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Impulsive Stabilization and Application to a Population Growth Model*

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Abstract: This paper studies the problem of impulsive stabilization of a system of autonomous ordinary differential equations. Necessary and sufficient conditions are established for a given state, which need not be an equilibrium point of the system, to be impulsively stabilizable. These results are applied to a three-species population growth model. In the population growth model, it is shown that by impulsively regulating one species, the population of all three species can be maintained at a positive level, which otherwise would drop to a level of extinction for one of the species.

Keywords: *Impulsive stabilization; control; population growth model.*

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1 Introduction

In this paper we shall investigate the problem of impulsively controlling a system of autonomous ordinary differential equations so as to keep solutions close to a given state, p, which need not be an equilibrium point of the system. Consider the following autonomous system

$$x' = f(x), \tag{1}$$

where $f \in C^1[D, \mathbb{R}^n]$, $D \subset \mathbb{R}^n$ is open. Let the space $X = \mathbb{R}^n$ be decomposed into the direct sum $X = Y \oplus Z$, where Y is an m-dimensional subspace of X, $1 \le m < n$, and $Z = Y^{\perp}$. We call Y and Z the impulsive and non-impulsive subspaces respectively. Any vector $x \in X$, or vector function $f(x)$, may be expressed uniquely as $x = y + z$,

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or $f(x) = f_y(x) + f_z(x)$, where y, $f_y(x) \in Y$ and $z, f_z(x) \in Z$. We shall utilize this decomposition throughout this chapter and remark that the subscripts y and z on any vector in x shall denote its unique portion in Y and Z respectively, while the vectors y and z shall represent the unique portions of x.

Let U be the set of admissible controls u, where $u = \{(t_k, \Delta y_k)\}_{k=1}^{\infty}$ and

(i) $0 \leq t_1 < t_2 < \ldots < t_k < \ldots$, and $t_k \to \infty$ as $k \to \infty$,

(ii) $\Delta y_k \in Y$, $k = 1, 2, \ldots$, $|\Delta y_k| \leq A$ for some positive constant A.

Consider the impulsive control system associated with system (1)

$$\begin{cases} y' = f_y(x), \\ z' = f_z(x), & t \neq t_k, \quad k = 1, 2, \ldots, \\ y(t_k^+) = y(t_k) + \Delta y_k, & k = 1, 2, \ldots, \\ x(0) = x_0. \end{cases} \tag{2}$$

For a comprehensive treatment of impulsive differential equations see Lakshmikantham, Bainov and Simeonov (1989). Let $\phi(t, x)$ be a solution of (1). Then for each $u \in U$, $x(t) = x(t, x_0, u)$ is a solution of (2) given by

$$x(t) = \phi(t - t_{k-1}, x_{k-1}^+), \qquad t \in (t_{k-1}, t_k], \quad k = 1, 2, \ldots, \tag{3}$$

where $t_0 = 0$, $x_0^+ = x_0$, and

$$x_k^+ = \phi(t_k - t_{k-1}, x_{k-1}^+) + \Delta y_k, \quad k = 1, 2, \ldots. \tag{4}$$

Note from (3) and (4) that since f is C^1, ϕ is a continuous function of t, and since $\Delta y_k \in Y$, it follows that $z(t)$ is continuous for all $t \geq 0$, while $y(t)$ is continuous on each interval $(t_{k-1}, t_k]$, where $y(t) + z(t)$ is the decomposition of $x(t)$.

2 Criteria for Stabilizability

We begin by stating the concept of impulsive stabilization of a point p. For $\alpha > 0$, let $B_\alpha(p) = \{x \in \mathbb{R}^n : |p - x| < \alpha\}$.

Definition 2.1 A point $p \in D$ is said to be

(S_1) *impulsively stabilizable* if for any given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for each $x_0 \in B_\delta(p)$ there exists a $u \in U$ such that $x(t) \in B_\epsilon(p)$, for all $t \geq 0$, where $x(t) = x(t, x_0, u)$ is any solution of (2);

(S_2) *asymptotically impulsively stabilizable* if for any given $\epsilon > 0$, there exists a $\sigma = \sigma(\epsilon) > 0$ such that for each $x_0 \in B_\sigma(p)$ there exists a $u \in U$ such that $x(t) \in B_\epsilon(p)$, for all $t \geq 0$, and $\lim_{t\to\infty} x(t) = p$;

(S_3) *impulsively unstabilizable* if (S_1) fails to hold.

It should be noted that the point p in the above definition is, in general, not an equilibrium point of the system, in contrast to those found in the standard control theory (see for example Sontag, 1990). The type of stability defined above may be considered a special case of stability in terms of two measures, a concept expounded by Lakshmikantham and Liu (1989), and Liu (1990).

2.1 Necessary conditions

It follows from the above definition that the vector field f must be tangent to the impulsive subspace at p for p to be impulsively stabilizable as indicated in the theorem below.

Theorem 2.1 *If $f(p) \notin Y$, then p is impulsively unstabilizable.*

Proof If $f(p) \notin Y$, then $f_z(p) = v \neq 0$. By continuity of f, there exists an $\epsilon > 0$ such that

$$|\mathrm{Proj}_v f_z(x)| > 0, \quad \forall\, x \in \overline{B_\epsilon(p)}, \tag{5}$$

where Proj_v denotes the orthogonal projection onto the one-dimensional subspace defined by span $\{v\}$. By continuity of the projection function and by the compactness of $\overline{B_\epsilon(p)}$, inequality (5) implies that there exists $m > 0$ such that

$$|\mathrm{Proj}_v f_z(x)| \geq m > 0, \quad \forall\, x \in \overline{B_\epsilon(p)}; \tag{6}$$

physically, m is the minimum speed in the positive v direction for all points in $\overline{B_\epsilon(p)}$. For any δ, $0 < \delta < \epsilon$, choose $x_0 = y_0 + z_0$ in $B_\delta(p)$. Since $z(t) = z(t, x_0, u)$ is continuous in t, (6) implies

$$|\mathrm{Proj}_v(z(t) - z_0)| \geq mt, \quad \forall\, t \geq 0, \quad \text{provided} \quad x(t) \in \overline{B_\epsilon(p)}. \tag{7}$$

Since the projection is orthogonal, (7) implies

$$|z(t) - z_0| \geq mt, \quad \forall\, t \geq 0, \quad \text{provided} \quad x(t) \in \overline{B_\epsilon(p)}, \tag{8}$$

but $|z(t) - z_0| \leq |x(t) - x_0| \leq |x(t) - p| + |p - x_0|$, hence from (8) we have

$$|x(t) - p| \geq mt - |p - x_0| \geq mt - \delta, \quad \forall\, t \geq 0,$$

so that $|x(t) - p| > \epsilon$ for t sufficiently large; consequently p is impulsively unstabilizable.

2.2 Impulsively invariant sets

To motivate and help illustrate our subsequent theorem we shall embark on a short discussion of the problem of finding a control u that will create an invariant set of system (2).

Consider a system in $\mathbb{R}^3$ and a point p for which $f(p)$ is aligned with the x-axis. Let $Y = \mathrm{span}\{(1,0,0)^T\}$ and $Z = \mathrm{span}\{(0,1,0)^T, (0,0,1)^T\}$. Such a system meets the necessary condition of Theorem 2.1. Consider a closed curve C, lying in the plane through p parallel to Z such that p is in the interior of C. Generate a "cylinder", S, by constructing, through each point of C, a line segment of length 2ℓ parallel to Y such that its midpoint lies on C, (see Figure 2.1). The boundary of the cylinder is composed of the cylinder's wall (the line segments) and its two ends which are surfaces parallel to Z.

Our aim is to make the cylinder, S, invariant with the application of impulses in the x_1-direction. Consider a point x_0, starting within the cylinder. The trajectory $\phi(t, x_0)$ will either stay within S or will reach the cylinder wall or the ends. Suppose $\phi(t^*, x_0) = A$ for some time t^*, where A is a point on one of the cylinder ends. It is then easy to see that an impulse of strength less than 2ℓ in the positive or negative x_1-direction as appropriate, will send the trajectory back into the interior of S. If however $\phi(t^*, x_0) = B$, where B is a point on the cylinder wall, then an impulse in Y can only carry the trajectory to some other point on the cylinder wall along the line through B parallel to Y. If along this line segment there is a point Q where the vector field f is moving into the cylinder then an

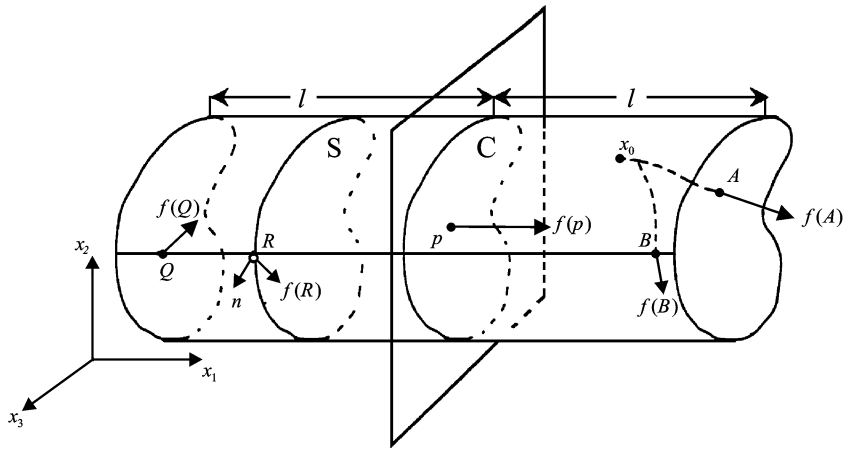

Figure 2.1. An invariant cylinder.

impulse to Q will keep $x(t)$ within S for at least a short time longer. If there is a point R along the line segment at which the velocity field is tangent to the cylinder wall and for which $\phi(t, R)$ lies either on or inside S for some positive time interval then an impulse to R would also keep $x(t)$ within S. If such points Q or R exist for each B on the cylinder wall then the cylinder S can be made invariant provided the sum of the time intervals between successive impulses is unbounded. We now formalize the preceding discussion on invariant sets.

Theorem 2.2 *Let $\Omega_z \subset Z$ be an open bounded region whose boundary is C^1, and let Ω_y be an open bounded region in Y. Define the "cylinder", Ω, and its wall, W, by $\Omega = \Omega_y \oplus \Omega_z$, and $W = \Omega_y \oplus \partial\Omega_z$. We assume $\Omega \subset D$. Let n be the unit outward normal to Ω defined on W, and define the set N as*

$$N = \{w \in W\colon f \cdot n\big|_w \leq 0\}^{O(W)},$$

where the superscript $O(W)$ denotes the interior of the set N with respect to W.

If $\mathrm{Proj}_z(N) = \partial\Omega_z$, then for any $x_0 \in \overline{\Omega}$, there exists a $u \in U$ such that $x(t, x_0, u) \in \overline{\Omega}$, for all $t \geq 0$.

Proof See Liu and Willms (1994).

2.3 Sufficient conditions

Sufficient conditions for impulsive stabilizability are given in the following theorem. Essentially, the conditions imposed assure that for any positive ϵ, there exists an impulsively invariant set contained within $B_\epsilon(p)$. The proof itself, although somewhat intuitive, is quite long and technical, for which reason the reader is referred to Liu and Willms (1994).

Theorem 2.3 *Let $p = p_y + p_z$ be a point in D and let $v \in C^1[Z, \mathbb{R}]$ be a positive* *definite function with respect to* p_z*. Define the sets*

$$
\begin{aligned}
I &= \{x \in D \mid \nabla v(z) \cdot f_z(x) \le 0\}^O , \\
I_\alpha &= \mathrm{Proj}_z \left(I \cap (B_\alpha^y(p_y) \oplus Z)\right) \cup \{p_z\}, \\
J &= \{x \in D \mid \nabla v(z) \cdot f_z(x) < 0\} , \\
J_\alpha &= \mathrm{Proj}_z \left(J \cap (B_\alpha^y(p_y) \oplus Z)\right) \cup \{p_z\},
\end{aligned}
$$

where the superscript O *denotes the interior of the set,* Proj_z *denotes the orthogonal projection onto the* Z *subspace, and* $B_\alpha^y(p_y)$ *is the* m*-dimensional* α*-ball around* p_y *in the* Y *subspace.*

(a) *If* I_α *is a neighbourhood of* p_z*, for all* $\alpha > 0$*, then* p *is impulsively stabilizable.*
(b) *If* J_α *is a neighbourhood of* p_z*, for all* $\alpha > 0$*, then* p *is asymptotically impulsively stabilizable.*

We remark that the openness of the set I in the above theorem is an essential requirement without which the theorem does not hold.

3 Application

In this section, we shall consider a fish population growth model. Suppose the owner of a resort on a small northern lake wishes to attract fishermen by increasing the population of two particular species of game fish in his lake. Upon looking into the matter he discovers that the cost of stocking his lake with these species is excessive while the cost of stocking his lake with the main prey species of these game fish is comparatively economical. The owner therefore wishes to determine how high he can keep the game species population by stocking the lake with the feeder fish. A model for this situation may be presented as below,

$$
\begin{aligned}
\dot N_1 &= N_1(b_1 - a_{11}N_1 - a_{12}N_2 - a_{13}N_3), \\
\dot N_2 &= N_2(b_2 - e_2 + a_{21}N_1 - a_{22}N_2 - a_{23}N_3), \\
\dot N_3 &= N_3(b_3 - e_3 + a_{31}N_1 - a_{32}N_2 - a_{33}N_3),
\end{aligned} \tag{9}
$$

where N_1 is the feeder fish population, N_2 and N_3 are the game fish populations, all of the b_i, e_i, a_{ij} are constants, $b_i - a_{ii}N_i$ is the per capita birth rate of the population N_i, $-a_{12}N_2 - a_{13}N_3$ represents the effect of the predation, $a_{21}N_1$, $a_{31}N_1$ represent the prey's contribution to the predator's growth rate, $-a_{23}N_3$, $-a_{32}N_2$ represent the effect of the competition between the predators, and e_2, e_3 are the fishing efforts applied by the anglers. From Theorem 2.1, the candidate positive points, p, that could be stabilized are those satisfying

$$
\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} b_2 - e_2 + a_{21}p_1 \\ b_3 - e_3 + a_{31}p_1 \end{pmatrix}. \tag{10}
$$

In what follows, we shall assume that all constants are positive, and furthermore, equation (10) tells us that $b_2 - e_2 > 0$ and $b_3 - e_3 > 0$, that is to say that the fishing efforts of the game fish e_2 and e_3 cannot exceed their birth rates b_2 and b_3.

Selecting p such that (10) is satisfied, we choose the Lyapunov function

$$v = N_2 - p_2 - p_2 \ln \frac{N_2}{p_2} + N_3 - p_3 - p_3 \ln \frac{N_3}{p_3}.$$

Differentiating with respect to t, substituting for $b_2 - e_2$ and $b_3 - e_3$ from (10) and rearranging gives

$$\begin{aligned}
\dot{v} &= (N_2 - p_2)[a_{21}(N_1 - p_1) - a_{22}(N_2 - p_2) - a_{23}(N_3 - p_3)] \\
&\quad + (N_3 - p_3)[a_{31}(N_1 - p_1) - a_{32}(N_2 - p_2) - a_{33}(N_3 - p_3)] \\
&= -(N_2 - p_2, N_3 - p_3) \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} N_2 - p_2 \\ N_3 - p_3 \end{pmatrix} \\
&\quad + (N_1 - p_1)[a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3)].
\end{aligned} \tag{11}$$

Let

$$A = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}.$$

We note that if det $A > 0$ then $\dot{v} < 0$ if the second term of (11) is negative. Likewise, if det $A = 0$ then $\dot{v} < 0$ if the second term is strictly negative.

Based on this, we can determine the regions in the positive orthant that belong to the set $\mathcal{J} = \{\dot{v} < 0\}$ and hence give impulses in the other regions to bring them to points in $\mathcal{J}$.

Case 1: Consider the case det $A > 0$. Here the set $\mathcal{J}$ is given by points in the positive orthant which make the second term of (11) negative, that is,

$$\begin{aligned}
\mathcal{J} &= \{N_1 \geq p_1, a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) < 0\} \\
&\quad \cup \{N_1 \leq p_1, a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) > 0\} \\
&\quad \cup \{a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) = 0,\ \ N_2 \neq p_2,\ \ N_3 \neq p_3\}.
\end{aligned} \tag{12}$$

Note that we must exclude the case $N_2 = p_2$ and $N_3 = p_3$ from $\mathcal{J}$, as otherwise, $\dot{v} = 0$.

Divide the positive orthant into the following regions:

$$\begin{aligned}
\Omega_1 &= \{N_1 \geq p_1,\ \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) < 0\}, \\
\Omega_2 &= \{N_1 > p_1,\ \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) > 0\}, \\
\Omega_3 &= \{N_1 \leq p_1,\ \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) > 0\}, \\
\Omega_4 &= \{N_1 < p_1,\ \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) < 0\}, \\
\Omega_5 &= \{a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) = 0,\ \ N_2 \neq p_2,\ \ N_3 \neq p_3\}, \\
\Omega_6 &= \{N_2 = p_2,\ \ N_3 = p_3\}.
\end{aligned} \tag{13}$$

Clearly, $\mathcal{J}$ is made up of Ω_1, Ω_3 and Ω_5, so, at most, we need to specify impulses in the other three regions.

Impulses in Ω_2 and Ω_4 are as follows:

$$\Delta N_1 = \begin{cases} c - N_1, & \mathbf{N} \in \Omega_2, \ 0 < c \le p_1, \\ d - N_1, & \mathbf{N} \in \Omega_4, \ d \ge p_1. \end{cases} \tag{14}$$

This ensures that points in Ω_2 are moved to points in $\Omega_3 \subset \mathcal{J}$ and points in Ω_4 are moved to points in $\Omega_1 \subset \mathcal{J}$.

We will now show that impulses are not required in region Ω_6. Our system of ODE's in Ω_6 reduces to

$$\begin{aligned} \dot{N}_1 &= N_1(b_1 - a_{11}N_1 - a_{12}p_2 - a_{13}p_3), \\ \dot{N}_2 &= p_2(b_2 - e_2 + a_{21}N_1 - a_{22}p_2 - a_{23}p_3) \\ &= p_2 a_{21}(N_1 - p_1) \qquad \text{(using (10))}, \\ \dot{N}_3 &= p_3(b_3 - e_3 + a_{31}N_1 - a_{32}p_2 - a_{33}p_3) \\ &= p_3 a_{31}(N_1 - p_1) \qquad \text{(using (10))}. \end{aligned} \tag{15}$$

It is clear that points where $\dot{N}_2 \neq 0$ and/or $\dot{N}_3 \neq 0$ will cause N_2 and/or N_3 to increase or to decrease and points will therefore leave Ω_6. We are therefore concerned with points where both $\dot{N}_2 = 0$ and $\dot{N}_3 = 0$. We see from (15) that this occurs only if $N_1 = p_1$. Now because both $\dot{N}_2$ and $\dot{N}_3$ are equal to zero, we must have $\dot{N}_1 \neq 0$, or else we have a positive equilibrium point, which we have assumed does not exist. It follows that N_1 must then either increase or decrease, in which case we move to regions in Ω_6 where either $N_1 > p_1$, or $N_1 < p_1$. At this new point, we no longer have $\dot{N}_2 = 0$ and $\dot{N}_3 = 0$, so that we, again, leave Ω_6, by the same reasoning as before. It follows that solutions passing through Ω_6 naturally leave there and move to one of the other five regions, where impulses have already been specified.

It follows that the required set of impulses in the case of det $A > 0$ is as specified by equation (14).

<u>Case 2:</u> Now consider the case det $A = 0$. In this case we must ensure that the second term of (11) is strictly negative, in which case we obtain the set $\mathcal{J}$ given by

$$\begin{aligned} \mathcal{J} = \{N_1 < p_1,\ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) > 0\} \\ \cup \{N_1 > p_1,\ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3)\}. \end{aligned} \tag{16}$$

Unless one further analyzes what happens on the plane $a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) = 0$, one cannot specify impulses in N_1 that guarantee survival of all three fish. The reason for this is that one does not know the behaviour of solutions on this plane, and since this plane is parallel to the N_1 axis, an impulse in N_1 does not move points off of the plane. A problem will certainly occur if solutions move along this plane towards extinction in N_2, or N_3, which is quite possible.

With this in mind, we try to find regions on this plane where solutions leave the plane, and thus we can specify impulses in the remaining regions to move to these "good" regions of the plane. It turns out that the entire plane without the line $\{N_2 = p_2,\ N_3 = p_3\}$ may be added to the set $\mathcal{J}$, provided certain conditions on the a_{ij} hold, which guarantee that both terms of (11) are not equal to zero at the same time. Call this set $\mathcal{S}$, that is, let

$$\mathcal{S} = \{a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) = 0,\ N_2 \neq p_2,\ N_3 \neq p_3\}.$$

We now derive these conditions on the a_{ij}, so that we may add the set $\mathcal{S}$ to $\mathcal{J}$.

The first term of (11) equal to zero gives

$$a_{22}(N_2-p_2)^2+(a_{23}+a_{32})(N_2-p_2)(N_3-p_3)+a_{33}(N_3-p_3)^2=0.$$

Using the quadratic formula, and that $a_{22}>0$ as given, we have

$$\begin{aligned} N_2-p_2 &= \frac{-(a_{23}+a_{32})\pm\sqrt{(a_{23}+a_{32})^2-4a_{22}a_{33}}}{2a_{22}}\,(N_3-p_3) \\ &= \frac{-(a_{23}+a_{32})\pm\sqrt{(a_{23}-a_{32})^2}}{2a_{22}}\,(N_3-p_3) \quad \text{(using det } A=0 \text{ and factoring)} \\ &= \begin{cases} -\dfrac{a_{32}}{a_{22}}\,(N_3-p_3), \\ -\dfrac{a_{23}}{a_{22}}\,(N_3-p_3). \end{cases} \end{aligned}$$

The second term of (11) equals zero for points inside the set $\mathcal{S}$ and they satisfy

$$N_2-p_2=-\frac{a_{31}}{a_{21}}\,(N_3-p_3).$$

It follows that if we have

$$\frac{a_{31}}{a_{21}}\neq\frac{a_{32}}{a_{22}}, \quad \text{and} \quad \frac{a_{31}}{a_{21}}\neq\frac{a_{23}}{a_{22}} \tag{17}$$

both terms are cannot equal zero at the same time, or equivalently, the first term is non-zero in the set $\mathcal{S}$. This means that $\dot{v}<0$ in this set, and we may, as a result, add it to $\mathcal{J}$.

As in the case det $A>0$, we split the positive orthant into the regions as shown, in which the regions Ω_1, Ω_3, and Ω_5 belong to $\mathcal{J}$.

$$\begin{aligned} \Omega_1 &= \{N_1>p_1,\ \ a_{21}(N_2-p_2)+a_{31}(N_3-p_3)<0\}, \\ \Omega_2 &= \{N_1\le p_1,\ \ a_{21}(N_2-p_2)+a_{31}(N_3-p_3)>0\}, \\ \Omega_3 &= \{N_1<p_1,\ \ a_{21}(N_2-p_2)+a_{31}(N_3-p_3)>0\}, \\ \Omega_4 &= \{N_1\ge p_1,\ \ a_{21}(N_2-p_2)+a_{31}(N_3-p_3)<0\}, \\ \Omega_5 &= \{a_{21}(N_2-p_2)+a_{31}(N_3-p_3)=0,\ \ N_2\neq p_2,\ \ N_3\neq p_3\}, \\ \Omega_6 &= \{N_2=p_2,\ \ N_3=p_3\}. \end{aligned} \tag{18}$$

The only thing different here from (13) are the inequalities in the N_1, p_1 terms.

One can show in exactly the same manner as in the Case 1, that solutions in Ω_6 naturally tend to one of the other regions.

Further, our set of impulses is the same as in (14), with the inequalities in the constants c and d changed, that is, the required set of impulses in the case $\det A=0$ is given by

$$\Delta N_1=\begin{cases} c-N_1, & \mathbf{N}\in\Omega_2,\ \ 0<c<p_1, \\ d-N_1, & \mathbf{N}\in\Omega_4,\ \ d>p_1. \end{cases} \tag{19}$$

Provided the conditions (17) are satisfied.

We now show that, even if (17) does not hold, one may still specify impulses in exactly the same manner as above.

Consider first the case $\dfrac{a_{31}}{a_{21}} = \dfrac{a_{23}}{a_{22}}$. Again, we try to add the set

$$\mathcal{S} = \{a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) = 0, \quad N_2 \neq p_2, \quad N_3 \neq p_3\}$$

to $\mathcal{J}$. Again, we may do this provided points in $\mathcal{S}$ don't remain there, possibly leading to extinction in one of the species. Consider what happens in $\mathcal{S}$ by looking at the system (9). We have

$$\begin{aligned}\dot{N}_1 &= N_1(b_1 - a_{11}N_1 - a_{12}N_2 - a_{13}N_3),\\ \dot{N}_2 &= N_2(b_2 - e_2 + a_{21}N_1 - a_{22}N_2 - a_{23}N_3)\\ &= N_2[a_{21}(N_1 - p_1) - \{a_{22}(N_2 - p_2) + a_{23}(N_3 - p_3)\}] \quad \text{(using (10))}\\ &= N_2a_{21}(N_1 - p_1)\end{aligned} \tag{20}$$

<u>*Aside:*</u>

$$\begin{aligned}a_{22}(N_2 - p_2) + a_{23}(N_3 - p_3) &= \frac{a_{21}a_{23}}{a_{31}}(N_2 - p_2) + a_{23}\,(N_3 - p_3)\\ &\qquad \Big(\text{using } \frac{a_{31}}{a_{21}} = \frac{a_{23}}{a_{22}}\Big)\\ &= \frac{a_{23}}{a_{31}}\,[a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3)]\\ &= 0 \qquad \text{(as we are in } \mathcal{S})\end{aligned}$$

and

$$\begin{aligned}\dot{N}_3 &= N_3(b_3 - e_3 + a_{31}N_1 - a_{32}N_2 - a_{33}N_3)\\ &= N_3[a_{31}(N_1 - p_1) - \{a_{32}(N_2 - p_2) + a_{33}(N_3 - p_3)\}] \quad \text{(using (q0))}\\ &= N_3a_{31}(N_1 - p_1)\end{aligned}$$

<u>*Aside:*</u>

$$\begin{aligned}a_{32}(N_2 - p_2) + a_{33}(N_3 - p_3) &= \frac{a_{22}a_{33}}{a_{23}}(N_2 - p_2) + a_{33}(N_3 - p_3)\\ &\qquad \text{(using } \det A = 0)\\ &= \frac{a_{33}}{a_{23}}[a_{22}(N_2 - p_2) + a_{23}(N_3 - p_3)]\\ &= 0. \qquad \text{(from above)}\end{aligned}$$

It follows that if $N_1 = p_1$, then $\dot{N}_2 = 0$ and $\dot{N}_3 = 0$. Further $\dot{N}_1 \neq 0$ or else we have a positive equilibrium point. If $\dot{N}_1 > 0$, then N_1 increases, so that at the next point, we have $N_1 > p_1$. This in turn implies $\dot{N}_2 > 0$ and $\dot{N}_3 > 0$. Similarly, if $\dot{N}_1 < 0$, then N_1 decreases, so at the next point we have $\dot{N}_2 < 0$ and $\dot{N}_3 < 0$. Note, that no matter what happens, $\dot{N}_2$ and $\dot{N}_3$ have the same sign, so that $a_{21}\dot{N}_2 + a_{31}\dot{N}_3 \neq 0$, that is, solutions will leave $\mathcal{S}$. It follows that all solutions in $\mathcal{S}$ will leave $\mathcal{S}$ and hence we will not encounter the problem of extinction.

In the same manner, one analyzes what happens in the second case $\dfrac{a_{31}}{a_{21}} = \dfrac{a_{32}}{a_{22}}$. Here, the system (9) reduces to

$$\begin{aligned}
\dot N_1 &= N_1(b_1 - a_{11}N_1 - a_{12}N_2 - a_{13}N_3),\\
\dot N_2 &= N_2(b_2 - e_2 + a_{21}N_1 - a_{22}N_2 - a_{23}N_3)\\
&= N_2[a_{21}(N_1 - p_1) - \{a_{22}(N_2 - p_2) + a_{23}(N_3 - p_3)\}] \qquad (21)\\
&\qquad \text{(using (10))},\\
\dot N_3 &= N_3(b_3 - e_3 + a_{31}N_1 - a_{32}N_2 - a_{33}N_3)\\
&= N_3[a_{31}(N_1 - p_1) - \{a_{32}(N_2 - p_2) + a_{33}(N_3 - p_3)\}]\\
&\qquad \text{(using (10))}\\
&= N_3\left[a_{31}(N_1 - p_1) - \left\{\frac{a_{22}a_{33}}{a_{23}}(N_2 - p_2) + a_{33}(N_3 - p_3)\right\}\right]\\
&\qquad \text{(using } \det A = 0 \text{ to substitute for } a_{32})\\
&= N_3\left[a_{31}(N_1 - p_1) - \frac{a_{33}}{a_{23}}\ \{a_{22}(N_2 - p_2) + a_{33}(N_3 - p_3)\}\right]. \qquad (22)
\end{aligned}$$

Solving for $a_{22}(N_2 - p_2) + a_{23}(N_3 - p_3)$ in (21) and substituting into (22) gives

$$\begin{aligned}
\dot N_3 &= N_3\left[a_{31}(N_1 - p_1) - \frac{a_{33}}{a_{23}}\left\{a_{21}(N_1 - p_1) - \frac{\dot N_2}{N_2}\right\}\right]\\
&= N_3\left[\left\{a_{31} - \frac{a_{33}a_{21}}{a_{23}}\right\}(N_1 - p_1) + \frac{a_{33}}{a_{23}}\frac{\dot N_2}{N_2}\right]\\
&= N_3\left[\frac{a_{31}a_{23} - a_{33}a_{21}}{a_{23}}\,(N_1 - p_1) + \frac{a_{33}}{a_{23}}\frac{\dot N_2}{N_2}\right]\\
&= N_3\,\frac{a_{33}}{a_{23}}\,\frac{\dot N_2}{N_2},
\end{aligned}$$

since

$$\begin{aligned}
\frac{a_{31}}{a_{21}} = \frac{a_{32}}{a_{22}} &\Rightarrow a_{21}a_{32} - a_{22}a_{31} = 0\\
&\Rightarrow a_{21}\,\frac{a_{22}a_{33}}{a_{23}} - a_{22}a_{31} = 0\\
&\qquad \text{(using } \det A = 0)\\
&\Rightarrow a_{21}a_{33} - a_{23}a_{31} = 0\\
&\Rightarrow a_{31}a_{23} - a_{33}a_{21} = 0.
\end{aligned}$$

It follows that in the set $\mathcal{S}$ the system of ODE's satisfies

$$\dot N_3 = \frac{a_{33}}{a_{23}}\,\frac{N_3}{N_2}\,\dot N_2.$$

Since we are concerned with points satisfying $a_{21}\dot{N}_2 + a_{31}\dot{N}_3 = 0$, we may encounter problems if

$$\frac{a_{33}}{a_{23}}\frac{N_3}{N_2} = -\frac{a_{21}}{a_{31}} \quad \Leftrightarrow \quad N_3 = -\frac{a_{21}a_{23}}{a_{31}a_{33}}\,N_2.$$

But we are given that all constants are positive, so that the above equation implies N_2 and N_3 are of opposite sign. This however, will never occur, since we are dealing with the positive orthant.

It follows that all points in $\mathcal{S}$ will leave $\mathcal{S}$ in this case, and we have no possibility of extinction.

In conclusion, we have shown that the points satisfying (10) of the system (9) can be globally asymptotically stabilized by giving the set of impulses as specified by (14) in the case of $\det A > 0$, and as specified by (19) in the case of $detA = 0$.

4 Conclusion

In this paper, we have established, respectively, necessary and sufficient conditions for a point p to be impulsively stabilizable, i.e. Theorem 2.1 and Theorem 2.3. These results are applied to a three-species population growth model and an impulsive control program is obtained to stabilize the point p. A constructive approach for actually determining an appropriate feedback control law may be generated from the notion of impulsively invariant sets described in Section 2.2 as follows. A Lyapunov function, v, is constructed so that it satisfies the conditions of Theorem 2.3. Then for any positive ϵ, we can choose a constant c such that the level set $v = c$ is contained within an ϵ-neighbourhood of p_z. The set $\{z\colon \|z\| \le \epsilon,\ v(z) < c\}$ defines Ω_z, the non-impulsive portion of our invariant cylinder. We then choose an α such that $\Omega \subset B_\epsilon(p)$, where $\Omega = B^y_\alpha(p_y) \oplus \Omega_z$. Since Ω is an invariant cylinder, for all points $q \in \partial\Omega$ from which the trajectory induced by f leaves $\overline{\Omega}$ there is at least one impulse $\Delta y(q)$ such that $q + \Delta y(q)$ is either in the interior of Ω or at a point on $\partial\Omega$ from which the vector field f will keep the trajectory within $\overline{\Omega}$ for some positive time interval. For each such q we choose one of these $\Delta y(q)$ and define our feedback control law as $q \mapsto q + \Delta y(q)$. This law is implemented by observing the trajectory of an initial point within Ω and firing the appropriate impulse $\Delta y(q)$ each time the trajectory reaches a point q. In this manner the system is kept within a ϵ-neighbourhood of the desired point p. In the population growth model, a control design procedure for asymptotic impulsive stabilization is derived. It is shown that by impulsively regulating one species, the population of all three species can be maintained at a positive level, which otherwise would drop to a level of extinction for one of the species. This control program enables an resort owner to lower his cost of maintaining the game fish in his lake by stocking the lake with the feeder fish. Consequently, the resort owner's profit will be maximized. The results developed in this paper may be applied to other real world problems.

References

[1] Lakshmikantham, V., Bainov, D.D. and Simeonov, P.S. *Theory of Impulsive Differential Equations*. World Scientific, Singapore, 1989.

[2] Lakshmikantham, V. and Liu, X.Z. Stability criteria for impulsive differential equations in terms of two measures. *J. Math. Anal. Appl.* **137** (1989) 591.

[3] Liu, X.Z. Stability theory for impulsive differential equations in terms of two measures. *Differential Equations Stability and Control.* (Ed.: S. Elaydi), Marcel Dekker, New York, 1990, p.61.

[4] Liu, X.Z. Practical stabilization of control systems with impulse effects. *J. Math. Anal. Appl.* **166** (1992) 563.

[5] Liu, X.Z. and Willms, A.R. Impulsive stabilizability of autonomous systems. *J. Math. Anal. Appl.* **187** (1994) 17.

[6] Sontag, E.D. *Mathematical Control Theory.* Springer-Verlag, New York, 1990.