, then one can consider the IVP

$$D_H U + MU = \tilde{F}(t,U), \quad U(0) = U_0,$$

where $\tilde{F}(t,X) = F(t,X) + MX$ to obtain the same conclusion as in (1). To see this, use the transformation $\tilde{U} = Ue^{Mt}$ so that

$$\begin{gathered} D_H\tilde{U} = [D_HU + MU]e^{Mt} = \tilde{F}(t, \tilde{U}e^{-Mt})e^{Mt} \equiv F_0(t,\tilde{U}), \\ \tilde{U}(0) = U_0. \end{gathered} \tag{3.17}$$

Clearly (3.17) has $\tilde{V} = Ve^{Mt}$ as a lower solution and $\tilde{W} = We^{Mt}$ as an upper solution and therefore we have the same conclusion as in (1).

(3) If $f(t,X) \equiv 0$ in Theorem 3.1, then we obtain the result for G nonincreasing.

(4) If in (3) above, G is not monotone but $\tilde{G}(t,Y) = G(t,Y) - MY$, $M > 0$ is nonincreasing in Y for each $t \in J$, then one can consider the IVP

$$D_HU - MU = \tilde{G}(t,U), \quad U(0) = U_0.$$

The transformation $\tilde{U} = Ue^{-Mt}$ gives the IVP

$$D_H\tilde{U} = G_0(t,\tilde{U}), \quad \tilde{U}(0) = U_0, \tag{3.18}$$

where $G_0(t,\tilde{U}) = \tilde{G}(t,\tilde{U}e^{Mt})e^{-Mt}$. In this case, we need to assume that (3.18) has coupled lower and upper solutions of (3.18) to get the same conclusion as in (3).

(5) Suppose that in Theorem 3.1, $G(t,Y)$ is nonincreasing in Y and $F(t,X)$ is not monotone but $\tilde{F}(t,X) = F(t,X) + MX$, $M > 0$ is nondecreasing in X. Then we consider the IVP

$$D_HU + MU = \tilde{F}(t,U) + G(t,U), \quad U(0) = U_0. \tag{3.19}$$

The transformation as in (2) yields the conclusion by Theorem 3.1 in this case as well.

(6) If F in Theorem 3.1 is nondecreasing and G is not monotone but $\tilde{G}_0(t,Y) = G(t,Y) - MY$, $M > 0$ is nonincreasing in Y for each $t \in J$, then we consider the IVP

$$D_HU - MU = F(t,U) + \tilde{G}(t,U), \quad U(0) = U_0,$$

and employ the same transformation as in (4) to obtain

$$D_H\tilde{U} = F_0(t,\tilde{U}) + G_0(t,\tilde{U}), \quad \tilde{U}(0) = U_0, \tag{3.20}$$

where $F_0(t,\tilde{U}) = F(t,\tilde{U}e^{Mt})e^{-Mt}$ and $G_0(t,\tilde{U}) = \tilde{G}(t,\tilde{U}e^{Mt})e^{-Mt}$. If we assume that (3.20) has coupled lower and upper solutions of type I then we get by Theorem 3.1 the same result in this case also.

(7) If both F and G are not monotone in Theorem 3.1 but $\tilde{F}(t,X) = F(t,X) + MX$, $M > 0$, $\tilde{G}(t,Y) = G(t,Y) - NY$, $N > 0$ are nondecreasing and nonincreasing respectively, then we consider the IVP

$$D_HU + (M-N)U = \tilde{F}(t,U) + \tilde{G}(t,U), \quad U(0) = U_0,$$

one can utilize a similar transformation to obtain

$$D_H\tilde{U} = F_0(t,\tilde{U}) + G_0(t,\tilde{U}), \quad \tilde{U}(0) = U_0, \tag{3.20*}$$

where F_0, G_0 are defined suitably as before. Assuming that (3.20*) has coupled lower and upper solutions of type I, one gets the same conclusion by Theorem 3.1.

Let us next consider utilizing the coupled lower and upper solutions of type II. In this case, we don't need to assume the existence of coupled lower and upper solutions of type II of (3.1) since one can construct them under the given assumptions. However, we have to pay a price to get monotone flows, by assuming certain conditions on the second iterates. Also, we get alternative sequences which are monotone but complicated.

Theorem 3.2 *Assume that* (A_2) *of Theorem 3.1 holds. Then for any solution* $U(t)$ *of (3.1) with* $V_0 \le U \le W_0$ *on* J*, we have the iterates* $\{V_n\}$, $\{W_n\}$ *satisfying*

$$V_0 \le V_2 \le \dots \le V_{2n} \le U \le V_{2n+1} \le \dots \le V_3 \le V_1 \quad on \quad J, \tag{3.21}$$

$$W_1 \le W_3 \le \dots \le W_{2n+1} \le U \le W_{2n} \le \dots \le W_2 \le W_0 \quad on \quad J, \tag{3.22}$$

Provided $V_0 \le V_2$, $W_2 \le W_0$ *on* J*, where the iterative schemes are given by*

$$D_H V_{n+1} = F(t, W_n) + G(t, V_n), \quad V_{n+1}(0) = U_0, \tag{3.23}$$

$$D_H W_{n+1} = F(t, V_n) + G(t, W_n), \quad W_{n+1}(0) = U_0, \quad on \quad J. \tag{3.24}$$

Moreover, the monotone sequences $\{V_{2n}\}$, $\{V_{2n+1}\}$, $\{W_{2n}\}$, $\{W_{2n+1}\} \in K_c(R^n)$ *converge to* ρ, R, ρ^*, R^* *in* $K_c(R^n)$ *respectively and verify*

$$\begin{aligned}
&D_H R = F(t, R^*) + G(t, \rho), && R(0) = U_0,\\
&D_H \rho = F(t, \rho^*) + G(t, R), && \rho(0) = U_0,\\
&D_H R^* = F(t, R) + G(t, \rho^*), && R^*(0) = U_0,\\
&D_H \rho^* = F(t, \rho) + G(t, R^*), && \rho^*(0) = U_0, \quad on \quad J.
\end{aligned}$$

Proof We shall first show that coupled lower and upper solutions V_0, W_0 of type II of (3.1) exist on J satisfying $V_0 \le W_0$ on J. For this purpose, consider the IVP

$$D_H Z = F(t, \theta) + G(t, \theta), \quad Z(0) = U_0. \tag{3.25}$$

Let $Z(t)$ be the unique solution of (3.25) which exists on J. Define V_0, W_0 by

$$R_0 + V_0 = Z \quad \text{and} \quad W_0 = Z + R_0,$$

where the positive vector $R_0 = (R_{01}, R_{02}, \dots, R_{0n})$ is chosen sufficiently large so that we have $V_0 \le \theta \le W_0$ on J. Then using the monotone character of F and G, we arrive at

$$\begin{gathered}
D_H V_0 = D_H Z = F(t, \theta) + G(t, \theta) \le F(t, W_0) + G(t, V_0),\\
V_0(0) = Z(0) - R_0 \le Z(0) = U_0.
\end{gathered}$$

Similarly, $D_H W_0 \ge F(t, V_0) + G(t, W_0)$, $W_0(0) \ge U_0$. Thus V_0, W_0 are the coupled lower and upper solutions of type II of (3.1).

Let $U(t)$ be any solution of (3.1) such that $V_0 \le U \le W_0$ on J. We shall show that

$$\begin{gathered}
V_0 \le V_2 \le U \le V_3 \le V_1,\\
W_1 \le W_3 \le U \le W_2 \le W_0 \quad \text{on} \quad J.
\end{gathered} \tag{3.26}$$

Since U is a solution of (3.1), we have using the monotone character of F and G and the fact $V_0 \leq U \leq W_0$,

$$D_H U = F(t, U) + G(t, U) \leq F(t, W_0) + G(t, V_0), \quad U(0) = U_0,$$

and V_1 satisfies

$$D_H V_1 = F(t, W_0) + G(t, V_0), \quad V_1(0) = U_0, \quad \text{on} \quad J. \tag{3.27}$$

Hence Corollary 2.1 yields $U \leq V_1$ on J. Similarly, $W_1 \leq U$ on J. Next we show that $V_2 \leq U$ on J. Note that

$$D_H V_2 = F(t, W_1) + G(t, V_1), \quad V_2(0) = U_0,$$

and because of monotonicity of F and G, we get

$$D_H U = F(t, U) + G(t, U) \geq F(t, W_1) + G(t, V_1), \quad U(0) = U_0 \quad \text{on} \quad J.$$

Corollary 2.1 therefore gives $V_2 \leq U$ on J. A similar argument shows that $U \leq W_2$ on J. Next we find utilizing the assumption $V_0 \leq V_2$, $W_2 \leq W_0$ on J and monotonicity of F and G,

$$D_H V_3 = F(t, W_2) + G(t, V_2) \leq F(t, W_0) + G(t, V_0), \quad V_3(0) = U_0 \quad \text{on} \quad J.$$

This together with (3.27) shows by Corollary 2.1 that $V_3 \leq V_1$, on J. In the same way one can show that $W_1 \leq W_3$ on J. Also, employing a similar reasoning, one can prove that $U \leq V_3$ and $W_3 \leq U$ on J, proving the relations (3.26).

Now assuming for some $n > 2$, the inequalities

$$\begin{gathered} V_{2n-4} \leq V_{2n-2} \leq U \leq V_{2n-1} \leq V_{2n-3}, \\ W_{2n-3} \leq W_{2n-1} \leq U \leq W_{2n-2} \leq W_{2n-4}, \quad \text{on} \quad J, \end{gathered}$$

to hold, it can be shown, employing similar arguments that

$$\begin{gathered} V_{2n-2} \leq V_{2n} \leq U \leq V_{2n+1} \leq V_{2n-1}, \\ W_{2n-1} \leq W_{2n+1} \leq U \leq W_{2n} \leq W_{2n-2}, \quad \text{on} \quad J. \end{gathered}$$

Thus by induction (3.21) and (3.22) are valid for all $n = 0, 1, 2, \ldots$.

Since V_n, $W_n \in K_c(R^n)$ for all n, employing a similar reasoning as in Theorem 3.1, we conclude that the limits

$$\begin{aligned} &\lim_{n\to\infty} V_{2n} = \rho, && \lim_{n\to\infty} V_{2n+1} = R, \\ &\lim_{n\to\infty} W_{n+1} = \rho^*, && \lim_{n\to\infty} W_{2n} = R^*, \end{aligned}$$

exist, in $K_c(R^n)$, uniformly on J. It therefore follows using the integral representations (3.23) and (3.24) suitably that ρ, ρ^*, R, R^* satisfy corresponding set differential equations given in Theorem 3.2 on J. Also, from (3.21)and (3.22), we arrive at

$$\rho \leq U \leq R, \quad \rho^* \leq U \leq R^* \quad \text{on} \quad J.$$

The proof is therefore complete.

Corollary 3.2 *Under the assumptions of Theorem 3.2 if F and G satisfy the assumptions of Corollary 3.1, then $\rho = \rho^* = R = R^* = U$ is the unique solution of (3.1).*

Proof Let $q_1 + \rho = R$, $q_2 + \rho^* = R^*$ so that $q_1,\ q_2 \geq 0$ on J, since $\rho \leq R$ and $\rho^* \leq R^*$ on J. It then follows using the assumptions, that

$$D_H(q_1 + q_2) \leq (N_1 + N_2)(q_1 + q_2), \quad q_1(0) + q_2(0) = 0 \quad \text{on} \quad J.$$

This implies that $q_1 + q_2 \leq 0$ on J and consequently, we get

$$U = \rho = R \quad \text{and} \quad \rho^* = R^* = U \quad \text{on} \quad J,$$

and this proves the claim of Corollary 3.2.

Theorem 3.2 also has several remarks which correspond to the remarks of Theorem 3.1. To avoid monotony we do not list them again. For similar results which unify monotone iterative technique refer to [5].

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