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Dynamics of Bidirectional Associative Memory Networks with Processing Delays

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Abstract: A mathematical model describing the dynamical interactions of the bidirectional associative memory networks, incorporating among other things processing time delays, has been proposed in this paper. The existence and stability characteristics of the equilibrium patterns have been discussed. Results on local asymptotic stability of the equilibrium patterns have been presented. Three sets of easily verifiable sufficient conditions describing the global stability of the equilibrium patterns of these networks are obtained.

Keywords: *Bidirectional associative memory networks; global stability.*

Mathematics Subject Classification (2000): 34K20, 34K15, 92B20, 94CXX.

1 Introduction

Mathematical models describing the dynamical interactions of the bidirectional associative memory (BAM for short) networks have been a subject of numerous investigations, Kosko [14−16], Simpson ([23] and the references there in). In particular, the following BAM network model, known as Hopfield network is expressed by the following system of equations:

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j) + I_i, \tag{1.1}$$

for $i = 1, 2, \dots, n$ (see, e.g. [11, 12, 16]). As may be seen this model describes the activation dynamics among the various neurons in one single neuronal field. In (1.1), a_i for $i = 1, 2, \dots, n$ represent the passive decay rates, b_{ij} denotes the synaptic connection weights between i-th and j-th neurons, $f_j(x_j)$, for $j = 1, 2, \dots, n$ denote signal

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propagation functions and I_i for $i = 1, 2, \ldots, n$ denote the exogenous inputs to the i-th neuron.

The Hopfield model illustrates an autoassociative BAM. Autoassociativity means that the network topology reduces to only one field, F_X of neurons. The synaptic connection matrix M symmetrically intraconnects the n neurons in Hopfield network. Thus $M = M^T$ and hence it is termed as a BAM model according to [7, 11, 16, 23]. Hopfield network is governed by feedback law. As an important generalization of the Hopfield equation, the following system of equations [7, 14−16, 23]:

$$
\begin{aligned}
x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(y_j) + I_i, \\
y_j'(t) &= -c_j y_j(t) + \sum_{i=1}^{m} d_{ji} g_i(x_i) + J_j,
\end{aligned}
\tag{1.2}
$$

in which $b_{ij} = d_{ji}$, have been proposed to describe the BAM network in the neuronal fields F_X and F_Y. Kohonen [13, 16] described these two layer networks as Heteroassociative networks. In (1.2), a_i for $i = 1, 2, \ldots, m$ and c_j for $j = 1, 2, \ldots, n$ denote the passive decay rates, b_{ij}, d_{ji} for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, are synaptic connection strengths, f_j for $j = 1, 2, \ldots, n$ and g_i for $i = 1, 2, \ldots, m$ denote propagational signal functions and I_i for $i = 1, 2, \ldots, m$ and J_j for $j = 1, 2, \ldots, n$ are exogenous inputs.

Evidently if $F_X = F_Y$, the system (1.2) includes (1.1) and hence the notion of BAM described by (1.2) reduces to that expressed by Hopfield network. In their studies, Cohen-Grossberg [2, 7, 16] assumed that the synaptic connection matrices are symmetric, as in the case of Hopfield networks. Kosko [16], expressed that when b_{ij} and d_{ji} differ, fixed point equilibrium tends not to occur, instead equilibrium behaviour may be oscillatory or a periodic. We disagree with this view by presenting various global stability criteria under the circumstances that b_{ij} and d_{ji} can differ.

We now consider the networks, in which the synaptic connection matrices B and D need not satisfy $B = D^T$ or vice versa and $B = B^T$ and regard these networks as BAM networks. However, if the matrices B and D satisfy $B = D$, $B = B^T$ our definition of BAM networks reduces to the earlier known definition. Clearly the above definition of BAM is more general and allows us to consider arbitrary connection matrices.

It is generally known that in the biological and artificial neural networks as well, time delays arise due to the propagation of information. More specifically, in the electronic implementation of analog neural networks, time delays occur in the communication and response of neurons due to the finite switching speed of amplifiers. Usually constant fixed time delays in models of delayed feedback systems serve as good approximations in simple circuits having a small number of cells. Due to the spatial nature of neural networks, resulting in the parallel pathways of a variety of axon sizes and lengths, in [24, 26] distributed time delays representing transmission of information have been considered.

In the present paper, we propose a BAM model incorporating a fixed discrete delay to represent the processing delays. It is known that in the mammals, the processing of information at the neuronal level is rather slow implying that the neuron is not very efficient, but when these neurons are connected in a network, their efficiency increases, see an der Heiden [1] and Kosko [16, p. 45]. The following model in a more general form

$$
\begin{aligned}
x_i' &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(\lambda_j, y_j(t)) + I_i, \\
y_i' &= -c_i y_i(t) + \sum_{j=1}^{n} d_{ij} g_j(\mu_j, x_j(t)) + J_i,
\end{aligned}
\tag{1.3}
$$

where $i = 1, 2, \dots, n$, has been suggested in [14−16] to describe the activation dynamics of the neurons in the absence of time delays. These equations lead us to the understanding that the neurons process the information instantaneously, contrary to what has been observed in the references [1, 16]. Thus the activational dynamics described by the system (1.3) seems unrealistic. It is also natural to think that any system (whether biological or man made), which responds instantaneously accumulates certain amount of strain over a period of time, which may result in its break down. From these considerations, it appears that a certain amount of delay (time-lag) in its performance is necessary for its well being. Thus, we modify the network equations (1.3) to include a discrete time delay in the signal response functions and accordingly, our model equations assume the form

$$
\begin{aligned}
x_i' &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(\lambda_j, y_j(t-\tau)) + I_i, \\
y_i' &= -c_i y_i(t) + \sum_{j=1}^{n} d_{ij} g_j(\mu_j, x_j(t-\tau)) + J_i,
\end{aligned}
\tag{1.4}
$$

for $i = 1, 2, \dots, n$ (see [22]). Our model (1.4) includes the earlier proposed models involving discrete time delays [7, 18−20]. It is important to mention here that in biological/man made systems increasing time delays always render the system attain instability, see Cushing [3], Mac Donald [17]. However, it has been established by Freedman and Sree Hari Rao [5] that a proper interplay between the time delay and various other parameters of the system, may help stabilize the otherwise unstable systems. Thus from this discussion, we certainly cannot neglect the time delays but rather like to study their influence on the stability behaviour of the system. To be more precise, we shall be interested mainly in examining the effect of time delays on the maintenance and preservation of stability/instability of the equilibrium. It is worth pointing out that the delay parameter τ, may be regarded as a mechanism to limit the strain in the performance of the network, particularly when it processes information instantaneously.

In (1.4) the parameter τ corresponds to the time delay arising due to the processing of information at neuronal level. In artificial neural networks, this time delay arises due to the finite switching speed of amplifiers. The passive decay rates a_i, c_i for $i = 1, 2, \dots, n$ are assumed to be positive constants. The numbers b_{ij}, d_{ij} for $i, j = 1, 2, \dots, n$ are synaptic connection strengths between the i-th and j-th neurons in the neuronal fields F_X and F_Y. I_i and J_i for $i = 1, 2, \dots, n$ are exogenous inputs. The functions f_i and g_i for $i = 1, 2, \dots, n$ are signal response functions. The initial functions associated with the system (1.4) are given by

$$
x_i(s) = p_i(s), \qquad y_i(s) = q_i(s) \tag{1.5}
$$

for $s \in [-\tau, 0]$ and each $i = 1, 2, \ldots, n$, where p_i and q_i, for each $i = 1, 2, \ldots, n$ are assumed to be continuous functions on $[-\tau, 0]$.

This paper is organized as follows. In Section 2, results on the existence of a unique equilibrium pattern are presented. The influence of processing delays on the stability behaviour of the network has been discussed in Section 3. Conditions for the length of delay for which stability has been maintained have been presented. A result on the preservation of stability/instability of the equilibrium has been presented in Section 3. Stability of bifurcating periodic solutions is discussed in Section 4. Three independent sets of sufficient conditions for the global asymptotic stability of the equilibrium patterns have been presented in Section 5. Examples illustrating the merit of our results have been presented in Section 6. Finally a discussion follows in Section 7.

2 Existence and Uniqueness of Equilibrium Pattern

It is easy to see that the equilibria of the system (1.4) are solutions of the following system of equations:

$$\begin{aligned} a_i x_i^* &= \sum_{j=1}^{n} b_{ij} f_j(\lambda_j, y_j^*) + I_i, \\ c_i y_i^* &= \sum_{j=1}^{n} d_{ij} g_j(\mu_j, x_j^*) + J_i, \end{aligned} \tag{2.1}$$

for $i = 1, 2, \ldots, n$.

We shall state the following conditions on the signal functions f_i and g_i for $i = 1, 2, \ldots, n$, which will be utilized in this work. There exist positive constants $\alpha_i(\lambda_i)$ and $\beta_i(\mu_i)$ for $i = 1, 2, \ldots, n$ such that

$$\begin{aligned} \|f_j(\lambda_j, u_j(t)) - f_j(\lambda_j, v_j(t))\| &\le \alpha_j(\lambda_j)\|u_j - v_j\|, \\ \|g_j(\mu_j, u_j(t)) - g_j(\mu_j, v_j(t))\| &\le \beta_j(\mu_j)\|u_j - v_j\| \end{aligned} \tag{2.2}$$

for λ, μ, u, $v \in R^n$ and $t \in [0, \infty)$ and $\|\cdot\|$ denotes any appropriate norm on R^n.

We now, present our first result on the existence of a unique equilibrium pattern (x^*, y^*).

Theorem 2.1 *Assume that (1.5), (2.2) are satisfied. In addition assume that the decay rates a_i, c_i, the synaptic weights b_{ij}, d_{ij} and the parameters α_i, β_i satisfy the following inequalities:*

$$\frac{\alpha_i \sum\limits_{j=1}^{n} |b_{ji}|}{a_i} < 1 \quad \text{and} \quad \frac{\beta_i \sum\limits_{j=1}^{n} |d_{ji}|}{c_i} < 1 \tag{2.3}$$

for each $i = 1, 2, \ldots, n$.

Then for any pair of input vectors (I, J) the system (1.4) has a unique positive equilibrium pattern (x^*, y^*) satisfying the equations (2.1).

In the following, we present a result guaranteeing the existence of a unique equilibrium pattern (x^*, y^*) to our model equations (1.4), at the expense of dropping the Lipschitzian hypotheses (2.2). This allows us to include non-Lipschitzian signal response functions in our equations. Here and henceforth the term “*Non-Lipschitz*” should be understood as *not necessarily Lipschitz*. Further, the decay rates a_i, c_i, synaptic connections b_{ij}, d_{ij} are accorded more freedom in the following result, than the restrictions placed on them in Theorem 2.1.

We now rewrite the system of equations (1.4) as:

$$X'(t) = F(X(t)) \tag{2.4}$$

in which

$$X(t) = (x_1(t), \dots, x_n(t),\, y_1(t), \dots, y_n(t))^T$$

and

$$F(X(t)) = \begin{pmatrix} -a_1x_1(t) + \sum\limits_{j=1}^{n} b_{1j} f_j(\lambda_j, y_j(t-\tau)) + I_1 \\ \vdots \\ -a_nx_n(t) + \sum\limits_{j=1}^{n} b_{nj} f_j(\lambda_j, y_j(t-\tau)) + I_n \\ -c_1y_1(t) + \sum\limits_{j=1}^{n} d_{1j} g_j(\mu_j, x_j(t-\tau)) + J_1 \\ \vdots \\ -c_ny_n(t) + \sum\limits_{j=1}^{n} d_{nj} g_j(\mu_j, x_j(t-\tau)) + J_n \end{pmatrix}. \tag{2.5}$$

We consider the initial value problem associated with the autonomous system (2.4), in which the initial functions are given by

$$x_i(s) = p_i(s), \qquad y_i(s) = q_i(s), \tag{2.6}$$

for $s \in (-\tau, 0]$ and for $i = 1, 2 \dots n$, where p_i and q_i are assumed to be continuous functions of bounded variation on $(-\tau, 0]$. Let S be an open subset of R^{2n}. For any $\xi \in R^{2n}$, we define $\|\xi\| = \sum\limits_{i=1}^{2n} |\xi_i|$.

We now present a lemma which is an application of the theorem in [21] and is useful in proving the next theorem.

Lemma 2.1 *Let $F\colon S \to R^{2n}$ be continuous and satisfy the following condition: Corresponding to each point $\xi \in S$, its neighbourhood U, there exists a constant $k > 0$, and functions h_j and Φ_l for $j = 1, 2, \dots, n$ and $l = 1, 2, \dots, n, n+1, \dots, 2n$ such that*

$$\|F(\xi) - F(\eta)\| \le k\|\xi - \eta\| + k \sum_{l=1}^{2n} |\Phi_l(h_j(\xi)) - \Phi_l(h_j(\eta))| \tag{2.7}$$

on U, where each $h_j\colon U \to R$ is a continuously differentiable function in ξ satisfying

$$\sum_{i=1}^{2n} \frac{\partial h_j(\xi)}{\partial \xi_i} F_i(\xi) \neq 0 \quad on \quad U \tag{2.8}$$

and each $\Phi_l\colon R \to R$, $l = 1, 2, \dots, n, n+1, \dots, 2n$ is continuous and of bounded variation on bounded sub intervals. Then there exists a unique solution for the initial value problem (2.4)–(2.5) on any interval containing the initial functions (2.6).

Theorem 2.2 *Assume that the hypotheses of Lemma 2.1 hold for the functions* f_i *and* g_i *for each* $i = 1, 2, \dots, n$. *Then the system of equations (2.1) admits a unique solution, yielding a unique equilibrium pattern for our model equations (1.4).*

It is easy to modify the results on existence and uniqueness of equilibrium in [24, 26] in proving these results.

3 Delay Dependent Stability Results

In this section, we examine the influence of the time lags on the stability of the equilibrium pattern of (1.4). The linearized system associated with (1.4) is given by

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij}\alpha_j y_j(t-\tau), \\ y_i'(t) &= -c_i y_i(t) + \sum_{j=1}^{n} d_{ij}\beta_j x_j(t-\tau) \end{aligned} \tag{3.1}$$

for $i = 1, 2, \dots, n$. It is convenient to represent the linearized system of $2n$ delay equations (3.1) as amplitudes u_i and v_i for $i = 1, 2, \dots, n$, along the $2n$ eigen values of the connection matrices respectively. Following the study in [18], we let r_i and s_i be the connection eigen values of b_{ij} and d_{ij} respectively for $i,\, j = 1, 2, \dots, n$ and accordingly, we have

$$\begin{aligned} u_i'(t) &= -a_i u_i(t) + r_i\alpha_i v_i(t-\tau), \\ v_i'(t) &= -c_i v_i(t) + s_i\beta_i u_i(t-\tau) \end{aligned} \tag{3.2}$$

for $i = 1, 2, \dots, n$. Since the neural gains α_i and β_i are positive for each $i = 1, 2, \dots, n$, then $r_i\alpha_i$ and $s_i\beta_i$ have the same sign as those of r_i and s_i respectively for $i = 1, 2, \dots, n$. We now introduce that the complex characteristic exponent λ_i and define $u_i(t) = u_i(0)e^{\lambda_i t}$, $v_i(t) = v_i(0)e^{\lambda_i t}$ for each $i = 1, 2, \dots, n$. Substituting this form into (3.2) yields

$$\lambda_i^2 + (a_i + c_i)\lambda_i + a_i c_i - \alpha_i\beta_i r_i s_i e^{-2\lambda_i\tau} = 0 \tag{3.3}$$

for $i = 1, 2, \dots, n$.

We now let

$$a_i + c_i = a, \quad a_i c_i = b, \quad -\alpha_i\beta_i r_i s_i = c, \quad \lambda_i = \lambda \tag{3.4}$$

for $i = 1, 2, \dots, n$. Clearly, both a and b are positive. Then, (3.3) yields,

$$\lambda^2 + a\lambda + b + ce^{-2\lambda\tau} = 0. \tag{3.5}$$

The equation (3.5) has roots with negative real parts, if

$$F(\lambda, \tau) = \lambda^2 + a\lambda + b + ce^{-2\lambda\tau} \neq 0 \quad \text{for} \quad \operatorname{Re}\lambda \geq 0. \tag{3.6}$$

In this section, we examine the following aspects related to the stability/instability of the equilibrium patterns of the system (1.4). Throughout this section we shall use the term *stability* to mean and imply *asymptotic stability* of the equilibrium patterns for the model equations.

(i) If the equation (3.5) is stable for $\tau = 0$, then for what other values of $\tau > 0$, it is stable? This amounts to determining an interval for τ, say, $0 \le \tau \le \tau_*$, such that for all values of τ in this interval, the equation (3.5.) is stable or (3.6) holds.

(ii) For $\tau > 0$ large, what type of stability for equation (3.5) prevails? More specifically, if the equation (3.5) is stable (or unstable) for $\tau = 0$, can it continue to be stable(or unstable) for all values of $\tau > 0$?

Clearly, for stability of the equations (3.5), one has to see that it has no pure imaginary zeros or zeros with positive real parts. We note that $\lambda \neq 0$ is a root of (3.5) if and only if $b + c \neq 0$.

Our first result provides us an estimate on the length of the delay parameter τ, say τ_* which ensures that the equilibrium pattern is asymptotically stable for all values of τ satisfying the inequalities, $0 \le \tau \le \tau_*$. For some biological models, methods to estimate τ_0, has been presented in Erbe, Freedman and Sree Hari Rao [4] and Freedman, Sree Hari Rao and Jayalakshmi [6]. We utilize these techniques to establish the following result.

Theorem 3.1 *Assume that the following inequality*

$$b + c > 0 \tag{3.7}$$

is satisfied. Then the equilibrium pattern is asymptotically stable for all values of τ *satisfying*

$$0 \le \tau < \tau_* = \frac{\sqrt{c^2(b+|c|) + 2a^2|c|(b+c)} - |c|\sqrt{b+|c|}}{2a|c|\sqrt{b+|c|}}. \tag{3.8}$$

Proof It is easy to see that the inequality (3.7) is a consequence of stability for $\tau = 0$, which follows from the application of Routh Hurwitz method. Since λ is a continuous function of the parameter τ, all eigen values will continue to have negative real parts for sufficiently small $\tau > 0$.

If $\lambda = \mu + i\nu$ satisfies (3.5), then μ and ν are real solutions of

$$\mu^2 - \nu^2 + a\mu + b + ce^{-2\mu\tau} \cos 2\tau\nu = 0, \tag{3.9}$$

$$2\mu\nu + a\nu - ce^{-2\mu\tau} \sin 2\tau\nu = 0. \tag{3.10}$$

Let $\hat{\tau}$ be such that $\mu(\hat{\tau}) = 0$, then from (3.9) and (3.10), we have

$$-\hat{\nu}^2 + b + c \cos 2\hat{\tau}\hat{\nu} = 0, \tag{3.11}$$

$$a\hat{\nu} - c \sin 2\hat{\tau}\hat{\nu} = 0. \tag{3.12}$$

We see that the conditions for the asymptotic stability of the equilibrium, following the procedure suggested in Freedman and Sree Hari Rao [5], are given by

$$\operatorname{Im} F(i\nu_0) > 0, \tag{3.13}$$

$$\operatorname{Re} F(i\nu_0) = 0, \tag{3.14}$$

where $F(s) = s^2 + as + b + ce^{-2\tau s}$ and ν_0 is the smallest positive root of (3.14). The conditions in our case become

$$-\nu_0^2 + b + c\cos 2\tau\nu_0 = 0, \tag{3.15}$$

$$a\nu_0 > c\sin 2\tau\nu_0. \tag{3.16}$$

To get our estimate on the length of delay we shall utilize the inequality (3.16) and the equation (3.15), which if simultaneously satisfied, are sufficient to guarantee stability. Rewriting the same, we get

$$\nu^2 - b = c\cos 2\tau\nu, \tag{3.17}$$

$$a\nu > c\sin 2\tau\nu. \tag{3.18}$$

We recall that the equilibrium will be stable if the inequality (3.18) holds at $\nu = \nu_0$, where ν_0 is the first positive root of the equation (3.17). Our technique is to find an upper bound ν_+ on ν_0, independent of τ, and then to estimate τ so that (3.18) holds for all values of ν, $0 \le \nu \le \nu_+$ and hence in particular at $\nu = \nu_0$.

Since the right hand side of (3.17) is less than or equal to $|c|$, the unique positive solution of

$$\nu^2 - b = |c|, \tag{3.19}$$

denoted by ν_+, is always greater than or equal to ν_0. Clearly,

$$\nu_+ = \sqrt{b + |c|}. \tag{3.20}$$

Note that ν_+ is independent of τ. We need an estimate on τ so that (3.18) holds for all $0 \le \nu \le \nu_+$.

Note that at $\tau = 0$, this inequality becomes $a\nu > 0$. However at $\tau = 0$, the solution of (3.17) is $\nu_0 = \sqrt{b + c}$. Hence (3.18) is valid at $\tau = 0$, $\nu = \nu_0$. So by continuity it will continue to hold for small enough $\tau > 0$ and $\nu = \nu_0$.

From (3.18), we get

$$a\nu^2 > c\nu\sin 2\tau\nu. \tag{3.21}$$

From (3.17), we get

$$c\nu\sin 2\tau\nu + 2ac\sin^2\tau\nu < a(b + c). \tag{3.22}$$

Denote the left hand side of (3.22) by $\phi(\tau,\nu)$. We now use the inequalities $\sin 2\tau\nu \le 2\tau\nu$ and $\sin^2\tau\nu \le \tau^2\nu^2$. Then

$$\phi(\tau,\nu) \le \psi(\tau,\nu) \equiv 2|c|\tau\nu^2 + 2a|c|\tau^2\nu^2.$$

We note that for $0 \le \nu \le \nu_+$, we have $\phi(\tau,\nu) \le \psi(\tau,\nu) \le \psi(\tau,\nu_+)$. Hence if $\psi(\tau,\nu_+) < a(b + c)$, then $\phi(\tau,\nu_0) < a(b + c)$.

Let τ_* denote the unique positive root of $\psi(\tau,\nu_+) = a(b + c)$. Then

$$2a|c|\tau^2\nu_+^2 + 2|c|\nu_+^2\tau = a(b + c).$$

Thus

$$\tau_* = \frac{\sqrt{c^2(b + |c|) + 2a^2|c|(b + c)} - |c|\sqrt{b + |c|}}{2a|c|\sqrt{b + |c|}}. \tag{3.23}$$

Then for $\tau < \tau_*$, the Nyquist criterion holds, and τ_* is the estimate for the length of delay for which stability is preserved.

Remark 3.1 Utilizing the relation (3.4) and evaluating the estimate for τ in the special case $b = \frac{a^2}{4}$ and $c = \frac{a^2}{2}$, we get $\tau_* = \frac{\sqrt{5}-1}{2(a_i+c_i)}$. Clearly, τ_* decreases with increasing a_i or c_i or *vice versa.* This special situation is interesting in the sense that there is an explicit relation between the delay τ and the passive decay rates a_i and c_i.

In the next result we shall improve the estimate on τ_* in this special case by employing a different technique which has been presented in detail in Sree Hari Rao and Phaneendra [25].

In the following, we utilize a method as suggested in [10], to find the stability interval, in which the equilibrium is asymptotically stable.

Theorem 3.2 *Assume that the hypothesis (3.7) is satisfied. In addition, let* $b = \frac{a^2}{4}$, $c = \frac{a^2}{2}$ *in (3.5) are satisfied. Then the equilibrium pattern is asymptotically stable for all* τ *satisfying*

$$0 \le \tau < \tau_* = \frac{\pi}{2a} = \frac{\pi}{2(a_i + c_i)}. \tag{3.24}$$

Proof Let $F(\lambda, \tau) \equiv 4\lambda^2 + 4a\lambda + a^2 + 2a^2 e^{-2\lambda\tau}$.

Now, $F_1(\lambda) = F(\lambda, 0) = 4\lambda^2 + 4a\lambda + 3a^2 = 0$ has roots with negative real parts since $a > 0$.

Now let $F_2(\lambda) \equiv 4\lambda^2 + 4a\lambda + 3a^2$, which is obtained by replacing $e^{-\lambda\tau}$ with -1.

Clearly, $F_2(\lambda) \neq 0$. Now, substituting, $e^{-\lambda\tau} = \frac{1-T\lambda}{1+T\lambda}$ for $T > 0$, in the equation

$$4\lambda^2 + 4a\lambda + a^2 + 2a^2 e^{-2\lambda\tau} = 0, \tag{3.25}$$

we obtain

$$\begin{aligned} F_3(\lambda) \equiv 4T^2\lambda^4 + 4T(2 + aT)\lambda^3 + (4 + 8aT + 3a^2T^2)\lambda^2 \\ + 2a(2 - aT)\lambda + 3a^2 = 0. \end{aligned} \tag{3.26}$$

Equation (3.26) has pure imaginary roots $\lambda = i\nu,\ \nu > 0$ if and only if

$$\tau = \frac{2}{\omega}\left[\tan^{-1}(\omega T) - k\pi\right], \qquad k = 0, \pm 1, \dots . \tag{3.27}$$

In order to obtain the pure imaginary roots for the equation (3.26), we set the Routh-Hurwitz determinant to zero. Then we obtain,

$$3a^4T^4 + 16a^3T^3 + 8a^2T^2 - 16 = 0. \tag{3.28}$$

Differentiating (3.25) with respect to τ and simplifying, we get

$$\frac{d\lambda}{d\tau} = \frac{a^2\lambda e^{-2\lambda\tau}}{2\lambda + a - a^2\tau e^{-2\lambda\tau}}. \tag{3.29}$$

Evaluating $\frac{d\lambda}{d\tau}$, $\lambda = i\nu$, we obtain

$$\operatorname{Re} \left.\frac{d\lambda}{d\tau}\right|_{\lambda = i\nu} = \frac{a^3\nu\, \sin 2\tau\nu + 2a^2\nu^2\, \cos 2\tau\nu}{\Delta}, \tag{3.30}$$

where

$$\Delta = a^2 + 4\nu^2 + a^4\tau^2 - 2a^3\tau \cos 2\tau\nu + 4a^2\nu\tau \sin 2\tau\nu.$$

Now, solving the equations (3.9) and (3.10) at $\mu = 0$ and $\tau = \tau^*$ for $\sin 2\tau_*\nu$ and $\cos 2\tau_*\nu$, we have

$$\sin 2\tau_*\nu = \frac{2\nu}{a}, \qquad \cos 2\tau_*\nu = \frac{4\nu^2 - a^2}{2a^2}. \tag{3.31}$$

Thus from (3.30) and (3.31), we have at $\tau = \tau_*$,

$$\mathrm{Re}\, \frac{d\lambda}{d\tau}\bigg|_{\lambda=i\nu} = \frac{4\nu^4 + a^2\nu^2}{\Delta}, \tag{3.32}$$

$$\Delta = a^2 + 4\nu^2 + a^4\tau^2 + 4a\tau\nu^2 + a^3\tau.$$

From (3.26) and (3.28), it follows that for $T = \frac{2\sqrt{2}-2}{a}$ and $\nu = \frac{a}{2}$, we have $F_3(i\nu) = 0$. Further, at $\tau = \tau_* = \frac{\pi}{2a}$,

$$\mathrm{Re}\, \frac{d\lambda}{d\tau}\bigg|_{\lambda=i\nu} = \frac{2a^2}{(8 + 4\pi + \pi^2)}, \tag{3.33}$$

which proves the theorem. Further, from (3.33), it is clear that the transversality condition of the Hopf bifurcation is satisfied.

Remark 3.2 Following the method used in the above theorem, when $b = c$, the inequality $a^2 - 2b < 0$ implies that (3.5) has a pair of pure imaginary zeros which upon substitution of the values of a and b lead us to the contradiction that $a_i^2 + c_i^2 < 0$. Thus, this situation (though mathematically acceptable for the characteristic equation (3.5) in its most general form involving the coefficients a, b, c) can not arise for this specific model under consideration.

Remark 3.3 Following the discussion in Section 1, we understand that the strain on the network arising out of its instantaneous processing of the information (the case where the time delay $\tau = 0$) may result in the break down in course of time. To avoid the unwanted breakdown in the nervous system, we have proposed the introduction of processing delays. Our Theorems 3.1 and 3.2 clearly ensure the stability of the network in the presence of processing delays so long as they lie in the interval $[0, \tau_*]$. Notice that the expression for τ_* in terms of the relations (3.4) may be written as

$$\tau_* = \frac{\sqrt{\alpha_i^2\beta_i^2 r_i^2 s_i^2 (a_i c_i + \alpha_i\beta_i|r_i s_i|) + 2\alpha_i\beta_i|r_i s_i|(a_i + c_i)^2(a_i c_i - \alpha_i\beta_i r_i s_i)}}{2\alpha_i\beta_i|r_i s_i|(a_i + c_i)\sqrt{a_i c_i + \alpha_i\beta_i|r_i s_i|}} - \frac{1}{2(a_i + c_i)}. \tag{3.34}$$

Clearly, τ_* is expressed among others mainly in terms of the passive decay rates a_i, c_i of the network. Also in the Remark 3.1, we have noted that in the special case where $b = \frac{a^2}{4}$ and $c = \frac{a^2}{2}$, the increase/decrease in τ_* is related to the passive decay rates being smaller/greater (respectively). But it is evident from biological considerations that if the decay rates are smaller then the network takes longer time to return to equilibrium and to process the subsequent inputs. On the other hand larger decay rates make τ_* very small, which according to our view strains the network and may result in an eventual breakdown. Further, one may understand, the vital decay rates, as the decay rates of the membrane potentials. Also the expression (3.34) for τ_* involves apart from decay rates, other network parameters as well. Thus a proper interplay between the processing delays and the various network parameters is essential for the network to have stability.

The next result describes a situation in which, instability at $\tau = 0$ will be preserved for all $\tau > 0$. That is no matter how large the processing delays be, the network continues to have instability if it starts with instability at $\tau = 0$. Thus the inequality (3.7) for maintenance of stability can not be relaxed.

Theorem 3.3 *Assume the conditions* $a^2+2c = 0$, $a^2 = 4b$ *hold. Then for all* $\tau \geq 0$, *the equilibrium is unstable. Further, at* $\tau = \tau_* = \frac{3\pi}{2a}$, *(3.5) has a pair of pure imaginary roots and*

$$\operatorname{Re} \left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\nu} = \frac{a^2}{53.2146} > 0.$$

Proof The conditions $a^2+2c = 0$, $a^2 = 4b$ imply that the condition (3.7) is violated. Now, following the lines of argument in Theorem 3.2, we get

$$3a^4T^4 + 56a^2T^2 - 64aT - 16 = 0. \tag{3.35}$$

A positive value of T is given by $\frac{2\sqrt{2}+2}{a}$ and $\nu = \frac{a}{2}$. Now

$$\tau = \frac{2}{\omega}\tan^{-1}(\omega T) = \frac{3\pi}{2a}. \tag{3.36}$$

Now, at $\tau = \tau^* = \frac{3\pi}{2a} = \frac{3\pi}{2(a_i+c_i)}$

$$\operatorname{Re} \left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\nu} = \frac{a^2}{53.2146} > 0,$$

which proves the theorem.

Observe that Theorem 3.3 presents conditions under which the instability of the equilibrium pattern would be maintained in the special case, in which $c = \frac{-a^2}{2}$, $b = \frac{a^2}{4}$. Now, our next result is a general result which explains the circumstances under which the network does not change its stability. More specifically, if the network is stable (unstable) in the absence of processing delays, it remains stable(unstable) in the presence of processing delays.

Theorem 3.4 *Assume condition* (H_1) *holds. Then if the equilibrium is stable (unstable) at* $\tau = 0$, *then the equilibrium remains stable (unstable) for all* $\tau > 0$.

Assume condition (H_2) *holds. Then if the equilibrium is unstable for any* $\tau = \tau_* \geq 0$, *then it will be unstable for all* $\tau \geq \tau_*$.

(H_1) (i) $b^2 - c^2 \geq 0$ *or*

(ii) $a^4 - 4a^2b + 4c^2 < 0$ *holds*

(H_2) $b^2 - c^2 < 0$

Proof Here we analyze the question of stability by examining the sign of the derivative of the real part of the eigen values with respect to τ, as the real part crosses 0. That is, we analyze $\frac{d\mu}{d\tau}(\hat{\tau})$, where $\mu(\hat{\tau}) = 0$. If this derivative is positive (negative), then clearly, a stabilization (destabilization) can not take place at that value of $\hat{\tau}$.

We first note that the imaginary part $\hat{\nu} = \nu(\hat{\tau})$ must satisfy the equation

$$\nu^4 + (a^2 - 2b)\nu^2 + b^2 - c^2 = 0. \tag{3.37}$$

We begin with equations (3.9) and (3.10) and differentiate with respect to τ. Then setting $\tau = \hat{\tau}$, $\mu = 0$ and $\nu = \hat{\nu}$ gives us the two equations in $\frac{d\mu}{d\tau}(\hat{\tau})$ and $\frac{d\nu}{d\tau}(\hat{\tau})$:

$$\begin{aligned} \xi \frac{d\mu}{d\tau}(\hat{\tau}) - \eta \frac{d\nu}{d\tau}(\hat{\tau}) &= C, \\ \eta \frac{d\mu}{d\tau}(\hat{\tau}) + \xi \frac{d\nu}{d\tau}(\hat{\tau}) &= D, \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} \xi &= a - 2c\hat{\tau}\,\cos 2\hat{\tau}\hat{\nu}, & \eta &= (2\hat{\nu} + 2c\hat{\tau}\,\sin 2\hat{\tau}\hat{\nu}), \\ C &= 2c\hat{\nu}\,\sin\hat{\tau}\hat{\nu}, & D &= 2c\hat{\nu}\,\cos 2\hat{\tau}\hat{\nu}. \end{aligned} \tag{3.39}$$

Solving (3.38) gives

$$\frac{d\mu}{d\tau} = \frac{\xi C + \eta D}{\xi^2 + \eta^2} \tag{3.40}$$

and clearly, $\frac{d\mu}{d\tau}(\hat{\tau})$ has the same sign as that of $\xi C + \eta D$. From (3.39), after some simplifications we get

$$\xi C + \eta D = 2\nu^2[2\nu^2 + (a^2 - 2b)]. \tag{3.41}$$

Let

$$F(z) = z^2 + (a^2 - 2b)z + (b^2 - c^2) \tag{3.42}$$

[which is the left hand side of (3.37) with $\hat{\nu}^2 = z$]; then $F(\hat{\nu}^2) = 0$ and we note that

$$\frac{dF}{dz}(\hat{\nu}^2) = \frac{\xi^2 + \eta^2}{2\hat{\nu}^2}\frac{d\mu}{d\tau}(\hat{\tau}). \tag{3.43}$$

Hence, we can describe the criteria for preservation of instability (stability) as follows:

(1) If the polynomial $w = F(z)$ has no positive roots, there can be no change of stability.
(2) If $w = F(z)$ is increasing(decreasing) at all of its positive roots, instability (stability) is preserved.

We now proceed to analyze $F(z)$. Since, $a^2 - 2b > 0$ and if $F(0) = b^2 - c^2 < 0$, then by Decarte's rule of signs, $F(z)$ has at most one positive root. If $F(0) = b^2 - c^2 \geq 0$, $F(z)$ has no positive roots. Similarly, when $a^4 - 4a^2b + 4c^2 < 0$, then $F(z)$ has no real roots.

To summarize, we state the following conditions: Condition (H_1) implies that $w = F(z)$ has no positive roots and condition (H_2) implies that $w = F(z)$ has at most one positive root.

Corollary 3.1 *If $b^2 < c^2$, and if the equilibrium is stable at $\tau = 0$, there exists $\hat{\tau} > 0$ for which this equilibrium is unstable for $\tau > \hat{\tau}$.*

We now present a bifurcation result. This theorem gives conditions under which the equilibrium that is asymptotically stable for $0 \le \tau < \tau_*$ bifurcates at $\tau = \tau_*$ into small amplitude periodic solutions.

Theorem 3.5 *Let $b + c > 0$ and $b - c < 0$ are satisfied. Then there exists a τ_*, the smallest value of τ for which the equations (3.9) and (3.10) have a solution such that $\mu = 0$. For $\tau < \tau_*$ the equilibrium is asymptotically stable. For $\tau > \tau_*$ the equilibrium is unstable. Further, as τ increases through τ_* the equilibrium bifurcates into small amplitude periodic solutions.*

Proof Suppose $\mu = 0$, $\nu = \nu_0$ at $\tau = \tau_*$. Then (3.9) and (3.10) yield

$$-\nu_0^2 + b + c\cos 2\tau_0\nu_0 = 0, \tag{3.44}$$

$$a\nu_0 - c\sin 2\tau_0\nu_0 = 0. \tag{3.45}$$

Eliminating τ_* by squaring and adding (3.44) and (3.45)

$$\nu_0^4 + (a^2 - 2b)\nu_0^2 + b^2 - c^2 = 0. \tag{3.46}$$

Since $a^2 - 2b > 0$ only positive root of (3.46) is given when $b - c < 0$. Accordingly,

$$\nu_0 = \pm\frac{\sqrt{2}}{2}\big\{-(a^2 - 2b) + \{a^4 - 4a^2b + 4c^2\}^{\frac{1}{2}}\big\}^{\frac{1}{2}}. \tag{3.47}$$

We now solve (3.44) and (3.45) for τ_*. Accordingly, we get

$$c^2\cos^2 2\tau_*\nu_0 - a^2c\cos 2\tau_*\nu_0 - (a^2b + c^2) = 0. \tag{3.48}$$

Let $f(z) = c^2z^2 - a^2cz - (a^2b + c^2)$. Clearly, (3.48) has a real solution of the form $\cos 2\tau_*\nu_0 = k$, where $|k| < 1$. From (3.45) this solution in τ_* is of the form

$$\tau_* = \frac{1}{2\nu_0}\sin^{-1}\Big(\frac{a\nu_0}{c}\Big) + \frac{n\pi}{\nu_0}, \qquad n = 0, 1, 2, \dots, \tag{3.49}$$

where the positive value of ν_0 is given by (3.47). Hence the τ_* required by the theorem is obtained by choosing $n = 0$.

Clearly, for $\tau = 0$, the equilibrium is stable. Hence by continuity it remains to be stable for $\tau < \tau_*$. We now show that $\frac{d\mu}{d\tau}\Big|_{\tau=\tau_*} > 0$ when $\nu = \nu_0$ and $n = 0, 1, 2, \dots$. This will imply that there is at least one eigenvalue with positive real part for $\tau > \tau_*$, $n = 0$ and hence the equilibrium is unstable for $\tau > \tau_*$. From (3.44) and (3.45) differentiating with respect to τ and after some simplifications, we get for $\nu = \nu_0$, $\tau = \tau_*$, $\mu = 0$,

$$\frac{d\mu}{d\tau}\Big|_{\tau=\tau_*} = \frac{1}{\Delta}\big[4\nu_0^4 + 2\nu_0^2(a^2 - 2b)\big] > 0.$$

This completes the proof.

4 Stability of Bifurcating Periodic Solutions

In this section, we discuss the stability behaviour of solutions for (1.4) in the neighbourhood of $\tau = \tau_*$ as given in Section 3. We scale the time so as to fix the delay equal to 1 and accordingly set $s = t\tau$, $\tilde{x_i}(s) = x_i(\tau s)$, $\tilde{y_i}(s) = y_i(\tau s)$. Equations (1.4) become after replacing $\tilde{x_i}$ by x_i, $\tilde{y_i}$ by y_i and s by t again,

$$\begin{aligned} x_i'(t) &= \tau\Big[-a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(\lambda_j, y_j(t-1))\Big], \\ y_i'(t) &= \tau\Big[-c_i y_i(t) + \sum_{j=1}^{n} d_{ij} g_j(\lambda_j, x_j(t-1))\Big] \end{aligned} \tag{4.1}$$

for $i = 1, 2, \ldots, n$.

Since r_i and s_i are eigen values of the matrices b_{ij} and d_{ij} respectively expressing along the amplitudes, the system (4.1) become

$$\begin{aligned} x_i'(t) &= \tau\big[-a_i x_i(t) + r_i f_i(\lambda_i, y_i(t-1))\big], \\ y_i'(t) &= \tau\big[-c_i y_i(t) + s_i g_i(\lambda_i, x_i(t-1))\big] \end{aligned} \tag{4.2}$$

for $i = 1, 2, \ldots, n$.

Expanding the above system around the equilibrium using Taylor's series and simplifying we get

$$\begin{aligned} x_i'(t) &= \tau\big[-a_i x_i(t) + r_i\alpha_i y_i(t-1) + r_i\gamma_i y_i^2(t-1) + o(3)\big], \\ y_i'(t) &= \tau\big[-c_i y_i(t) + s_i\beta_i x_i(t-1) + s_i\delta_i x_i^2(t-1) + o(3)\big] \end{aligned} \tag{4.3}$$

for $i = 1, 2, \ldots, n$, where $\alpha_i = f'(y^*)$, $\gamma_i = \frac{f''}{2}(y^*)$, $\beta_i = g'(x^*)$, $\delta_i = \frac{g''}{2}(x^*)$ and $o(3)$ denotes the terms of third order and above.

Now we present our theorem following the method suggested in [9] on stability of bifurcating periodic solutions.

Theorem 4.1 *The system (4.2) has a Hopf bifurcation near $\tau = \tau_*$, obtained in Section 3. The direction of bifurcation is determined by*

$$\mu_2 = \frac{-\operatorname{Re} C_1(0)}{\alpha'(0)} \tag{4.4}$$

and the stability of bifurcating periodic solutions is determined by $\beta_2 = 2\operatorname{Re} C_1(0)$, where

$$\alpha'(0) = \operatorname{Re} \frac{d\lambda}{d\tau}\Big|_{\lambda = i\omega_0,\, \tau=\tau_0}$$

and

$$C_1(0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}. \tag{4.5}$$

The quantities in the above expression are given by the following:

$$
\begin{aligned}
\frac{g_{20}}{2} &= \bar{D}\tau_* r_i \gamma_i B^2 q^2(-1) + \bar{D}\bar{C}\tau_* s_i \delta_i q^2(-1),\\
\frac{g_{02}}{2} &= \bar{D}\tau_* r_i \gamma_i \bar{B}^2 \bar{q}^2(-1) + \bar{D}\bar{C}\tau_* s_i \delta_i \bar{q}^2(-1),\\
g_{11} &= 2\bar{D}\tau_* r_i \gamma_i B\bar{B} q(-1)\bar{q}(-1) + 2\tau_*\bar{D}\bar{C} s_i\delta_i q(-1)\bar{q}(-1)\\
\frac{g_{21}}{2} &= \bar{D}\tau_* r_i \gamma_i (W_{20}^{(2)}(-1)\bar{B}q(-1) + 2W_{20}^{(2)}(-1)Bq(-1),\\
&\quad + \bar{D}\bar{C}\tau_* s_i \delta_i (W_{20}^{(1)}(-1)\bar{q}(-1) + 2W_{20}^{(1)}(-1)q(-1)
\end{aligned}
$$

and

$$
\begin{aligned}
W_{20}^{(1)}(\theta) &= \sigma_1\, e^{i\omega_0\theta} + \sigma_2\, e^{-i\omega_0\theta} + \sigma_f\, e^{2i\omega_0\theta},\\
W_{20}^{(2)}(\theta) &= \mu_1\, e^{i\omega_0\theta} + \mu_2\, e^{-i\omega_0\theta} + \mu_f\, e^{2i\omega_0\theta},\\
W_{11}^{(1)}(\theta) &= \rho_1\, e^{i\omega_0\theta} + \rho_2\, e^{-i\omega_0\theta} + \rho_f,\\
W_{11}^{(2)}(\theta) &= \chi_1\, e^{i\omega_0\theta} + \chi_2\, e^{-i\omega_0\theta} + \chi_f
\end{aligned}
$$

in which

$$
\begin{aligned}
\sigma_1 &= \frac{2\tau_*\bar{D}T_1 i}{\omega_0}, \quad \sigma_2 = \frac{2\tau_*\bar{D}T_2 i}{3\omega_0}, \quad \mu_1 = \sigma_1 B, \quad \mu_2 = \sigma_2\bar{B},\\
\rho_1 &= \frac{-\tau_*\bar{D}T_3 i}{\omega_0}, \quad \rho_2 = \frac{\tau_*\bar{D}T_4 i}{\omega_0}, \quad \chi_1 = \rho_1 B, \quad \chi_2 = \rho_2\bar{B},\\
\sigma_f &= \frac{C_{20}^{(1)}(2i\omega_0 + \tau_* c_i) + \tau_* r_i\alpha_i C_{20}^{(2)}}{(2i\omega_0 + \tau_* a_i)(2i\omega_0 + \tau_* c_i) - \tau_*^2 r_i s_i \alpha_i \beta_i e^{-2i\omega_0}},\\
\mu_f &= \frac{C_{20}^{(1)} s_i\beta_i\, e^{-2i\omega_0} + C_{20}^{(2)}(2i\omega_0 + \tau_* a_i)}{(2i\omega_0 + \tau_* a_i)(2i\omega_0 + \tau_* c_i) - \tau_*^2 r_i s_i \alpha_i \beta_i e^{-2i\omega_0}},\\
\rho_f &= \frac{r_i\alpha_i C_{11}^{(2)} + C_{11}^{(1)}}{\tau_*(a_i c_i - r_i\alpha_i s_i\beta_i)},\\
\chi_f &= \frac{s_i\beta_i C_{11}^{(1)} + a_i C_{11}^{(2)}}{\tau_*(a_i c_i - r_i\alpha_i s_i\beta_i)}.
\end{aligned}
$$

Further

$$
\begin{aligned}
C_{20}^{(1)} &= H_{20}^{(0)} + \tau_* r_i\alpha_i(\mu_1 + \mu_2) - (2i\omega_0 + \tau_* a_i)(\sigma_1 + \sigma_2),\\
C_{20}^{(2)} &= H_{20}^{(0)} + \tau_* s_i\beta_i(\sigma_1 e^{-i\omega_0}) - (2i\omega_0 + \tau_* c_i)(\mu_1 + \mu_2),\\
C_{11}^{(1)} &= H_{11}^{(0)} - \tau_* a_i(\rho_1 + \rho_2) + \tau_* r_i\alpha_i(\lambda_1 + \lambda_2),\\
C_{11}^{(2)} &= H_{11}^{(0)} - \tau_* c_i(\lambda_1 + \lambda_2) + \tau_* s_i\beta_i(\rho_1\, e^{-i\omega_0} + \rho_2 e^{i\omega_0}),
\end{aligned}
$$

$$H_{20}(\theta) = -2\tau_*\left[\bar{D}T_1 q(\theta) + DT_1\bar{q}(\theta)\right] + 2\tau_* \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 1 < \theta < 0, \\ \begin{bmatrix} r_i\gamma_i q^2(-1)B^2 \\ s_i\delta_i q^2(-1) \end{bmatrix} \theta = 0, \end{cases}$$

$$H_{11}(\theta) = -\tau_*\left[\bar{D}T_3 q(\theta) + DT_4\bar{q}(\theta)\right] + 2\tau_* \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 1 < \theta < 0, \\ \begin{bmatrix} r_i\gamma_i B\bar{B}q(-1)\bar{q}(-1) \\ s_i\delta_i q(-1)\bar{q}(-1) \end{bmatrix} \theta = 0. \end{cases}$$

The expressions T_1, T_2, T_3 and T_4 are given by the following.

$$\begin{aligned} T_1 &= r_i\gamma_i q^2(-1)B^2 + \bar{C}s_i\delta_i q^2(-1), \\ T_2 &= r_i\gamma_i q^2(-1)B^2 + Cs_i\delta_i q^2(-1), \\ T_3 &= 2r_i\gamma_i q(-1)\bar{q}B\bar{B} + 2\bar{C}s_i\delta_i q(-1)\bar{q}(-1), \\ T_4 &= 2r_i\gamma_i q(-1)\bar{q}B\bar{B} + 2Cs_i\delta_i q(-1)\bar{q}(-1). \end{aligned}$$

The terms B, C and D are components in the following

$$q(\theta) = \begin{bmatrix} 1 \\ B \end{bmatrix} e^{i\omega_0\theta}, \quad -1 < \theta \leq 0,$$

$$q^*(\theta) = D\begin{bmatrix} 1 \\ C \end{bmatrix} e^{i\omega_0\theta}, \quad 0 \leq \theta < 1,$$

where

$$B = \frac{-s_i\beta_i\tau_*}{c_i\tau_* + i\omega_0}, \qquad C = \frac{-r_i\alpha_i\tau_*}{c_i\tau_* - i\omega_0},$$
$$\bar{D} = \frac{1}{(1 - e^{-i\omega_0})(1 + \bar{C}B)}.$$

5 Global Stability Results

In this section, we present results dealing with the circumstances under which the equilibrium pattern (x^*, y^*) of (1.4) relative to a given input pair (I, J), is globally asymptotically stable. Our results are analogous to those obtained in [24].

Theorem 5.1 *Assume that the hypotheses (2.2) are satisfied. Then the equilibrium pattern of (1.4) is globally asymptotically stable, provided the following inequalities hold:*

$$\frac{\beta_i \sum_{j=1}^{n} |d_{ji}|}{a_i} \leq 1 \quad \textit{and} \quad \frac{\alpha_i \sum_{j=1}^{n} |b_{ji}|}{c_i} \leq 1. \tag{5.1}$$

Proof The following change of variables,

$$u_i(t) = x_i(t) - x_i^*, \quad v_i(t) = y_i(t) - y_i^*$$

transform the system (1.4) to

$$\dot{u}_i(t) = -a_i u_i(t) + \sum_{j=1}^{n} b_{ij}\big\{f_j(\lambda_j, v_j + y_j^*) - f_j(\lambda_j, y_j^*)\big\},$$
$$\dot{v}_i(t) = -c_i v_i(t) + \sum_{j=1}^{n} d_{ij}\big\{g_j(\mu_j, u_j + x_j^*) - g_j(\mu_j, x_j^*)\big\}$$

for $i = 1, 2, \dots, n$. In view of (2.2) this may be written as

$$\begin{aligned} \dot{u}_i(t) &\le -a_i u_i(t) + \sum_{j=1}^{n} |b_{ij}|\alpha_j |v_j(t-s)|, \\ \dot{v}_i(t) &\le -c_i v_i(t) + \sum_{j=1}^{n} |d_{ij}|\beta_j |u_j(t-s)| \end{aligned} \tag{5.2}$$

for $i = 1, 2, \dots, n$.

Employing the Lyapunov functional,

$$\begin{aligned} V(u(t), v(t)) = \sum_{i=1}^{n} \Bigg[&|u_i(t)| + |v_i(t)| + \sum_{j=1}^{n} |b_{ij}|\alpha_j \int_{t-\tau}^{t} |v_j(z)|\, dz \\ &+ \sum_{j=1}^{n} |d_{ij}|\beta_j \int_{t-\tau}^{t} |u_j(z)|\, dz \Bigg] \end{aligned} \tag{5.3}$$

and proceeding along the lines of the proof of Theorem 3.1 (see [24]) the remaining proof of this theorem may be completed.

We now present our next result on the global asymptotic stability.

Theorem 5.2 *Assume that the inequalities*

$$a_i > \frac{1}{2}\sum_{j=1}^{n} |b_{ij}|, \quad c_i > \frac{1}{2}\sum_{j=1}^{n} |d_{ij}| \tag{5.4}$$

for $i = 1, 2, \dots, n$ *are satisfied. Further, assume that there exists constants* $\gamma_i > 0$ *and* $\delta_i > 0$ *satisfying*

$$\frac{\beta_i^2 \sum\limits_{j=1}^{n} |d_{ji}|}{a_i - \frac{1}{2}\sum\limits_{j=1}^{n} |b_{ij}|} \le \frac{\gamma_i}{\delta_i} \le \frac{c_i - \frac{1}{2}\sum\limits_{j=1}^{n} |d_{ij}|}{\alpha_i^2 \sum\limits_{j=1}^{n} |b_{ji}|} \tag{5.5}$$

for $i = 1, 2, \dots, n$.

Then the equilibrium pattern (x^*, y^*) of (1.4) is globally asymptotically stable.

Proof We consider the following functional

$$\begin{aligned} V(x(t),y(t)) = \sum_{i=1}^{n}\Bigg[\gamma_i\,\frac{(x_i(t)-x_i^*)^2}{2} + \delta_i\,\frac{(y_i(t)-y_i^*)^2}{2} \\ + \gamma_i\sum_{j=1}^{n}|b_{ij}|\int_{t-\tau}^{t}\big\{f_j(\lambda_j,y_j(u)) - f_j(\lambda_j,y_j^*)\big\}^2\,du \\ + \delta_i\sum_{j=1}^{n}|d_{ij}|\int_{t-\tau}^{t}\big\{g_j(\mu_j,x_j(u)) - g_j(\mu_j,x_j^*)\big\}^2\,du\Bigg]. \end{aligned} \tag{5.6}$$

It is easy to see that $V(x^*, y^*) = 0$ and $V(x, y) \geq \omega(\|z\|)$, where

$$\omega(\|z\|) = \sum_{i=1}^{n}\frac{\epsilon_i}{2}\big[(x_i - x_i^*)^2 + (y_i - y_i^*)^2\big]$$

for $z = (x - x^*,\, y - y^*)$ with $\epsilon_i = \min\{\gamma_i, \delta_i\}$. Clearly $\omega(\|z\|)$ is positive definite.

Now, the derivative of V along the solutions of system (1.4) may be written as

$$\begin{aligned} V'(x(t),y(t)) = \sum_{i=1}^{n}\Bigg\{\gamma_i\Bigg[(x_i(t)-x_i^*)\Bigg(-a_ix_i(t) \\ + \sum_{j=1}^{n}b_{ij}f_j(\lambda_j,y_j(t-\tau)) + I_i\Bigg) + \sum_{j=1}^{n}|b_{ij}|\big\{f_j(\lambda_j,y_j(t)) - f_j(\lambda_j,y_j^*)\big\}^2 \\ - \sum_{j=1}^{n}|b_{ij}|\big\{f_j(\lambda_j,y_j(t-\tau)) - f_j(\lambda_j,y_j^*)\big\}^2\Bigg] \\ + \delta_i\Bigg[(y_i(t)-y_i^*)\Bigg(-c_iy_i(t) + \sum_{j=1}^{n}d_{ij}g_j(\mu_j,x_j(t-\tau)) + J_i\Bigg) \\ + \sum_{j=1}^{n}|d_{ij}|\big\{g_j(\mu_j,x_j(t)) - g_j(\mu_j,x_j^*)\big\}^2 \\ - \sum_{j=1}^{n}|d_{ij}|\big\{g_j(\mu_j,x_j(t-\tau)) - g_j(\mu_j,x_j^*)\big\}^2\Bigg]\Bigg\}. \end{aligned} \tag{5.7}$$

Proceeding along the lines of argument of Theorem 3.3 (see [24]), we obtain,

$$V'(x,y) < -\Phi(\|z\|),$$

where

$$\Phi(\|z\|) = \sum_{i=1}^{n}k_i\big[(x_i - x_i^*)^2 + (y_i - y_i^*)^2\big],$$

$k_i = \min\{\xi_i, \eta_i\}$, for $i = 1, 2, \dots, n$. The numbers ξ_i, η_i for $i = 1, 2, \dots, n$ are given by

$$\xi_i = \gamma_i\biggl(a_i - \frac{1}{2}\sum_{j=1}^{n}|b_{ij}|\biggr) - \delta_i\beta_i^2\sum_{j=1}^{n}|d_{ji}|$$

and

$$\eta_i = \delta_i\biggl(c_i - \frac{1}{2}\sum_{j=1}^{n}|d_{ij}|\biggr) - \gamma_i\alpha_i^2\sum_{j=1}^{n}|b_{ji}|.$$

Clearly, one may see that ξ_i, η_i are non-negative for each $i = 1, 2, \dots, n$ and so is k_i.

Now the conclusion follows from [8].

Theorem 5.3 *Assume that the neuronal gains α_i, β_i, the synaptic connection weights b_{ij}, d_{ij} and the decay rates a_i, c_i of the neuronal fields F_X and F_Y respectively, satisfy the inequality*

$$\sum_{j=1}^{n}(\alpha_j|b_{ij}| + \beta_j|d_{ij}|) < k_i = \min\{a_i, c_i\} \tag{5.8}$$

for $i = 1, 2, \dots, n$.

Then the equilibrium solution (x^*, y^*) of (1.4) is globally asymptotically stable.

Proof For each $i = 1, 2, \dots, n$, define

$$\begin{gathered} Q_i(t) = \|(x_i(t) - x_i^*,\, y_i(t) - y_i^*)\| = |x_i(t) - x_i^*| + |y_i(t) - y_i^*| \\ \text{for} \quad t \in [-\tau, \infty). \end{gathered} \tag{5.9}$$

Then using (1.4), (2.2) we get

$$\begin{aligned} D^+Q_i(t) \le \biggl[- a_i|x_i(t) - x_i^*| &+ \sum_{j=1}^{n}\alpha_j|b_{ij}|\,|y_j(t-\tau) - y_j^*| \\ &- c_i|y_i(t) - y_i^*| + \sum_{j=1}^{n}\beta_j|d_{ij}|\,|x_j(t-\tau) - x_j^*|\biggr] \end{aligned}$$

which in turn yields,

$$D^+Q_i(t) \le \biggl[- k_iQ_i(t) + \sum_{j=1}^{n}(\alpha_j|b_{ij}| + \beta_j|d_{ij}|)Q_j(t-\tau)\biggr] \tag{5.10}$$

for each $i = 1, 2, \dots, n$.

Since, $Q_i(t) \le M$ for $t \in [-\tau, 0]$ and for each $i = 1, 2, \dots, n$, we now claim that

$$Q_i(t) \le M = \max_{1\le i\le n}\biggl[\sup_{-\tau\le s\le 0}\bigl(|\phi_i(s) - x_i^*| + |\psi_i(s) - y_i^*|\bigr)\biggr] \tag{5.11}$$

for $t \ge 0$.

If the inequality (5.11) does not hold for all $t \geq 0$, then there must exist a $t_1 > 0$ and some i such that

$$Q_i(t_1) = M, \qquad Q_j(t) \begin{cases} < M, & \text{for} \quad i = j, \; -\tau \leq t < t_1, \\ \leq M, & \text{for} \quad i \neq j, \; -\tau \leq t \leq t_1. \end{cases}$$

It is easy to see that

$$D^+ Q_i(t_1) \geq 0. \tag{5.12}$$

But, from (5.8) and (5.10), we have

$$\begin{aligned} D^+ Q_i(t_1) &\leq \Big[-k_i M + \sum_{j=1}^{n} \big(\alpha_j |b_{ij}| + \beta_j |d_{ij}| \big) M \Big] \\ &= -\Big[k_i - \sum_{j=1}^{n} \big(\alpha_j |b_{ij}| + \beta_j |d_{ij}| \big) \Big] M < 0 \end{aligned}$$

which contradicts (5.12) and thus (5.11) holds for all $t \geq 0$.

Now, for each $i = 1, 2, \ldots, n$, let

$$\lim_{t \to \infty} \sup Q_i(t) = \bar{\sigma}_i \quad \text{and} \quad \lim_{t \to \infty} \inf Q_i(t) = \underline{\sigma_i}.$$

Clearly $0 \leq \underline{\sigma_i} \leq \bar{\sigma}_i < \infty$, for $i = 1, 2, \ldots, n$. Without loss of generality, assume that $\bar{\sigma}_1 \geq \bar{\sigma}_2 \geq \cdots \geq \bar{\sigma}_n$. We shall prove that $\bar{\sigma}_1 = 0$. Suppose that $\bar{\sigma}_1 > 0$.

Now, choose $\epsilon > 0$ in such a way that the inequality

$$0 < \epsilon \leq \frac{k_1 - \sum\limits_{j=1}^{n} (\alpha_j |b_{1j}| + \beta_j |d_{1j}|)}{2[k_1 + \sum\limits_{j=1}^{n} (\alpha_j |b_{1j}| + \beta_j |d_{1j}|)(1 + M)]} \, \bar{\sigma}_1 \tag{5.13}$$

is satisfied.

Since $\lim\limits_{t \to \infty} \sup Q_i(t) = \bar{\sigma}_i$ by definition (for this ϵ) there exists a $t_2 > 0$ such that for $t \geq t_2$, we have

$$Q_i(t - \tau) \leq \bar{\sigma}_i + \epsilon \leq \bar{\sigma}_1 + \epsilon,$$

for $i = 1, 2, \ldots, n$ and $\tau > 0$.

Then, from (5.10), for $t \geq t_2$, it follows that

$$D^+ Q_1(t) \leq \Big[-k_1 Q_1(t) + \sum_{j=1}^{n} (\alpha_j |b_{1j}| + \beta_j |d_{1j}|)(\bar{\sigma}_1 + \epsilon) \Big]. \tag{5.14}$$

We first prove $\bar{\sigma}_1 = \underline{\sigma}_1$. If $\bar{\sigma}_1 > \underline{\sigma}_1$, then there are infinite number of intervals on which $Q_1(t)$ is non decreasing. We can choose $t_4 > t_3 \geq t_2$ such that $Q_1(t)$ is non decreasing on (t_3, t_4) and

$$Q_1(t) > \bar{\sigma}_1 - \epsilon \quad \text{for} \quad t \in (t_3, t_4).$$

From (5.14), for $t \in (t_3, t_4)$, we have

$$
\begin{aligned}
D^+Q_1(t) &\le -\left[k_1(\bar\sigma_1 - \epsilon) - (\bar\sigma_1 + \epsilon)\sum_{j=1}^{n}(\alpha_j|b_{1j}| + \beta_j|d_{1j}|)\right] \\
&= -\left[\left\{k_1 - \sum_{j=1}^{n}(\alpha_j|b_{1j}| + \beta_j|d_{1j}|)\right\}\bar\sigma_1 \right. \\
&\qquad \left. - \left\{k_1 + \sum_{j=1}^{n}(\alpha_j|b_{1j}| + \beta_j|d_{1j}|\right\}\epsilon\right]
\end{aligned}
$$

and using (5.13) one can see that

$$
D^+Q_1(t) \le -\frac{\bar\sigma_1}{2}\left[k_1 - \sum_{j=1}^{n}(\alpha_j|b_{1j}| + \beta_j|d_{1j}|)\right] < 0 \tag{5.15}
$$

which is a contradiction to the statement that $Q_1(t)$ is non-decreasing over (t_3, t_4). Accordingly we must have $\bar\sigma_1 = \underline{\sigma}_1 = \sigma$ (say).

Since $\bar\sigma_1 = \underline{\sigma}_1 = \sigma > 0$, there must exist a $t_5 \ge t_2$ such that for $t \ge t_5$ we have

$$
\sigma - \epsilon < Q_1(t) < \sigma + \epsilon
$$

and

$$
Q_i(t) \le \sigma + \epsilon \quad \text{for} \quad i = 2, \dots, n.
$$

For $t \ge t_5$, from (5.15) we have

$$
0 \le Q_1(t) \le Q_1(t_5) - \frac{\sigma}{2}\left[k_1 - \sum_{j=1}^{n}(\alpha_j|b_{1j}| + \beta_j|d_{1j}|)\right](t - t_5)
$$

which is a contradiction. Hence $\sigma = 0$ and thus

$$
\lim_{t\to\infty}\|(x_i(t) - x_i^*,\, y_i(t) - y_i^*)\| = 0
$$

for $i = 1, 2, \dots, n$, implying that (x^*, y^*) is globally asymptotically stable.

6 Examples

In this section, we present several examples illustrating our results. Further, we establish that the various stability criteria are independent.

Example 6.1 [24] Consider the network described by the system

$$
\begin{aligned}
\dot x_i &= -a_i x_i(t) + \sum_{j=1}^{2} b_{ij} f_j(\lambda_j, y_j(t-\tau)) + I_i, \\
\dot y_i &= -c_i y_i(t) + \sum_{j=1}^{2} d_{ij} g_j(\mu_j, x_j(t-\tau)) + J_i.
\end{aligned} \tag{6.1}
$$

where $i = 1, 2$.

Now choose

$$a_1 = 3, \quad a_2 = 3, \quad c_1 = 3, \quad c_2 = 4,$$
$$\lambda_1 = 5/6, \quad \lambda_2 = 4/5, \quad \mu_1 = 1/2, \quad \mu_2 = 3/4,$$
$$I_1 = -1/2, \quad I_2 = 1, \quad J_1 = 1/4, \quad J_2 = -2,$$
$$[b_{ij}] = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \quad [d_{ij}] = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}.$$

Further, choose the signal functions f_i and g_i for $i = 1, 2$ as follows:

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \tanh(\lambda_1 y_1) \\ \tanh(\lambda_2 y_2) \end{pmatrix},$$
$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \tanh(\mu_1 x_1) \\ \tanh(\mu_2 x_2) \end{pmatrix}.$$

Observe that all the hypotheses of Theorem 5.1 are satisfied while the hypotheses (5.5) of Theorem 5.2 and (5.8) of Theorem 5.3 are violated. From this it is clear that the equilibrium pattern is globally asymptotically stable by virtue of Theorem 5.1.

Example 6.2 [24] Consider the system (6.1) in which

$$a_1 = 3/2, \quad a_2 = 6, \quad c_1 = 7, \quad c_2 = 2.3,$$
$$\lambda_1 = 8, \quad \lambda_2 = 16, \quad \mu_1 = 11/6, \quad \mu_2 = 3/2,$$
$$I_1 = 3, \quad I_2 = 1, \quad J_1 = 2, \quad J_2 = 4,$$
$$[b_{ij}] = \begin{bmatrix} 1/5 & 1/3 \\ 1/2 & 1/4 \end{bmatrix}, \quad [d_{ij}] = \begin{bmatrix} -1/2 & 1/4 \\ -1/3 & -1/5 \end{bmatrix}$$

and the signal functions are given by

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+e^{-\lambda_1 y_1}} \\ \frac{1}{1+e^{-\lambda_2 y_2}} \end{pmatrix},$$
$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \tanh(\mu_1 x_1) \\ \tanh(\mu_2 x_2) \end{pmatrix}$$

for $i = 1, 2$.

Notice that all conditions of Theorem 5.2 are satisfied while some of the inequalities (5.1) of Theorem 5.1 and inequalities (5.8) of Theorem 5.3 are violated.

Example 6.3 [24] Again consider the system (6.1) and choose

$$a_1 = 1.4, \quad a_2 = 1.3, \quad c_1 = 1.25, \quad c_2 = 1.35,$$
$$\lambda_1 = 1/3, \quad \lambda_2 = 1/9, \quad \mu_1 = 1/2, \quad \mu_2 = 1,$$
$$I_1 = 1/2, \quad I_2 = 1/4, \quad J_1 = -1/3, \quad J_2 = 1/5,$$

and

$$[b_{ij}] = \begin{bmatrix} 5/2 & 1/2 \\ 3/2 & 1/2 \end{bmatrix}, \quad [d_{ij}] = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 5/2 \end{bmatrix}$$

and the signal functions are

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \tanh(\lambda_1 y_1) \\ \tanh(\lambda_2 y_2) \end{pmatrix}, g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+e^{-\mu_1 x_1}} \\ \frac{1}{1+e^{-\mu_2 x_2}} \end{pmatrix}.$$

It follows easily that all the requirements of Theorem 5.3 are satisfied while some of the inequalities of the hypotheses (5.1) of Theorem 5.1 and those of (5.4) of Theorem 5.2 are not satisfied.

Remark 6.1 Note that from Examples 6.1, 6.2 and 6.3, it follows that the global asymptotic stability criteria guaranteed by Theorems 5.1, 5.2 and 5.3 are independent.

7 Discussion

In this paper, a model describing the activation dynamics of neurons in a bidirectional associative memory network involving processing delays has been presented. The importance of the necessity of introducing processing delays in the model has been highlighted. Results on maintenance and preservation of stability together with circumstances leading to instability have been presented. We have established that a proper interplay between the processing delays and the various system parameters is highly essential to have global stability behaviour of the network.

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