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On a New Approach to Some Problems of Classical Calculus of Variations

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Abstract: Employing the contemporary theory of functional differential equations, we propose an effective test on the existence of a minimum for a wide class of functionals in various Banach spaces.

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1 Introduction

The classical calculus of variations assumes that the functional is defined on a specific set and has a very characteristic form. Thus, some problems of minimization prove to be unsolvable in the frame of the classical calculus of variations. In the case of such an "unsolvable" situation D.Hilbert proposed to define the functional on a suitable set such that the functional under consideration obtains a point of a minimum on the set [4]. But what should be done if the "proper" set does not comply with the requirement of the known methods? The main idea of the contemporary theory of functional differential equations is that "any problem needs its proper space of functions" [5]. Using some given below elements of the mentioned theory, we are able to propose a new approach to certain problems of minimization.

2 Preliminaries

Let R^n be the space of vectors $\alpha = \mathrm{col}\{\alpha^1, \dots, \alpha^n\}$ with real components α^i, L_2 be the Banach space of square integrable functions $z\colon [a,b] \to R^1$ under the norm $\|z\|_{L_2} =$

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$\left(\int\limits_a^b z^2(s)\,ds\right)^{\frac{1}{2}}$, and D be a linear space of functions $x\colon [a,b] \to R^1$. Denote by $r = \operatorname{col}\{r^1,\dots,r^n\}\colon D \to R^n$ a system of linearly independent linear functionals on D. Let further $\mathcal{L}\colon D \to L_2$ be a linear operator and the system

$$\begin{aligned} \mathcal{L}x &= z, \\ rx &= \alpha \end{aligned} \tag{1}$$

have a unique solution $x \in D$ for each pair $\{z,\alpha\} \in L_2 \times R^n$. We define the norm $\|x\|_D = \|\mathcal{L}x\|_{L_2} + \|rx\|_{R^n}$. Then D becomes a Banach space. The solution x of (1) has the representation

$$x = Gz + Y\alpha. \tag{2}$$

Here the "Green operator" $G\colon L_2 \to \{x \in D\colon rx = 0\}$ is an integral one:

$$(Gz)(t) = \int\limits_a^b G(t,s)z(s)\,ds$$

whenever the space D is continuously embedded into the space C of continuous functions $x\colon [a,b] \to R^1$ under the norm $\|x\|_C = \max\limits_{t\in[a,b]} |x(t)|$; the finite-dimensional $Y\colon R^n \to D$ is defined by

$$(Y\alpha)(t) = \sum_{k=1}^n \alpha^k y_k(t),$$

where $y_i,\ i = 1,\dots,n$, are the solutions of the semi-homogeneous problems

$$\mathcal{L}x = 0, \qquad r^k x = \begin{cases} 1 & \text{if} \quad k = i, \\ 0 & \text{if} \quad k \neq i, \end{cases} \qquad k = 1,\dots,n.$$

According to (2) any pair $\{z,\alpha\} \in L_2 \times R^n$ defines an element $x \in D$ as well as any $x \in D$ defines a pair $\{z,\alpha\} \in L_2 \times R^n$ with $z = \mathcal{L}x,\ \alpha = rx$. Thus there exists an isomorphism $\mathcal{J} = \{G,Y\}\colon L_2 \times R^n \to D$ ($\mathcal{J}^{-1} = [\mathcal{L},r]\colon D \to L_2 \times R^n$) between D and the direct product $L_2 \times R^n$. We will denote the fact by $D \simeq L_2 \times R^n$. Below we consider functionals on $D \simeq L_2 \times R^n$. The first example of $D \simeq L_2 \times R^n$ is the space of continuous functions $x\colon [a,b] \to R^1$ such that $x^{(i)},\ i = 0,1,\dots,n-1$, are absolutely continuous and $x^{(n)} \in L_2$. In this case

$$(Gz)(t) = \int\limits_a^t \frac{(t-s)^{n-1}}{(n-1)!} z(s)\,ds,$$

$$Y(t) = \left(1, t-a, \dots, \frac{(t-a)^{n-1}}{(n-1)!}\right),$$

$$\mathcal{L}x = x^{(n)}, \quad rx = \{x(a), \dot{x}(a), \dots, x^{(n-1)}(a)\}.$$

In the capacity of a more complicated example consider the space of continuous functions $x\colon [0,1] \to R^1$ such that $\dot{x}$ is absolutely continuous on any $[c,d] \subset (0,1)$ and $t(1-t)\ddot{x}$ is square integrable on $[0,1]$. The space of such functions is isomorphic to $L_2 \times R^2$,

$$(Gz)(t) = \int\limits_0^1 G(t,s)z(s)\,ds, \tag{3}$$

$$G(t,s) = \begin{cases} -\dfrac{1-t}{1-s} & \text{if} \quad 0 \le s \le t \le 1, \\ -\dfrac{t}{s} & \text{if} \quad 0 \le t < s \le 1, \end{cases} \tag{4}$$

$$(Y\alpha)(t) = \alpha^1(1-t) + \alpha^2 t. \tag{5}$$

Some other examples can be found in [3, 5].

3 Main Assertions

Let $D \simeq L_2 \times R^n$, $\mathcal{J} = \{G, Y\}$, $\mathcal{J}^{-1} = [\mathcal{L}, r]$, the linear operator $T\colon D \to L_2$ be bounded, $D_\alpha \overset{def}{=} \{x \in D\colon rx = \alpha\}$. Consider the functional

$$J(x) = \int\limits_a^b \left(\frac{1}{2}(\mathcal{L}x)^2(t) - f(t,(Tx)(t))\right) dt$$

defined on an open set $\Omega \subset D_\alpha$. We will say that a point $x_0 \in D_\alpha$ such that $J(x) \ge J(x_0)$ for any x from a neighbourhood of x_0 is a point of a local minimum of J. The problem of the existence of such a point is denoted by

$$J(x) \to \min, \quad rx = \alpha. \tag{6}$$

In what follows $Q = TG$, $Q^*\colon L_2 \to L_2$ is the adjoint to $Q\colon L_2 \to L_2$, $\varphi(t,\theta) = \frac{\partial}{\partial\theta} f(t,\theta)$, $(F(y))(t) = \varphi(t, y(t))$, $\Psi(x) = GQ^*F(Tx) + Y\alpha$.

Theorem 3.1 *Let $\Psi\colon \Omega \to L_2$ be continuous and bounded. Then any point $x_0 \in \Omega$ of a local minimum of the functional $J(x)$ satisfies the equation*

$$x = \Psi(x).$$

Proof Using the substitution $x = Gz + Y\alpha$, we get the auxiliary functional $J_1(z)$ on L_2:

$$J_1(z) = J(Gz + Y\alpha) = \int\limits_a^b \left(\frac{1}{2}(\mathcal{L}(Gz + Y\alpha))^2(t) - f(t, T(Gz + Y\alpha)(t))\right) dt$$

$$= \int\limits_a^b \left(\frac{1}{2}z^2(t) - f(t,(Qz)(t) + (TY)(t)\alpha)\right) dt.$$

An element $x_0 = Gz_0 + Y\alpha$ is a point of a local minimum of $J(x)$ if z_0 is a point of a local minimum of $J_1(z)$. By virtue of the generalized Fermat theorem, the point z_0 such that $J_1(z) \geq J_1(z_0)$ for each z from a neighbourhood of z_0 satisfies the equality $\delta J_1(z_0, \xi) = 0$, where

$$J_1'(z_0)\xi = \delta J_1(z_0, \xi) \stackrel{def}{=} \lim_{\tau \to 0} \frac{J_1(z_0 + \tau\xi) - J_1(z_0)}{\tau}.$$

The differential $J_1'(z)\xi$ at the point z by the increment ξ has the form

$$J_1(z)\xi = \int_a^b \Big(z(t)\xi(t) - (F(Qz + TY\alpha))(t)(Q\xi)(t) \Big)\, dt.$$

Using the definition

$$\int_a^b y_1(t)(By_2)(t)\, ds = \int_a^b (B^* y_1)(t) y_2(t)\, dt,$$

we obtain

$$J_1(z)\xi = \int_a^b \Big(z(t) - (Q^* F(Qz + TY\alpha))(t) \Big) \xi(t)\, dt.$$

Therefore the point $z_0 \in L_2$ of a minimum of $J_1(z)$ satisfies the equation

$$z_0 = Q^* F(Qz_0 + TY\alpha).$$

Thus, the solution $x_0 \in D$ of problem (6) satisfies the equation

$$x = \Psi(x).$$

Theorem 3.2 *Let $M \subseteq \Omega$ be a nonempty closed convex set and the operator $\Psi \colon M \to M$ be contractive. Then there exists a unique point $x_0 \in M$ such that $J(x) \geq J(x_0)$ for any $x \in M_\alpha \stackrel{def}{=} \{x \in M \colon rx = \alpha\}$.*

To prove Theorem 3.2 we use the following well-known result [6, p.376].

Lemma 3.1 *Let functional ω be differentiable on a convex set M and, besides,*

$$\|\omega'(x_1) - \omega'(x_2)\| \leq q\|x_1 - x_2\|$$

on M. Then

$$|\omega(x_1) - \omega(x_2) - \omega'(x_2)(x_2 - x_1)| \leq \frac{q}{2}\|x_1 - x_2\|^2. \tag{7}$$

Proof of Theorem 3.2 By the conditions, the operator Ψ maps the set M_α into the set M_α. By the Banach principle, there exists a unique solution $x_0 \in M_\alpha$ to the equation $x_0 = \Psi(x_0)$. The Frechét differential of

$$\varphi(z) = \int_a^b f(t, (Qz)(t) + A(t)\alpha)\, dt,$$

where $A = TY$, is defined by

$$\varphi'(z)\xi = \int_a^b f_1'(t,(Qz)(t) + A(t)\alpha)(Q\xi)(t)\,dt = \int_a^b (Q^* f_1'(\cdot, Qz + A\alpha))(t)\xi(t)\,dt.$$

Take arbitrary points $x_1,\, x_2 \in M_\alpha,\ x_1 = \Lambda z_1 + Y\alpha,\ x_2 = \Lambda z_2 + Y\alpha$. We have

$$\begin{aligned}\|\varphi'(z_1) - \varphi'(z_2)\|_{L_2} &= \|Q^* f_1'(\cdot, Tx_1) - Q^* f_1'(\cdot, Tx_2)\|_{L_2} = \|\Psi(x_1) - \Psi(x_2)\|_D \\ &= \|\Psi(\Lambda z_1 + Y\alpha) - \Psi(\Lambda z_2 + Y\alpha)\|_D \le \|\Lambda(z_1 - z_2)\|_D = q\|z_1 - z_2\|_{L_2},\end{aligned}$$

where $q \in (0,1)$ is the contraction constant of Ψ on the set M_α. So, the operator φ' is a contraction on the set $S = \{z \in L_2 \colon \Lambda z + Y\alpha \in M_\alpha\}$ with the constant q. Thus estimate (7) is valid for the functional φ on S.

Let $z_0 = \delta x_0$. Note that the equality $x_0 = \Psi(x_0)$ implies $z_0 = Q^* F(T(\Lambda z_0 + Y\alpha)) = \varphi'(z_0)$. The equality

$$\begin{gathered}J(x) - J(x_0) = J(\Lambda z + Y\alpha) - J(\Lambda z_0 + Y\alpha) \\ = \frac{1}{2}\int_a^b (z^2(t) - z_0^2(t))\,dt - \varphi(z) + \varphi(z_0) = \frac{1}{2}(\|z\|^2 - \|z_0\|^2) - \varphi(z) + \varphi(z_0)\end{gathered}$$

is fulfilled for all $x \in M_\alpha$. Using Lemma 3.1, we get

$$\varphi(z) - \varphi(z_0) \le \varphi'(z_0)(z - z_0) + \frac{q}{2}\,\|z - z_0\|^2.$$

Then

$$\begin{gathered}J(x) - J(x_0) \ge \frac{1}{2}\,(\|z\|^2 - \|z_0\|^2) - \varphi'(z_0)(z - z_0) - \frac{1}{2}\,q\|z - z_0\|^2 \\ \ge \frac{1}{2}\,(\|z\|^2 - \|z_0\|^2) - \int_a^b z_0(t)z(t)\,dt + \|z_0\|^2 - \frac{1}{2}\,q\|z - z_0\|^2 \\ \ge \frac{1}{2}\Big(\|z\|^2 - 2\int_a^b z_0(t)z(t)\,dt + \|z_0\|^2 - q\|z - z_0\|^2\Big) \ge \frac{1}{2}\,(1 - q)\|z - z_0\|^2 \ge 0.\end{gathered}$$

Hence the functional J takes on its minimal value on the set M_α at the point x_0.

This completes the proof.

4 Examples

Example 4.1 The problem

$$\int_0^b \left(\frac{1}{2}((\mathcal{L}x)^2(t) - p(t)f((Tx)(t))\right) dt \longrightarrow \min, \qquad x(b) - x(0) = 0, \tag{8}$$

where $\mathcal{L}x = \dot{x} + x(b)$ and $f(x) = \frac{1}{2}x^2$, was investigated in [1] for $Tx = x$ and in [5] for an arbitrary linear T. Consider the case with

$$(Tx)(t) = \begin{cases} x(h(t)) & \text{if} \quad h(t) \in [0, b], \\ 0 & \text{if} \quad h(t) \notin [0, b], \end{cases} \tag{9}$$

where the real function h is measurable.

Here

$$(Gz)(t) = \int_0^b G(t, s)z(s)\, ds,$$

$$G(t, s) = \begin{cases} \dfrac{1 + b - t}{b} & \text{if} \quad 0 \le s \le t \le b, \\ \dfrac{1 - t}{b} & \text{if} \quad 0 \le t < s \le b. \end{cases}$$

Then

$$(\Psi x)(t) = \int_0^b K(t, s)f'(x(h(s)))\, ds,$$

where $x(\xi) = 0$ for $\xi \notin [0, b]$;

$$K(t, s) = p(s) \begin{cases} \dfrac{1 + (b - t)h(s)}{b} & \text{if} \quad 0 \le h(s) \le t \le b, \\ \dfrac{1 + (b - h(s))t}{b} & \text{if} \quad 0 \le t < h(s) \le b, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\mu(r) = \sup_{|x| \le r} |f'(x)|, \qquad q(r) = \sup_{|x_1| \le r,\, |x_2| \le r,\, x_1 \ne x_2} \frac{|f'(x_1) - f'(x_2)|}{|x_1 - x_2|},$$

$$(Kx)(t) = \int_0^b K(t, s)x(s)\, ds.$$

Then the operator Ψ maps the set $M(r) = \{x \in D\colon \|x\|_C \le r\}$ into itself if $\mu(r)\|K\|_{C \to C} \le r$, and Ψ is a contractive operator on $M(r)$ if $q(r)\|K\|_{L_2 \to L_2} < 1$. So, problem (8) has a solution under the condition

$$\int_0^b (1 + (b - s)s)^2 p^2(s)\, ds < \frac{b}{\left(q(r) + \frac{|f'(0)|}{r}\right)^2}.$$

Example 4.2 Consider the singular problem (see the quadratic case with a linear ordinary differential operator T in [2] and with a functional differential T in [5])

$$\int_0^1 \left(\frac{t^2(1-t)^2}{2}\,\ddot{x}^2(t) - p(t)f((Tx)(t))\right) ds \longrightarrow \min, \qquad x(0)=\alpha^1, \quad x(1)=\alpha^2, \tag{10}$$

where T is defined by (9) for $b=1$. Let $(\mathcal{L}x)(t) = t(1-t)\ddot{x}(t)$, $rx = \{x(0), x(1)\}$. Define the elements of the space D by equalities (2)−(5).

Then

$$(\Psi x)(t) = \int_0^1\int_0^1 G(t,s)G(h(\tau),s)p(\tau)f'(x(h(\tau)))\,ds\,d\tau + \alpha^1(1-t) + \alpha^2 t,$$

where $x(\xi)=0$ for $\xi \notin [0,1]$.

Assume that there exists a non-decreasing function $\tilde{f}$ such that

$$|f'(x) - f'(y)| \le |x-y|\tilde{f}(\max\{|x|,|y|\}), \quad |f'(x)| \le \gamma|x|\tilde{f}(|x|).$$

Let $M = \{x \in D\colon rx = \alpha,\ |t(1-t)\ddot{x}(t)| \le r\}$ and denote $v(t) = (t-1)\ln(1-t) - t\ln t$, $u(t) = |\alpha^1|(1-t) + |\alpha^2|t$.

Then the operator Ψ maps the set M into itself if

$$\gamma\int_0^1 |p(t)|(v(t)r + u(t))\tilde{f}(v(t)r+u(t))\,dt \le r,$$

and Ψ is a contractive operator on M if

$$\sqrt{2}\int_0^1 |p(t)|\tilde{f}(v(t)r+u(t))\,dt < 1.$$

Hence problem (10) is solvable if $\gamma < \sqrt{2}$,

$$\int_0^1 |p(t)|\tilde{f}\left(\frac{\max\{|\alpha^1|,|\alpha^2|\}\gamma}{\sqrt{2}-\gamma}v(t) + u(t)\right) dt < \frac{1}{\sqrt{2}}$$

or

$$\int_0^1 |p(t)|\,dt < \frac{1}{\sqrt{2}\tilde{f}\left(\frac{\sqrt{2}\max\{|\alpha^1|,|\alpha^2|\}}{\sqrt{2}-\gamma}\right)}.$$

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