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Stochastic Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control of Time-Varying Delay Systems

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Abstract: This paper deals with the class of continuous-time linear systems with Markovian jumps and time-delay. The delay in the system dynamics is assumed to be time-varying. Under norm-bounded uncertainties and based on the Lyapunov method, a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller that minimizes the $\mathcal{H}_2$ performance measure when satisfying a prescribed $\mathcal{H}_\infty$ norm bound on the closed-loop system is proposed. LMI-based sufficient conditions for the existence of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller and the upper bound of the performance measure are developed.

Keywords: *Jump linear system; linear matrix inequality; stochastic stability; stochastic stabilizability; norm bounded uncertainty; state feedback.*

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1 Introduction

During the last decades the state feedback control that meets desired performance and/or robustness specifications has attracted a lot of researchers from the control community and different types of controllers were proposed. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state feedback controller belongs to this class of controllers and it consists of determining a state feedback gain that achieves a certain nominal (suboptimal) performance measure subject to a robustness constraint. This feedback controller satisfying simultaneously the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ specifications is interesting since it gives robust stability and nominal performance.

Bernstein and Haddad [1] were the first to introduc the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem. Their approach consists of minimizing an auxiliary cost function subject to the $\mathcal{H}_\infty$ norm constraint and this cost provides an upper bound on the $\mathcal{H}_2$ norm. The work of Berstein and Haddad has been extended to other mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem (see for instance the work in [2, 3]). For other related works on the design of $\mathcal{H}_2/\mathcal{H}_\infty$ controllers by state feedback or output feedback, we refer the reader to Haddad, *et al.* [4], Limbeer, *et al.* [5], Mustapha [6], Rotea and Khargonekar [7] and Saberi, *et al.* [8] and Leibfritz [14]. For results using the LMI formalism we quote the works of Geromel, *et al.* [9], Giusto, *et al.* [10], Kaminer, *et al.* [11], Khargonekar and Rotea [12], and Rotea and Khargonekar [13].

For the time-delay system there exists only one reference that deals with the robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$. This work was done by Kim [15]. The paper considers the norm bounded uncertainties. The time was considered to be time-varying. Kim developed some LMI-based sufficient conditions that solve the robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for uncertain linear systems with time-delay.

As it was mentioned by different papers reported in the literature, there exist some plants that can not be modelled by deterministic time invariant model as it is the case in the work of Kim due maybe to abrupt changes in the dynamics for instance or to any equivalent phenomena that makes the dynamics switches instantaneously and randomly between some finite number of models. This behavior was shown to be adequately represented by the class of Markovian jumping parameters that has recently attracted a lot of researchers due to its power to model different practical situations that the standard time-invariant linear model doesn't do. For more details of this class of systems we refer the reader to Mariton [16] and the references therein. For the class of systems with time-delay and all the connected works we refer the reader to Boukas and Liu [17].

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for the class of linear systems with Markovian Jumping parameters was studied by Costa and Marques for the discrete-time case [18] and Aliyu and Boukas [19] for the continuous-time case. In these references, the given results are not in LMI-based. The problem of $\mathcal{H}_\infty$ control of the class of Markovian jumping parameters systems with time-delay has been tackled by some authors among them we quote the works of [17, 20, 21].

To the best of our knowledge, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of the class of systems we are considering in this paper has never been studied. The extension of the results on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ to the class of Markovian jumping parameters is of great interest for the control community due to the importance of this class of systems in practice. The problem we are addressing here consists of determining a mean-square stabilizing controller that minimizes the upper bound of the $\mathcal{H}_2$ performance measure under the restriction that the $\mathcal{H}_\infty$ performance measure is less than a prescribed value $\gamma > 0$ for all $\omega \in \mathcal{L}_2[0, \infty)$. We are interested by LMI-based conditions that can be easily solved using the existing LMI tools. In this paper we will address the design of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller with or without uncertainties in the dynamics of the class of Markovian jumping parameters with time-varying delay.

The rest of this paper is organized as follows. In Section 2, the problem is stated and the goal of the paper is presented. In Section 3, the main results are given and they include results on stochastic stabilizability and its robustness. A memoryless controller is used in this paper and a design algorithm in terms of the solutions of linear matrix inequalities is proposed to synthesize the controller gains we are using.

Notation. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscript "$^\top$" denotes matrix transposition and the notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrices with compatible dimensions. Y is a constant matrix associated with the controller. $\mathbb{E}\{\cdot\}$ denotes the expectation operator with respective to some probability measure $\mathcal{P}$. L_2 is the space of integral vector over $[0, \infty)$. $\|\cdot\|$ will refer to the Euclidean vector norm whereas $\|\cdot\|$ denotes the L_2-norm over $[0,\infty)$ defined as $\|f\|^2 = \mathbb{E}\Big[\int\limits_0^\infty f^\top(t)f(t)\,dt\Big]$.

2 Problem Statement

Consider a hybrid linear continuous-time system with N modes, i.e., $\mathbf{S} = \{1, 2, \cdots, N\}$ and assume that the mode switching is governed by a continuous-time Markov process $\{r_t, t \geq 0\}$ taking values in the state space $\mathbf{S}$ and having the following infinitesimal generator:

$$\Lambda = (\lambda_{ij}), \quad i, j \in \mathbf{S},$$

where $\lambda_{ij} \geq 0$, $\forall\, j \neq i$, $\lambda_{ii} = -\sum\limits_{j\neq i} \lambda_{ij}$.

The mode transition probabilities are described as follows:

$$P[r_{t+\Delta} = j \mid r_t = i] = \begin{cases} \lambda_{ij}\Delta + o(\Delta), & j \neq i, \\ 1 + \lambda_{ii}\Delta + o(\Delta), & j = i, \end{cases} \tag{1}$$

where $\lim\limits_{\Delta\to 0} o(\Delta)/\Delta = 0$.

Let $x(t) \in \mathbb{R}^n$ be the physical state of the system, which satisfies the following dynamics:

$$\begin{cases} \dot{x}(t) = A(r_t,t)x(t) + A_1(r_t,t)x(t-h(t)) + B(r_t,t)u(t) + B_1(r_t)\omega(t), \\ x(s) = \phi(s), -\tau \leq s \leq 0, \\ z_1(t) = C_1(r_t)x(t) + D_1(r_t)u(t), \\ z_2(t) = C_2(r_t)x(t) + D_2(r_t)u(t), \end{cases} \tag{2}$$

where $u(t) \in \mathbb{R}^m$ is the control input system, $\omega(t) \in \mathbb{R}^l$ is the disturbance to be rejected and/or reference to be tracked, which we assume to belong to $L_2[0,\infty)$, $z_i(t) \in \mathbb{R}^p$, $i = 1, 2$ is the controlled (regulated) signal, $A(r_t,t) = A(r_t) + D_A(r_t)F_1(r_t,t)E_A(r_t) \in \mathbb{R}^{n\times n}$, $A_1(r_t,t) = A_1(r_t) + D_{A1}(r_t)F_2(r_t,t)E_{A1}(r_t) \in \mathbb{R}^{n\times n}$, and $B(r_t,t) = B(r_t) + D_B(r_t)F_3(r_t,t)E_B(r_t) \in \mathbb{R}^{n\times m}$ with $A(r_t)$, $A_1(r_t)$, $B(r_t)$, $B_1(r_t)$, $D_A(r_t)$, $D_{A1}(r_t)$, $D_B(r_t)$, $E_A(r_t)$, $E_{A1}(r_t)$, and $E_B(r_t)$, are known real matrices with appropriate dimensions for each $r_t \in \mathbf{S}$, and $F_k(r_t)$, $k = 1, 2, 3$ are unknown real time-varying matrices with appropriate dimensions satisfying the following:

$$F_k^\top(r_t,t)F_k(r_t,t) \leq I, \quad \forall\, r_t \in \mathbf{S}, \tag{3}$$

$h(t) > 0$ represents the system delay, that satisfies $0 \leq h(t) \leq \tau$, $\dot{h}(t) \leq \beta < 1$, and $\phi(t)$ is a smooth vector-valued initial function in $[-\tau, 0]$.

The initial condition of the system is specified as $(r_0, \phi(\cdot))$ with r_0 is the initial mode and $\phi(.)$ is the initial functional such that

$$x(s) = \phi(s) \in L_2[-\tau, 0] \stackrel{\Delta}{=} \{f(\cdot) | \int\limits_0^\infty f^\top(t)f(t)\,dt < \infty\}.$$

Remark 2.1 The uncertainties that satisfies the conditions (3) are referred to be admissible. The uncertainties we are considering here are time and mode system dependent only. The results we are developing here will remain valid for systems with uncertainties that may depend on time, modes and states systems.

For system (2) with $u(.) \stackrel{\Delta}{=} 0$ for $t \geq 0$, we have the following definitions:

Definition 2.1 *System* (2) with $u(.) \stackrel{\Delta}{=} 0$, $\forall\, t \geq 0$ and all the uncertainties equal to zero, is said to be

(i) *stochastically stable* (SS) if there exists a positive constant $T(r_0, \phi(\cdot))$ such that the following holds for any initial condition $(r_0, \phi(.))$:

$$\mathbb{E}\left[\int_0^\infty \|x(t)\|^2\,dt \,\middle|\, r_0,\ x(s) = \phi(s),\ s \in [-\tau, 0]\right] \leq T(r_0, \phi(\cdot)); \tag{4}$$

(ii) *mean square stable* (MSS) if the following holds for any initial condition $(r_0, \phi(.))$:

$$\lim_{t \to \infty} \mathbb{E}\|x(t)\|^2 = 0;$$

(iii) *mean exponentially stable* (MES) if there exist constants $\alpha(r_0, \phi(\cdot)) > 0$, $\beta > 0$ such that the following holds for any initial condition $(r_0, \phi(.))$:

$$\mathbb{E}[\|x(t)\|^2] \leq \alpha(r_0, \phi(\cdot))e^{-\beta t}. \tag{5}$$

Obviously, MES implies MSS and SS.

Definition 2.2 *System* (2) with $u(.) \stackrel{\Delta}{=} 0$ for $t \geq 0$, is said to be

(i) *robustly stochastically stable* (RSS) if there exists a positive constant $T(r_0, \phi(\cdot))$ such that (4) holds for any initial condition $(r_0, \phi(.))$ and for all admissible uncertainties;

(ii) *robustly mean exponentially stable* (RMES) if there exist constants $\alpha(r_0, \phi(\cdot)) > 0$, $\beta > 0$ such that (5) holds for any initial condition $(r_0, \phi(.))$ and for all admissible uncertainties.

Obviously, we can show that RMES implies RSS.

In the rest of this paper, we will be interested by the design of a state feedback control law in the following form:

$$u(t) = K(r_t)x(t), \tag{6}$$

where $x(t)$ is the system state, and $K(i)$, $i \in \mathbf{S}$ is a constant gain matrix that has to be determined and which constitutes one of our main goal in this paper.

In the rest of this paper, we will assume that we have complete access to the state vector, $x(t)$, and to the mode, r_t at nay time $t \geq 0$.

Definition 2.3 *System* (2) with all the uncertainties equal to zero, is said to be *stabilizable in the stochastic sense* if there exists a control law of the form (6) such that the closed-loop system is stochastically stable for any initial condition $(r_0, \phi(.))$.

Definition 2.4 *System* (2) *is said to be robustly stabilizable in the stochastic sense* if there exists a state feedback controller of the form (6) such that the closed-loop system is robustly stochastically stable for any initial condition $(r_0, \phi(.))$ and for all admissible uncertainties.

Remark 2.2 Notice that the stability in each mode doesn't imply the stochastic stability of the global system. It is the same for the stabilization problem. The stability and the stabilization problems of the class of system we are considering is not a trivial one and more care should be taken when working with this class of systems.

The $\mathcal{H}_2$ performance and $\mathcal{H}_\infty$ performance measures used in the rest of this paper are defined as follows:

$$J_{\mathcal{H}_2} = \mathbb{E}\left[\int_0^\infty z_1^\top(t) z_1(t)\,dt\right]: \ \ \mathcal{H}_2 \text{ performance measure,} \tag{7}$$

$$J_{H_\infty} = \mathbb{E}\left[\int_0^\infty z_2^\top(t) z_2(t) - \gamma^2 \omega^\top(t)\omega(t)\,dt\right]: \ \ \mathcal{H}_\infty \text{ performance measure.} \tag{8}$$

The goal of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control can be summarized as follows: Given the dynamical system (2) find a controller (6) that achieves the minimization of $\mathcal{H}_2$ performance measure and satisfying $\mathcal{H}_\infty$ norm bound within γ (a given real positive constant) for all $\omega(t) \in \mathcal{L}_2[0,\infty)$. In other words, the aim of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control is to minimize the output energy of $z_1(t)$ and at the same time satisfy the prescribed $\mathcal{H}_\infty$ norm bound of the closed-loop system from $\omega(t)$ to $z_2(t)$.

Plugging the controller (6) in the dynamics (2) we get:

$$\begin{cases} \dot{x}(t) = A_K(r_t,t)x(t) + A_1(r_t,t)x(t-h(t)) + B_1(r_t,t)\omega(t), \\ x(s) = \phi(s), \quad -\tau \le s \le 0, \\ z_1(t) = C_{1K}(r_t)x(t), \\ z_2(t) = C_{2K}(r_t)x(t), \end{cases} \tag{9}$$

where $A_K(r_t,t) = A(r_t,t) + B(r_t,t)K(r_t)$, $C_{1K}(r_t) = C_1(r_t) + D_1(r_t)K(r_t)$ and $C_{2K}(r_t) = C_{2K}(r_t) + D_2(r_t)K(r_t)$.

Let us now give the following lemmas that we will use extensively in proving our results in the rest of this paper. The proofs of the results of these lemmas can be found in Boukas and Liu [17] or any equivalent reference.

Lemma 2.1 *Let Y be a given symmetric and positive-definite matrix, $x(t)$ and $y(t)$ be two given vectors of appropriate dimensions, and $F(t)$ a matrix with appropriate dimension satisfying $F^\top(t)F(t) \le I$. Then, for any $\epsilon > 0$ we have:*

$$pm2x^\top(t)F(t)y(t) \le \epsilon x^\top(t)Yx(t) + \epsilon^{-1}y^\top(t)Y^{-1}y(t), \quad \forall\, r_t \in \mathbf{S}.$$

Lemma 2.2 *Let A, D, F, E be real matrices of appropriate dimensions with $\|F\| \le 1$. Then, we have*

(i) *for any matrix $P > 0$ and scalar $\varepsilon > 0$ satisfying $\varepsilon I - EPE^\top > 0$,*

$$(A + DFE)P(A + DFE)^\top \le APA^\top + APE^\top(\varepsilon I - EPE^\top)^{-1}EPA^\top + \varepsilon DD^\top; \tag{10}$$

(ii) *for any matrix $P > 0$ and scalar $\varepsilon > 0$ satisfying $P - \varepsilon DD^\top > 0$,*

$$(A + DFE)^\top P^{-1}(A + DFE) \le A^\top(P - \varepsilon DD^\top)^{-1}A + \frac{1}{\varepsilon}E^\top E. \tag{11}$$

Lemma 2.3 *The linear matrix inequality*

$$\begin{bmatrix} H & S^\top \\ S & R \end{bmatrix} > 0$$

is equivalent to

$$R > 0, \quad H - S^\top R^{-1} S > 0,$$

where $H = H^\top$, $R = R^\top$ and S is a constant matrix.

3 Main Results

The main goal of this paper is to develop an LMI-based design procedure for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller for the class of systems we are considering. The rest of this section will treat the nominal system first and then consider the case of uncertain systems with norm bounded uncertainties. In both cases, we will establish LMI-based conditions for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller design.

3.1 Nominal system

Let us now assume that the uncertainties in the dynamics (2) are equal to zero for all time and for all modes. In this case, the previous closed-loop dynamics becomes:

$$\begin{cases} \dot{x}(t) = A_K(r_t)x(t) + A_1(r_t)x(t-h(t)) + B_1(r_t)\omega(t), \\ x(s) = \phi(s), \quad -\tau \le s \le 0, \\ z_1(t) = C_{1K}(r_t)x(t), \\ z_2(t) = C_{2K}(r_t)x(t), \end{cases} \tag{12}$$

where $A_K(r_t) = A(r_t) + B(r_t)K(r_t)$, $C_{1K}(r_t) = C_1(r_t) + D_1(r_t)K(r_t)$ and $C_{2K}(r_t) = C_{2K}(r_t) + D_2(r_t)K(r_t)$.

When the external disturbance $\omega(t)$ is equal to zero for all $t \ge 0$, the following theorem gives the conditions that controller (6) should satisfy to stabilize the class of systems under consideration.

Theorem 3.1 *Let the disturbance input be equal to zero, i.e. $\omega(t) = 0$ for $t \ge 0$. The controller (6) is an $\mathcal{H}_2$ optimal controller satisfying the minimization of the $\mathcal{H}_2$ performance measure (7) if there exist symmetric and positive-definite matrices $P = (P(1), \dots, P(N))$, Q and a controller gain $K = (K(1), \dots, K(N))$ that the following holds for every mode $i \in \mathbf{S}$:*

$$\Theta(i) \stackrel{\Delta}{=} \begin{bmatrix} J(i) & P(i)A_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q \end{bmatrix} < 0 \tag{13}$$

with $J(i) = A_K^\top(i)P(i) + P(i)A_K(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) + Q + C_{1K}^\top(r_t)C_{1K}(r_t)$. The $\mathcal{H}_2$ performance measure is bounded by a positive scalar, i.e.:

$$J_{H_2} \le J^\star \stackrel{\Delta}{=} \left[x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds \right]^{\frac{1}{2}}. \tag{14}$$

Proof Let $\mathbb{C}[-\tau, 0]$ be a space of continuous functions on the interval $[-\tau, 0]$ and for any $x(t)$, $t \in \mathbb{C}[-\tau, 0]$, define $\|x\| = \sup\limits_{-\tau \le s \le 0} \|x(s)\|$. Obviously, the evolution of $x(t)$ depends on $x(s)$, $t - \tau \le s \le t$, which means that $\{(x(t), r_t), t \ge 0\}$ is not a Markov process. To cast our model into the framework of Markov system, let us define a process $\mathbf{x}(t)$ taking values in $\mathbb{C}[-\tau, 0]$ by

$$\mathbf{x}_s(t) = x(s + t), \quad t - \tau \le s \le t$$

then, $\{(\mathbf{x}(t), r_t), t \ge 0\}$ is a strong Markov process. Consider now the Lyapunov functional candidate with the following form:

$$V(\mathbf{x}(t), r_t) = x^\top(t) P(r_t) x(t) + \int\limits_{t-h(t)}^{t} x^\top(\theta) Q x(\theta)\, d\theta, \tag{15}$$

where $P(r_t)$ and Q are symmetric and positive-definite matrices.

Let $\mathcal{A}$ be the infinitesimal generator of the process $\{(\mathbf{x}(t), r_t), t \ge 0\}$. Then, we get:

$$\begin{aligned}
\mathcal{A}V(\mathbf{x}(t), r_t) &= \dot{x}^\top(t) P(r_t) x(t) + x^\top(t) P(r_t) \dot{x}(t) + x^\top(t) Q x(t) \\
&\qquad - (1 - \dot{h}(t)) x^\top(t - h(t)) Q x(t - h(t)) + \sum_{j=1}^{N} \lambda_{r_t j} x^\top(t) P(j) x(t) \\
&= [(A(r_t) + B(r_t) K(r_t)) x(t) + A_1(r_t) x(t - h(t))]^\top P(r_t) x(t) \\
&\quad + x^\top(t) P(r_t) [(A(r_t) + B(r_t) K(r_t)) x(t) + A_1(r_t) x(t - h(t))] \\
&\quad + x^\top(t) Q x(t) - (1 - \dot{h}(t)) x^\top(t - h(t)) Q x(t - h(t)) \\
&\quad + \sum_{j=1}^{N} \lambda_{r_t j} x^\top(t) P(j) x(t)
\end{aligned}$$

which gives the following:

$$\begin{aligned}
\mathcal{A}V(\mathbf{x}(t), r_t) \le x^\top(t) &\left[A_K^\top(r_t) P(r_t) + P(r_t) A_K(r_t) + Q + \sum_{j=1}^{N} \lambda_{r_t j} P(j) \right] x(t) \\
&+ 2x^\top(t) P(r_t) A_1(r_t) x(t - h(t)) - (1 - \beta) x^\top(t - h(t)) Q x(t - h(t))
\end{aligned}$$

Notice that (13) can be rewritten as follows:

$$\begin{bmatrix} A_K^\top(i) P(i) + P(i) A_K(i) + Q + \sum\limits_{j=1}^{N} \lambda_{ij} P(j) & P(i) A_1(i) \\ A_1^\top(i) P(i) & -(1 - \beta) Q \end{bmatrix} + \begin{bmatrix} C_{1K}^\top(i) \\ 0 \end{bmatrix} [C_{1K}(i) \;\; 0] < 0$$

which gives in turn:

$$\begin{bmatrix} A_K^\top(i) P(i) + P(i) A_K(i) + Q + \sum\limits_{j=1}^{N} \lambda_{ij} P(j) & P(i) A_1(i) \\ A_1^\top(i) P(i) & -(1 - \beta) Q \end{bmatrix} < 0.$$

This implies that the system is stochastically stable under the control law (6) (see Boukas and Liu [17] for the details of the proof).

Using now Dynkin's formula, we get:

$$\mathbb{E}[V(\mathbf{x}(t), r_t)] - V(\mathbf{x}(0), r_0) = \mathbb{E}\Big[\int_0^t \mathcal{A}V(\mathbf{x}(s), r_s)\, ds\Big].$$

Combining this with (13) we have:

$$\mathbb{E}[V(\mathbf{x}(t_f), r_{t_f})] - V(\mathbf{x}(0), r_0) \leq \mathbb{E}\Big[\int_0^{t_f} \zeta^\top(s)\Theta(r_s)\zeta(s)\, ds\Big]$$

with $\zeta(s) = \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}$.

Using the fact that system is stable, this implies the following when letting t_f goes to infinity:

$$\mathbb{E}\Big[\int_0^\infty z_1^\top(s) z_1(s)\, ds\Big] \leq V(\mathbf{x}(0), r_0)$$

$$= x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} x^\top(\theta)Qx(\theta)\, d\theta,$$

i.e.:

$$\|z_1\| \leq \Big[x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} x^\top(\theta)Qx(\theta)\, d\theta\Big]^{\frac{1}{2}}$$

which gives an upper bound for the $\mathcal{H}_2$ performance measure for the class of systems we are dealing with. This ends the proof of Theorem 3.1.

Let us now put the condition of Theorem 3.1 in the LMI formalism since it is now nonlinear in $P(r_t)$ and $K(R_t)$. From (13) we get the following using Schur complement:

$$\begin{bmatrix} A_K^\top(i)P(i) + P(i)A_K(i) + \sum_{j=1}^N \lambda_{ij}P(j) & P(i)A_1(i) & I & C_{1K}^\top(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q & 0 & 0 \\ I & 0 & -Q^{-1} & 0 \\ C_{1K}(i) & 0 & 0 & -I \end{bmatrix} < 0.$$

Letting $\bar{Q} = (1-\beta)Q$, the previous condition becomes:

$$\begin{bmatrix} A_K^\top(i)P(i)+P(i)A_K(i)+\sum_{j=1}^N \lambda_{ij}P(j)+P(i)A_1(i)\bar{Q}^{-1}A_1^\top(i)P(i) & I & C_{1K}^\top(i) \\ I & -Q^{-1} & 0 \\ C_{1K}(i) & 0 & -I \end{bmatrix} < 0.$$

Letting now $X(i) = P^{-1}(i)$ and pre and post-multiplying the previous condition by $\operatorname{diag}(X(i), I, I)$ we get:

$$\begin{bmatrix} J_0(i) & X(i) & X(i)K^\top(i)D_1^\top(i) + X(i)C_1^\top(i) \\ X(i) & -Q^{-1} & 0 \\ D_1(i)K(i)X(i) + C_1^\top(i)X(i) & 0 & -I \end{bmatrix} < 0$$

with $J_0(i) = X(i)A_K^\top(i) + A_K(i)X(i) + X(i)\Big[\sum_{j=1}^N \lambda_{ij}X^{-1}(j)\Big]X(i) + A_1(i)\bar{Q}^{-1}A_1^\top(i)$.

Putting

$$\begin{aligned} U &= Q^{-1} \\ Y(i) &= K(i)X(i) \\ S_i(X) &= \Big(\sqrt{\lambda_{i1}}X(i), \dots \sqrt{\lambda_{ii-1}}X(i), \sqrt{\lambda_{ii}}X(i), \dots, \sqrt{\lambda_{iN}}X(i)\Big) \\ \mathcal{X}_i &= \operatorname{diag}(X(1), \dots, X(i-1), X(i+1), \dots, X(N)) \end{aligned}$$

and noticing that:

$$X(i)A_K^\top(i) = X(i)(A(i) + B(i)K(i))^\top = X(i)A^\top(i) + Y^\top(i)B^\top(i),$$

$$X(i)\Bigg[\sum_{j=1}^N \lambda_{ij}X^{-1}(j)\Bigg]X(i) = \lambda_{ii}X(i) + S_i(X)\mathcal{X}_i^{-1}S_i^\top(X)$$

the previous condition becomes:

$$\begin{bmatrix} J_1(i) & X(i) & Y^\top(i)D_1^\top(i) + X(i)C_1^\top(i) & S_i(X) \\ X(i) & -U & 0 & 0 \\ D_1(i)Y(i) + C_1(i)X(i) & 0 & -I & 0 \\ S_i^\top(X) & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0$$

with

$$\begin{aligned} J_1(i) &= X(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) \\ &+ B(i)Y(i) + \lambda_{ii}X(i) + (1-\beta)^{-1}A_1(i)UA_1^\top(i). \end{aligned}$$

This condition can be solved using the LMI toolbox of Matlab or any equivalent tool to get the controller gain, $K(r_t)$ for every $r_t \in \mathbf{S}$.

Let us now consider that the external disturbance is not equal to zero. The controller (6) in this case is a $\mathcal{H}_\infty$ controller and the following theorem gives the associated results.

Theorem 3.2 *Let γ be a given positive constant. The controller (6) will stabilize the system and guarantee the disturbance rejection of level γ for all $\omega(t) \in \mathcal{L}_2[0,\infty)$ if there exist symmetric and positive-definite matrices $P = (P(1), \dots, P(N))$ and Q, and a controller gain $K = (K(1), \dots, K(N))$ such that the following holds for every $i \in \mathcal{S}$:*

$$\Theta_{H_\infty} = \begin{bmatrix} \tilde{J}_2(i) & P(i)A_1(i) & P(i)B_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q & 0 \\ B_1^\top(i)P(i) & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{16}$$

with $\tilde{J}_2(i) = A_K^\top(i)P(i) + P(i)A_K(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) + Q + C_{2K}^\top(i)C_{2K}(i)$. *In this case we have:*

$$\|z_2\| = \left[\gamma^2\|\omega\|^2 + x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds\right]^{\frac{1}{2}}. \tag{17}$$

Proof To prove this theorem, let us assume that the controller exists and show that it stochastically stabilizes the class of system we are considering. For this purpose notice that (16) implies the following:

$$\begin{bmatrix} \tilde{J}_2(i) & P(i)A_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q \end{bmatrix} < 0. \tag{18}$$

This inequality can be rewritten as:

$$\begin{bmatrix} A_K^\top(i)P(i) + P(i)A_K(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) + Q & P(i)A_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q \end{bmatrix} + \begin{bmatrix} C_2^\top(i) \\ 0 \end{bmatrix} [C_2(i) \;\; 0] < 0$$

which gives in turn:

$$\begin{bmatrix} A_K^\top(i)P(i) + P(i)A_K(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) + Q & P(i)A_1(i) \\ A_1^\top(i)P(i) & -(1-\beta)Q \end{bmatrix} < 0.$$

This implies in turn that the system is stochastically stable under the controller (6) (for more details on the rest of the proof, we refer the reader to Boukas and Liu [17]).

Let us now, show that the $\mathcal{H}_{H_\infty}$ performance measure is bounded. For this purpose, let us define the performance function:

$$\mathcal{J}_T = \mathbb{E}\left[\int_0^T \left(z_2^\top(t)z_2(t) - \gamma^2\omega^\top(t)\omega(t)\right) dt\right].$$

To prove that $\mathcal{H}_{H_\infty}$ performance measure is bounded, it suffices to establish

$$\mathcal{J}_\infty \leq V(\mathbf{x}(0), r_0) = x_0^\top P(r_0)x_0 + \int_{-h(0)}^{0} \phi(s)Q\phi(s)\,ds.$$

Using Dynkin's formula, we have

$$\mathbb{E}\left[\int_0^T \mathcal{A}V(\mathbf{x}(t), r_t)]\,dt\right] = \mathbb{E}[V(\mathbf{x}(T), r_T)] - V(\mathbf{x}(0), r_0),$$

where $V(\mathbf{x}(t), r_t)$ is given by equation (15).

Noticing that:

$$z_2^\top(t)z_2(t) - \gamma^2\omega^\top(t)\omega(t) = x^\top(t)C_{2K}^\top(r_t)C_{2K}(r_t)x(t) - \gamma^2\omega^\top(t)\omega(t)$$

and

$$\begin{aligned}\mathcal{A}V(\mathbf{x}(t),r_t)] \le x^\top(t)\Big[A_K^\top(r_t)P(r_t) + P(r_t)A_K(r_t) + Q + \sum_{j=1}^{N}\lambda_{ij}P(j)\Big]x(t)\\ + 2x^\top(t)P(r_t)A_1(r_t)x(t-h(t)) + 2x^\top(t)P(r_t)B_1(r_t)\omega(t)\\ - (1-\beta)x^\top(t-h(t))Qx(t-h(t))\end{aligned}$$

we get:

$$z_2^\top(t)z_2(t) - \gamma^2\omega^\top(t)\omega(t) + \mathcal{A}V(\mathbf{x}(t),r_t) \le \eta^\top(t)\Theta_{H_\infty}(r_t)\eta(t),$$

where $\eta^\top(t) = \big(x^\top(t)\;\; x^\top(t-h(t))\;\; \omega^\top(t)\big)$. Therefore,

$$\begin{aligned}\mathcal{J}_T &= \mathbb{E}\Big[\int_0^T [z_2^\top(t)z_2(t) - \gamma^2\omega^\top(t)\omega(t) + \mathcal{A}V(\mathbf{x}(t),r_t)]\,dt\Big]\\ &\quad - \mathbb{E}\Big[\int_0^T \mathcal{A}V(\mathbf{x}(t),r_t)]\,dt\Big] \\ &\le \mathbb{E}\Big[\int_0^T \eta^\top(t)\Theta_{H_\infty}(r_t)\eta(t)\,dt\Big] - \mathbb{E}[V(\mathbf{x}(T),r_T)] + V(\mathbf{x}(0),r_0).\end{aligned} \tag{19}$$

Since $\Theta_{H_\infty}(i) < 0$ and $\mathbb{E}[V(\mathbf{x}(T),r_T)] \ge 0$, (19) implies

$$\mathcal{J}_T \le V(\mathbf{x}(0),r_0),$$

yielding

$$\mathcal{J}_\infty \le V(\mathbf{x}(0),r_0),$$

i.e.,

$$\|z_2\|_2^2 - \gamma^2\|w\|_2^2 \le x_0^\top P(r_0)x_0 + \int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds.$$

This yields

$$\|z_2\|_2^2 \le \gamma^2\|w\|_2^2 + x_0^\top P(r_0)x_0 + \int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds,$$

which gives the bound we are looking for.

This ends the proof of Theorem 3.2.

In a similar way we can put the condition (16) in the LMI formalism in the design parameters. The new conditions becomes:

$$\begin{bmatrix} J_2(i) & X(i) & B_1(i) & Y^\top(i)D_2^\top(i) + X(i)C_2^\top(i) & S_i(X) \\ X(i) & -U & 0 & 0 & 0 \\ B^\top(i) & 0 & -\gamma^{-2}I & 0 & 0 \\ C_2(i)X(i) + D_2(i)Y(i) & 0 & 0 & -I & 0 \\ S_i^\top(X) & 0 & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0 \quad (20)$$

with

$$J_2(i) = X(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) + B(i)Y(i) + \lambda_{ii}X(i) + (1-\beta)^{-1}A_1(i)UA_1^\top(i).$$

Notice that in (14) and (31) we have the following common term:

$$x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds$$

that we should minimize to guarantee good performances. By doing so, simultaneously we will guarantee a minimum upper bound for the $\mathcal{H}_2$ performance measure and a good disturbance rejection with a level γ for all $\omega(t) \in \mathcal{L}_2[0,\infty)$. Before giving the optimization problem that will allow us to reach our goal, let us formulate the cost function.

First of all, notice that $x^\top(0)P(i)x(0)$ for all $i \in \mathbf{S}$ can be bounded by a real positive constant that we should minimize:

$$x^\top(0)P(i)x(0) \le \alpha$$

with $\alpha = \max(x^\top(0)P(1)x(0), \ldots, x^\top(0)P(N)x(0))$; which we can rewrite as follows:

$$-\alpha + \phi^\top(0)X^{-1}(i)\phi(0) < 0,$$

where $X(i) = P^{-1}(i)$.

This can be rewritten in matrix form as:

$$\begin{bmatrix} -\alpha & \phi^\top(0) \\ \phi(0) & -X(i) \end{bmatrix} < 0. \quad (21)$$

For the second term of the common term for the two performance measures, notice that:

$$\int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds = \int_{-h(0)}^{0} \mathrm{tr}\,\left(\phi^\top(s)U^{-1}\phi(s)\right) ds$$

$$= \mathrm{tr}\,\left(\mathcal{N}\mathcal{N}^\top U^{-1}\right) = \mathrm{tr}\,\left(\mathcal{N}^\top U^{-1}\mathcal{N}\right) < \mathrm{tr}\,(\mathcal{Q})$$

with $\mathcal{N}\mathcal{N}^\top = \int_{-h(0)}^{0} \phi(s)\phi^\top(s)\,ds$. This gives:

$$-Q_1 + \mathcal{N}^\top U^{-1}\mathcal{N} < 0.$$

In matrix form we get:

$$\begin{bmatrix} -Q_1 & \mathcal{N}^\top \\ \mathcal{N} & -U \end{bmatrix} < 0. \quad (22)$$

The following theorem gives the optimization that we could solve to get the controller gain.

Theorem 3.3 *Let γ be a given positive constant. If there exist symmetric and positive-definite matrices $X = (X(1), \dots, X(N))$, U and $\mathcal{Q}$ and a positive scalar α, and a matrix $Y = (Y(1), \dots, Y(N))$, solution of the following optimization problem:*

$$\min\left(\alpha + \operatorname{tr}(\mathcal{Q})\right)$$

$$s.t:$$

$$\begin{bmatrix} J_1(i) & X(i) & Y^\top(i)D_1^\top(i) + X(i)C_1(i) & S_i(X) \\ X(i) & -U & 0 & 0 \\ C_1(i)X(i) + D_1(i)Y(i) & 0 & -I & 0 \\ S_i^\top(X) & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} J_2(i) & X(i) & B_1(i) & Y^\top(i)D_2^\top(i) + X(i)C_2^\top(i) & S_i(X) \\ X(i) & -U & 0 & 0 & 0 \\ B_1^\top(i) & 0 & -\gamma^{-2}I & 0 & 0 \\ C_2(i)X(i) + D_2(i)Y(i) & 0 & 0 & -I & 0 \\ S_i^\top(X) & 0 & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0, \tag{24}$$

$$\begin{bmatrix} -\alpha & \phi^\top(0) \\ \phi(0) & -X(i) \end{bmatrix} < 0, \tag{25}$$

$$\begin{bmatrix} -Q_1 & \mathcal{N}^\top \\ \mathcal{N} & -U \end{bmatrix} < 0, \tag{26}$$

then the controller (6) is a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller satisfying the control objective (8). The controller gain is given by $K(r_t) = Y(r_t)X^{-1}(r_t)$, for every mode $r_t \in \mathbf{S}$.

This theorem gives a procedure to design the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller for the nominal class of systems we are dealing with. The optimization that we propose is a convex one that we can solve using the existing tools like the one of Matlab or any equivalent one.

In the next subsection we will see how we can modify the results on this subsection to handle the case of uncertain systems.

3.2 Uncertain system

Let us now assume that uncertainties are not equal to zero and suppose that they satisfy the conditions (3). In this case the closed-loop dynamics becomes:

$$\begin{cases} \dot{x}(t) = A_K(r_t)x(t) + A_1(r_t,t)x(t-h(t)) + B_1(r_t)\omega(t), \\ x(s) = \phi(s), -\tau \le s \le 0, \\ z_1(t) = C_{1K}(r_t)x(t), \\ z_2(t) = C_{2K}(r_t)x(t), \end{cases} \tag{27}$$

where $A_K(r_t) = A(r_t,t) + B(r_t,t)K(r_t)$, $C_{1K}(r_t) = C_1(r_t) + D_1(r_t)K(r_t)$ and $C_{2K}(r_t) = C_{2K}(r_t) + D_2(r_t)K(r_t)$.

If we apply the results of Theorem 3.1 to the uncertain system (27), we get:

$$\begin{bmatrix} J(r_t,t) & P(r_t)A_1(r_t,t) \\ A_1^\top(r_t,t)P(r_t) & -(1-\beta)Q \end{bmatrix} < 0$$

with $J(r_t,t) = A_K^\top(r_t,t)P(r_t) + P(r_t)A_K(r_t,t) + \sum_{j=1}^{N} \lambda_{r_t j}P(j) + Q + C_{1K}^\top(r_t)C_{1K}(r_t)$.

This gives in turn the following:

$$J(r_t,t) + (1-\beta)^{-1}P(r_t)A_1(r_t,t)Q^{-1}A_1^\top(r_t,t)P(r_t) < 0 \tag{28}$$

since

$$\begin{aligned} A_K^\top(r_t,t)P(r_t) + P(r_t)A_K(r_t,t) &= A_K^\top(r_t)P(r_t) + P(r_t)A_K(r_t) \\ &+ 2P(r_t)\Delta A(r_t,t) + 2P(r_t)\Delta B(r_t,t)K(r_t) \end{aligned}$$

using Lemma 2.1, we get:

$$\begin{aligned} A_K^\top(r_t,t)P(r_t) + P(r_t)A_K(r_t,t) &\le A_K^\top(r_t)P(r_t) + P(r_t)A_K(r_t) \\ &+ \varepsilon_A P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t) + \varepsilon_A^{-1}E_A^\top(r_t)E_A(r_t) \\ &+ \varepsilon_B P(r_t)D_B(r_t)D_B^\top(r_t)P(r_t) + \varepsilon_B^{-1}K^\top(r_t)E_B^\top(r_t)E_B(r_t)K(r_t). \end{aligned}$$

For the term $(1-\beta)^{-1}P(r_t)A_1(r_t,t)Q^{-1}A_1^\top(r_t,t)P(r_t)$ notice that using Lemma 2.2, we have:

$$\begin{aligned} A_1(r_t,t)Q^{-1}A_1^\top(r_t,t) &\le A_1(r_t)Q^{-1}A_1^\top(r_t) + A_1(r_t)Q^{-1}E_{A_1}^\top(r_t) \\ &\times \Big(\varepsilon_{A1}I - E_{A_1}(r_t)Q^{-1}E_{A_1}^\top(r_t)\Big)^{-1}E_{A_1}(r_t)Q^{-1}A_1^\top(r_t) + \varepsilon_{A1}D_{A_1}(r_t)D_{A_1}^\top(r_t) \end{aligned}$$

which gives the following when we replace Q^{-1} by $(1-\beta)^{-1}Q^{-1}$:

$$\begin{aligned} A_1(r_t,t)(1-\beta)^{-1}Q^{-1}A_1^\top(r_t,t) &\le A_1(r_t)(1-\beta)^{-1}Q^{-1}A_1^\top(r_t) \\ &+ A_1(r_t)(1-\beta)^{-1}Q^{-1}E_{A_1}^\top(r_t)\Big(\varepsilon_{A1}I - E_{A_1}(r_t)(1-\beta)^{-1}Q^{-1}E_{A_1}^\top(r_t)\Big)^{-1} \\ &\times E_{A_1}(r_t)(1-\beta)^{-1}Q^{-1}A_1^\top(r_t) + \varepsilon_{A1}D_{A_1}(r_t)D_{A_1}^\top(r_t). \end{aligned}$$

Based on all these transformations the condition (28) becomes:

$$\begin{aligned} &A_K^\top(r_t)P(r_t) + P(r_t)A_K(r_t) + \varepsilon_A P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t) + Q + \sum_{j=1}^{N}\lambda_{r_t j}P(j) \\ &+ C_{1K}^\top(r_t)C_{1K}(r_t) + \varepsilon_A^{-1}E_A^\top(r_t)E_A(r_t) + \varepsilon_B P(r_t)D_B(r_t)D_B^\top(r_t)P(r_t) \\ &+ \varepsilon_B^{-1}K^\top(r_t)E_B^\top(r_t)E_B(r_t)K(r_t) + P(r_t)A_1(r_t)(1-\beta)^{-1}Q^{-1}A_1^\top(r_t)P(r_t) \\ &+ P(r_t)A_1(r_t)(1-\beta)^{-1}Q^{-1}E_{A_1}^\top(r_t)\Big(\varepsilon_{A1}I - E_{A_1}(r_t)(1-\beta)^{-1}Q^{-1}E_{A_1}^\top(r_t)\Big)^{-1} \\ &\times E_{A_1}(r_t)(1-\beta)^{-1}Q^{-1}A_1^\top(r_t)P(r_t) + \varepsilon_{A1}P(r_t)D_{A_1}(r_t)D_{A_1}^\top(r_t)P(r_t) < 0. \end{aligned}$$

In matrix form we get:

$$\begin{bmatrix} \tilde{J}_3(r_t) & E_A^\top(r_t) & K^\top(r_t)E_B^\top(r_t) & \frac{P(r_t)A_1(r_t)Q^{-1}E_{A1}(r_t)}{(1-\beta)} & I & C_{1K}^\top(r_t) \\ E_A(r_t) & \varepsilon_A I & 0 & 0 & 0 & 0 \\ E_B(r_t)K(r_t) & 0 & \varepsilon_B I & 0 & 0 & 0 \\ \frac{E_{A1}^\top(r_t)Q^{-1}A_1^\top(r_t)P(r_t)}{(1-\beta)} & 0 & 0 & -\varepsilon_{A1}I + \frac{E_A(r_t)Q^{-1}E_A^\top(r_t)}{(1-\beta)} & 0 & 0 \\ I & 0 & 0 & 0 & -Q^{-1} & 0 \\ C^{1K}(r_t) & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0$$

with

$$\tilde{J}_3(r_t) = A_K^\top(r_t)P(r_t) + P(r_t)A_K^\top(r_t) + \sum_{j=1}^{N} \lambda_{r_t j}P(j)$$
$$+ \varepsilon_A P(r_t)D_A(r_t)D_A^\top(r_t)P(r_t) + \varepsilon_B P(r_t)D_B(r_t)D_B^\top(r_t)P(r_t)$$
$$\times \varepsilon_{A1} P(r_t)D_{A1}(r_t)D_{A1}^\top(r_t)P(r_t) + (1-\beta)^{-1}P(r_t)A_1(r_t)Q^{-1}A_1^\top(r_t)P(r_t).$$

Now if we pre and post-multiplying the right hand side term by diag $(X(i), I, I, I, I)$ with $X(i) = P^{-1}(i)$ and by following the same steps as we followed to transform (13) in LMI form, we get:

$$\begin{bmatrix} J_3(r_t) & X(r_t)E_A^\top(r_t) & Y^\top(r_t)E_B^\top(r_t) & \frac{A_1(r_t)UE_{A1}^\top(r_t)}{(1-\beta)} & X(r_t) & Y^\top(r_t)D_1^\top(r_t) + X(r_t)C_1^\top(r_t) & S_{r_t}(X) \\ E_A(r_t)X(r_t) & \varepsilon_A I & 0 & 0 & 0 & 0 & 0 \\ E_B(r_t)Y(r_t) & 0 & \varepsilon_B I & 0 & 0 & 0 & 0 \\ \frac{E_{A1}^\top(r_t)UA_1(r_t)}{(1-\beta)} & 0 & 0 & -\varepsilon_{A1}I + \frac{E_A(r_t)UE_A^\top(r_t)}{(1-\beta)} & 0 & 0 & 0 \\ X(r_t) & 0 & 0 & 0 & -U & 0 & 0 \\ C_1(r_t)X(r_t) + D_1(r_t)Y(r_t) & 0 & 0 & 0 & 0 & -I & 0 \\ S_{r_t}^\top(X) & 0 & 0 & 0 & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0 \tag{29}$$

with

$$J_3(r_t) = A(r_t)X(r_t) + X(r_t)A^\top(r_t) + \lambda_{r_t r_t}X(r_t) + B(r_t)Y(r_t) + Y^\top(r_t)B^\top(r_t)$$
$$+ \varepsilon_A D_A(r_t)D_A^\top(r_t) + \varepsilon_B D_B(r_t)D_B^\top(r_t) + \varepsilon_{A1}D_{A1}(r_t)D_{A1}^\top(r_t)$$
$$+ (1-\beta)^{-1}A_1(r_t)UA_1^\top(r_t).$$

If this condition is satisfied, we can easily prove following the steps of Theorem 3.1's proof that the system is stable under the control law (6) when the external disturbance is equal to zero and that the $\mathcal{H}_2$ performance measure is bounded, i.e.:

$$\|z_1\| \le \left[x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds\right]^{\frac{1}{2}}.$$

The following theorem summarizes the corresponding results.

Theorem 3.4 *Let the disturbance input be equal to zero, i.e. $\omega(t) = 0$ for $t \geq 0$. The controller (6) is an $\mathcal{H}_2$ optimal controller satisfying the minimization of the $\mathcal{H}_2$ performance measure (7) if there exist symmetric and positive-definite matrices $P = (P(1), \ldots, P(N))$, Q and a matrix $Y = (Y(1), \ldots, Y(N))$ that (29) holds for every mode $i \in \mathbf{S}$. The $\mathcal{H}_2$ performance measure is bounded by a positive scalar, i.e.:*

$$J_{H_2} \leq J^\star \triangleq \left[x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds \right]^{\frac{1}{2}}.$$

The controller gain $K(i) = Y(i)X^{-1}(i)$ for every $i \in \mathbf{S}$.

When the external disturbance is not equal to zero we can easily follow the same step as for Theorem 3.4 to establish the results of Theorem 3.5.

Theorem 3.5 *Let γ be a given positive constant. The controller (6) will stabilize the system and guarantee the disturbance rejection of level γ if there exist symmetric and positive-definite matrices $P = (P(1), \ldots, P(N))$, Q and positive constants ε_A, ε_B and ε_{A_1}, and a matrix $Y = (Y(1), \ldots, Y(N))$ such that the following holds for every $i \in \mathbf{S}$:*

$$\begin{bmatrix} J_5(r_t) & X(r_t)E_A^\top(r_t) & Y^\top(r_t)E_B^\top(r_t) & \frac{A_1(r_t)UE_{A1}^\top(r_t)}{(1-\beta)} & X(r_t) & B_1(r_t) & Y^\top(r_t)D_2^\top(r_t) + X(r_t)C_2^\top(r_t) & S_{r_t}(X) \\ E_A(r_t)X(r_t) & \varepsilon_A I & 0 & 0 & 0 & 0 & 0 & 0 \\ E_B(r_t)Y(r_t) & 0 & \varepsilon_B I & 0 & 0 & 0 & 0 & 0 \\ \frac{E_{A1}^\top(r_t)UA_1(r_t)}{(1-\beta)} & 0 & 0 & -\varepsilon_{A1}I + \frac{E_A(r_t)UE_A^\top(r_t)}{(1-\beta)} & 0 & 0 & 0 & 0 \\ X(r_t) & 0 & 0 & 0 & -U & 0 & 0 & 0 \\ B^\top(r_t) & 0 & 0 & 0 & 0 & -\gamma^{-2}I & 0 & 0 \\ C_2(r_t)X(r_t) + D_2(r_t)Y(r_t) & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ S_{r_t}^\top(X) & 0 & 0 & 0 & 0 & 0 & 0 & -\mathcal{X}_i \end{bmatrix} < 0 \tag{30}$$

with

$$\begin{aligned} J_5(i) &= X(i)A^\top(i) + A(i)X(i) + Y^\top(i)B^\top(i) + B(i)Y(i) + \lambda_{ii}X(i) \\ &\quad + \varepsilon_A D_A(i)D_A^\top(i) + \varepsilon_B D_B(i)D_B^\top(i) + \varepsilon_{A1}D_{A1}(i)D_{A1}^\top(i) \\ &\quad + (1-\beta)^{-1}A_1(i)UA^\top(i) + \gamma^{-2}B_1(i)B_1^\top(i). \end{aligned}$$

In this case we have:

$$\|z_2\| = \left[\gamma^2\|\omega\|^2 + x^\top(0)P(r_0)x(0) + \int_{-h(0)}^{0} \phi^\top(s)Q\phi(s)\,ds \right]^{\frac{1}{2}}.$$

For the same reasons as before, if we combine the two previous theorems we get the following one that gives the optimization that we could solve to get the controller gains in each mode for the uncertain class of systems we are dealing with.

Theorem 3.6 *Let γ be a given positive constant. If there exist symmetric and positive-definite matrices $X = (X(1), \dots, X(N))$, U and $\mathcal{Q}$ and positive scalars α, ε_A, ε_B and ε_{A_1}, and a matrix $Y = (Y(1), \dots, Y(N))$, solution of the following optimization problem:*

$$
\begin{gathered}
\min(\alpha + \operatorname{tr}(\mathcal{Q})) \\
s.t:
\end{gathered}
$$

$$
\left[\begin{matrix}
J_4(r_t) & X(r_t)E_A^\top(r_t) & Y^\top(r_t)E_B^\top(r_t) & \frac{A_1(r_t)UE_{A1}^\top(r_t)}{(1-\beta)} \\
E_A(r_t)X(r_t) & \varepsilon_A I & 0 & 0 \\
E_B(r_t)Y(r_t) & 0 & \varepsilon_B I & 0 \\
\frac{E_{A1}^\top(r_t)UA_1(r_t)}{(1-\beta)} & 0 & 0 & -\varepsilon_{A1}I + \frac{E_A(r_t)UE_A^\top(r_t)}{(1-\beta)} \\
X(r_t) & 0 & 0 & 0 \\
C_1(r_t)X(r_t) + D_1(r_t)Y(r_t) & 0 & 0 & 0 \\
S_{r_t}^\top(X) & 0 & 0 & 0
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
X(r_t) & Y^\top(r_t)D_1^\top(r_t) + X(r_t)C_1^\top(r_t) & S_{r_t}(X) \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-U & 0 & 0 \\
0 & -I & 0 \\
0 & 0 & -\mathcal{X}_i
\end{matrix}\right] < 0, \tag{31}
$$

$$
\left[\begin{matrix}
J_5(r_t) & X(r_t)E_A^\top(r_t) & Y^\top(r_t)E_B^\top(r_t) & \frac{A_1(r_t)UE_{A1}^\top(r_t)}{(1-\beta)} \\
E_A(r_t)X(r_t) & \varepsilon_A I & 0 & 0 \\
E_B(r_t)Y(r_t) & 0 & \varepsilon_B I & 0 \\
\frac{E_{A1}^\top(r_t)UA_1(r_t)}{(1-\beta)} & 0 & 0 & -\varepsilon_{A1}I + \frac{E_A(r_t)UE_A^\top(r_t)}{(1-\beta)} \\
X(r_t) & 0 & 0 & 0 \\
B^\top(r_t) & 0 & 0 & 0 \\
C_2(r_t)X(r_t) + D_2(r_t)Y(r_t) & 0 & 0 & 0 \\
S_{r_t}^\top(X) & 0 & 0 & 0
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
X(r_t) & B_1(r_t) & Y^\top(r_t)D_2^\top(r_t) + X(r_t)C_2^\top(r_t) & S_{r_t}(X) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-U & 0 & 0 & 0 \\
0 & -\gamma^{-2}I & 0 & 0 \\
0 & 0 & -I & 0 \\
0 & 0 & 0 & -\mathcal{X}_i
\end{matrix}\right] < 0, \tag{32}
$$

$$
\begin{bmatrix} -\alpha & \phi^\top(0) \\ \phi(0) & -X(i) \end{bmatrix} < 0, \tag{33}
$$

$$\begin{bmatrix} -\mathcal{Q} & \mathcal{N}^\top \\ \mathcal{N} & -U \end{bmatrix} < 0, \tag{34}$$

then the controller (6) is a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller satisfying the control objective (8). The controller gain is given by $K(r_t) = Y(r_t)X^{-1}(r_t)$, for every mode $r_t \in \mathbf{S}$.

This theorem provides a procedure to design a memoryless state feedback controller of the form (6) that stabilizes system (2) in the robust SS sense. The advantage of these results is that we can use the LMI tools to solve it for any dynamical system of the class we are considering in this paper.

4 Conclusion

This paper deals with the class of continuous-time linear systems with Markovian jumps and time-delays. The time-delay is assumed to be time-varying. Results on stochastic stabilizability and its robustness are developed. The LMI framework is used to establish the different results on stabilizability. The conditions we developed can easily be solved using any LMI toolbox like the one of Matlab or the one of Scilab. These results we can be extended to other type of controller and also to the case where the time-delay is mode dependent as it was developed in Boukas and Liu [17]. This will be the subject of our future research.

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