Subharmonic Solutions of a Class of Hamiltonian Systems

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Abstract: In this paper, we prove the existence of subharmonic solutions for the non autonomous Hamiltonian system: \( \dot{u}(t) = J \nabla H(t, u(t)) \) when \( H \) is convex and non coercive.

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1 Introduction and Statement of Results

Let \( G \in C^1(\mathbb{R}^n, \mathbb{R}) \) be a convex function, \( A, B \in C(\mathbb{R}, \mathcal{M}_n(\mathbb{R})) \) be periodic with minimal period \( T \) \((T > 0)\), \( B(t) \) be invertible for all \( t \in \mathbb{R} \) and \( h = (f, g) \in C(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^n) \) be \( T \)-periodic with mean value zero.

Let \( H(t, (r,p)) = G(A(t)r + B(t)p) + h(t), (r, p) > , \forall (r, p) \in \mathbb{R}^n \times \mathbb{R}^n, \forall t \in \mathbb{R} \).

In this paper we consider the Hamiltonian system of ordinary differential equations

\[
\dot{u}(t) = J \nabla H(t, u(t)),
\]

\((\mathcal{H}_h)\)

where \( \nabla H \) is the first derivative of the Hamiltonian \( H \) with respect to \( (r, p) \) and \( J \) is the standard symplectic \((2n \times 2n)\)-matrix

\[
J = \begin{pmatrix}
0 & -I_n \\
I_n & 0
\end{pmatrix}.
\]

The motion of a relativist particle submitted to an electromagnetic field is governed by a noncoercive Hamiltonian system. However, most of results proving the existence of solutions to systems like \((\mathcal{H}_h)\) have been made use of a coercivity assumption on \( H \), i.e.,

\[
\lim_{|x| \to +\infty} H(t, x) = \infty,
\]

see for example [5, 8, 9, 12] and references therein.

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Timoumi investigates the case of non coercivity when $H$ is convex (see [10, 11]). The purpose of this paper is to improve and complete the results obtained in [10, 11] dealing with this problem.

In the first theorem we establish the existence of subharmonic solutions, i.e., periodic solutions with minimal period in the set $\{kT, k \in \mathbb{N}, k \geq 2\}$ for the Hamiltonian system of ordinary differential equations $(H_0)$.

The problem of search for subharmonics is classical, it has been dealt with using various methods, especially index theories in different settings, see [3, 5, 6, 12].

In [10], Timoumi studied the question when the Hamiltonian has the form

$$H(t, (r, p)) = f(|p - A(t)r|),$$

where $f: \mathbb{R}_+ \to \mathbb{R}_+$ such that:

$$\exists \lambda, \mu > 0 / f(t) \leq \lambda t + \mu \quad \forall \, t \geq 0$$

and the matrix $A(t)$ satisfies

1. $A^*(t) = -A(t) \quad \forall \, t \in \mathbb{R}$
2. $\int_0^T A(t) \, dt \neq 0.$

Here, we try to conserve the same results when the Hamiltonian is subquadratic and $A(t)$ belongs to a larger set of matrices.

Precisely, we assume

$(H_1)$ $\lim_{|x| \to +\infty} G(x) = +\infty$;

$(H_2)$ $\lim_{|x| \to \infty} \frac{G(x)}{|x|^2} = 0$;

$(H_3)$ $G'$ is one to one;

$(H_4)$ $C_0 = \int_0^T B^{-1}(t)A(t) \, dt$ is non symmetric.

**Theorem 1.1** Under the above assumptions, for all $k \in \mathbb{N}^*$, $(H_0)$ possesses a $kT$ periodic solution $u_k = (r_k, p_k)$ satisfying

(i) $\lim_{k \to +\infty} \|Ar_k + Bp_k\|_{\infty} = +\infty.$

(ii) The minimal period of $u_k$ is $kT$ for any sufficiently large and prime integer $k$.

**Corollary 1.1** Under the assumptions $(H_2)$, $(H_4)$ and $(H_5)$ $G$ is strictly convex;

$(H_6)$ $\lim_{|x| \to \infty} \frac{G(x)}{|x|} = +\infty$

the conclusion of Theorem 1.1 holds.

The second result concerns the forced case ($h \neq 0$), where $h$ is interpreted as exterior forcing term. Here we prove the existence of a non constant $T$-periodic solution for $(H_h)$ without the following assumption, needed in [11]

$$\forall \, r \in \mathbb{R}^n \setminus \{0\} \quad t \mapsto A(t)r \quad \text{is non constant}.$$

Assume that

$(H_7)$ $G(x) > G(0), \forall x \in \mathbb{R}^n \setminus \{0\}$;

$(H_8)$ $(B^{-1}A)^* g \neq f.$
Theorem 1.2 Under assumptions \((H_1), (H_2), (H_7), (H_8)\), the problem \((\mathcal{H}_h)\) possesses a non constant \(T\)-periodic solution.

Remark 1.1 The assumption \((H_8)\) is technical, it will be used only to guarantee the non constancy of solution for \((\mathcal{H}_h)\).

2 Proof of Theorem 1.1

Proof of the first part:

We use the dual action of Clarke-Ekeland.

Denote \(H_0(t, r, p) = G(A(t)r + B(t)p)\). \(H_0\) is convex with respect to \((r, p)\) and its Fenchel’s conjugate \(H_0^*\) is given by

\[
\forall (s, q) \in \mathbb{R}^n \times \mathbb{R}^n, \quad H_0^*(t, s, q) = \begin{cases} 
G^*(B^{-1}*q) & \text{if } s = (B^{-1}A)^*q, \\
\infty & \text{otherwise.}
\end{cases}
\]

For all \(k \in \mathbb{N}^*\) we consider the functional

\[
\Phi_k(w) = \frac{1}{2} \int_0^{kT} \langle Jw, \pi w \rangle \, dt + \int_0^{kT} H_0^*(t, w) \, dt
\]

defined on the space

\[
L^2_0(0, kT, \mathbb{R}^{2n}) = \left\{ w \in L^2(0, kT, \mathbb{R}^{2n}) \, \middle| \, \int_0^{kT} w(t) \, dt = 0 \right\},
\]

where \(\pi w\) is the primitive of \(w\) with mean value zero.

Also, for all \(v \in L^2_0(0, kT, \mathbb{R}^n)\) we define

\[
\Psi_k(v) = \int_0^{kT} \langle B^{-1}A\pi v, v \rangle \, dt + \int_0^{kT} G^*(B^{-1}*v) \, dt.
\]

Obviously, we have \(\Phi_k(w) = \Psi_k(v)\) for all \(w = ((B^{-1}A)^*v, v) \in L^2_0(0, kT, \mathbb{R}^{2n})\).

Hence, we use the functional \(\Psi_k\) on the space \(E_k = L^2_0(0, kT, \mathbb{R}^n)\).

For \(v \in E_k\) we set

\[
g(v) = \int_0^{kT} G^*(B^{-1}*v) \, dt
\]

and

\[
Q(v) = \int_0^{kT} \langle B^{-1}A\pi v, v \rangle \, dt.
\]
Lemma 2.1 \( \Psi_k \) has a global minimum on \( E_k \) attained in \( \bar{v}_k \).

Proof Using Wirtinger’s inequality, there exists a constant \( \alpha_0 > 0 \) such that

\[
Q(v) \geq -\alpha_0 \|v\|_{L^2}^2, \quad \forall v \in E_k.
\]

(1)

By \((H_2)\), for all \( \alpha > 0 \) there exists \( \beta > 0 \) such that

\[
G(x) \leq \alpha |x|^2 + \beta, \quad \forall x \in \mathbb{R}^n
\]

(2)

and by going to the conjugate, we get

\[
G^*(y) \geq \frac{1}{4\alpha} |y|^2 - \beta, \quad \forall y \in \mathbb{R}^n
\]

so

\[
g(v) \geq \frac{1}{4\alpha} \|B^{-1}v\|_{L^2}^2 - \beta kT, \quad \forall v \in E_k.
\]

(3)

From (1) and (3) there exists a constant \( \gamma > 0 \) such that

\[
\Psi_k(v) \geq \gamma \|v\|_{L^2}^2 - \beta kT, \quad \forall v \in E_k.
\]

(4)

Let \((v_n) \in E_k\) be a minimizing sequence of \( \Psi_k \). From (4), \((v_n)\) is bounded and since \( E_k \) is reflexive, there exists a subsequence \((v_{n_j})\) weakly convergent to \( \bar{v}_k \).

Moreover, \( g \) is weakly lower semi-continuous, so

\[
\lim_{j \to +\infty} \int_0^{kT} G^*(B^{-1}v_{n_j}) \, dt \geq \int_0^{kT} G^*(B^{-1}\bar{v}_k) \, dt.
\]

Since the operator \( \pi \) is compact then

\[
\pi v_{n_j} \rightharpoonup \pi \bar{v}_k
\]

and so

\[
\lim_{j \to +\infty} \int_0^{kT} \langle B^{-1}A\pi v_{n_j}, v_{n_j} \rangle \, dt = \int_0^{kT} \langle B^{-1}A\pi \bar{v}_k, \bar{v}_k \rangle \, dt.
\]

Consequently

\[
\min_{E_k} \Psi_k = \Psi_k(\bar{v}_k).
\]

Lemma 2.2 For all \( v \in E_k \) on which \( g \) is finite we have

\[
\tilde{\partial}g(v) = \left\{ u \in L^2(0,kT,\mathbb{R}^n) / \exists \xi \in \mathbb{R}^n : B(t)(u(t) + \xi) \in \partial G^*(B^{-1}v) \text{ a.e.} \right\},
\]

where \( \tilde{\partial}g \) denotes the restriction of \( g \) on \( E_k \).

Proof Let \( u \in L^2(0,kT,\mathbb{R}^n) \) and \( \xi \in \mathbb{R}^n \) such that

\[
B(t)(u(t) + \xi) \in \partial G^*(B^{-1}v) \text{ a.e.}
\]
so it’s easy to show that \( u \in \bar{\partial}g(v) \).

Conversely, it’s clear that for \( v \in E_k \)

\[
\bar{\partial}g(v) = \partial(g + \delta_{E_k})(v),
\]

where

\[
\delta_{E_k}(v) = \begin{cases} 
0 & \text{if } v \in E_k, \\
+\infty & \text{otherwise}.
\end{cases}
\]

Since

\[
\partial g(v) = \{u \in L^2(0,kT, \mathbb{R}^n)/B(t)u(t) \in \partial G^*(B^{-1}v) \text{ a.e.}\}
\]

and

\[
\partial \delta_{E_k} = \mathbb{R}^n
\]

the result will be proved if

\[
\partial(g + \delta_{E_k}) = \partial g + \partial \delta_{E_k}
\]

The functionals \( g \) and \( \delta_{E_k} \) are proper convex and l.s.c., it suffices to prove that the inf-convolute \( g^*\nabla \delta_{E_k}^* \) is exact (i.e., the infimum is attained).

Indeed, we have

\[
(g^*\nabla \delta_{E_k}^*)(v) = \inf_{x \in \mathbb{R}^n} \frac{kT}{0} G(B(t)v + B(t)x) dt.
\]

The function

\[
F(x) = \int_0^{kT} G(B(t)v + B(t)x) dt, \quad \forall x \in \mathbb{R}^n
\]

is continuous on \( \mathbb{R}^n \), so by \((H_1)\) and the fact that \( B(t) \) is invertible it’s clear that

\[
\lim_{|x| \to +\infty} F(x) = +\infty
\]

and consequently \( F \) attains its minimum on \( \mathbb{R}^n \).

**Conclusion of the first part:**

Let \( \bar{v}_k \in E_k \), where \( \Psi_k \) attains its minimum, we have

\[
0 \in Q'(\bar{v}_k) + \bar{\partial}g(\bar{v}_k)
\]

which implies that

\[
-Q'(\bar{v}_k) \in \bar{\partial}g(\bar{v}_k).
\]

By Lemma 2.2, there exists \( \xi_k \in \mathbb{R}^n \) such that

\[
B(-B^{-1}A\pi \bar{v}_k + \pi(B^{-1}A)^*\bar{v}_k + \xi_k) \in \partial G^*(B^{-1}v_k) \text{ a.e.}
\]

Setting

\[
r_k = -\pi \bar{v}_k, \quad p_k = \pi(B^{-1}A)^*\bar{v}_k + \xi_k, \quad u_k = (r_k, p_k).
\]

We get, by Fenchel’s reciprocity

\[
B^{-1}v_k = \nabla G(Ar_k + Bp_k)
\]
and
\[
\begin{align*}
    \dot{r}_k &= -\bar{v}_k = -B^*\nabla G(Ar_k + Bp_k) = -\frac{\partial H_0}{\partial p}(t, u_k(t)) \\
    \dot{p}_k &= (B^{-1}A)^*\bar{v}_k = A^*\nabla G(Ar_k + Bp_k) = \frac{\partial H_0}{\partial r}(t, u_k(t)).
\end{align*}
\]
Therefore \( u_k \) is a solution of \((H_0)\), moreover since \( \bar{v}_k \in E_k \), \( r_k \) is \( kT \) periodic.

In the other hand \( r_k \) is \( C^1 \) so \( \dot{r}_k \) is \( kT \) periodic. By \((H_3)\) and \((6)\), we have
\[
p_k = B^{-1}[\nabla G^{-1}(-B^{-1}A^*\dot{r}_k) - Ar_k]
\]
so \( p_k \) is \( kT \) periodic and then \( u_k \) is \( kT \) periodic.

**Proof of the second part:**

By \((H_1)\) and the convexity assumption of \( G \) there exist two constants \( m, M > 0 \) such that
\[
G(x) \geq m|x| - M, \quad \forall x \in \mathbb{R}^n
\]
so for all \( y \in \mathbb{R}^n \) such that \( |y| \leq m \) we have
\[
-G(0) \leq G^*(y) \leq M.
\]
Let
\[
q(t) = a \cos \left( \frac{2\pi}{kT} t \right) + b \sin \left( \frac{2\pi}{kT} t \right)
\]
with any \((a, b) \in \mathbb{R}^{2n}\).

It’s clear that \( q \in E_k \) and a simple computation gives for all \( k \geq 3 \)
\[
Q(q) = \frac{k^2T^2}{4\pi} \prec (C_0 - C_0^*)a, b \succ .
\]
By the assumption \((H_4)\), we can choose \((a, b)\) such that
\[
\begin{align*}
    \prec (C_0 - C_0^*)a, b \succ &< 0 \\
    \|B^{-1}q\|_{\infty} &\leq m.
\end{align*}
\]
Setting \( \delta = -\frac{T}{4\pi} \prec (C_0 - C_0^*)a, b \succ \), we have
\[
Q(q) = -\delta Tk^2, \quad \text{with} \quad \delta > 0 \text{ independent of } k.
\]
Now, by \((8)\) and \((9)\) we have
\[
\Psi_k(\bar{v}_k) \leq \Psi_k(q) \leq -\delta Tk^2 + MkT, \quad \forall k \geq 3
\]
and
\[
Q(\bar{v}_k) \leq -\delta Tk^2 + MkT + G(0)kT \leq 0
\]
for all \( k \geq k_0 \) sufficiently large.
In the other hand, by duality we have
\[ G(Ar_k + Bp_k) + G^*(B^{-1}v_k) = \langle Ar_k + Bp_k, B^{-1}v_k \rangle \]
and by integration, we obtain
\[ \int_0^{kT} G(Ar_k + Bp_k) \, dt + \int_0^{kT} G^*(B^{-1}v_k) \, dt = -2 \int_0^{kT} \langle B^{-1}A\pi \bar{v}_k, \bar{v}_k \rangle \, dt. \]

Then it follows from (10) and (11) that
\[ \frac{1}{kT} \int_0^{kT} G(Ar_k + Bp_k) \, dt \geq \delta k - M, \quad \forall k \geq k_0. \]

Hence by (2) we obtain
\[ \delta k - M \leq \frac{\alpha}{kT} \int_0^{kT} |Ar_k + Bp_k|^2 \, dt + \beta \leq \alpha \|Ar_k + Bp_k\|^2_\infty + \beta, \quad \forall k \geq k_0 \]
and consequently
\[ \lim_{k \to +\infty} \|Ar_k + Bp_k\|_\infty = +\infty. \]

To prove (ii) of Theorem 1.1, we need the following lemma:

**Lemma 2.3** For all T-periodic solution \( u = (r, p) \) of \( (\mathcal{H}_0) \) we have
1. \[ \int_0^T |\dot{u}|^2 \, dt \leq \frac{2s(\beta + M)\pi T}{\pi - \alpha T}, \]
2. \[ \frac{1}{T} \int_0^T |Ar + Bp| \, dt \leq \frac{(\beta + M)\pi}{m(\pi - \alpha T)}. \]

**Proof** By \( (H_2) \) and \( (7) \), for all \( \alpha \in \] 0, \( \frac{T}{\pi} \] there exists \( \beta > 0 \) only dependent on \( \alpha \) such that
\[ -M \leq H_0(t, x) \leq \frac{\alpha}{2} |x|^2 + \beta, \quad \forall x \in \mathbb{R}^{2n}, \quad \forall t \in [0, T]. \]

A result of convex analysis gives
\[ \frac{1}{2\alpha} |\nabla H_0(t, x)|^2 \leq \langle \nabla H_0(t, x), x \rangle + \beta + M, \quad \forall x \in \mathbb{R}^{2n}. \]

It follows from \( (\mathcal{H}_0) \) that
\[ \frac{1}{2\alpha} \int_0^T |\dot{u}|^2 \, dt + \int_0^T \langle J\dot{u}, u \rangle \, dt \leq (\beta + M)T. \]
so
\[
\left(\frac{1}{2\alpha} - \frac{T}{2\pi}\right) \int_0^T |\dot{u}|^2 dt \leq (\beta + M)T
\]
and therefore
\[
\int_0^T |\dot{u}|^2 dt \leq \frac{2\alpha(\beta + M)\pi T}{\pi - \alpha T}.
\]
(12)

By convexity and (7), for all \(T\)-periodic solution \(u = (r, p)\) of \((H_0)\) we have
\[
m \int_0^T |Ar + Bp| dt - MT \leq TG(0) + \frac{T}{2\pi} \int_0^T |\dot{u}|^2 dt.
\]
(13)

By (12) and (13), we deduce the desired result.

Now, we shall prove that the minimal period of \(u_k\) tends to \(+\infty\) as \(k\) tends to \(+\infty\). If not, there exists \(\tau > 0\) and a subsequence \((k_n)\) such that the minimal period \(T_{k_n}\) of \(u_{k_n}\) satisfies \(T_{k_n} \leq \tau, \forall n \in \mathbb{N}\). By Lemma 2.3, with \(T\) replaced by \(T_{k_n}\), we get
\[
\int_0^{T_{k_n}} |\dot{u}_{k_n}|^2 dt \leq \frac{2\alpha(\beta + M)\pi T_{k_n}}{\pi - \alpha T_{k_n}} \leq \frac{2\alpha(\beta + M)\pi \tau}{\pi - \alpha \tau}
\]
(14)

and
\[
\frac{1}{T_{k_n}} \int_0^{T_{k_n}} |Ar_{k_n} + Bp_{k_n}| dt \leq \frac{\pi(\beta + M)}{m(\pi - \alpha \tau)}.
\]
(15)

Writing \(u_k = \bar{u}_k + \tilde{u}_k\) with \(\bar{u}_k = \frac{1}{T_k} \int_0^{T_k} u_k(t) dt\).

By Sobolev’s inequality and (14), we obtain
\[
||\tilde{u}_{k_n}||_\infty \leq \frac{\tau}{12} \left(\frac{2\alpha(\beta + M)\pi \tau}{\pi - \alpha \tau}\right)
\]
thus \(||\tilde{u}_{k_n}||_\infty\) is bounded. By (5) we have
\[
\bar{u}_{k_n} = (\bar{r}_{k_n}, \bar{p}_{k_n}) = (0, \xi_{k_n}).
\]

Since \(||u_{k_n}||_\infty \to +\infty\) and \(||\bar{u}_{k_n}||_\infty\) is bounded so \(||\xi_{k_n}|| \to +\infty\).

In the other hand, by (15) we deduce that
\[
\frac{1}{T} \int_0^T |B(t)\xi_{k_n}| dt = \frac{1}{T_{k_n}} \int_0^{T_{k_n}} |A(t)\bar{r}_{k_n} + B(t)\bar{p}_{k_n}| dt
\]
is bounded, but this is in contradiction with the fact that
\[
|B(t)\xi_{k_n}| \to +\infty, \quad \forall t \in [0, T].
\]
Then, the minimal period $T_k$ of $u_k$ tends to $+\infty$ as $k$ tends to $+\infty$ and so for sufficiently large prime integer $k$, the minimal period of $u_k$ is $kT$.

### 3 Proof of Theorem 1.2

We consider the functional $\Phi$ defined on the space $L_0^2 = L_0^2(0, T, \mathbb{R}^{2n})$ by

$$
\Phi(w) = \frac{1}{2} \int_0^T \langle Jw, \pi w \rangle dt + \int_0^T H_0^*(t, w - h) dt.
$$

Let for $w \in L_0^2$

$$
Q(w) = \frac{1}{2} \int_0^T \langle Jw, \pi w \rangle dt \quad \text{and} \quad \psi(w) = \int_0^T H_0^*(t, w - h) dt.
$$

We follow the same ideas of the proof of Theorem 1.1.

**Lemma 3.1** $\Phi$ achieves its minimum over $L_0^2$ in $\bar{v}$.

**Proof** By $(H_2)$, for all $\alpha \in (0, \frac{2\pi}{T})$ there exists $\beta > 0$ such that

$$
H_0(t, x) \leq \frac{\alpha}{2} |x|^2 + \beta, \quad \forall x \in \mathbb{R}^{2n}, \quad \forall t \in [0, T],
$$

and by going to the conjugate, we get

$$
H_0^*(t, y) \geq \frac{1}{2\alpha} |y|^2 - \beta, \quad \forall y \in \mathbb{R}^{2n}, \quad \forall t \in [0, T]
$$

so

$$
\int_0^T H_0^*(t, w) dt \geq \frac{1}{2\alpha} \|w\|_{L^2}^2 - \beta T, \quad \forall w \in L_0^2.
$$

Moreover, by Wirtinger’s inequality, we get for all $w \in L_0^2$

$$
\Phi(w) \geq \frac{1}{2} \left( \frac{1}{\alpha} - \frac{T}{2\pi} \right) \|w\|_{L^2}^2 + \frac{1}{2\alpha} \|h\|_{L^2}^2 - \frac{1}{\alpha} \|w\|_{L^2} \|h\|_{L^2} - \beta T. \quad (16)
$$

Let $(v_n) \in L_0^2$ be a minimizing sequence of $\Phi$. From (16), $(v_n)$ is bounded and since $L_0^2$ is reflexive, there exists a subsequence $(v_{n_k})$ weakly convergent to $\bar{v}$.

Moreover, $\psi$ is weakly l.s.c., so

$$
\lim \int_0^T H_0^*(t, v_{n_k} - h) dt \geq \int_0^T H_0^*(t, \bar{v} - h) dt
$$

and

$$
\lim_{k \to +\infty} \int_0^T \langle Jv_{n_k}, \pi v_{n_k} \rangle dt = \int_0^T \langle J\bar{v}, \pi \bar{v} \rangle dt.
$$

Consequently

$$
\min_{L_0^2} \Phi = \Phi(\bar{v}).
$$
Lemma 3.2 For every $v \in L^2_0$ on which $\psi$ is finite, we have

$$\bar{\partial} \psi(v) = \{ u \in L^2 : \exists \xi \in \mathbb{R}^{2n} : u(t) + \xi \in \partial H_0^*(t, v(t) - h(t)) \text{ a.e.} \}.$$

Proof Let $I(v) = \int_0^T H_0^*(t, v) \, dt$, $\forall v \in L^2$, then $\psi(v) = I(v - h)$.

For $u, v \in L^2_0$ and $\xi \in \mathbb{R}^{2n}$ such that

$$u(t) + \xi \in \partial H_0^*(t, v(t)) \text{ a.e.,}$$

we can prove easily that $u \in \bar{\partial} I(v)$.

Conversely, it’s clear that for $v \in L^2_0$ we have

$$\bar{\partial} I(v) = \partial (I + \delta_{L^2_0})(v),$$

where

$$\delta_{L^2_0}(v) = \begin{cases} 0 & \text{if } v \in L^2_0, \\ +\infty & \text{otherwise.} \end{cases}$$

Arguing as in proof of Lemma 2.2, it suffices to prove that the inf-convolution $I^* \nabla \delta^*_{L^2_0}$ is exact.

In fact, for $u = (r, p) \in L^2$ we have

$$I^* \nabla \delta^*_{L^2_0}(u) = \inf_{x \in \mathbb{R}^n} \int_0^T H_0(t, u(t) + x) \, dt$$

$$= \inf_{(a, b) \in \mathbb{R}^{2n}} \int_0^T G[A(t)r + B(t)p + A(t)a + B(t)b] \, dt.$$

We need the following lemma:

Lemma 3.3 The function

$$F(a, b) = \int_0^T G(A(t)r + B(t)p + A(t)a + B(t)b) \, dt, \quad \forall (a, b) \in \mathbb{R}^{2n}$$

attains its minimum on $\mathbb{R}^{2n}$.

Proof Let

$$E = \left\{ a \in \mathbb{R}^n : B^{-1}(t)A(t)a = B^{-1}(0)A(0)a, \ \forall \ 0 \leq t \leq T \right\},$$

$E$ is a linear subspace of $\mathbb{R}^n$, so for all $a \in \mathbb{R}^n$ there exists $a_0 \in \mathbb{R}^n$ such that $a - a_0 \in E^\perp$.

Notice that

$$F(a, b) = F(a - a_0, b + B^{-1}A(0)a_0) \in F(E^\perp \times \mathbb{R}^n)$$
so
\[ \inf_{\mathbb{R}^{2n}} F = \inf_{E^\perp \times \mathbb{R}^n} F. \]

Arguing by contradiction, we suppose that \( \inf_{E^\perp \times \mathbb{R}^n} F \) is not attained so there exists a sequence \( (a_n, b_n) \in E^\perp \times \mathbb{R}^n \) such that
\[ \lim_{n \to +\infty} (a_n^2 + b_n^2) = +\infty \quad \text{and} \quad \lim_{n \to +\infty} F(a_n, b_n) = \inf F. \]

It follows that
\[ \lim_{n \to +\infty} \frac{F(a_n, b_n)}{\sqrt{a_n^2 + b_n^2}} = 0. \]

In the other hand, by convexity of \( G \), we have for \( n \) large enough
\[ \int_{0}^{T} G \left( \frac{A(t)r + B(t)p + A(t)a_n + B(t)b_n}{\sqrt{a_n^2 + b_n^2}} \right) dt \leq \frac{F(a_n, b_n)}{\sqrt{a_n^2 + b_n^2}} + \left( 1 - \frac{1}{\sqrt{a_n^2 + b_n^2}} \right) G(0)T. \]

The sequence
\[ \left( \frac{a_n}{\sqrt{a_n^2 + b_n^2}}, \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \right) \in E^\perp \times \mathbb{R}^n \]
is bounded, then by going to the limit in the above inequality through a subsequence, we obtain
\[ \int_{0}^{T} G(A(t)a + B(t)b) dt \leq G(0)T \]

for some \( (a, b) \in E^\perp \times \mathbb{R}^n \) such that \( a^2 + b^2 = 1 \). Then
\[ \int_{0}^{T} [G(A(t)a + B(t)b) - G(0)] dt \leq 0 \]

and by \((H_7)\) we obtain
\[ A(t)a + B(t)b = 0, \quad \forall t \in [0, T] \]

which is equivalent to
\[ B^{-1}(t)A(t)a + b = 0, \quad \forall t \in [0, T], \]

but this is in contradiction with \( a \in E^\perp \) and \( a^2 + b^2 = 1 \).

**Conclusion of the proof**

Let \( \bar{v} \in L_0^2 \) where \( \Phi \) attains its minimum so
\[ 0 \in J\pi \bar{v} + \partial\psi(\bar{v}). \]

By Lemma 3.2, there exists \( \xi \in \mathbb{R}^{2n} \) such that
\[ J\pi \bar{v} + \xi \in \partial H_0^*(t, \bar{v}(t) - h(t)) \quad \text{a.e.} \]
Let $u = J\pi \tilde{v} + \xi$, by Fenchel’s reciprocity, we get

$$\dot{u} = J\ddot{v} = J\nabla H(t, u(t))$$

and it’s clear that $u(0) = u(T)$.

It remains to prove that $u$ is not constant.

Setting $u = (r, p), (H_h)$ is equivalent to

$$\dot{u}(t) = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = J \begin{pmatrix} A^* \\ B^* \end{pmatrix} \nabla G(Ar + Bp) + \begin{pmatrix} f \\ g \end{pmatrix}$$

but $\dot{u} = 0$ gives

$$- \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} A^* \\ B^* \end{pmatrix} \nabla G(Ar + Bp)$$

and then $(B^{-1}A)^* g = f$, which is in contradiction with the assumption $(H_8)$.

References