Nonlinear Dynamics and Systems Theory, **2**(2) (2002) 203–213

Analysis of Time-Controlled Switched Systems by Stability Preserving Mappings

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Received: November 22, 2001; Revised: June 17, 2002

Abstract: In this paper, we study the stability of a wide class of switched systems using stability preserving mappings. By considering an existing result and extending it to a general class of switched systems, we show that stability preserving mappings constitute an important and practical tool in stability analysis and design of switched systems.

Keywords: *Switched systems; stability analysis; stability preserving mappings.*

Mathematics Subject Classification (2000): 34A37, 34D20, 93D30.

1 Introduction

We study the stability of a large class of switched systems using stability preserving mappings [1 – 4]. By a switched system, we mean a hybrid dynamical system that is composed of a family of continuous-time subsystems and a rule orchestrating the switching between the subsystems. Recently, there has been increasing interest in the stability analysis and switching control design of such systems (for recent progress in this field, see the survey papers [5, 6] and the references cited therein). It is known that when considering the switching method among several given subsystems, there are two main approaches for stability analysis or design: in one the switching depends only on time while in the other, the switching depends on the state and/or output of the system. In this paper, we focus our attention on the case of switching among subsystems determined by time, and in this sense we use the term *time-controlled switched system*. For such switched

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systems, there are several existing results. The paper [7] shows that when all subsystems are linear time-invariant and all subsystem matrices are Hurwitz stable, we can choose each subsystem's activation time interval (called *dwell time*) sufficiently large so that the switched system is exponentially stable. In [8], a dwell time scheme is analyzed for local asymptotic stability of nonlinear switched systems with the activation time being used as a dwell time. In [9, 10], Hespanha extends the concept of "*dwell time*" to "*average dwell time*", by showing that when the average time interval between consecutive switchings is sufficiently large, the switched system is exponentially stable. In the recent papers [11, 12], the authors extended the above stability results to the case where both Hurwitz stable and unstable subsystems exist, by showing that if the average dwell time is chosen sufficiently large and the total activation time of unstable subsystems is relatively small compared with that of Hurwitz stable subsystems, then global exponential stability of a desired decay rate is guaranteed.

In this paper, we aim to extend the above results for a wide class of switched systems. The switched system under consideration is composed of N subsystems and is described by

$$\begin{cases} \dot{x}(t) = f_{i_k}(t, x(t), x(\tau_k)), & \tau_k \leq t < \tau_{k+1}, \\ x(t) = g_{i_{k+1}}(t, x(t^-), x(\tau_k)), & t = \tau_{k+1}, \end{cases} \tag{1}$$

where $x(t) \in \Re^n$ is the state, $\tau_0, \tau_1, \tau_2, \cdots, \tau_k, \cdots$ are the switching points, and $i_k \in I_N = \{1, 2, \cdots, N\}$ denotes the number of the subsystem that is activated during the time interval $\tau_k \leq t < \tau_{k+1}$. For all $i \in I_N$, it is assumed that $f_i \in C^1[\Re^+ \times \Re^n \times \Re^n, \Re^n]$ and $f_i(t, 0, 0) = 0$, $g_i \in C[\Re^+ \times \Re^n \times \Re^n, \Re^n]$ and $g_i(t, 0, 0) = 0$. Clearly, the differential equation in (1) determines the dynamical behavior of the system over the indicated time intervals while the second equation specifies the amount of the state change when switching occurs.

It is well known that the switched system (1) can be regarded as a discontinuous dynamical system. There are many results within this framework (for example, [1−4, 8]). In these references, the notion of stability preserving mapping is very important and effective in analyzing discontinuous dynamical systems. In this paper, we first use stability preserving mappings to recall an existing result and then extend our considerations to more general switched systems. In contrast with the general results given in [1−4, 8], we will in this paper take advantage of specific properties of switched systems to obtain some practical results.

The remainder of the present paper is organized as follows. In Section 2, we review some of the stability preserving mapping theory for discontinuous/hybrid dynamical systems established in [1−4, 8]. In Section 3, stability results for various cases of the switched system (1) are established. Finally, in Section 4 we make some concluding remarks.

2 Preliminaries

In the interests of completeness and clarity, we summarize in this section some of the stability preserving mappings theory developed in [1−4, 8]. To do this, we need to recall the definition of dynamical system and discontinuous/hybrid dynamical system.

Dynamical systems are families of motions determined by evolutionary processes (see, e.g., [1]). The evolution of such processes takes place over time which we denote by T. Every motion of a dynamical system depends on *initial data* (t_0, a), where $t_0 \in T$ is called *initial time* and $a \in A \subset X$ is called *initial point*, where X, the *state space*, is a *metric space* with metric d (i.e., (X, d) is a metric space), and A is an appropriate subset of X. For a given (t_0, a), we denote a *motion*, if it exists, by $p(t, t_0, a)$, $t \in T_{t_0,a}$, where $T_{t_0,a} = [t_0, t_1) \cap T$, and where t_1 may be finite or infinite. Thus, a motion is a mapping $p(\cdot, t_0, a)\colon T_{t_0,a} \to X$ with $p(t_0, t_0, a) = a$, and the family of motions which makes up the dynamical system is obtained by varying the initial point a over the set A and the initial time t_0 over T_0, an appropriate subset of T (the set of initial times). If we denote such a family by S, then the dynamical system is signified by the quintuple $\{T, X, A, S, T_0\}$. When $T = T_0$, we simply write $\{T, X, A, S\}$, and when all is clear from context, we will simply speak of a dynamical system S (rather than a dynamical system $\{T, X, A, S, T_0\}$). If $T = R^+ = [0, \infty)$, we speak of a *continuous-time dynamical system* and when $T = N = \{0, 1, 2, \cdots\}$ we speak of a *discrete-time dynamical system*. If $T = R^+$ and all $p \in S$ are continuous with respect to t, we speak of a *continuous dynamical system*. If $T = R^+$ and the elements of S are not continuous with respect to t, we speak of a *discontinuous dynamical system* (DDS). Most frequently, the system motions are determined by means of the solutions of initial-value problems.

Hybrid dynamical systems are capable of exhibiting simultaneously several kinds of dynamic behavior in different parts of the system (e.g., continuous-time dynamics, discrete-time dynamics, logic commands, discrete events, jump phenomena, and the like). For such systems, a general model which appears to be suitable for the *qualitative analysis* of *general* hybrid dynamical systems was introduced in [1−4]. This model incorporates a concept of *generalized time*. If we generalize the dynamical systems considered in the above paragraph by replacing the usual concept of time with the general time space (T, ρ), we end up with a notion of *hybrid dynamical system* $\{T, X, A, S, T_0\}$ (HDS) which includes most of the specific classes of dynamical systems considered in the literature as special cases. Presently, T is a totally ordered space with relation "$\prec$" which is bounded from below by $t_{min} \in T$ and for the metric ρ, triangle inequality is replaced by "triangle equality".

For $\{T, X, A, S, T_0\}$, a set $M \subset A$ is said to be *invariant with respect to system S* if $a \in M$ implies that $p(t, a, t_0) \in M$ for all $t \in T_{a,t_0}$, all $t_0 \in T_0$ and all $p(\cdot, a, t_0) \in S$. We will state the above more compactly by saying that M *is an invariant set of* S, or (S, M) *is invariant*. If in particular, $M = \{x_0\}$, then x_0 is called an *equilibrium*.

In the following, d denotes the metric on X (i.e., (X, d) is a metric space).

Let $\{T, X, A, S, T_0\}$ be an HDS and let $M \subset A$ be an invariant set for S. We say that (S, M) is *stable* if for every $\epsilon > 0$ and $t_0 \in T_0$, there exists $\delta = \delta(\epsilon, t_0) > 0$ such that $d(p(t, a, t_0), M) < \epsilon$ for all $t \in T_{a,t_0}$ and for all $p(\cdot, a, t_0) \in S$, whenever $d(a, M) < \delta$. We say that (S, M) is *uniformly stable* if $\delta = \delta(\epsilon)$. If (S, M) is stable and if for any $t_0 \in T_0$ there exists an $\eta = \eta(t_0) > 0$ such that $\lim\limits_{t\to\infty} d(p(t, a, t_0), M) = 0$ (i.e., for every $\epsilon > 0$, there exists a $t_\epsilon \in T$ such that $d(p(t, a, t_0), M) < \epsilon$ whenever $t \in T$ and $t_\epsilon \prec t$) for all $p(\cdot, a, t_0) \in S$ whenever $d(a, M) < \eta$, then (S, M) is said to be *asymptotically stable*. We call (S, M) *uniformly asymptotically stable* if (S, M) is uniformly stable and if there exists a $\delta > 0$ and for every $\epsilon > 0$ there exists a $\tau = \tau(\epsilon) > 0$ such that $d(p(t, a, t_0), M) < \epsilon$ for all $t \in \{t \in T_{a,t_0}\colon \rho(t, t_0) \geq \tau\}$ and all $p(\cdot, a, t_0) \in S$ whenever $d(a, M) < \delta$. We call (S, M) *exponentially stable* if there exists $\alpha > 0$, and for every $\epsilon > 0$ and $t_0 \in T_0$, there exists $\delta = \delta(\epsilon) > 0$ such that $d(p(t, a, t_0), M) < \epsilon e^{-\alpha\cdot\rho(t,t_0)}$ for all $t \in T_{a,t_0}$ and for all $p(\cdot, a, t_0) \in S$, whenever $d(a, M) < \delta$. The notions of *uniform asymptotic stability in the large*, and *global exponential stability* are defined similarly. Finally, we call (S, M) *unstable* if (S, M) is not stable. It has been shown (for example, [1−4]) that, *by using the isometric mapping* $e\colon T \to R^+$ *given by* $e(t) = \rho(t, t_{min})$, *the qualitative analysis of invariant sets of hybrid dynamical systems defined on abstract time space T can be reduced to the qualitative analysis of the same invariant sets of the corresponding discontinuous dynamical systems defined on* R^+ (t_{min} is the minimum element on T determined by the relation "$\prec$"). Thus, in the qualitative analysis of such hybrid systems, we can confine ourselves to the qualitative analysis of appropriate DDS. For further details, see [1].

We now introduce the concept of stability preserving mapping between two discontinuous dynamical systems $\{R^+, X_1, A_1, S_1\}$ (with invariant set M_1) and $\{R^+, X_2, A_2, S_2\}$ (with invariant set M_2). Such mappings will serve as a basis for developing a general comparison (stability) theory for discontinuous dynamical systems. For example, if a stability preserving mapping has been established between S_1 and S_2, and if the stability properties of (S_2, M_2) are well understood, then it will be possible to deduce the stability properties of (S_1, M_1) from those of (S_2, M_2).

Definition 2.1 Let $\{R^+, X_1, A_1, S_1\}$ and $\{R^+, X_2, A_2, S_2\}$ be two discontinuous dynamical systems with invariant sets $M_1 \subset A_1$ and $M_2 \subset A_2$, respectively. We say that $V: X_1 \times R^+ \to X_2$ is a *stability preserving mapping* from S_1 to S_2 (or more explicitly, from (S_1, M_1) to (S_2, M_2)) if V satisfies the following conditions:

(i) $S_2 = \mathcal{V}(S_1) \stackrel{\triangle}{=} \{q(\cdot, b, t_0)\colon q(t, b, t_0) = V(p(t, a, t_0), t)$, with $b = V(a, t_0)$ and $T_{b,t_0} = T_{a,t_0}$, $a \in A_1$, $t_0 \in R^+\}$;
(ii) $M_2 = V(M_1 \times R^+) \stackrel{\triangle}{=} \{x \in X_2\colon x = V(x_1, t')$ for some $x_1 \in M_1$ and $t' \in R^+\}$;
(iii) the invariance of (S_1, M_1) is *equivalent* to the invariance of (S_2, M_2), i.e., (S_1, M_1) is invariant if and only if (S_2, M_2) is invariant; and
(iv) the stability, uniform stability, asymptotic stability, uniform asymptotic stability, exponential stability, uniform asymptotic stability in the large, and exponential stability in the large of (S_1, M_1) and (S_2, M_2) are equivalent, respectively (i.e., (S_1, M_1) is stable if and only if (S_2, M_2) is stable; (S_1, M_1) is uniformly stable if and only if (S_2, M_2) is uniformly stable; and so forth.)

The above definition states that the function V from $X_1 \times R^+$ into X_2 induces a mapping $\mathcal{V}\colon S_1 \to S_2$ and that under $\mathcal{V}$ several stability properties of (S_1, M_1) and $(S_2, V(M_1 \times R^+))$ are preserved.

Lemma 2.1 [1] *Let $\{R^+, X_i, A_i, S_i\}$, $i = 1, 2$, be two discontinuous dynamical systems and let $M_i \subset A_i$, $i = 1, 2$, be closed sets. Assume there exists $V\colon X_1 \times R^+ \to X_2$ which satisfies*

(i) *$\mathcal{V}(S_1) \subset S_2$, where $\mathcal{V}(S_1)$ and M_2 are defined as in Definition 2.1;*
(ii) *there exist ψ_1, $\psi_2 \in K$ defined on R^+ such that*

$$\psi_1(d_1(x, M_1)) \le d_2(V(x, t), M_2) \le \psi_2(d_1(x, M_1)) \tag{2}$$

for all $x \in X_1$, and $t \in R^+$, where d_1, d_2 are the metrics defined on X_1 and X_2, respectively. ($\psi \in K$ means that $\psi \in C[R^+, R^+]$, $\psi(0) = 0$, and $\psi(r)$ is monotonically increasing in r.)

Then,

(a) *the invariance of (S_2, M_2) implies the invariance of (S_1, M_1);*
(b) *the stability, uniform stability, asymptotic stability, and uniform asymptotic stability of (S_2, M_2) imply the same corresponding types of stability for (S_1, M_1); and*

(c) *if in (2),* $\psi_1(r) = ar^b$, $a > 0$, $b > 0$, *then the exponential stability of* (S_2, M_2) *implies the exponential stability for* (S_1, M_1)*; and*

(d) *if in (2),* $\lim_{r\to\infty} \psi_1(r) = \infty$ *and if* M_1 *and* M_2 *are bounded and closed, then the global uniform asymptotic stability of* (S_2, M_2) *implies the global uniform asymptotic stability for* (S_1, M_1)*; and*

(e) *if in (c),* M_1 *and* M_2 *are bounded and closed, then the global exponential stability of* (S_2, M_2) *implies the global exponential stability for* (S_1, M_1)*.*

There may be a temptation to view the notions of *stability preserving mapping* and *Lyapunov function* as being identical concepts. This, however, is not correct, as can be seen by considering, e.g., for $\{T, X, A, S, T_0\}$ with $X = R^n$ and $M = \{0\}$, the function $V(x) = x \in R^n$. This function is clearly a stability preserving mapping. However, by any standards, it hardly qualifies as being a Lyapunov function.

3 Stability Analysis of Several Classes of Switched Systems

In the present section, we apply the stability preserving mapping theory in the analysis of several classes of switched systems described by (1).

First, we consider the linear case of the system (1) given by

$$\begin{cases} \dot{x}(t) = A_{i_k} x(t), & \tau_k \le t < \tau_{k+1}, \\ x(t) = B_{i_{k+1}} x(t^-), & t = \tau_{k+1}, \quad k \in N, \end{cases} \tag{3}$$

where A_{i_k}, $B_{i_{k+1}} \in \Re^{n\times n}$, and $E \overset{\triangle}{=} \{\tau_0, \tau_1, \dots : \tau_0 < \tau_1 < \cdots\}$ is a fixed, unbounded, closed, discrete set. For this switched system, we obtain the following result according to [4].

Lemma 3.1 *Assume that*

(i) *there exists a constant* $\alpha > 0$ *such that for all* $i \in I_N$, $\|A_i\| < \alpha$, *where* $\|\cdot\|$ *denotes the matrix norm induced by the Euclidean vector norm;*

(ii) $\sup_k \{\tau_{k+1} - \tau_k\} \le \lambda < \infty$, *where* λ *is a constant;*

(iii) $\|B_{i_{k+1}} e^{A_{i_k}(\tau_{k+1}-\tau_k)}\| < q < 1$, *where* q *is a constant,* $\forall\, k \in N$.

Then, the equilibrium $x_e = 0$ *of the switched system (3) is uniformly asymptotically stable in the large.*

We now show how one could obtain the above result by the stability preserving mapping theory (Lemma 2.1). Let $S_1 = S_{(3)}$ ($S_{(3)}$ denotes the dynamical system determined by the solutions of (3)), $M_1 = \{0\}$, and choose $y(t) = V(x(t)) = (x^T x)^{1/2}$. Along the solutions of (3), we have for $\tau_k \le t < \tau_{k+1}$,

$$\begin{aligned} \dot{y}(t) &= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}} (x(t)^T \dot{x}(t) + \dot{x}(t)^T x(t)) \\ &= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}} (x(t)^T (A_{i_k} + A_{i_k}^T) x(t)) \\ &\le \alpha y(t). \end{aligned} \tag{4}$$

Also, at $t = \tau_{i_{k+1}}$, we have

$$y(t) = |x(t)| \le \|B_{i_{k+1}} e^{A_{i_k}(\tau_{k+1}-\tau_k)}\| |x(\tau_k)| \le q y(\tau_k). \tag{5}$$

Now consider the discontinuous dynamical system described by the scalar-valued inequalities

$$\begin{cases} \dot{y}(t) \le \alpha y(t), & \alpha > 0 - \text{constant}, \quad \tau_k \le t < \tau_{k+1}, \\ y(\tau_{k+1}) \le q|y(\tau_k)|, & 0 < q < 1, \quad \forall\, k \in N. \end{cases} \tag{6}$$

Let $S_2 = S_{(6)}$ denote the dynamical system determined by (6), $M_2 = \{0\}$, and let d_1, d_2 be the metrics determined by the Euclidean norms on $\Re^n$ and $\Re^1$. Then, the function $V(x)$ induces a mapping $\mathcal{V}$ from S_1 to S_2 (see Definition 2.1), and $\mathcal{V}(S_1) \subset S_2$. Since the equilibrium $y_e = 0$ of (6) is uniformly asymptotically stable in the large (as can be verified by solving (6) directly), and since all the conditions of Lemma 2.1 are satisfied, we conclude that the equilibrium $x_e = 0$ of (3) is uniformly asymptotically stable in the large.

From Lemma 3.1, we obtain the following result, which is an extension of the results that appeared in [7] and [8].

Theorem 3.1 *Assume that A_i is Hurwitz stable for all $i \in I_N$. Then, there exists a constant $T > 0$ such that if every subsystem is activated over a time interval larger than T, then the switched system (3) is exponentially stable.*

Proof Since every A_i is Hurwitz stable, there exist positive scalars K and η such that $\|e^{A_i t}\| \le K e^{-\eta t}$. Also, we can always find a positive scalar β such that $\|B_i\| \le \beta$ for all $i \in I_N$. For any positive scalar $q < 1$, we choose

$$T > \frac{1}{\eta} \ln\left(\frac{\beta K}{q}\right). \tag{7}$$

When $\tau_{k+1} - \tau_k > T$, we let $l_k = \left\lfloor \dfrac{\tau_{k+1} - \tau_k}{T} \right\rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, and we let

$$\tau_{k,i} = \begin{cases} \tau_k + (i-1)T & \text{for} \quad 1 \le i \le l_k, \\ \tau_{k+1} & \text{for} \quad i = l_k + 1. \end{cases} \tag{8}$$

Obviously, during the interval $[\tau_{k,i}, \tau_{k,i+1})$ $(i = 1, \cdots, l_k)$ the i_k-th subsystem is activated. Now, since $T \le \tau_{k,i+1} - \tau_{k,i} < 2T$, and since

$$\|B_{i_{k,i+1}} e^{A_{i_k}(\tau_{k,i+1}-\tau_{k,i})}\| \le \beta K e^{-\eta T} < q < 1, \tag{9}$$

it follows from Lemma 3.1 that the equilibrium $x_e = 0$ of the switched system (3) is uniformly asymptotically stable in the large, and thus exponentially stable in this case.

Next, we consider the more general class of switched systems

$$\begin{cases} \dot{x}(t) = A_{i_k} x(t) + M_{i_k} x(\tau_k), & \tau_k \le t < \tau_{k+1}, \\ x(t) = B_{i_{k+1}} x(t^-) + N_{i_{k+1}} x(\tau_k), & t = \tau_{k+1}, \quad k \in N, \end{cases} \tag{10}$$

where A_{i_k}, M_{i_k}, $B_{i_{k+1}}$, $N_{i_{k+1}} \in \Re^{n \times n}$. For such systems, we now prove the following *new result.*

Theorem 3.2 *Assume that*

(i) *there exist two constants* $\alpha > 0$, $\gamma > 0$ *such that for all* $i \in I_N$, $\|A_i\| < \alpha$, $\|M_i\| < \gamma$;

(ii) $\sup\limits_k \{\tau_{k+1} - \tau_k\} \le \lambda < \infty$, *where* λ *is a constant;*

(iii) $\left\| B_{i_{k+1}} \left(e^{A_{i_k}(\tau_{k+1}-\tau_k)} + \left[\int\limits_0^{\tau_{k+1}-\tau_k} e^{A_{i_k}\tau}\, d\tau \right] M_{i_k} \right) + N_{i_{k+1}} \right\| < q < 1$, *where* q *is a constant,* $\forall\, k \in N$.

Then, the equilibrium $x_e = 0$ *of the switched system (10) is uniformly asymptotically stable in the large.*

Proof Let $S_1 = S_{(10)}$, $M_1 = \{0\}$, and choose $y(t) = V(x(t)) = (x^T x)^{1/2}$. Along the solutions of (10), we have for $\tau_k \le t < \tau_{k+1}$,

$$
\begin{aligned}
\dot y(t) &= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}} (x(t)^T \dot x(t) + \dot x(t)^T x(t)) \\
&= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}} \left(x(t)^T (A_{i_k} + A_{i_k}^T) x(t) + x(t)^T M_{i_k} x(\tau_k) + x(\tau_k)^T M_{i_k}^T x(t) \right) \\
&\le \alpha y(t) + \gamma y(\tau_k).
\end{aligned}
\tag{11}
$$

Also, at $t = \tau_{i_{k+1}}$, we have according to the condition (iii)

$$
\begin{aligned}
y(t) &= |x(t)| \\
&\le \left\| B_{i_{k+1}} \left(e^{A_{i_k}(\tau_{k+1}-\tau_k)} + \left[\int\limits_0^{\tau_{k+1}-\tau_k} e^{A_{i_k}\tau}\, d\tau \right] M_{i_k} \right) + N_{i_{k+1}} \right\| |x(\tau_k)| \le q y(\tau_k).
\end{aligned}
\tag{12}
$$

Similarly as in the proof of Lemma 3.1, the function $V(x)$ induces a mapping $\mathcal{V}$ from S_1 to the discontinuous dynamical system $S_2 = S_{(13)}$ determined by the scalar-valued inequalities

$$
\begin{cases}
\dot y(t) \le \alpha y(t) + \gamma y(\tau_k), & \alpha > 0, \quad \gamma > 0, \quad \tau_k \le t < \tau_{k+1}, \\
y(\tau_{k+1}) \le q |y(\tau_k)|, & 0 < q < 1, \quad \forall\, k \in N.
\end{cases}
\tag{13}
$$

Since the equilibrium $y_e = 0$ of (13) is uniformly asymptotically stable in the large, and since all the conditions of Lemma 2.1 are satisfied, we conclude that the equilibrium $x_e = 0$ of (10) is uniformly asymptotically stable in the large.

From Theorem 3.2, we obtain the following interesting result.

Lemma 3.2 *Assume that* A_i*'s,* $i \in I_N$, *are Hurwitz stable, and thus there exist constants* $K > 0$, $\eta > 0$ *such that* $\|e^{A_i t}\| \le K e^{-\eta t}$ *for all* $t \ge 0$, *and assume that* $\|B_i\| < \beta$, $\|N_i\| < \mu < 1$, $\|M_i\| < \gamma$, *where* β, μ, γ *are positive constants. If* $\dfrac{\beta\gamma K}{\eta} + \mu = q_0 < 1$, *then there exist constants* $T_2 > T_1 > 0$ *such that when every subsystem is activated over a time interval of duration* T *satisfying* $T_1 < T < T_2$, *the entire switched system is exponentially stable.*

Proof The condition (iii) of Theorem 3.2 is calculated as

$$\begin{aligned}
&\left\| B_{i_{k+1}} \left(e^{A_{i_k}(\tau_{k+1}-\tau_k)} + \left[\int_0^{\tau_{k+1}-\tau_k} e^{A_{i_k}\tau}\, d\tau \right] M_{i_k} \right) + N_{i_{k+1}} \right\| \\
&\qquad \le \beta K e^{-\eta(\tau_{k+1}-\tau_k)} + \beta\gamma K \int_0^{\tau_{k+1}-\tau_k} e^{-\eta\tau}\, d\tau + \mu \qquad (14)\\
&\qquad \le \beta K e^{-\eta(\tau_{k+1}-\tau_k)} + \frac{\beta\gamma K}{\eta} + \mu \\
&\qquad = \beta K e^{-\eta(\tau_{k+1}-\tau_k)} + q_0.
\end{aligned}$$

Therefore, for any $0 < q < q_0$, we can always choose $T_1 > 0$ such that when $\inf_k(\tau_{k+1} - \tau_k) \ge T_1$, we have $\beta K e^{-\eta(\tau_{k+1}-\tau_k)} + q_0 \le q < 1$. Pick any $T_2 > T_1$. Then, if every subsystem is activated over a time interval of magnitude T satisfying $T_1 < T < T_2$, then by Theorem 3.2 we can conclude that the entire switched system is uniformly asymptotically stable in the large, and thus exponentially stable in this case.

Note that in Theorem 3.1, τ_k can be ∞ (because the system is autonomous), while in Lemma 3.2 this case must be excluded since the term $M_{i_k}x(\tau_k)$ in the system (10) depends specifically on $x(\tau_k)$. Therefore, an upper bound T_2 is required to avoid the case that only one subsystem is activated after some time instant.

Finally, we consider the nonlinear switched systems determined by equations of the form

$$\begin{cases} \dot{x}(t) = A_{i_k}x(t) + M_{i_k}x(\tau_k) + F_{i_k}(t, x(t), x(\tau_k)), & \tau_k \le t < \tau_{k+1}, \\ x(t) = B_{i_{k+1}}x(t^-) + N_{i_{k+1}}x(\tau_k) + G_{i_{k+1}}(t, x(t^-), x(\tau_k)), & t = \tau_{k+1},\ k \in N, \end{cases} \qquad (15)$$

where $A_{i_k}, M_{i_k}, B_{i_{k+1}}, N_{i_{k+1}} \in \Re^{n\times n}$, $F_{i_k}, G_{i_{k+1}} \in C[\Re^+ \times \Re^n \times \Re^n, \Re^n]$, $F_{i_k}(t,0,0) = 0$, $G_{i_{k+1}}(t,0,0) = 0$ for all $t \in R^+$, and

$$\lim_{x\to 0,\ z\to 0} \frac{F_{i_k}(t,x,z)}{\sqrt{\|x\|^2 + \|z\|^2}} = 0, \qquad \lim_{x\to 0,\ z\to 0} \frac{G_{i_{k+1}}(t,x,z)}{\sqrt{\|x\|^2 + \|z\|^2}} = 0 \qquad (16)$$

hold uniformly for all $t \in \Re^+$, $k \in N$. Obviously, the system (15) may be a consequence of a *linearization process* of the system (1) about the point $x_e = 0$. We now prove the following *new* result.

Theorem 3.3 *Assume that*

(i) *there exist three positive constants α, γ, β such that for all $i \in I_N$, $\|A_i\| < \alpha$, $\|M_i\| < \gamma$, $\|B_i\| < \beta$;*

(ii) $\sup_{k\in N}\{\tau_{k+1} - \tau_k\} \le \lambda < \infty$, *where λ is a constant;*

(iii) $\left\| B_{i_{k+1}} \left(e^{A_{i_k}(\tau_{k+1}-\tau_k)} + \left[\int_0^{\tau_{k+1}-\tau_k} e^{A_{i_k}\tau}\, d\tau \right] M_{i_k} \right) + N_{i_{k+1}} \right\| < q < 1$, *where q is a constant, for all $k \in N$.*

Then, the equilibrium $x_e = 0$ of the switched system (15) is uniformly asymptotically stable.

To prove Theorem 3.3, we need the following preliminary result.

Lemma 3.3 *For any given* $\epsilon > 0$, *there exists a* $\delta_1 = \delta_1(\epsilon) > 0$ *such that*

$$\begin{cases} \left\| \int\limits_{\tau_k}^{\tau_{k+1}} e^{A_{i_k}(\tau_{k+1}-\tau)} F_{i_k}(t, x(t), x(\tau_k))\, d\tau \right\| \le \epsilon \|x(\tau_k)\|, \\ \|G_{i_{k+1}}(\tau_{k+1}, x(\tau_{k+1}^-), x(\tau_k))\| \le \epsilon(\|x(\tau_{k+1}^-)\| + \|x(\tau_k)\|), \\ \|x(t)\| \le c_0 \|x(\tau_k)\|, \end{cases} \tag{17}$$

whenever $\|x(\tau_k)\| \le \delta_1$ *for* $k \in N$, $t \in [\tau_k, \tau_{k+1})$, *where* $c_0 = (1+\lambda+\lambda\gamma)e^{(\alpha+1)\lambda}$.

Proof By continuity, there exists a $\delta_2 > 0$ such that $\|F_{i_k}(t, x(t), x(\tau_k))\| \le \|x(t)\| + \|x(\tau_k)\|$ whenever $\|x(t)\| \le \delta_2$, $\|x(\tau_k)\| \le \delta_2$. We show that there exists a $\delta_3 > 0$ $\left(\delta_3 < \dfrac{\delta_2}{c_0}\right)$ such that $\|x(t)\| \le \delta_2$ for all $t \in [\tau_k, \tau_{k+1})$ whenever $\|x(\tau_k)\| \le \delta_3$. Otherwise, since $\delta_3 < \delta_2$, there exists a $t_0 \in (\tau_k, \tau_{k+1})$ such that $\|x(t)\| \le \delta_2$ for all $t \in [\tau_k, t_0)$ and $\|x(t_0)\| = \delta_2$. From the first equation of (15), we have

$$x(t) = x(\tau_k) + M_{i_k} x(\tau_k)(t-\tau_k) + \int\limits_{\tau_k}^{t} (A_{i_k} x(\tau) + F_{i_k}(\tau, x(\tau), x(\tau_k)))\, d\tau. \tag{18}$$

Now for $t \in [\tau_k, t_0)$, we have

$$\|x(t)\| \le (1+\lambda+\lambda\gamma)\|x(\tau_k)\| + \int\limits_{\tau_k}^{t} (\alpha+1)\|x(\tau)\|\, d\tau. \tag{19}$$

By the Grownwall inequality, we obtain

$$\|x(t)\| \le (1+\lambda+\lambda\gamma)e^{(\alpha+1)(t-\tau_k)}\|x(\tau_k)\|, \tag{20}$$

and hence

$$\|x(t_0)\| \le (1+\lambda+\lambda\gamma)e^{(\alpha+1)\lambda}\delta_3 = c_0\delta_3 < \delta_2, \tag{21}$$

which is a contradiction. Thus, our conclusion follows. In addition, we know that whenever $\|x(\tau_k)\| \le \delta_3$, $\|x(t)\| \le c_0\|x(\tau_k)\|$ for $t \in [\tau_k, \tau_{k+1})$.

Now, for given $\epsilon > 0$, let $\epsilon = \epsilon_1 \lambda e^{\alpha\lambda}(c_0+1)$. There exists a $\delta_4 > 0$ such that

$$\begin{aligned} \|F_{i_k}(t, x(t), x(\tau_k))\| &\le \epsilon_1(\|x(t)\| + \|x(\tau_k)\|), \\ \|G_{i_{k+1}}(t, x(t^-), x(\tau_k))\| &\le \epsilon_1(\|x(t^-)\| + \|x(\tau_k)\|), \end{aligned} \tag{22}$$

whenever $\|x(t)\| + \|x(\tau_k)\| \le \delta_4$. Let $\delta_1 = \min\left\{\delta_3, \dfrac{\delta_4}{c_0+1}\right\}$. When $\|x(\tau_k)\| \le \delta_1$, we have for all $t \in [\tau_k, \tau_{k+1})$, $\|x(t)\| \le c_0\|x(\tau_k)\|$. Thus $\|x(t)\| + \|x(\tau_k)\| \le \delta_4$, and then (22) is true. Furthermore, we obtain by (22) that

$$\begin{aligned} \left\| \int\limits_{\tau_k}^{\tau_{k+1}} e^{A_{i_k}(\tau_{k+1}-\tau)} F_{i_k}(t, x(t), x(\tau_k))\, d\tau \right\| &\le \epsilon_1 e^{\alpha\lambda} \int\limits_{\tau_k}^{\tau_{k+1}} (\|x(t)\| + \|x(\tau_k)\|)\, d\tau \\ &\le \epsilon_1 e^{\alpha\lambda}\lambda(c_0+1)\|x(\tau_k)\| = \epsilon\|x(\tau_k)\|. \end{aligned} \tag{23}$$

Since $\epsilon_1 < \epsilon$, we get from the second inequality of (22) that

$$\|G_{i_{k+1}}(t_{k+1}, x(t_{k+1}^-), x(\tau_k))\| \le \epsilon(\|x(t_{k+1}^-)\| + \|x(\tau_k)\|. \tag{24}$$

This completes the proof of the lemma.

We are now in a position to prove Theorem 3.3.

Proof of Theorem 3.3 Let $S_1 = S_{(15)}$, and let $V(x) = (x^T x)^{1/2}$. For any solution $x(t)$ of (15), let $y(t) = V(x(t))$. Then for $t \in [\tau_k, \tau_{k+1})$,

$$\begin{aligned} \dot{y}(t) &= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}}(x(t)^T \dot{x}(t) + \dot{x}(t)^T x(t)) \\ &= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}}(x(t)^T (A_{i_k} + A_{i_k}^T)x(t) + x(t)^T M_{i_k} x(\tau_k) + x(\tau_k)^T M_{i_k}^T x(t) \\ &\quad + x(t)^T F_{i_k}(t, x(t), x(\tau_k)) + F_{i_k}(t, x(t), x(\tau_k))^T x(t)). \end{aligned} \tag{25}$$

Assumption (16) implies that for $\epsilon > 0$ ($\epsilon < 1$, which will be specified later) there exists $\delta = \delta(\epsilon) > 0$ such that

$$\begin{aligned} \|F_{i_k}(t, x(t), x(\tau_k))\| &\le \epsilon(\|x(t)\| + \|x(\tau_k)\|), \\ \|G_{i_{k+1}}(t, x(t^-), x(\tau_k))\| &\le \epsilon(\|x(t^-)\| + \|x(\tau_k)\|) \end{aligned} \tag{26}$$

for all $x \in B(\delta) = \{x \in R^n\colon \|x\| < \delta\}$ and $t \in R^+,\ k \in N$. According to Lemma 3.3, for the given $\epsilon > 0$, there exists a $\delta_1 > 0$ such that (17) holds. Updating δ with $\min\{\delta, \delta_1\}$ and combining (25), (26), we obtain for $t \in [\tau_k, \tau_{k+1})$,

$$\begin{aligned} \dot{y}(t) &\le \frac{1}{2}\,(x(t)^T x(x))^{-\frac{1}{2}}((2\alpha + 2\epsilon)\|x(t)\|^2 + (2\gamma + 2\epsilon)\|x(t)\|\|x(\tau_k)\|) \\ &\le (\alpha + 1)y(t) + (\gamma + 1)y(\tau_k). \end{aligned} \tag{27}$$

We now apply Lemma 2.1. We let $X_1 = B(\delta)$ to derive local stability results. For $t = \tau_{k+1}$, we have

$$\begin{aligned} y(\tau_{k+1}) &= \|x(\tau_{k+1})\| = \|B_{i_{k+1}} x(\tau_{k+1}^-) + N_{i_{k+1}} x(\tau_k) + G_{i_{k+1}}(\tau_{k+1}, x(\tau_{k+1}^-), x(\tau_k))\| \\ &\le \left\| B_{i_{k+1}}\left(e^{A_{i_k}(\tau_{k+1}-\tau_k)} + \left[\int_0^{\tau_{k+1}-\tau_k} e^{A_{i_k}\tau}\, d\tau \right] M_{i_k} \right) + N_{i_{k+1}} \right\| \|x(\tau_k)\| \\ &\quad + \|B_{i_{k+1}}\| \left\| \int_{\tau_k}^{\tau_{k+1}} e^{A_{i_k}(\tau_{k+1}-\tau)} F_{i_k}(t, x(t), x(\tau_k))\, d\tau \right\| + \epsilon(\|x(\tau_{k+1}^-)\| + \|x(\tau_k)\|) \\ &\le (q + \epsilon(\beta + c_0 + 1))\|x(\tau_k)\|. \end{aligned} \tag{28}$$

Since $q < 1$, there exists an $\epsilon_0 > 0$ such that

$$q_0 = q + \epsilon_0(\beta + c_0 + 1) < 1. \tag{29}$$

Clearly $B(\delta(\epsilon_0))$ is included in the region of attraction.

Now consider the discontinuous system determined by

$$\begin{cases} \dot{y}(t) \le (\alpha+1)y(t) + (\gamma+1)y(\tau_k), & \tau_k \le t < \tau_{k+1}, \\ y(\tau_{k+1}) \le q_0|y(\tau_k)|, & \forall\, k \in N. \end{cases} \tag{30}$$

The function $V(x(t))$ induces a mapping $\mathcal{V}$ from $S_1 = S_{(15)}$ to $S_2 = S_{(30)}$ which satisfies $\mathcal{V}(S_1) \subset S_2$. Since the equilibrium $y_e = 0$ of (30) is uniformly asymptotically stable, it now follows from Lemma 2.1 that the equilibrium $x_e = 0$ of system (15) is uniformly asymptotically stable.

4 Concluding Remarks

In this paper, we have analyzed the stability properties of a large class of switched systems by using the stability preserving mapping theory. By first considering an existing result and then analyzing more general switched systems, we have shown that the stability preserving mapping theory is very practical in the stability analysis and design of switched systems. We suggest that the same idea applies also to logic-based switched systems or discrete event systems, provided that we can model the state change when switchings occur.

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