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The Relationship between Pullback, Forward and Global Attractors of Nonautonomous Dynamical Systems

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Abstract: Various types of attractors are considered and compared for non-autonomous dynamical systems involving a cocycle state space mapping that is driven by an autonomous dynamical system on a compact metric space. In particular, conditions are given for a uniform pullback attractor of the cocycle mapping to form a global attractor of the associated autonomous skew-product semi-dynamical system. The results are illustrated by several examples that are generated by differential equations on a Banach space with a uniformly dissipative structure induced by a monotone operator.

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1 Introduction

Nonautonomous dynamical systems can often be formulated in terms of a cocycle mapping for the dynamics in the state space that is driven by an autonomous dynamical system in what is called a parameter or base space. Traditionally the driving system

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is topological and the resulting cartesian product system forms an autonomous semi-dynamical system that is known as a skew-product flow. Results on global attractors for autonomous semi-dynamical systems can thus be adapted to such nonautonomous dynamical systems via the associated skew-product flow [5, 7, 8, 11, 13, 22, 34, 38].

A new type of attractor, called a pullback attractor, was proposed and investigated for nonautonomous deterministic dynamical systems and for random dynamical systems [12, 15, 19, 29, 31]. Essentially, it consists of a parametrized family of nonempty compact subsets of the state space that are mapped onto each other by the cocycle mapping as the parameter is changed by the underlying driving system. Pullback attraction describing such attractors to a component subset for a fixed parameter value is achieved by starting progressively earlier in time, that is, at parameter values that are carried forward to the fixed value. A deeper reason for this procedure is that a cocycle can be interpreted as a mapping between the fibers of a fiber bundle. For the pullback convergence the *image* fiber is fixed. (The kernels of a global attractor of the skew-product flows considered in [13] are very similar). This differs from the more conventional forward convergence where the parameter value of the limiting object also evolves with time, in which case the parametrized family could be called a forward attractor.

Pullback attractors and forward attractors can, of course, be defined for nonautonomous dynamical systems with a topological driving system [25 − 27]. In fact, when the driving system is the shift operator on the real line, forward attraction to a time varying solution, say, is the same as the attraction in Lyapunov asymptotic stability. The situation of a compact parameter space is dynamically more interesting as the associated skew-product flow may then have a global attractor. The relationship between the global attractor of the skew-product system and the pullback and forward attractor of the cocycle system is investigated in this paper. It will be seen that forward attractors are stronger than global attractors when a *compact* set of nonautonomous perturbations is considered. In addition, an example will be presented in which the cartesian product of the component subsets of a pullback attractor is not a global attractor of the skew-product flow. This set is, however, a maximal compact invariant subset of the skew-product flow. By a generalization of some stability results of Zubov [39] it is asymptotically stable. Thus a pullback attractor always generates a local attractor of the skew-product system, but this need not be a global attractor. If, however, the pullback attractor generates a global attractor in the skew-product flow and if, in addition, its component subsets depend lower continuously on the parameter, then the pullback attractor is also a forward attractor.

Several examples illustrating these results are presented in the final section.

In concluding this introduction, we note that although our assumption of the compactness of the parameter space P is a restriction, it nevertheless occurs in many important and interesting applications such as for nonautonomous differential equations with temporally almost periodic vector fields, where P is a compact subset of a function space defined by the hull of the vector field; see the example following Definition 2.2 in the next section. More generally, the vector fields could be almost automorphic in time [35] or be generated by an affine control system [14], in which case P is the space of measurable control taking values in a compact convex set, or the driving system could itself be an autonomous differential equation on a compact manifold P.

2 Nonautonomous Dynamical Systems and Their Attractors

A general nonautonomous dynamical system is defined here in terms of a cocycle mapping ϕ on a state space U that is driven by an autonomous dynamical system σ acting on a base space P, which will be called the parameter space. It is based on definitions in [4, 24]. In particular, let (U, d_U) be a complete metric space, let (P, d_P) be a compact metric space and let $\mathbb{T}$, the time set, be either $\mathbb{R}$ or $\mathbb{Z}$.

An autonomous dynamical system $(P, \mathbb{T}, \sigma)$ on P consists of a continuous mapping $\sigma\colon \mathbb{T} \times P \to P$ for which the $\sigma_t = \sigma(t, \cdot)\colon P \to P$, $t \in \mathbb{T}$, form a group of homeomorphisms on P under composition over $\mathbb{T}$, that is, satisfy

$$\sigma_0 = \mathrm{id}_P, \qquad \sigma_{t+\tau} = \sigma_t \circ \sigma_\tau$$

for all $t, \tau \in \mathbb{T}$. In addition, a continuous mapping $\phi\colon \mathbb{T}^+ \times U \times P \to U$ is called a cocycle with respect to an autonomous dynamical system $(P, \mathbb{T}, \sigma)$ if it satisfies

$$\phi(0, u, p) = u, \qquad \phi(t + \tau, u, p) = \phi(t, \phi(\tau, u, p), \sigma_\tau p)$$

for all $t, \tau \in \mathbb{T}^+$ and $(u, p) \in U \times P$.

Definition 2.1 The *triple* $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ is called a *nonautonomous dynamical system on the state space* U.

Let (X, d_X) be the cartesian product of (U, d_U) and (P, d_P). Then the mapping $\pi\colon \mathbb{T}^+ \times X \to X$ defined by

$$\pi(t, (u, p)) := (\phi(t, u, p), \sigma_t p)$$

forms a semi-group on X over $\mathbb{T}^+$ [33].

Definition 2.2 The *autonomous semi-dynamical system* $(X, \mathbb{T}^+, \pi) = (U \times P, \mathbb{T}^+, (\phi, \sigma))$ is called the *skew-product dynamical system associated with the cocycle dynamical system* $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$.

For example, let U be a Banach space and let the space $C = C(\mathbb{R}\times U, U)$ of continuous functions $f\colon \mathbb{R} \times U \to U$ be equipped with the compact open topology. Consider the autonomous dynamical system $(C, \mathbb{R}, \sigma)$, where σ is the shift operator on C defined by $\sigma_t f(\cdot, \cdot) := f(\cdot + t, \cdot)$ for all $t \in \mathbb{T}$. Let P be the hull $H(f)$ of a given functions $f \in C$, that is,

$$P = H(f) := \overline{\bigcup_{t\in\mathbb{R}} \{f(\cdot + t, \cdot)\}},$$

and denote the restriction of $(C, \mathbb{R}, \sigma)$ to P by $(P, \mathbb{R}, \sigma)$. Let $F\colon P \times U \to U$ be the continuous mapping defined by $F(p, u) := p(0, u)$ for $p \in P$ and $u \in U$. Then, under appropriate restrictions on the given function $f \in C$ (see Sell [33]) defining P, the differential equation

$$u' = p(t, u) = F(\sigma_t p, u) \tag{1}$$

generates a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{R}, \sigma)\rangle$, where $\phi(t, p, u)$ is the solution of (1) with the initial value u at time $t = 0$.

Let dist_Y denote the Hausdorff distance (semi-metric) between two nonempty sets of a metric space (Y, d_Y), that is,

$$\mathrm{dist}_Y(A,B) = \sup_{a\in A}\inf_{b\in B} d_Y(A,B),$$

and let $\mathcal{D}(U)$ be either $\mathcal{D}_c(U)$ or $\mathcal{D}_b(U)$, classes of sets containing either the compact subsets or the bounded subsets of the metric space (U, d_U).

The definition of a global attractor for an autonomous semi-dynamical system $(X, \mathbb{T}^+, \pi)$ is well known. Specifically, a nonempty compact subset $\mathcal{A}$ of X which is π-invariant, that is, satisfies

$$\pi(t,\mathcal{A}) = \mathcal{A} \quad \text{for all} \quad t \in \mathbb{T}^+, \tag{2}$$

is called a *global attractor* for $(X, \mathbb{T}^+, \pi)$ with respect to $\mathcal{D}(X)$ if

$$\lim_{t\to\infty} \mathrm{dist}_X(\pi(t,D),\mathcal{A}) = 0 \tag{3}$$

for every $D \in \mathcal{D}(X)$. Conditions for the existence of such global attractors and examples can be found in [3, 8, 21, 37, 38]. Of course, semi-dynamical systems need not be a skew-product systems. When they are and when P is compact (in which case it suffices to consider the convergence (3) just for sets in $\mathcal{D}(U)\times\{P\}$, that is of the form $D\times P \subset X$, where $D \in \mathcal{D}(U)$), then the following definition will be used.

Definition 2.3 The *global attractor* $\mathcal{A}$ with respect to $\mathcal{D}(U)\times\{P\}$ of the skew-product dynamical system $(X, \mathbb{T}^+, \pi) = (U\times P, \mathbb{T}^+, (\phi,\sigma))$ with P compact will be called the *global attractor with respect to* $\mathcal{D}(U)$ *of the associated nonautonomous dynamical system* $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$.

Other types of attractors, in particular pullback attractors, that consist of a family of nonempty compact subsets of the state space of the cocycle mapping have been proposed for nonautonomous or random dynamical systems [15, 16, 27, 31, 32]. The main objective of this paper is to investigate the relationships between these different types of attractors.

Definition 2.4 Let $\widehat{A} = \{A(p)\}_{p\in P}$ be a family of nonempty compact sets of U for which $\bigcup_{p\in P} A(p)$ is pre-compact and let $\widehat{A}$ be ϕ-invariant with respect to a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$, that is, satisfies

$$\phi(t, A(p), p) = A(\sigma_t p) \quad \text{for all} \quad t \in \mathbb{T}^+, \quad p \in P. \tag{4}$$

The *family* $\widehat{A}$ is called a *pullback attractor* of $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ with respect to $\mathcal{D}(U)$ if

$$\lim_{t\to\infty} \mathrm{dist}_U(\phi(t, D, \sigma_{-t}p), A(p)) = 0 \tag{5}$$

for any $D \in \mathcal{D}(U)$ and $p \in P$, or a *uniform pullback attractor* if the convergence (5) is uniform in $p \in P$, that is, if

$$\lim_{t\to\infty}\sup_{p\in P} \mathrm{dist}_U(\phi(t, D, \sigma_{-t}p), A(p)) = 0.$$

The *family* $\widehat{A}$ is called a *forward attractor* if the forward convergence

$$\lim_{t\to\infty} \mathrm{dist}_U(\phi(t,D,p), A(\sigma_t p)) = 0$$

holds instead of the pullback convergence (5), or a *uniform forward attractor* if this forward convergence is uniform in $p \in P$, that is, if

$$\lim_{t\to\infty} \sup_{p\in P} \mathrm{dist}_U(\phi(t,D,p), A(\sigma_t p)) = 0.$$

It follows directly from the definition that a pullback attractor is unique. Obviously, any uniform pullback attractor is also a uniform forward attractor, and vice versa.

If $\widehat{A}$ is a forward attractor for the nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$, then ([8], Lemma 4.2) the subset

$$\mathcal{A} = \bigcup_{p\in P} \big(A(p) \times \{p\}\big) \tag{6}$$

of X is the global attractor for the skew-product dynamical system $(X, \mathbb{T}^+, \pi)$; since the global attractor is unique so is the forward attractor. A weaker result holds when $\widehat{A}$ is a pullback attractor, but the inverse property is not true in general.

Although we could formulate our results with weaker assumptions we restrict our attention here to the case, which arises in certain important applications, where $\bigcup_{p\in P} A(p)$ is pre-compact and $\mathcal{D}(U)$ consists of compact or bounded sets. A further generalization which we will not consider here involves pullback attractors with a general domain of attraction $\mathcal{D}$ consisting of family of sets $D = \{D(p)\}_{p\in P}$ such that $\bigcup_{p\in P} D(p)$ is pre-compact or bounded in U, see [32].

The following existence result for pullback attractors is adapted from [16, 23].

Theorem 2.1 *Let $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$, with P compact be a nonautonomous dynamical system and suppose that there exists a family of nonempty sets $C = \{C(p)\}_{p\in P}$, $\bigcup_{p\in P} C(p)$ pre-compact such that*

$$\lim_{t\to\infty} \mathrm{dist}_U(\phi(t,D,\sigma_{-t}p), C(p)) = 0$$

for any bounded subset D of U and any $p \in P$. Then there exists a pullback attractor.

A related result is given by Theorem 4.3.4 in [8]: if the skew-product system $(X, \mathbb{T}^+, \pi)$ has a global attractor $\mathcal{A}$, then the nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ has a pullback attractor. The proof is based on the fact that the identical sets $C(p) \equiv \mathrm{pr}_U\, \mathcal{A}$ satisfy the assumptions of the previous theorem.

Alternatively, conditions can be given on the nonautonomous dynamical system to ensure the existence of a global attractor of the associated skew-product system. The following theorem is from [8].

Theorem 2.2 *Let $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ be a nonautonomous dynamical system with P* `compact` *for which*

(i) *ϕ is asymptotically compact, that is, for every bounded positive invariant set D and $p \in P$, there exists a compact set C such that*

$$\lim_{t\to\infty} \mathrm{dist}_U(\phi(t,D,p), C) = 0,$$

(ii) *there exists a bounded set* B_0 *that absorbs bounded subsets, that is, for every* $p \in P$ *and* $D \in \mathcal{D}_b(U)$ *there exists a* $T_{p,D} \geq 0$ *such that*

$$\phi(t, D, p) \subset B_0 \quad \text{for all} \quad t \geq T_{p,D}.$$

Then the skew-product system $(X, \mathbb{T}^+, \pi)$ *has a unique global attractor that attracts sets from* $\mathcal{D}_b(U)$.

We now continue to derive properties of pullback attractors and the associated skew-product dynamical systems.

Lemma 2.1 *If* $\widehat{A}$ *is a pullback attractor of a nonautonomous dynamical system* $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$*, where* P *is compact, then the subset* $\mathcal{A}$ *of* X *defined by (6) is the maximal* π*-invariant compact set of the associated skew-product dynamical system* $(X, \mathbb{T}^+, \pi)$.

Proof The π-invariance follows from the ϕ-invariance of $\widehat{A}$ via

$$\pi(t, \mathcal{A}) = \bigcup_{p \in P} (\phi(t, A(p), p), \sigma_t p) = \bigcup_{p \in P} (A(\sigma_t p), \sigma_t p) = \mathcal{A}.$$

Now $\mathcal{A} \subset \bigcup_{p \in P} A(p) \times P$, where P is compact and $\bigcup_{p \in P} A(p)$ is pre-compact, so $\mathcal{A}$ is pre-compact. Hence $\mathcal{B} := \bar{\mathcal{A}}$ is compact, from which it follows that

$$B(p) := \{u \colon (u, p) \in \mathcal{B}\}$$

is a compact set in U for each $p \in P$ and that the set

$$\bigcup_{p \in P} B(p) \subset \mathrm{pr}_1 \mathcal{B}$$

is pre-compact. On the other hand, $\mathcal{B}$ is π-invariant since

$$\pi(t, \mathcal{B}) = \pi(t, \bar{\mathcal{A}}) = \overline{\pi(t, \mathcal{A})} = \bar{\mathcal{A}} = \mathcal{B}$$

for the continuous mapping $\pi(t, \cdot)$. In addition, $\phi(t, B(p), p) = B(\sigma_t p)$ holds, that is, the $B(p)$ are ϕ-invariant, since

$$\pi(t, \mathcal{B}) = \bigcup_{p \in P} (\phi(t, B(p), p), \sigma_t p) = \mathcal{B} = \bigcup_{p \in P} (B(\sigma_t p), \sigma_t p)$$

and $\sigma_t p = \sigma_t \hat{p}$ implies that $p = \hat{p}$ for the homeomorphism σ_t. The construction shows $B(p) \supset A(p)$. By the ϕ-invariance of the $B(p)$ and the pullback attraction property it follows then that $B(p) = A(p)$ such that $\mathcal{A} = \mathcal{B}$. Hence $\mathcal{A}$ is compact.

To prove that the compact invariant set $\mathcal{A}$ is maximal, let $\mathcal{A}'$ be any other compact invariant set the of skew-product dynamical system $(X, \mathbb{T}^+, \pi)$. Then $\widehat{A}' = \{A'(p)\}_{p \in P}$ is a family of compact ϕ-invariant subsets of U and by pullback attraction

$$\begin{aligned} \mathrm{dist}_U(A'(p), A(p)) &= \mathrm{dist}_U(\phi(t, A^{'}(\sigma_{-t} p), \sigma_{-t} p), A(p)) \\ &\leq \mathrm{dist}_U(\phi(t, K, \sigma_{-t} p), A(p)) \to 0 \end{aligned}$$

as $t \to +\infty$, where $K = \overline{\bigcup_{p \in P} A'(p)}$ is compact. Hence $A'(p) \subseteq A(p)$ for every $p \in P$, i.e. $\widehat{A}' \subseteq \widehat{A}$, which means $\mathcal{A}$ is maximal for $(X, \mathbb{T}^+, \pi)$.

A set valued mapping M with $p \to M(p) \subset U$ for each $p \in P$ is called is *upper semi-continuous* if

$$\lim_{p \to p_0} d_U(M(p), M(p_0)) = 0 \quad \text{for any} \quad p_0 \in P,$$

lower semi-continuous if

$$\lim_{p \to p_0} d_U(M(p_0), M(p)) = 0 \quad \text{for any} \quad p_0 \in P,$$

and *continuous* if it is both upper and lower semi-continuous. Note that M is upper semi-continuous if and only if the its graph in $P \times U$ is closed in $P \times U$, see ([2, Proposition 1.4.8]). Then it follows straightforwardly from Lemma 2.1 that

Corollary 2.1 *The set valued mapping $p \to A(p)$ formed with the components sets of a pullback attractor $\widehat{A} = \{A(p)\}_{p \in P}$ of a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ with P compact is upper semi-continuous.*

The following example shows that, in general, a pullback attractor need not also be a forward attractor nor form a global attractor of the associated skew-product dynamical system.

Example 2.1 Let f be the function on $\mathbb{R}$ defined by

$$f(t) = -\left(\frac{1+t}{1+t^2}\right)^2, \qquad t \in \mathbb{R},$$

and let $(P, \mathbb{R}, \sigma)$ be the autonomous dynamical system $P = H(f)$, the hull of f in $C(\mathbb{R}, \mathbb{R})$, with the shift operator σ. Note that

$$P = H(f) = \bigcup_{h \in \mathbb{R}} \{f(\cdot + h)\} \cup \{0\}.$$

Finally, let E be the evaluation functional on $C(\mathbb{R}, \mathbb{R})$, that is $E(p) = p(0) \in \mathbb{R}$.

Lemma 2.2 *The functional*

$$\gamma(p) = -\int_0^\infty e^{-\tau} E(\sigma_\tau p)\, d\tau = -\int_0^\infty e^{-\tau} p(\tau)\, d\tau$$

is well defined and continuous on P, and the function of $t \in \mathbb{R}$ given by

$$\gamma(\sigma_t p) = -e^t \int_t^\infty e^{-\tau} p(\tau)\, d\tau = \begin{cases} \dfrac{1}{1+(t+h)^2} & : \quad p = \sigma_h f \\ 0 & : \quad p = 0 \end{cases}$$

is the unique solution of the differential equation

$$x' = x + E(\sigma_t p) = x + p(t)$$

that exists and is bounded for all $t \in \mathbb{R}$.

The proof is by straightforward calculation, so will be omitted.

Consider now the nonautonomous differential equation

$$u' = g(\sigma_t(p), u), \tag{7}$$

where

$$g(p,u) := \begin{cases} -u - E(p)u^2 & : \quad 0 \le u\gamma(p) \le 1,\ p \ne 0, \\ -\dfrac{1}{\gamma(p)}\left(1 + \dfrac{E(p)}{\gamma(p)}\right) & : \quad 1 < u\,\gamma(p), \quad p \ne 0, \\ -u & : \quad 0 \le u, \quad p = 0. \end{cases}$$

It is easily seen that this equation has a unique solution passing through any point $u \in U = \mathbb{R}^+$ at time $t = 0$ defined on $\mathbb{R}$. These solutions define a cocycle mapping

$$\phi(t, u_0, p) = \begin{cases} \dfrac{u_0}{e^t(1 - u_0\gamma(p)) + u_0\gamma(\sigma_t p)} & : \quad 0 \le u_0\gamma(p) \le 1,\ p \ne 0, \\ u_0 + \dfrac{1}{\gamma(\sigma_t p)} - \dfrac{1}{\gamma(p)} & : \quad 1 < u_0\gamma(p),\ p \ne 0, \\ e^{-t}u_0 & : \quad 0 \le u_0,\ p = 0. \end{cases} \tag{8}$$

According to the construction, the cocycle mapping ϕ admits as its only invariant sets $A(p) = \{0\}$ for $p \in P$. To see that the $A(p) = \{0\}$ form a pullback attractor, observe that

$$\phi(t, u_0, \sigma_{-t}p) = \begin{cases} \dfrac{u_0}{e^t(1 - u_0\gamma(\sigma_{-t}p)) + u_0\gamma(p)} & : \quad 0 \le u_0\gamma(\sigma_{-t}p) \le 1,\ p \ne 0, \\ u_0 + \dfrac{1}{\gamma(p)} - \dfrac{1}{\gamma(\sigma_{-t}p)} & : \quad 1 < u_0\gamma(\sigma_{-t}p),\ p \ne 0, \\ e^{-t}u_0 & : \quad 0 \le u_0,\ p = 0. \end{cases}$$

In particular, note that $t \to \gamma(\sigma_t p)^{-1}$ is a solution of the differential equation (7). Since $\gamma(\sigma_{-t}p)^{-1}$ tends to $+\infty$ subexponentially fast for $t \to \infty$, it follows that

$$\phi(t, u, \sigma_{-t}p) \le \frac{1}{2}\, L\, e^{-\frac{1}{2}t}$$

for any $u \in [0, L]$, for any $L \ge 0$ and $p \in P$ provided t is sufficiently large. Consequently $\widehat{A} = \{A(p)\}_{p \in P}$ with $A(p) = \{0\}$ for all $p \in P$ is a pullback attractor for ϕ. In view of (8), the stable set $W^s(\mathcal{A}) := \{x \in X \mid \lim_{t \to +\infty} \mathrm{dist}_X(\pi(t,x), \mathcal{A}) = 0\}$ of $\mathcal{A}$, that is, the set of all points in X that are attracted to $\mathcal{A}$ by π, is given by

$$W^s(\mathcal{A}) = \{(u,p) \colon p \in P,\ u \ge 0,\ u\gamma(p) < 1\} \ne X.$$

Hence the cocycle mapping ϕ in this example admits a pullback attractor that is neither a forward attractor for ϕ nor a global attractor of the associated skew-product flow.

Other examples for different kinds of attractors are given by Scheutzow [30] for random dynamical systems generated by one dimensional stochastic differential equations. However, these considerations are based on the theory of Markov processes.

3 Asymptotic Stability in α-condensing Semi-dynamical Systems

To continue our investigation of the general relations between pullback attractors and skew-product flows we first need to derive some results from general stability theory. We start with some definitions.

Let $(X, \mathbb{T}^+, \pi)$ be a semi-dynamical system. The *ω-limit set* of a set M is defined to be

$$\omega(M) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \pi(t, M)}.$$

A set M is called *Lyapunov stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\pi(t, \mathcal{U}_\delta(M)) \subset \mathcal{U}_\varepsilon(M)$ for $t \geq 0$. M is called a *local attractor* if there exists a neighborhood $\mathcal{U}(M)$ of M such that $\mathcal{U}(M) \subset W^s(M)$. A set M which is Lyapunov stable and a local attractor is said to be *asymptotically stable*. Note that any asymptotically stable compact set M also attracts compact sets contained in $\mathcal{U}(M)$.

Recall that a π-invariant compact set M is said to be *locally maximal* if there exists a number $\delta > 0$ such that any π-invariant compact set contained in the open δ-neighborhood $U_\delta(M)$ of M is in fact contained in M. In addition, a mapping $\gamma^x \colon \mathbb{T} \to X$ is called an *entire trajectory through* x of the semi-dynamical system $(X, \mathbb{T}^+, \pi)$ if

$$\pi(t, \gamma^x(\tau)) = \gamma^x(t + \tau) \quad \text{for all} \quad t \in \mathbb{T}^+, \;\; \tau \in \mathbb{T}, \;\; \gamma^x(0) = x.$$

Finally, the *alpha limit set* of an entire trajectory γ^x is defined by

$$\alpha_{\gamma^x} = \{y \in X \colon \exists\, \tau_n \to -\infty,\, \gamma^x(\tau_n) \to y\}.$$

Let α be a measure of noncompactness on the bounded subsets of a complete metric space (Y, d_Y) ([21], P.13 ff.). Then $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$ for all nonempty bounded subsets A and B with $\alpha(A) = 0$ whenever A is pre-compact. An example is the Kuratowski measure of noncompactness defined by

$$\alpha(A) = \inf\{d \colon A \quad \text{has a finite cover of diameter} \quad < d\}.$$

An autonomous semi-dynamical system $(X, \mathbb{T}^+, \pi)$ is called *α-condensing* if $\pi(t, B)$ is bounded and

$$\alpha(\pi(t, B)) < \alpha(B)$$

for all $t > 0$ for any bounded set B of X with $\alpha(B) > 0$.

Remark 3.1 Examples of α-condensing systems can be found in Hale [21].

Theorem 3.1 *Let M be a locally maximal compact set for an α-condensing semi-dynamical system $(X, \mathbb{T}^+, \pi)$. Then M is Lyapunov stable if and only if there exists a $\delta > 0$ such that*

$$\alpha_{\gamma^x} \cap M = \emptyset$$

for any entire trajectory γ^x through any $x \in U_\delta(M) \setminus M$.

Proof A proof of the necessity direction was given by Zubov in [39, Theorem 7] for a locally compact space X. This proof remains also true for a nonlocally compact space under consideration here. Indeed, let M be a compact invariant Lyapunov stable set for $(X, \mathbb{T}^+, \pi)$, but suppose that the assertion of the theorem is not true. Then there exist $x \notin M$, and a sequence $\tau_n \to -\infty$ such that $\rho(\gamma^x(\tau_n), M) \to 0$ for $n \to \infty$. Let $0 < \varepsilon < \rho(x, M)$ and $\delta(\varepsilon) > 0$ the corresponding positive number for the Lyapunov stability of set M, then for sufficiently large n we have $\rho(\gamma^x(\tau_n), M) < \delta(\varepsilon)$. Consequently, $\rho(\pi^t \gamma^x(\tau_n), M) < \varepsilon$ for all $t \geq 0$. In particular for $t = -\tau_n$ we have $\rho(x, M) = \rho(\pi^{-\tau_n}\gamma^x(\tau_n), M) < \varepsilon$. The obtained contradiction proves our assertion.

For the sufficiency direction, consider first the case $\mathbb{T}^+ = \mathbb{Z}^+$ and let $U_{\delta_0}(M)$ be a neighborhood such that M is locally maximal in $U_{\delta_0}(M)$. Suppose that M is not Lyapunov stable, but that the other condition of the theorem holds. Then there exist an $\varepsilon_0 > 0$ and sequences $\delta_n \to 0$, $x_n \in U_{\delta_n}(M)$, $k_n \to \infty$ such that $\pi(k, x_n) \in U_{\varepsilon_0}(M)$ for $0 \leq k \leq k_n - 1$ and $\pi(k_n, x_n) \notin U_{\varepsilon_0}(M)$. This ε_0 has to be chosen sufficiently small such that

$$\operatorname{dist}_X(\pi(1, U_{\varepsilon_0}(M)), M) < \frac{\delta_0}{2}.$$

Define $A = \{x_n\}$ and $B = \bigcup_{n \in \mathbb{N}} \{\pi(k, x_n) | 0 \leq k \leq k_n - 1\}$. Then $\alpha(A) = 0$ since A is pre-compact. In addition, $\pi(1, B) \subset U_{\delta_0}(M)$, so $\pi(1, B)$ is bounded. Suppose that B is not pre-compact, so $\alpha(B) > 0$. It follows by the properties of the measure of noncompactness for the non pre-compact set B that

$$\alpha(B) = \alpha(A \cup \pi(1, B) \cap B) \leq \max(\alpha(A), \pi(1, B)) = \alpha(\pi(1, B)) < \alpha(B)$$

which is a contradiction. This shows that B is pre-compact. Hence there exist subsequences (denoted with the same indices for convenience) such that

$$\pi(k_n - 1, x_n) \to \bar{x}, \quad \pi(k_n, x_n) \to \pi(1, \bar{x}) = \tilde{x} \in X \setminus U_{\varepsilon_0}(M) \quad \text{for} \quad n \to \infty,$$

the limit $\tilde{\gamma}^{\tilde{x}}(m) := \lim_{n' \to \infty} \gamma(n', m)$ exists for any $m \in \mathbb{Z}$ and some subsequence n' given by the diagonal procedure, where

$$\gamma(n, m) = \begin{cases} \pi(k_n + m, x_n) & : \quad -k_n \leq m < +\infty, \\ x_n & : \quad m < -k_n. \end{cases}$$

Note that $\tilde{\gamma}^{\tilde{x}}$ is an entire trajectory of the discrete-time semi-dynamical system above with $\tilde{\gamma}^{\tilde{x}}(0) = \tilde{x}$ and $\overline{\tilde{\gamma}^{\tilde{x}}(\mathbb{Z}^-)} \subset \bar{B}$. Thus the alpha limit set $\alpha_{\tilde{\gamma}^{\tilde{x}}}$ is nonempty, compact and invariant. In addition, $\alpha_{\tilde{\gamma}^{\tilde{x}}} \subset U_{\varepsilon_0}(M)$, hence $\alpha_{\tilde{\gamma}^{\tilde{x}}} \subset M$ because M is a locally maximal invariant compact set. On the other hand, $\tilde{\gamma}^{\tilde{x}}(0) = \tilde{x} \in U_{\varepsilon_0}(M) \setminus M$, so $\alpha_{\tilde{\gamma}^{\tilde{x}}} \cap M = \emptyset$ holds by the assumptions. This contradiction proves the sufficiency of the condition in the discrete-time case.

Now let $\mathbb{T}^+ = \mathbb{R}^+$ and suppose that $\alpha_{\gamma^x} \cap M = \emptyset$, where $x \notin M$ holds for the continuous-time system. Then it also holds for the restricted discrete-time system generated by $\pi_1 := \pi(1, \cdot)$ because any entire trajectory γ^x of the restricted discrete-time system can be extended to an entire trajectory of the continuous-time system via

$$\gamma^x(t) = \pi(\tau, \gamma^x(n)), \quad n \in \mathbb{Z}, \quad t = n + \tau, \quad 0 < \tau < 1.$$

Consequently, the set M is Lyapunov stable with respect to the restricted discrete-time dynamical system generated by π_1. Since M is compact, for every $\varepsilon > 0$ there exists a $\mu > 0$ such that

$$d_X(\pi(t, x), M) < \varepsilon \quad \text{for all} \quad t \in [0, 1], \; x \in U_\mu(M).$$

In view of the first part of the proof above, there is a $\delta > 0$ such that

$$d_X(\pi(n,x), M) < \min(\mu, \varepsilon) \quad \text{for} \quad x \in U_\delta(M) \quad \text{for} \quad n \in \mathbb{Z}^+.$$

The Lyapunov stability of M for the continuous dynamical system $(X, \mathbb{R}^+, \pi)$ then follows from the semi-group property of π.

The next lemma will be needed to formulate the second main theorem of this section.

Lemma 3.1 *Let M be a compact subset of X that is positively invariant for a semi-dynamical system $(X, \mathbb{T}^+, \pi)$. Then M is asymptotically stable if and only if $\omega(M)$ is locally maximal and asymptotically stable.*

Proof Suppose that M is asymptotically stable. Then there exists a closed positively invariant bounded neighborhood C of M contained in its stable set $W^s(M)$. The mapping π can be restricted to the complete metric space (C, d_X) to form a semi-dynamical system $(C, \mathbb{T}^+, \pi)$. Since M is a locally attracting set it attracts compact subsets of C. The assertion then follows by Theorems 2.4.2 and 3.4.2 in [21] because $\omega(M) = \bigcap_{t \in \mathbb{T}^+} \pi(t, M)$.

Suppose instead that $\omega(M)$ is asymptotically stable and locally maximal. Since M is compact, $\omega(M) = \bigcap_{t \geq 0} \pi(t, M)$. Hence there exist $\eta > 0$ and $\tau \in \mathbb{T}^+$ such that

$$\pi(\tau, M) \subset U_\eta(\omega(M)) \subset W^s(\omega(M)).$$

Now $\pi^{-1}(\tau, U_\eta(\omega(M)))$, where π^{-1} denotes the pre-image of $\pi(\tau, \cdot)$ for fixed τ, is an open neighborhood of M and $\pi(\tau, \pi^{-1}(\tau, U_\eta(\omega(M)))) \subset W^s(\omega(M))$. Hence for any $x \in \pi^{-1}(\tau, U_\eta(\omega(M))) \subset W^s(\omega(M))$ we have that $\pi(t,x)$ tends to $\omega(M)$ as $t \to \infty$, from which it follows that $\pi(t,x)$ also tends to M because $M \supset \omega(M)$.

Then, if M were not Lyapunov stable, there would exist $\varepsilon_0 > 0$, $\delta_n \to 0$, $x_n \in U_{\delta_n}(M)$ and $t_n \to \infty$ such that

$$\operatorname{dist}_X(\pi(t_n, x_n), M) \geq \varepsilon_0. \tag{9}$$

For sufficiently large n_0, the set $\overline{\{x_n\}_{n \geq n_0}}$ would then be contained in the pre-image $\pi^{-1}(1, U_\eta(\omega(M)))$. Since M is compact, so is the set $\overline{\{x_n\}_{n \geq n_0}}$. This set would thus be attracted by $\omega(M) \subset M$, which contradicts (9).

Lemma 3.2 *Let M be a compact subset of X that is a positively invariant set for an asymptotically compact semi-dynamical system $(X, \mathbb{T}^+, \pi)$. Then the set M is asymptotically stable if and only if $\omega(M)$ is locally maximal and Lyapunov stable.*

Proof The necessity follows by Lemma 3.1. Suppose instead that $\omega(M)$ is locally maximal and Lyapunov stable; it is automatically π-invariant since it is an omega limit set. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\pi(t, U_\delta(\omega(M))) \subset U_\varepsilon(\omega(M)) \quad \text{for all} \quad t \geq 0.$$

By the assumption of asymptotical compactness, $\omega(U_\delta(\omega(M)))$ is nonempty and compact with

$$\lim_{t \to \infty} \operatorname{dist}_X(\pi(t, U_\delta(\omega(M))), \omega(U_\delta(\omega(M)))) = 0$$

(see [21, Corollary 2.2.4]). Since $\omega(M)$ is locally maximal, $\omega(U_\delta(\omega(M))) \subset \omega(M)$ for sufficiently small $\delta > 0$, which means that $\omega(M)$ is asymptotically stable. The conclusion then follows by Lemma 3.1.

Corollary 3.1 *Let $(X, \mathbb{T}^+, \pi)$ be asymptotically compact and let M be a compact π-invariant set. Then M is asymptotically stable if and only if M is locally maximal and Lyapunov stable.*

Indeed, $M = \omega(M)$ here, so just apply Lemma 3.2.

The next theorem is a generalization to infinite dimensional spaces and α-condensing systems of Theorem 8 of Zubov [39] characterizing the asymptotic stability of a compact set.

Theorem 3.2 *Let $(X, \mathbb{T}^+, \pi)$ be an α-condensing semi-dynamical system and let $M \subset X$ be a compact invariant set. Then the set M is asymptotically stable if and only if*

(i) *M is locally maximal, and*
(ii) *there exists a $\delta > 0$ such that $\alpha_{\gamma^x} \cap M = \emptyset$ for any entire trajectory γ^x through any $x \in U_\delta(M) \setminus M$.*

Proof By Lemma 2.3.5 in [21] any α-condensing semi-dynamical system is asymptotically compact, so the assertion follows easily from Theorem 3.1 and Corollary 3.1.

A cocycle mapping ϕ of a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ will be called α-condensing if the set $\phi(t, B, P)$ is bounded and

$$\alpha(\phi(t, B, P)) < \alpha(B)$$

for all $t > 0$ for any bounded subset B of U with $\alpha(B) > 0$.

Lemma 3.3 *Suppose that the cocycle mapping ϕ of a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ is α-condensing. Then the mapping π of the associated skew-product flow $(P, \mathbb{T}, \sigma)$ is also α-condensing.*

Proof Let $M = \bigcup_{p \in P} (M(p) \times \{p\})$ be a bounded set in X. Then M can be covered by finitely many balls $M_i \subset X$, $i = 1, \ldots, n$, of largest radius $\alpha(M) + \varepsilon$ for an arbitrary $\varepsilon > 0$. The sets $\mathrm{pr}_1 M_i \subset U$, $i = 1, \ldots, n$, cover $\mathrm{pr}_1 M$. The sets M_i are balls so $\alpha(\mathrm{pr}_1 M_i) = \alpha(M_i) < \alpha(M) + \varepsilon$ for $i = 1, \ldots, n$. It is easily seen that

$$\pi(t, M) = \bigcup_{p \in P} \{\pi(t, (M(p), p))\} = \bigcup_{p \in P} \{(\phi(t, M(p), p), \sigma_t p)\} \subset \phi(t, \mathrm{pr}_1 M, P) \times P.$$

Since ϕ is α-condensing, the set $\phi(t, \mathrm{pr}_1 M, P)$ is bounded. Hence

$$\begin{aligned} \alpha(\pi(t, M)) &\leq \alpha(\phi(t, \mathrm{pr}_1 M, P) \times P) \\ &\leq \alpha(\phi(t, \mathrm{pr}_1 M, P)) < \alpha(\mathrm{pr}_1 M) \leq \alpha(M) \quad \text{for each} \quad t > 0. \end{aligned} \tag{10}$$

The second inequality above is true by the compactness of P. Indeed, P can be covered by finitely many open balls P_i of arbitrarily small radius. Hence

$$\alpha(\phi(t, \mathrm{pr}_1 M, P) \times P) \leq \max_i \alpha(\phi(t, \mathrm{pr}_1 M, P) \times P_i) \leq \alpha(\phi(t, \mathrm{pr}_1 M, P)) + \varepsilon$$

for arbitrarily small $\varepsilon > 0$. The conclusion of the Lemma follows by (10).

4 Uniform Pullback Attractors and Global Attractors

It was seen earlier that the set $\bigcup_{p\in P}(A(p)\times\{p\})\subset X$ which was defined in terms of the pullback attractor $\widehat{A}=\{A(p)\}_{p\in P}$ of a nonautonomous dynamical system $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$ is the maximal π-invariant compact subset of the associated skew-product system $(X,\mathbb{T}^+,\pi)$, but need not be a global attractor. However, this set is always a *local* attractor under the additional assumption that the cocycle mapping ϕ is α-condensing.

Theorem 4.1 *Let $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$ with P compact be an α-condensing dynamical system with a pullback attractor $\widehat{A}=\{A(p)\}_{p\in P}$ and define $\mathcal{A}=\bigcup_{p\in P}(A(p)\times\{p\})$. Then*

(i) *The α-limit set α_{γ^x} of any entire trajectory γ^x passing through $x\in X\setminus\mathcal{A}$ is empty.*
(ii) *$\mathcal{A}$ is asymptotically stable with respect to π.*

Proof Suppose that there exists an entire trajectory γ^x through $x=(u,p)\in X\setminus\mathcal{A}$ such that $\alpha_{\gamma^x}\neq\emptyset$. Then there exists a subsequence $-\tau_n\to\infty$ such that $\gamma^x(\tau_n)$ converges to a point in α_{γ^x}. The set $K=\mathrm{pr}_1\overline{\bigcup_{n\in\mathbb{N}}\gamma^x(\tau_n)}$ is compact since $\overline{\bigcup_{n\in\mathbb{N}}\gamma^x(\tau_n)}$ is compact. Also $\widehat{A}=\{A(p)\}_{p\in P}$ is a pullback attractor, so

$$\lim_{n\to\infty}\mathrm{dist}_U(\phi(-\tau_n,K,\sigma_{\tau_n}p),A(p))=0$$

from which it follows that $u\in A(p)$. Hence $(u,p)\in\mathcal{A}$, which is a contradiction. This proves the first assertion.

By Lemma 3.3 $(X,\mathbb{T}^+,\pi)$ is α-condensing. According to Lemma 2.1 $\mathcal{A}$ is a maximal compact invariant set of $(X,\mathbb{T}^+,\pi)$ since $\widehat{A}$ is a pullback attractor of the cocycle ϕ. The second assertion then follows from Theorem 3.2 and from the first assertion of this theorem.

Remark 4.1 (i) The skew-product system in the example in Section 2 has only a *local* attractor associated with the pullback attractor.

(ii) If in addition to the assumptions of Theorem 4.1 the stable set $W^s(\mathcal{A})$ of $\mathcal{A}$ satisfies $W^s(\mathcal{A})=X$, then $\mathcal{A}$ is in fact a global attractor ([6, Lemma 7]).

Theorem 4.2 *Suppose that $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$ with P compact is a nonautonomous dynamical system with a pullback attractor $\widehat{A}=\{A(p)\}_{p\in P}$ and suppose that $W^s(\mathcal{A})=X$, where $\mathcal{A}=\bigcup_{p\in P}(A(p)\times\{p\})$.*

If the mapping $p\to A(p)$ is lower semi-continuous, then $\widehat{A}$ is a uniform pullback attractor and hence a uniform forward attractor.

Proof Suppose that the uniform convergence

$$\lim_{t\to\infty}\sup_{p\in P}\mathrm{dist}_U(\phi(t,D,p),A(\sigma_t p))=0$$

is not true for some $D\in\mathcal{D}_c$. Then there exist $\varepsilon_0>0$, a set $D_0\in\mathcal{D}_c$ and sequences $t_n\to\infty$, $p_n\in P$ and $u_n\in D_0$ such that:

$$\mathrm{dist}_U(\phi(t_n,u_n,p_n),A(\sigma_{t_n}p))\geq\varepsilon_0. \tag{11}$$

Now P is compact and $\mathcal{A}$ is a global attractor by Remark 4.1 (ii), so it can be supposed that the sequences $\{\phi(t_n, u_n, p_n)\}$ and $\{\sigma_{t_n} p\}$ are convergent. Let $\bar{u} = \lim_{n\to\infty} \phi(t_n, u_n, p_n)$ and $\bar{p} = \lim_{n\to\infty} \sigma_{t_n} p_n$. Then $\bar{u} \in A(\bar{p})$ because $\bar{x} = (\bar{u}, \bar{p}) \in \mathcal{A}$. On the other hand, by (11),

$$\begin{aligned}\varepsilon_0 \leq\ & \mathrm{dist}_U(\phi(t_n, p_n, x_n), A(\sigma_{t_n} p_n)) \\ \leq\ & \mathrm{dist}_U(\phi(t_n, p_n, x_n), A(\bar{p})) + \mathrm{dist}_U(A(\bar{p}), A(\sigma_{t_n} p_n)).\end{aligned}$$

By the lower semi-continuity of $p \to A(p)$ it follows then that $\bar{u} \notin A(\bar{p})$, which is a contradiction.

Remark 4.2 The example in Section 2 shows that Theorem 4.2 is in general not true without the assumption that $W^s(\mathcal{A}) = X$. In view of Corollary 2.1, the set valued mapping $p \to A(p)$ will, in fact, then be continuous here.

5 Examples of Uniform Pullback Attractors

Several examples illustrating the application of the above results, in particular of Theorem 4.2, are now presented. More complicated examples will be discussed in another paper.

5.1 Periodic driving systems

Consider a periodical dynamical system $(P, \mathbb{T}, \sigma)$, that is, for which there exists a minimal positive number T such that $\sigma_T p = p$ for any $p \in P$.

Theorem 5.1 *Suppose that a nonautonomous α-condensing dynamical system $\langle U, \phi, (P, \mathbb{R}, \sigma)\rangle$ with a periodical dynamical system $(P, \mathbb{R}, \sigma)$ has a pullback attractor $\widehat{A} = \{A(p)\}_{p\in P}$. Then $\widehat{A}$ is a uniform forward attractor for $\langle U, \phi, (P, \mathbb{R}, \sigma)\rangle$.*

Proof Consider a sequence $p_n \to p$. By the periodicity of the driving system there exists a sequence $\tau_n \in [0, T]$ such that $p_n = \sigma_{\tau_n} p$. By compactness, there is a convergent subsequence (indexed here for convenience like the full one) $\tau_n \to \tau \in [0, T]$. Hence

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} \sigma_{\tau_n} p = \sigma_\tau p$$

which means $\tau = 0$ or T. Suppose that $\tau = T$. Then

$$\begin{aligned}\lim_{n\to\infty} \mathrm{dist}_U(A(p), A(p_n)) =\ & \lim_{n\to\infty} \mathrm{dist}_U(A(p), \phi(\tau_n, A(p), p)) \\ =\ & \mathrm{dist}_U(A(p), \phi(T, A(p), p)) \\ =\ & \mathrm{dist}_U(A(p), A(\sigma_T p)) = 0\end{aligned}$$

since ϕ is continuous and $A(p_n) = A(\sigma_{\tau_n} p) = \phi(\tau_n, A(p), p)$ by the ϕ-invariance of $\widehat{A}$. Hence the set valued $p \to A(p)$ is lower semi-continuous.

Now $\phi(nT, u_0, p) = \phi(nT, u_0, \sigma_{-nT} p)$ since $p = \sigma_{-nT} p$ by the periodicity of the driving system $(P, \mathbb{R}, \sigma)$. Hence from pullback convergence

$$\lim_{n\to\infty} \mathrm{dist}_U(\phi(nT, u_0, p), A(p)) = \lim_{n\to\infty} \mathrm{dist}_U(\phi(nT, u_0, \sigma_{-nT} p), A(p)) = 0$$

for any $(u_0, p) \in U \times P$. On the other hand

$$\sup_{s\in[0,T]} \mathrm{dist}_U(\phi(s+nT, u_0, p), A(\sigma_{s+nT}p))$$
$$= \sup_{s\in[0,T]} \mathrm{dist}_U(\phi(s, \phi(nT, u_0, p), \sigma_{nT}p), \phi(s, A(\sigma_{nT}p), \sigma_{nT}p)) = 0$$

by the cocycle property of ϕ and the ϕ-invariance of $\widehat{A}$. Hence

$$\lim_{t\to\infty} \mathrm{dist}_X((\phi(t, u_0, p), \sigma_t p), \mathcal{A}) = 0,$$

where $\mathcal{A} = \bigcup\limits_{p\in P} (A(p) \times \{p\})$. This shows that $W^s(\mathcal{A}) = X$. The result then follows by Theorem 4.2.

Consider the 2-dimensional Navier–Stokes equation in the operator form

$$\frac{du}{dt} + \nu Au + B(u) = f(t), \qquad u(0) = u_0 \in H, \tag{12}$$

which can be interpreted as an evolution equation on the rigged space $V \subset H \subset V'$, where V and H are certain Banach spaces. In particular, here $U = H$, which is in fact a Hilbert space, for the phase space. Then, from [36, Chapter 3],

Lemma 5.1 *The 2-dimensional Navier–Stokes equation (12) has a unique solution $u(\cdot, u_0, f)$ in $C(0,T;H)$ for each initial condition $u_0 \in H$ and forcing term $f \in C(0,T;H)$ for every $T > 0$. Moreover, $u(t, u_0, f)$ depends continuously on (t, u_0, f) as a mapping from $\mathbb{R}^+ \times H \times C(\mathbb{R}, H)$ to H.*

Now suppose that f is a periodic function in $C(\mathbb{R}, H)$ and define $\sigma_t f(\cdot) := f(\cdot + t)$. Then $P = \bigcup\limits_{t\in\mathbb{R}} \sigma_t f$ is a compact subset of $C(\mathbb{R}, H)$. By Lemma 5.1 the mapping $(t, u_0, p) \to \phi(t, u_0, p)$ from $\mathbb{R}^+ \times H \times C(\mathbb{R}, H) \to H$ defined by $\phi(t, u_0, p) = u(t, u_0, p)$ is continuous and forms a cocycle mapping with respect to σ on P. By [36, Theorem III.3.10] the mapping ϕ is completely continuous and hence α-condensing.

Lemma 5.2 *The nonautonomous dynamical system $\langle H, \phi, (P, \mathbb{R}, \sigma)\rangle$ generated by the Navier–Stokes equation (12) with periodic forcing term in $C(\mathbb{R}, H)$ has a pullback attractor.*

Proof The solution of the Navier–Stokes equation satisfies an energy inequality

$$\|u(t)\|_H^2 + \lambda_1\nu \int_0^t \|u(\tau)\|_H^2 d\tau \le \|u_0\|_H^2 + \frac{1}{\nu}\int_0^t \|p(\tau)\|_{V'}^2\, d\tau,$$

where λ_1 is the smallest eigenvalue of A. It follows that the balls $B(p)$ in H with center zero and square radii

$$R^2(p) = \frac{1}{\nu}\int_{-\infty}^0 e^{\nu\lambda_1\tau}\|p(\tau)\|_{V'}^2\, d\tau$$

is a pullback attracting family of sets in the sense of Theorem 2.1. In particular, $C(p) := \phi(1, B(\sigma_{-1}p), \sigma_{-1}p)$ satisfies all of the required properties of Theorem 2.1 because $\phi(1, \cdot, p)$ is completely continuous.

This theorem and Theorem 5.1 give

Theorem 5.2 *The nonautonomous dynamical system $\langle H, \phi, (P, \mathbb{R}, \sigma)\rangle$ generated by the Navier–Stokes equation (12) with periodic forcing term in $C(\mathbb{R}, H)$ has a uniform pullback attractor which is also a uniform forward attractor.*

Remark 5.1 See [18] for a related result involving a different type of nonautonomous attractor.

5.2 Pullback attractors with singleton component sets

Now pullback attractors with singleton component sets, that is with

$$A(p) = \{a(p)\}, \quad a(p) \in U,$$

will be considered.

Lemma 5.3 *Let $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ a nonautonomous dynamical system and let $\widehat{A} = \{A(p)\}_{p\in P}$ be a pullback attractor with singleton component sets. Then the mapping $p \to A(p)$ is continuous, hence lower semi-continuous.*

Proof This follows from Corollary 2.1 since the upper semi-continuity of a set valued mapping $p \to A(p)$ reduces to continuity when the $A(p)$ are single point sets.

It follows straightforwardly from this lemma and Theorem 4.2 that

Theorem 5.3 *Suppose that $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ with compact P has at pullback attractor $\widehat{A} = \{A(p)\}_{p\in P}$ with singleton component sets which generates a global attractor $\mathcal{A} = \bigcup\limits_{p\in P} A(p) \times \{p\}$. Then $\widehat{A}$ is a uniform pullback attractor and, hence, also a uniform forward attractor.*

The previous theorem can be applied to differential equations on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ of the form

$$u' = F(\sigma_t p, u), \tag{13}$$

where $F \in C(P \times H, H)$ is uniformly dissipative, that is, there exist $\nu \geq 2$, $\alpha > 0$

$$\langle F(p, u_1) - F(p, u_2), u_1 - u_2\rangle \leq -\alpha\|u_1 - u_2\|^\nu \tag{14}$$

for any u_1, $u_2 \in H$ and $p \in P$.

Theorem 5.4 [10] *The nonautonomous dynamical system $\langle H, \varphi, (P, \mathbb{T}, \sigma)\rangle$ generated by (13) has a uniform pullback attractor that consists of singleton component subsets.*

For example, the equation

$$u' = F(\sigma_t p, x) = -u|u| + g(\sigma_t p)$$

with $g \in C(P, \mathbb{R})$ satisfies

$$\langle u_1 - u_2, F(\sigma_t p, u_1) - F(\sigma_t p, u_2)\rangle \leq -\frac{1}{2}\,|u_1 - u_2|^2(|u_1| + |u_2|) \leq -\frac{1}{2}\,|u_1 - u_2|^3,$$

which is condition (14) with $\alpha = \frac{1}{2}$ and $\nu = 3$.

The above considerations apply also to nonlinear nonautonomous partial differential equations with a uniform dissipative structure, such as the dissipative quasi-geostrophic equations

$$\omega_t + J(\psi, \omega) + \beta\psi_x = \nu\Delta\omega - r\omega + f(x, y, t), \tag{15}$$

with relative vorticity $\omega(x, y, t) = \Delta\psi(x, y, t)$, where $J(f, g) = f_x g_y - f_y g_x$ is the Jacobian operator. This equation can be supplemented by homogeneous Dirichlet boundary conditions for both ψ and ω

$$\psi(x, y, t) = 0, \quad \omega(x, y, t) = 0 \quad \text{on} \quad \partial D, \tag{16}$$

and an initial condition,

$$\omega(x, y, 0) = \omega_0(x, y) \quad \text{on} \quad D,$$

where D is an arbitrary bounded planar domain with area $|D|$ and piecewise smooth boundary. Let U be the Hilbert space $L_2(D)$ with norm $\|\cdot\|$.

Theorem 5.5 *Assume that*

$$\frac{r}{2} + \frac{\pi\nu}{|D|} > \frac{1}{2}\beta\left(\frac{|D|}{\pi} + 1\right)$$

and that the wind forcing $f(x, y, t)$ is temporally almost periodic with its $L^2(D)$-norm bounded uniformly in time $t \in \mathbb{R}$ by

$$||f(\cdot, \cdot, t)|| \le \sqrt{\frac{\pi r}{3|D|}}\left[\frac{r}{2} + \frac{\pi\nu}{|D|} - \frac{1}{2}\beta\left(\frac{|D|}{\pi} + 1\right)\right]^{\frac{3}{2}}.$$

Then the dissipative quasigeostrophic model (15) – (16) has a unique temporally almost periodic solution that exists for all time $t \in \mathbb{R}$.

The proof in [17] involves explicitly constructing a uniform pullback and forward absorbing ball in $L^2(D)$ for the vorticity ω, hence implying the existence of a uniform pullback attractor as well as a global attractor for the associate skew-product flow system for which the component sets are singleton sets. The parameter set P here is the hull of the forcing term f in $L^2(D)$ and a completely continuous cocycle mapping $\phi(t, u_0, p) = \omega(t, u_0, p)$ with respect to the shift operator σ on P that is continuous in all variables.

5.3 Distal dynamical systems

A function $\gamma^{(u,p)}\colon \mathbb{R} \to U$ represents an entire trajectory $\gamma^{(u,p)}$ of a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ if $\gamma^{(u,p)}(0) = u \in U$ and $\phi(t, \gamma^{(u,p)}(\tau), \sigma_\tau p) = \gamma^{(u,p)}(t+\tau)$ for $t \ge 0$ and $\tau \in \mathbb{R}$. A nonautonomous dynamical system is called *distal on* $\mathbb{T}^-$ if

$$\inf_{t\in\mathbb{T}^-} d_U\left(\gamma^{(u_1,p)}(t), \gamma^{(u_2,p)}(t)\right) > 0$$

for any entire trajectories $\gamma^{(u_1,p)}$ and $\gamma^{(u_2,p)}$ with $u_1 \ne u_2 \in U$ and any $p \in P$. A nonautonomous dynamical system is said to be *uniformly Lyapunov stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$d_U(\phi(t, u_1, p), \phi(t, u_2, p)) < \delta$$

for all u_1, $u_2 \in U$ with $d_U(u_1, u_2) < \varepsilon$, $p \in P$ and $t \ge 0$. Finally, an autonomous dynamical system $(P, \mathbb{T}, \sigma)$ is called *minimal* if P does not contain proper compact subsets which are σ-invariant.

The following lemma is due to Furstenberg [20] (see also [4, Chapter 3] or [28, Chapter 7, Proposition 4]).

Lemma 5.4 *Suppose that a nonautonomous dynamical system $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$ is distal on $\mathbb{T}^-$ and that $(P,\mathbb{T},\sigma)$ is minimal. In addition suppose that a compact subset $\mathcal{A}$ of X is π-invariant with respect to the skew-product system $(X,\mathbb{T}^+,\pi)$. Then the mapping $p\to A(p):=\{u\in U\colon (u,p)\in\mathcal{A}\}$ is continuous.*

The following theorem gives the existence of uniform forward attractors.

Theorem 5.6 *Suppose that the nonautonomous dynamical system $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$ is uniformly Lyapunov stable and that the skew-product system $(X,\mathbb{T}^+,\pi)$ has a global attractor $\mathcal{A}=\bigcup_{p\in P}A(p)\times\{p\}$. Then $\widehat{A}=\{A(p)\}_{p\in P}$ is a uniform forward attractor for $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$.*

Proof Suppose that the nonautonomous dynamical system $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$ is not distal. Then there is a $p_0\in P$, a sequence $t_n\to\infty$ and entire trajectories $\gamma^{(u_1,p_0)}$, $\gamma^{(u_2,p_0)}$ with $u_1\neq u_2$ such that

$$\lim_{n\to\infty} d_U\Big(\gamma^{(u_1,p_0)}(-t_n),\gamma^{(u_2,p_0)}(-t_n)\Big)=0.$$

Let $\varepsilon=d_U(u_1,u_2)>0$ and choose $\delta=\delta(\varepsilon)>0$ from the uniformly Lyapunov stability property. Then

$$d_U\Big(\gamma^{(u_1,p_0)}(-t_n),\gamma^{(u_2,p_0)}(-t_n)\Big)<\delta$$

for sufficiently large n. Hence

$$d_U\Big(\phi(t,\gamma^{(u_1,p_0)}(-t_n),\sigma_{-t_n}p_0),\phi(t,\gamma^{(u_2,p_0)}(-t_n),\sigma_{-t_n}p_0)\Big)<\varepsilon$$

for $t\geq 0$ and, in particular,

$$\varepsilon=d_U(u_1,u_2)=d_U\Big(\phi(t_n,\gamma^{(u_1,p_0)}(-t_n),\sigma_{-t_n}p_0),\phi(t_n,\gamma^{(u_2,p_0)}(-t_n),\sigma_{-t_n}p_0)\Big)<\varepsilon$$

for $t=t_n$, which is a contradiction. The nonautonomous dynamical system is thus distal, so $p\to A(p)$ is continuous by Lemma 5.4. The result then follows from Theorem 4.2 since $\{A(p)\}_{p\in P}$ generates a pullback attractor.

This theorem will now be applied to the nonautonomous differential equation (13) on a Hilbert space H, which is assumed to generate a cocycle ϕ that is continuous on $\mathbb{T}^+\times P\times H$ and asymptotically compact.

Theorem 5.7 *Suppose that $F\in C(H\times P,H)$ satisfies the dissipativity conditions*

$$\langle F(u_1,p)-F(u_2,p),u_1-u_2\rangle\leq 0, \tag{17}$$

$$\langle F(u,p),u\rangle\leq-\mu(|u|) \tag{18}$$

for u_1, u_2, $u\in H$ and $p\in P$, where $\mu\colon[R,\infty)\to\mathbb{R}^+\setminus\{0\}$. Suppose also that (13) generates a cocycle ϕ that is continuous and asymptotically compact. Finally, suppose that $(P,\mathbb{T},\sigma)$ is a minimal dynamical system.

Then the nonautonomous dynamical system $\langle H,\phi,(P,\mathbb{T}^+,\sigma)\rangle$ has a uniform pullback attractor.

Proof It follows by the chain rule applied to $\|u\|^2$ for a solution of (13) that

$$\|\phi(t,u,p)\|<\|u\|\quad\text{for}\quad |u|>R,\quad t>0\quad\text{and}\quad p\in P.$$

Hence the nonautonomous dynamical system $(X, \mathbb{T}^+, \pi)$ has a global attractor [9]. On the other hand, by (17),

$$\|\phi(t, u_1, p) - \phi(t, u_2, p)\| \leq \|u_1 - u_2\|$$

for $t \geq 0$, $p \in P$ and u_1, $u_2 \in H$. Theorem 5.6 then gives the result.

The above theorem holds for a differential equation (13) on $H = \mathbb{R}$ with

$$F(p, u) = \begin{cases} -(u+1) + g(p) & : \quad u < -1, \\ g(p) & : \quad |u| \leq 1, \\ -(u-1) + g(p) & : \quad u > 1, \end{cases}$$

where $g \in C(P, \mathbb{R})$.

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