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# Hamilton's Action Function in Stability Problem of Conservative Systems

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**Abstract:** In the paper the equilibrium stability of conservative systems both holonomic and nonholonomic in case when the appropriate force function of a system has not a local maximum in the equilibrium state is considered. For the investigation of stability the Hamilton action is used as a function of phase variables and time.

**Keywords:** *Hamilton's action function; stability.*

**Mathematics Subject Classification (2000):** 70H05, 70H14.

## 0 Introduction

The most vulnerable area of researches on stability of systems based on the application of the Liapunov second method is the problem of finding a Liapunov function (including its analogues and modifications), especially if the problem is not solved in the framework of linear approximation. For this reason relevant stability (instability) theorems involving auxiliary functions, whose construction remains a problem, are actually inefficient or even useless.

In view of the above, the approach has become of theoretical and practical significance when beforehand the class of considered systems (for example conservative, reversible, with constant phase volume, etc.) is defined, for which the construction of a Liapunov function or its analogue is possible. Thus, the solution of the problem about stability (instability) passes into constructive direction at once for the whole class of systems. Just such idea can be realized concerning the class of conservative systems

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \tag{0.1}$$

whose Lagrangian $L$ can retain its sign at least on some set of motions.

 

Within the framework of the proposed approach the Liapunov second method should be interpreted somewhat wider in comparison with the classical approach. In particular, the ideas incorporated in the second method are put in the forefront instead of the specific theorems covered by it. Also allowing for the peculiarities of the examined systems takes an important place.

## 1 About Hamilton Action as a Function of Phase Variables

In the construction of a Liapunov function analogue for conservative systems it is proposed to use the Hamilton action $S$ as a function of phase coordinates and time. The possibility to obtain the equations of conservative systems from the condition for the Hamilton action $S$ being stationary

$$\delta S = \delta \int\limits_0^{t_1} L(q, \dot{q})\, d\tau = 0, \tag{1.1}$$

enables the action $S$ to be recognized as a carrier of information on conservative systems. In view of this fact, we replace the fixed value $t_1$ by the current value $t$ in the expression for the action $S$ and consider $S$ as a magnitude which characterizes the true motion of the system, i.e. as the action function

$$S = \int\limits_0^{t} L(q, \dot{q})\, d\tau. \tag{1.2}$$

It means that the values

$$\begin{aligned} q = q(t, q_0, \dot{q}_0), \qquad \dot{q} = \dot{q}(t, q_0, \dot{q}_0), \\ q_0 = q(t=0), \qquad \dot{q}_0 = \dot{q}(t=0) \end{aligned} \tag{1.3}$$

in the integrand of equality (1.2) satisfy the equations (0.1). Let us assume further that the Lagrangian $L(q, \dot{q}) \in C^2(D_q \times R^n_{\dot{q}})$ and

$$\begin{aligned} L(q, \dot{q}) &= L_2(q, \dot{q}) + L_1(q, \dot{q}) + L_0(q) \\ &= \frac{1}{2}\, \dot{q}^T A(q)\dot{q} + f(q)^T \dot{q} + L_0(q), \end{aligned} \tag{1.4}$$

where the quadratic form $L_2(0, \dot{q})$ is positive definite, the point $q = \dot{q} = 0$ corresponds to the equilibrium state of system (0.1), (1.4), $f(0) = 0$, $L_0(0) = 0$. Besides, let the solution (1.3) satisfy the definition of a flow [1]. This does not limit generality of the consideration, since the instability of the equilibrium state is dealt with below.

Replacing $t$ by $\tau$ in (1.3) and carrying out integration in equality (1.2), we obtain

$$S = \tilde{S}(\tau, q_0, \dot{q}_0)|_0^t \in C^{(1,1,1)}_{tq_0\dot{q}_0}(R \times s_\delta), \tag{1.5}$$

where the vector $(q_0, \dot{q}_0)$ belongs to the neighborhood $s_\delta = \{(q_0, \dot{q}_0) \in D_q \times R^n_{\dot{q}}, \|q_0 \oplus \dot{q}_0\| < \delta\}$ of the point $q = \dot{q} = 0$. Taking into account that the solution (1.3) defines a flow and thus

$$q_0 = q(-t, q, \dot{q}), \qquad \dot{q}_0 = \dot{q}(-t, q, \dot{q}), \tag{1.6}$$

we have from (1.5)

$$S = S^*(\tau, q(\tau), \dot{q}(\tau))|_0^t \in C^{(1,1,1)}_{tq\dot{q}}(R \times D_q \times R^n_{\dot{q}}). \tag{1.7}$$

The use of the Hamilton action function $S$ in the form of (1.7) as an analogue of the Liapunov function provides greater possibility for more complete representation of the internal properties of the system in question than the standard approach within the Liapunov second method. Actually, this is confirmed by the investigations of the inversion of the Lagrange-Dirichlet and Routh theorems [2 − 8]. Sufficient conditions of instability obtained in these investigations are more general in comparison with the ones known earlier (see the reviews [7, 9 − 11]. In particular, the following result is true

**Theorem 1.1** [6] *Let a number* $\varepsilon > 0$ $(D \supset \overline{s^*_\varepsilon})$ *exist such that:*

(1) $\omega = \{q \in s^*_\varepsilon \colon L_0(q) > 0\} \neq \emptyset,\ 0 \in \partial\omega;$
(2) $\partial L_0/\partial q \neq 0\ \ \forall q \in \omega;$
(3) $L_0 - \frac{1}{2} f^T A^{-1} f \geq 0\ \ \forall q \in \omega.$

*Then the equilibrium state* $q = \dot{q} = 0$ *of system (0.1), (1.4) is unstable.*

**Corollary 1.1** *Let the system be natural* $(L_2 = T,\ L_1 \equiv 0,\ L_0 = -\Pi,$ *where the functions* $T$ *and* $\Pi$ *are kinetic energy and potential energy of system respectively) and let a number* $\varepsilon > 0$ $(D \supset \overline{s^*_\varepsilon})$ *exist such that:*

(1) $\omega = \{q \in s^*_\varepsilon \colon \Pi(q) < 0\} \neq \emptyset,\ 0 \in \partial\omega;$
(2) $\partial\Pi/\partial q \neq 0\ \ \forall q \in \omega.$

*Then the equilibrium state* $q = \dot{q} = 0$ *of the system is unstable.*

**Corollary 1.2** [2] *The isolated equilibrium state* $q = \dot{q} = 0$ *of a natural system is unstable if in this state the potential energy* $\Pi(q)$ *has not a local minimum.*

**Corollary 1.3** [3] *Let the Lagrangian* $L$ *in the neighborhood of the point* $q = \dot{q} = 0$ *be analytical function. Then the equilibrium state* $q = \dot{q} = 0$ *of a natural system is unstable if the potential energy* $\Pi(q)$ *at the point* $q = 0$ *has not a local minimum.*

*Remark 1.1* In special case when the expression $L_0 - \frac{1}{2} f^T A^{-1} f$ has the local minimum (not necessarily strict) at the point $q = 0$, the restriction (2) in Theorem 1.1 (and the restriction (2)) in Corollary 1.1 respectively) can be omitted.

*Remark 1.2* The statement of the problem about the equilibrium instability of a natural system under the assumptions of Corollary 1.3 is due to Liapunov [12].

## 2 The Application of the Hamilton's Action Function $S$ to the Investigation of Stability of Conservative Nonholonomic Systems

The use of the Hamilton's action function $S$ can also appear to be useful in analysing the equilibrium stability of nonholonomic systems. As it is well known [11, 13], the equilibria set of a nonholonomic system is larger than a set of critical points of the appropriate Lagrangian $L$. Restricting the investigation to stability analysis of the equilibria of nonholonomic systems which are critical points of the Lagrangian $L$, Whittaker [14] somewhat narrowed the class of considered nonholonomic systems. However this more narrow class is of interest first of all because many of the approaches characteristics of stability investigation of holonomic systems [15 − 19] are applicable to it. It turns out that for the systems in question, the application of the Hamilton's action function $S$ is efficient as well as for the holonomic ones.

So, we shall consider a nonholonomic system written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = B^T(q)\lambda, \qquad \lambda = (\lambda_1, \dots, \lambda_l)^T, \tag{2.1}$$

$$B(q)\,\frac{dq}{dt} = 0, \tag{2.2}$$

where $B(q) = (b_{ij}(q))$ is a $l \times n$ matrix $(i = 1, \dots, l,\ j = 1, \dots, n,\ l < n)$, $\lambda$ is the $l$-vector of the Lagrange multipliers, $L(q, \dot q)$, $B(q) \in C^2(D_q \times R^n_{\dot q})$ and the Lagrangian $L$ is defined by expression (1.4).

Let the quadratic form $\dot q^T A(0)\dot q$ be positive definite as before, the point $q = \dot q = 0$ correspond to the equilibrium state of system (2.1), (2.2), (1.4) and $f(0) = 0$, $L_0(0) = 0$.

Nonintegrable relations (2.2) which restrict the generalized velocities of the system $(\operatorname{rank} B(q) = l)$ are nonholonomic constraints.

It is well known [20, 21] that the equations of motion of nonholonomic system can be obtained on the basis of the Hamilton principle in the Hölder form

$$\int_0^{t_1} \delta L(q, \dot q)\, d\tau = 0. \tag{2.3}$$

In contrast to the holonomic systems, the Hamilton principle in the form of (2.3) is no longer the principle of stationary action when equality (1.1) is valid. Apparently, it was Hertz [22], who first called his attention to the fact that equality (1.1) ceases to be true for nonholonomic systems. Nevertheless the indisputable fact is that in the case of nonholonomic systems the Lagrangian $L(q, \dot q)$ is still the key characteristics of the system. Taking this into account, we shall consider the action function of the form of (1.2) for nonholonomic system too.

By analogy with the case of holonomic systems it is assumed that the solution of the considered nonholonomic system is extendable on the whole axis $t \in R$ and so in view of the assumptions on smoothness of $L(q, \dot q)$ and $B(q)$ satisfies the definition of a flow. This fact, as above, does not restrict generality of the consideration and enables the action function $S$ to be represented in the form of (1.7).

In what follows system (2.1), (2.2), (1.4) will be written as

$$\begin{aligned} \dot q &= \frac{\partial H}{\partial p}, \\ \dot p &= -\frac{\partial H}{\partial q} + B^T(q)\lambda, \\ B(q)\,\frac{\partial H}{\partial p} &= 0, \end{aligned} \tag{2.4}$$

where

$$H(q,p) = \frac{1}{2}\,p^T A^{-1} p - p^T A^{-1} f - L_0 + \frac{1}{2}\,f^T A^{-1} f = h = \text{const}\,. \tag{2.5}$$

Let us consider in $R^n$ the set $\pi$ which is defined by the equations

$$B(q)q = 0. \tag{2.6}$$

Since $\operatorname{rank} B(q) = l$, the equations (2.6) can always be solved with respect to any $l$ components of the vector of generalized coordinates $q$. Let us designate by $\widehat{\Psi}(q)$ the restriction of an arbitrary function $\Psi(q)$ to the set $\pi$.

Alongside the set $\pi$ we shall also define the sets

$$\Omega = \big\{(q,p) \in s_\varepsilon = \{(q,p) \in D_q \times R^n_p,\ \|q \oplus p\| < \varepsilon\}\colon\ H = h = 0\big\},$$
$$\Omega^+ = \big\{(q,p) \in s_\varepsilon\colon\ H = h > 0\big\},$$
$$\Omega_1^+ = \left\{(q,p) \in s_\varepsilon\colon\ H = h > 0,\ H - q\,\frac{\partial H}{\partial q} + (B(q)q)\lambda > 0\right\}.$$

**Theorem 2.1** *Let the function $L_0 - \frac{1}{2}\,f^T A^{-1} f$ have a local minimum (not necessarily strict) at the point $q = 0$ and besides:*

$$\omega^* = \big\{q \in s^*_\varepsilon = \{q \in D_q,\ \|q\| < \varepsilon\}\colon\ \widehat{L}_0(q) > 0\big\} \neq \emptyset.$$

*Then the equilibrium state $q = \dot q = 0$ of system (2.1), (2.2), (1.4) is unstable.*

Before proving this theorem we need the following lemma.

**Lemma 2.1** *Under the assumptions of Theorem 2.1 the set*

$$\Omega^0 = \left\{(q,p) \in s_\varepsilon\colon\ H = 0, -q\,\frac{\partial H}{\partial q} > 0,\ B(q)q = 0\right\}$$

*is not empty.*

*Proof* On the basis of the theorem about the mean [23] we have the equality

$$H(q,p) - H(0,0) = q\,\frac{\partial H}{\partial q}\,(\theta q, \theta p) + p\,\frac{\partial H}{\partial p}\,(\theta q, \theta p), \qquad \theta \in (0,1). \tag{2.7}$$

First let us assume that $L_0 - \frac{1}{2}\,f^T A^{-1} f > 0$. Then, taking into account the relation

$$L = p\dot q - H = \frac{1}{2}\,p^T A^{-1} p + L_0 - \frac{1}{2}\,f^T A^{-1} f = p\,\frac{\partial H}{\partial p}$$
$$\forall\,(q,p) \in \Omega, \quad q \in \omega^*,$$

we conclude that the term $q\,\partial H/\partial q$ on the right-hand side of equality (2.7) is negative $\forall\,(q,p) \in \Omega,\ q \in \omega^*$. Since the vectors $q$ and $\theta q$ are collinear, on the basis of (2.7) and the definition of $\omega^*$ we conclude that $\Omega^0 \neq \emptyset$.

If $L_0 - \frac{1}{2}\,f^T A^{-1} f \equiv 0$, then representing the Hamiltonian $H$ as

$$H = \frac{1}{2}\,p^T A^{-1}(p - 2f) - (L_0 - \frac{1}{2}\,f^T A^{-1} f)$$

and assuming $p = 2f$ $(\|f\| \neq 0)$, we come to the similar conclusion.

**Corollary 2.1** *Under the assumptions of lemma the set $\Omega_1^+$ is not empty.*

*Proof* Let us fix a point $(q^0, p^0) \in \Omega^0$. Taking into account the fact that $\Omega$ is the boundary for set $\Omega^+$, we carry out small perturbation of the point $(q^0, p^0)$:

$$\|(q^* - q^0) \oplus (p^* - p^0)\| < \eta, \quad 0 < \eta = \text{const}$$

such that the point $(q^*, p^*)$ becomes the element of the set $\Omega^+$. Then owing to the continuity of the product $q\,\partial H/\partial q$ and the equality (2.6) $\forall (q,p) \in \Omega^0$ the number $\eta$ (and consequently the appropriate perturbation) can be chosen small so that the inequalities

$$H(q^*, p^*) - q\,\frac{\partial H}{\partial q}\bigg|_{q=q^*,\,p=p^*} + (B(q^*)q^*)\lambda > 0, \qquad H(q^*, p^*) > 0$$

are satisfied. This implies the validity of the corollary.

*Proof of Theorem 2.1* Let us assume that the equilibrium state $q = \dot{q} = 0$ of the initial system (2.1), (2.2), (1.4) is stable.

Following [6], we consider the function

$$V = \frac{qp}{S_1^2 + 1},$$

where

$$S_1 = S^*\Big(t, q, \tfrac{\partial H}{\partial p}\Big) = S_1^*(t, q, p) \in C_{tqp}^{(1,1,1)}(R \times D_q \times R_p^n).$$

Its time derivative along the vector field defined by the equations (2.4) is of the form

$$\begin{gathered} \frac{dV}{dt} = \frac{L}{S_1^2 + 1}\,(1 - \mu) + \frac{(H - q\,\partial H/\partial q + (B(q)q)\lambda)}{S_1^2 + 1}, \\ \mu = 2qp\,\frac{S_1}{S_1^2 + 1}. \end{gathered} \tag{2.8}$$

According to the assumption about equilibrium stability, there always exists the positive semitrajectory $\gamma_1^+ \subset s_\varepsilon$ of the considered system passing through the point $(q^*, p^*) \in \Omega_1^+$. Let us integrate equality (2.8) over a segment of the semitrajectory $\gamma_1^+$ which corresponds to the interval $[t_1, t_2]$, where the numbers $t_1$ and $t_2$ are such that

$$\gamma_1^+\big|_{t_1}^{t_2} \subset \Omega_1^+. \tag{2.9}$$

We notice that as $\overline{\gamma_1^+}$ is a compact set, the absolute value of the velocity of the appropriate representing point $(q(t, q^*, p^*),\, p(t, q^*, p^*))$ moving along $\gamma_1^+$ is uniformly bounded. Consequently, it is possible to specify a number $a > 0$ such that $t_2 - t_1 \geq a$ irrespective of how large the values $t_1,\, t_2 \in R$ are. In result of integration of equality (2.8) we have

$$\frac{qp}{S_1^2 + 1}\bigg|_{t_1}^{t_2} = \arctan\, S_1\big|_{t_1}^{t_2} + o\big(\arctan\, S_1\big|_{t_1}^{t_2}\big) + \int\limits_{t_1}^{t_2} \frac{(H - q\,\partial H/\partial q + (B(q)q)\lambda)}{S_1^2 + 1}\, dt. \tag{2.10}$$

Function $\arctan S_1$ appearing in the right-hand side of equality (2.10) is a multifunction with branch points $S_1 = \pm\infty$. At the same time the intersection of the level set of the Hamiltonian $H = h > 0$ and the small neighborhood of the equilibrium state is the manifold, in any point of which the function $S_1$ does not become infinite. Further, without loss of generality, it is possible to consider that the inequality $S_1|_{t=t_1} > 1$ is valid.

Under condition (2.9) we shall choose the length of the interval $[t_1, t_2]$ small enough to deal with the domain of principal values of the function $\arctan S_1$, using, for example, the representation of the latter as

$$\arctan S_1 = \frac{\pi}{2} - \frac{1}{S_1} + \frac{1}{3S_1^3} - \dots .$$

Then from equality (2.10) when $\varepsilon > 0$ is sufficiently small we obtain

$$\frac{1}{S_1}\bigg|_{t_1}^{t_2} + O\bigg(\frac{1}{3S_1^3}\bigg|_{t_1}^{t_2}\bigg) + o\bigg(\frac{1}{S_1}\bigg|_{t_1}^{t_2}\bigg) = \int\limits_{t_1}^{t_2} \frac{(H - q\,\partial H/\partial q + (B(q)q)\lambda)}{S_1^2 + 1}\,dt. \tag{2.11}$$

Noticing that by (2.9) the right-hand side of equality (2.11) is positive, we come to the contradiction, because according to structure (1.5) of the Lagrangian $L$:

$$\frac{dS_1}{dt} = p\dot{q} - H = \frac{1}{2}\,p^T A^{-1} p + L_0 - \frac{1}{2}\,f^T A^{-1} f > 0 \quad \forall\,(q,p) \in \gamma_1^+\big|_{t_1}^{t_2}$$

the expression in its left-hand side is negative. Thus the assumption about stability of the examined equilibrium state is false. Theorem 2.1 is proved.

**Corollary 2.2** *Let* $L_0 - \frac{1}{2}\,f^T A^{-1} f \geq 0\ \forall\, q \in s_\varepsilon^*$ *and besides the function* $L_0(q)$ *have a strict local minimum at the point* $q = 0$.

*Then the equilibrium state* $q = \dot{q} = 0$ *of system (2.1), (2.2), (1.4) is unstable.*

**Corollary 2.3** *Under the assumptions of Corollary 2.2 the manifold* $M$ *of the equilibrium states of system (2.1), (2.2), (1.4) defined by the equations*

$$\frac{\partial L_0(q)}{\partial q} + B^T(q)\lambda = 0, \qquad \dot{q} = 0$$

*is unstable.*

*Proof* According to Corollary 2.2 the equilibrium state $q = \dot{q} = 0$ of system (2.1), (2.2), (1.4) is unstable. Let us show that the instability holds true relating to both variables $q$ and variables $\dot{q}$. For this purpose we shall assume on the contrary that instability of the equilibrium state under consideration motivates only the leave of $q$-vector. Then, irrespective of the smallness of the initial perturbation, there is an orbit of the system, whose representing point in a small enough neighborhood of the point $q = 0$ reaches some sphere $\|q\|^2 = \eta^2$, $\eta = \text{const}$. Let integer $\xi$ $(0 < \xi = \text{const})$ correspond to the minimum of function $L_0(q)$ on this sphere. It is clear that $\xi$ does not depend on the smallness of perturbation.

On the basis of equality

$$L_2(q, \dot{q}) - L_0(q) = h = \text{const} \tag{2.12}$$

we have

$$\frac{1}{2}\,\dot{q}^T A(q)\dot{q} \geq h + \xi. \tag{2.13}$$

Since according to the proof of Theorem 2.1 the equilibrium state $q = \dot{q} = 0$ of the examined system is unstable when $h > 0$, it follows from (2.13) that $\|\dot{q}\| \geq \varepsilon > 0$. In this case the number $\varepsilon$ does not depend on the smallness of the initial perturbation and thus the instability of the equilibrium state is accompanied by leaving of $\dot{q}$-vector. Taking into account that $\dot{q} = 0$ on the manifold $M$ of the equilibrium states of the system, according to definition [24, p.34] we make conclusion about the validity of Corollary 2.3.

The sense of Corollary 2.3 is that it establishes a relationship between instability of the fixed equilibrium state of nonholonomic system and that of the whole manifold of the equilibrium states, the existence of which is a distinguishing property of nonholonomic systems [25].

**Corollary 2.4** *In special case when the nonholonomic constraints are absent:* $B(q) = 0$*, it is possible to omit the condition* $\omega^* \neq \emptyset$ *in the Theorem 2.1 (cf. [6, 26, 27]).*

As we see, if a system is nonholonomic then this fact is embodied in the character of instability conditions, however it should be remembered that the latter are only sufficient. Therefore, unfortunately, the question about the real influence of nonholonomic constraints on the equilibrium stability remains still open.

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