Asymptotic Behaviour of Feedback Controlled Systems and the Ubiquity of the Brockett-Krasnosel'skii-Zabreiko Property

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Abstract: A well-known topological barrier – the Brockett-Krasnosel'skii-Zabreiko necessary condition on the underlying vector field – to stability of equilibria (or stabilizability of equilibria by regular feedback) of ordinary differential equations (or controlled differential equations) is shown to persist in a wider context of differential inclusions (encompassing controlled differential equations with nonsmooth feedback) that exhibit attracting compacta.

Keywords: Brockett-Krasnosel'skii-Zabreiko condition; feedback controlled system.

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1 Introduction

Let \( f : \mathbb{R}^N \to \mathbb{R}^N \) be locally Lipschitz and consider the system

\[
\dot{x} = f(x).
\]

By [1, Theorem 52.1], if (1) has an asymptotically stable (that is, Lyapunov stable and attractive) equilibrium \( \xi \), then the (isolated) zero \( -f \) has index \( \text{ind}(-f, \xi) = 1 \) and so, for all \( \epsilon > 0 \) sufficiently small, \( \deg_B(-f, B_\epsilon(\xi), 0) = 1 \), where \( \deg_B \) denotes Brouwer degree and \( B_\epsilon(\xi) \) denotes the open ball of radius \( \epsilon \) centred at \( \xi \). Therefore,

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by properties of Brouwer degree, \( f(\mathbb{R}^N) \) contains an open neighbourhood of 0. Now let \( f: \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N \) be locally Lipschitz and consider the controlled system

\[
\dot{x} = f(x, u).
\]

If (2) is stabilizable in the sense that there exists a time-invariant locally Lipschitz feedback \( u = k(x) \) that renders some point of \( \mathbb{R}^N \) an asymptotically stable equilibrium of the feedback system \( \dot{x} = f(x, k(x)) \), then, by the above result, the image of \( f \) contains an open neighbourhood of 0. This is Brockett’s necessary condition for stabilizability, originally proved in \([2, \text{Theorem 1}]\); for discussions on variants and ramifications of Brockett’s condition, see, for example, \([3–11]\). In either case of an uncontrolled (1) or controlled (2) system, if \( f: D \to \mathbb{R}^N \) is such that \( f(D) \) contains an open neighbourhood of 0, we say that \( f \) has the BKZ (Brockett-Krasnosel’skiǐ-Zabreiko) property.

In this paper, the necessity of the BKZ property is investigated in a wider context of differential inclusions under hypotheses weaker than asymptotic stability/stabilizability of equilibria. For example, amongst other consequences for (1), the results of the paper imply that, if any of the following hold, then \( f \) has the BKZ property:

(a) some compact set \( C \) is globally attractive for solutions of (1);
(b) some closed ball is a locally asymptotically stable (Lyapunov stable and locally attractive) set for (1);
(c) (1) is \( L^p \)-stable for some \( 1 \leq p < \infty \) (in the sense that every maximal solution has interval of existence \( \mathbb{R}_+ \) and is of class \( L^p \)).

Within the control framework of (2), these observations have natural counterparts: \( f \) has the BKZ property if there exists a (possibly discontinuous) feedback \( k \) such that the feedback-controlled system (a) has a globally attractive compact set, or (b) has a locally asymptotically stable closed ball, or (c) is \( L^p \)-stable (in the above sense).

2 Notation and Terminology

For a Banach space \( X \) and non-empty \( C \subset X \), \( d_C \) denotes the distance function given by

\[
d_C(x) := \inf_{c \in C} \|x - c\| \quad \forall \ x \in X.
\]

For non-empty \( B, C \subset X \),

\[
d(B, C) := \sup_{b \in B} d_C(b).
\]

The open ball of radius \( r \geq 0 \) centred at \( z \in \mathbb{R}^N \) is denoted \( \mathbb{B}_r(z) \) (with closure \( \overline{\mathbb{B}}_r(z) \)), to which the conventions \( \mathbb{B}_0(z) := \emptyset \) and \( \overline{\mathbb{B}}_0(z) := \{z\} \) apply; if \( z = 0 \), then we simply write \( \mathbb{B}_r \) (respectively, \( \overline{\mathbb{B}}_r \)) in place of \( \mathbb{B}_r(0) \) (respectively, \( \overline{\mathbb{B}}_r(0) \)). The boundary of a set \( \Omega \) is denoted \( \partial \Omega \). We write \( \mathbb{R}_+ := [0, \infty) \).

Throughout, a sequence \( (x_n) \) is regarded as synonymous with a map \( n \mapsto x_n \) with domain \( \mathbb{N} \). We shall frequently extract subsequences of sequences. In order to avoid proliferation of subscripts, the notation \( (x_{\sigma(n)}) \), where \( \sigma: \mathbb{N} \to \mathbb{N} \) is a strictly increasing map, is adopted to indicate a subsequence of \( (x_n) \). If \( \{(x_{\sigma_k(n)})\}_{k \in \mathbb{N}} \) is a sequence of subsequences of \( (x_n) \) nested in the following sense

\[
(x_n) \supset (x_{\sigma_1(n)}) \supset \cdots \supset (x_{\sigma_k(n)}) \supset \cdots,
\]
then $\sigma_k$ is to be interpreted as a $k$-fold composition of strictly increasing maps $\mathbb{N} \to \mathbb{N}$, with $\sigma_k = \delta_k \circ \sigma_{k-1}$ for all $k \geq 2$: the sequence $(x_{\sigma_k(n)}) \subset (x_n)$ will be referred to as the diagonal sequence.

$AC(I; \mathbb{R}^N)$ denotes the space of functions $I \to \mathbb{R}^N$ defined on an interval $I$ and absolutely continuous on compact subintervals thereof.

$\mathcal{U}(D)$ denotes the space of upper semicontinuous maps $x \mapsto F(x) \subset \mathbb{R}^N$, defined on $D \subset \mathbb{R}^N$, with non-empty convex compact values: if $D = \mathbb{R}^N$, then we simply write $\mathcal{U}$.

We record the following well-known facts (see, for example, [12]):

**Proposition 2.1** Let $F \in \mathcal{U}(D)$.

(i) If $K \subset D$ is compact, then $F(K)$ is compact.

(ii) For each $\epsilon > 0$, there exists locally Lipschitz $f_\epsilon: D \to \mathbb{R}^N$ such that

$$d(\text{graph}(f_\epsilon), \text{graph}(F)) < \epsilon$$

(any such $f_\epsilon$ is said to be an $\epsilon$-approximate selection for $F$).

### 3 Set-Valued Maps: Degree and the BKZ Property

If $F \in \mathcal{U}(D)$ is such that $F(D)$ contains an open neighbourhood of 0, then $F$ is said to have the BKZ property.

Let $\mathcal{M} := \{(F, \Omega, p) \mid F \in \mathcal{U}(D), \Omega \text{ an open bounded subset of } D, \ p \in \mathbb{R}^N \setminus F(\partial \Omega)\}$. As discussed in [8] within the framework of [13] (see, also, [14–16]), there exists a map $\deg: \mathcal{M} \to \mathbb{Z}$ with the properties:

P1. $\deg(F, \Omega, p) = \deg_B(f_\epsilon, \Omega, p)$ for all $\epsilon > 0$ sufficiently small, where $\deg_B$ denotes Brouwer degree and $f_\epsilon: \overline{\Omega} \to \mathbb{R}^N$ is any $\epsilon$-approximate selection for $F$.

P2. if $q: [0,1] \to \mathbb{R}^N \setminus F(\partial \Omega)$ is continuous, then $\deg(F, \Omega, q(t))$ is independent of $t$.

P3. if $\deg(F, \Omega, p) \neq 0$, then $p \in F(x)$ for some $x \in \Omega$.

**Lemma 3.1** Let $(F, \Omega, 0) \in \mathcal{M}$. If $\deg(F, \Omega, 0) \neq 0$, then $F$ has the BKZ property.

**Proof** Since $0 \notin F(\partial \Omega)$, $d_{F(x)}(0) > 0$ for all $x \in \partial \Omega$. Let $(x_n) \subset \partial \Omega$ be a convergent sequence with limit $x \in \partial \Omega$. Let $(x_{\sigma(n)})$ be a subsequence with

$$\lim_{n \to \infty} d_{F(x_{\sigma(n))}}(0) = \liminf_{n \to \infty} d_{F(x_n)}(0).$$

For each $n$, let $z_n$ be a minimizer of $\| \cdot \|$ over compact $F(x_{\sigma(n)}$ (and so $\|z_n\| = d_{F(x_{\sigma(n))}}(0)$). By upper semicontinuity of $F$, for each $\epsilon > 0$,

$$z_n \in F(x_{\sigma(n)}) \subset F(x) + B_\epsilon.$$

By compactness of $F(x)$ and since $\epsilon > 0$ is arbitrary, we may conclude that $(z_n)$ has a convergent subsequence (which we do not relabel) with limit $z \in F(x)$. Therefore,

$$d_{F(x)}(0) \leq \|z\| = \lim_{n \to \infty} \|z_n\| = \liminf_{n \to \infty} d_{F(x_n)}(0)$$

and so $x \mapsto d_{F(x)}(0)$ is lower semicontinuous and positive-valued on compact $\partial \Omega$. It follows that there exists $\mu > 0$ such that $p \notin F(\partial \Omega)$ for all $p \in B_\mu$. By properties P2 and P3,

$$p \in B_\mu \implies p \in F(x) \quad \text{for some } x \in \Omega.$$
Therefore, $F$ has the BKZ property.

4 Differential Inclusions

Let $F \in \mathcal{U}$ and consider the differential inclusion (subsuming (1))

$$\dot{x}(t) \in F(x(t)).$$

(3)

By an $F$-arc, we mean a function $x \in AC(I; \mathbb{R}^N)$ that satisfies (3) for almost all $t \in I$.

The following is a particular case of [17, Theorem 3.1.7].

**Proposition 4.1** Let $F \in \mathcal{U}$, let $K \subset \mathbb{R}^N$ be compact, let $I := [a, b]$, let $(\epsilon_n) \subset (0, \infty)$ be a decreasing sequence with $\epsilon_n \downarrow 0$ as $n \to \infty$ and, for each $n \in \mathbb{N}$, define $F_n: x \mapsto F(x) + \mathbb{E}_{\epsilon_n}$.

Let sequence $(x_n) \subset AC(I; \mathbb{R}^N)$ be such that, for each $n \in \mathbb{N}$, $x_n$ is an $F_n$-arc with $x_n(I) \subset K$. Then $(x_n)$ has a subsequence that converges uniformly to an $F$-arc $x \in AC(I; \mathbb{R}^N)$.

Next, we prove (by arguments similar to those used in establishing [18, Lemma 5 (p.8)], see also remarks on page 78 therein) a variant of the above, tailored to our later purposes.

**Proposition 4.2** Let $F \in \mathcal{U}$ and let $(s_n) \subset [a, b]$ be a convergent sequence with limit $s \in (a, b)$. If $(x_n) \subset AC([a, b]; \mathbb{R}^N)$ is a sequence of $F$-arcs and there exists $r > 0$ such that, for all $n \in \mathbb{N}$, $\|x_n(t)\| \leq r$ for all $t \in [a, s_n]$, then $(x_n)$ has a subsequence $(x_{s(n)})$ such that $(x_{s(n)}|_{[a, s]})$ converges to an $F$-arc $x \in AC([a, s]; \mathbb{R}^N)$.

**Proof** Let $(\delta_k) \subset (0, s - a)$ be a decreasing sequence with $\delta_k \downarrow 0$ as $k \to \infty$. Write $I_k := [a, s - \delta_k]$. By Proposition 4.1, the sequence $(x_n)$ has a subsequence, which we label $(x_{s(n)})$, such that $(x_{s(n)}|_{I_k})$ converges uniformly to an $F$-arc $x^1 \in AC(I_1; \mathbb{R}^N)$. Again by Proposition 4.1, the sequence $(x_{s(n)})$ has a subsequence, which we label $(x_{s(n)})$, such that $(x_{s(n)}|_{I_2})$ converges uniformly to an $F$-arc $x^2 \in AC(I_2; \mathbb{R}^N)$ (with $x^2|_{I_1} = x^1$).

By induction, we generate a sequence of subsequences of $(x_n)$,

$$(x_n) \supset (x_{s_1(n)}) \supset \cdots \supset (x_{s_{n}(n)}) \supset \cdots$$

such that, for all $k$, $(x_{s_k(n)}|_{I_k})$ converges to an $F$-arc $x^k \in AC(I_k; \mathbb{R}^N)$ with $x^k|_{I_{k-1}} = x^{k-1}$ for all $k \geq 2$. Therefore, the diagonal sequence of restrictions to $[a, s]$, that is, the sequence $(x_{s_n(n)}|_{[a, s]})$, converges to the $F$-arc $x: [a, s] \to \mathbb{R}^N$, defined by the property:

$$\forall k \in \mathbb{N} \quad x(t) = x^k(t) \quad \forall t \in I_k = [a, s - \delta_k].$$

By compactness of $F(\mathbb{R}^N)$, it follows that the bounded $F$-arc $x$ is uniformly continuous and so extends to an $F$-arc on the closed interval $[a, s]$ by defining $x(s) := \lim_{t \uparrow s} x(t)$.

4.1 The initial-value problem

Let $F \in \mathcal{U}$. For each $x^0 \in \mathbb{R}^N$, the initial-value problem

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x^0$$

(4)

has a solution and every solution can be extended to a maximal solution. By a solution, we mean an $F$-arc $x \in AC([0, \omega); \mathbb{R}^N)$, with $0 < \omega \leq \infty$ and $x(0) = x^0$; by a maximal solution, we mean a solution having no proper right extension which is also a solution. Moreover, if $x: [0, \omega) \to \mathbb{R}^N$ is maximal and $\omega < \infty$, then $\limsup_{t \uparrow \omega} \|x(t)\| = +\infty$. 


Proposition 4.3 Let non-empty $K \subset \mathbb{R}^N$ be compact. Assume that, for each $x^0 \in K$, every maximal solution of (4) has interval of existence $\mathbb{R}_+$. For $T > 0$, define
\[
\Sigma_T(K) := \bigcup_{t \in [0, T]} \{x(t) \mid x \in AC([0, T]; \mathbb{R}^N) \}
\]
and write $\Sigma_\infty(K) := \bigcup_{T > 0} \Sigma_T(K)$.

(a) For all $T > 0$, the set $\Sigma_T(K)$ is compact.

(b) Let non-empty $C_1, C_2 \subset \mathbb{R}^N$ be compact, with $C_1 \subset C_2 \subset K$ and $C_1 \cap \partial C_2 = \emptyset = K \cap \partial C_2$. Assume that, for every maximal solution $x$ of (4) with $x^0 \in K$, $d_{C_1}(x(t)) \to 0$ as $t \to \infty$. Then there exists $T > 0$ such that $\Sigma_T(K) = \Sigma_\infty(K)$ and, for all $x^0 \in \Sigma_\infty(K)$, every maximal solution $x$ of (4) has interval of existence $\mathbb{R}_+$ and has the properties:
\begin{enumerate}
\item $x(\mathbb{R}_+) \subset \Sigma_\infty(K)$;
\item $x(t) \in C_2$ for some $t \in [0, T]$.
\end{enumerate}

Proof (a) Let $T > 0$ be arbitrary. Seeking a contradiction, suppose that $\Sigma_T(K)$ is unbounded. Then there exist a constant $\delta > 0$, a sequence $(t_n) \subset [0, T]$ and a sequence $(x_n)$ of maximal solutions of (4) such that
\[
x_n(0) \in K \quad \text{and} \quad \|x_n(t_n)\| > (n + 1)\delta \quad \forall n \in \mathbb{N}.
\]
By continuity of the solutions, it follows that, for each $n \in \mathbb{N}$, there exist $s_n^k$, $k = 1, \ldots, n$, such that
\[
\|x_n(s_n^k)\| = (k + 1)\delta \quad \text{and} \quad \|x_n(t)\| < (k + 1)\delta \quad \forall t \in [0, s_n^k)
\]
and $s_1^1 < s_2^2 < \cdots < s_n^n$ for all $n \geq 2$.

From $(s_n^k)$, extract a convergent subsequence $(s_{\sigma_1(n)}^1)$ with limit $s^1 \in [0, T]$. By compactness of $F(\mathbb{R}_{2\delta}(0))$, $s^1 > 0$. Write $I_1 := [0, s^1]$. By Proposition 4.2, and passing to a subsequence if necessary, we may assume that $(x_{\sigma_1(n)}|_{I_1)}$ converges uniformly to an $F$-arc $x^1 \in AC(I_1; \mathbb{R}^N)$; moreover, by (5), $\|x^1(s^1)\| = 2\delta$. From $(s_{\sigma_2(n)}^2)$, extract a subsequence $(s_{\sigma_2(n)}^2)$ with limit $s^2 \in [0, T]$. By compactness of $F(\mathbb{R}_{3\delta}(0))$, $s^2 > s^1$. Write $I_2 := [0, s^2]$. By Proposition 4.2, and passing to a subsequence if necessary, we may assume that $(x_{\sigma_2(n)}|_{I_2})$ converges uniformly to an $F$-arc $x^2 \in AC(I_2; \mathbb{R}^N)$ with $x^2|_{I_1} = x^1$; moreover, by (5), $\|x^2(s^2)\| = 3\delta$. By induction, we generate a strictly increasing sequence $(s^k) \subset [0, T)$, with limit $s \in [0, T]$, and a sequence of subsequences of $(x_n)$,
\[
(x_n) \supset (x_{\sigma_1(n)}) \supset \cdots \supset (x_{\sigma_k(n)}) \supset \cdots
\]
such that the diagonal sequence of restricted functions $(x_{\sigma_k(n)}|_I)$, where $I := [0, s)$, converges to the $F$-arc $x \in AC(I; \mathbb{R}^N)$ defined by the property that, for each $k \in \mathbb{N}$,
\[
x(t) = x^k(t) \quad \forall t \in I_k := [0, s^k].
\]
Clearly, $x(0) \in K$. Furthermore, $\|x(s^k)\| = (k + 1)\delta$ for all $k \in \mathbb{N}$ and so $x$ has no proper right extension that is also an $F$-arc. This contradicts the hypothesis that all
maximal solutions of (4), with \( x^0 \in K \), have interval of existence \( \mathbb{R}_+ \). Therefore, \( \Sigma_T(K) \) is bounded.

Let \( (y_n) \subset \Sigma_T(K) \) be a convergent sequence with limit \( y \). Then \( y_n = x_n(t_n) \) for some sequence \( (t_n) \subset [0,T] \) and some sequence of \( F \)-arcs \( (x_n) \subset AC([0,T]; \mathbb{R}^N) \) with \( x_n(0) \in K \) for all \( n \). Without loss of generality, we may assume that \( (t_n) \) is convergent, with limit \( t \in [0,T] \). By boundedness of \( \Sigma_T(K) \), there exists compact \( C \) such that \( x_n([0,T]) \subset C \) for all \( n \). By Proposition 4.1, passing to a subsequence if necessary, we may assume that \( (x_n) \) converges uniformly to an \( F \)-arc \( x \in AC([0,T]; \mathbb{R}^N) \), with \( x(0) \in K \). Therefore,

\[
y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} x_n(t_n) = x(t) \in \Sigma_T(K),
\]

and so \( \Sigma_T(K) \) is closed.

(b) It suffices to show that there exists \( T > 0 \) such that, for every maximal solution \( x \) of (4), with \( x^0 \in K \), \( x(t) \in C_2 \) for some \( t \in [0,T] \) (in which case, \( \Sigma_T(K) = \Sigma_\infty(K) \)). Seeking a contradiction, suppose that no such \( T \) exists. Then there is a sequence \( (x_n) \subset AC(\mathbb{R}_+; \mathbb{R}^N) \) such that, for each \( n \in \mathbb{N} \), \( x_n(0) \in K \) and \( d_{C_2}(x_n(t)) > 0 \) for all \( t \in I_n := [0,n] \). By part (a) above, for each \( k \in \mathbb{N} \), the sequence \( (x_n|_{I_k}) \) is bounded. Therefore, repeated application of Proposition 4.1 yields a sequence of subsequences \( (x_n) \supset (x_{\sigma_1(n)}) \supset (x_{\sigma_2(n)}) \cdots \) such that, for each \( k \in \mathbb{N} \), the sequence \( (x_{\sigma_k(n)}|_{I_k}) \) converges uniformly to an \( F \)-arc \( x^k \in AC(I_k; \mathbb{R}^N) \) with \( d_{C_2}(x^k(t)) \geq 0 \) for all \( t \in I_k \). It follows that the diagonal sequence \( (x_{\sigma_1(n)}) \) converges to the \( F \)-arc \( x \in AC(\mathbb{R}_+; \mathbb{R}^N) \) defined by the property that, for each \( k \in \mathbb{N} \), \( x(t) = x^k(t) \) for all \( t \in I_k \). Therefore, \( d_{C_2}(x(t)) \geq 0 \) for all \( t \in \mathbb{R}_+ \), which contradicts the hypothesis that every maximal solution approaches \( C_1 \subset C_2 \) (recall that \( C_1 \cap \partial C_2 = \emptyset \)).

**Remark 4.1** Proposition 4.3(a) is closely akin to [18, Theorem 3 (p.79)]. Proposition 4.3(b-i) is essentially an assertion that \( \Sigma_\infty(K) \) is compact and is an invariant set for (4) in the sense that, for each \( x^0 \in \Sigma_\infty(K) \), every maximal solution of (4) has trajectory in \( \Sigma_\infty(K) \). A similar observation occurs in the proof of [7, Theorem 11].

### 4.2 Persistence of the BKZ property

The following is essentially Theorem 1 of [8].

**Theorem 4.1** Let \( F \in \mathcal{U} \). If there exist \( 0 < \tau < \delta < \rho \) and \( T > 0 \) such that

\[
\|x^0\| \leq \delta \quad \Rightarrow \quad \left\{ \begin{array}{ll} 
\|x(t)\| \leq \rho & \forall t \in [0,T] \\
\|x(t)\| \leq \tau & \forall t \in [T,2T]
\end{array} \right.
\]

for every maximal solution \( x \) of (4), then \( F \) has the BKZ property.

In view of Lemma 3.1, to prove this result it suffices to show that \( \deg(F, \mathbb{B}_3, 0) \neq 0 \). In the Appendix, we provide a proof which incorporates minor corrections to the proof in [8].

In what follows, several specific consequences of the above result are highlighted: simply stated, the first of these (Theorem 4.2) asserts that, if there exists a compact set that attracts all maximal solutions of (4), then \( F \) has the BKZ property.
A non-empty set $C \subset \mathbb{R}^N$ is said to be attractive for (4) if there exists an open neighbourhood $\mathcal{N}$ of $C$ (that is, an open set containing the closure of $C$) with the property that, for each $x^0 \in \mathcal{N}$, every maximal solution $x: [0, \omega) \to \mathbb{R}^N$ of (4) is such that $d_C(x(t)) \to 0$ as $t \uparrow \omega$ (if $C$ is compact, then $\omega = \infty$): $C$ is globally attractive if the latter property holds with $\mathcal{N} = \mathbb{R}^N$. Non-empty $C$ is said to be stable for (4) if, for each open neighbourhood $\mathcal{N}_1$ of $C$, there is an open neighbourhood $\mathcal{N}_2$ of $C$ such that, for each $x^0 \in \mathcal{N}_2$, every maximal solution of (4) has trajectory in $\mathcal{N}_1$.

**Theorem 4.2** Let $F \in \mathcal{U}$. Let $C \subset \mathbb{R}^N$ be non-empty and compact. If $C$ is globally attractive for (4), then $F$ has the BKZ property.

**Proof** By global attractivity of compact $C$, every maximal solution of (4) has interval of existence $\mathbb{R}_+$. Fix $r > 0$ such that $\overline{B}_r \supset C$. By Proposition 4.3, the set $\Sigma_\infty(\overline{B}_3r)$ is compact and positively invariant.

Let $\tau > 3r$ be sufficiently large so that $\Sigma_\infty(\overline{B}_3r) \subset \overline{B}_\tau$ and choose $\delta > \tau$. By Proposition 4.3(b), there exists $T > 0$ such that, for every $F$-arc $x \in AC(\mathbb{R}_+: \mathbb{R}^N)$ with $\|x(0)\| \leq \delta$, $\|x(t)\| \leq 3r$ for some $t \in [0, T]$. Since $\overline{B}_3r \subset \Sigma_\infty(\overline{B}_3r)$, it follows that, for each $x^0$,

$$\|x^0\| \leq \delta \quad \Rightarrow \quad x(t) \in \Sigma_\infty(\overline{B}_3r) \quad \text{for some} \quad t \in [0, T]$$

for every maximal solution of (4). Therefore, by (positive) invariance of $\Sigma_\infty(\overline{B}_3r) \subset \overline{B}_\tau$,

$$\|x^0\| \leq \delta \quad \Rightarrow \quad \|x(t)\| \leq \tau \quad \forall t \in [T, \infty)$$

for every maximal solution of (4).

By Proposition 4.3(a), there exists $\rho > \delta$ such that

$$\|x^0\| \leq \delta \quad \Rightarrow \quad \|x(t)\| \leq \rho \quad \forall t \in [0, T].$$

Therefore, the hypotheses of Theorem 4.1 hold and so the result follows.

Next, we highlight a further consequence of the above theorem which, for example, implies that, if (1) generates a global semiflow and is $L^p$ stable in the sense that all solutions are of class $L^p$ for some $1 \leq p < \infty$, then $f$ has the BKZ property.

**Corollary 4.1** Let $F \in \mathcal{U}$. Let $g: \mathbb{R}^N \to \mathbb{R}_+$ be lower semicontinuous with the properties:

(a) $C := g^{-1}(0)$ is compact;

(b) $\inf_{z \in K} g(z) > 0$ for any closed set $K \subset \mathbb{R}^N$ with $K \cap C = \emptyset$.

If, for each $x^0 \in \mathbb{R}^N$, every maximal solution of (4) has interval of existence $\mathbb{R}_+$ and

$$\int_0^\infty g(x(t)) \, dt < \infty,$$

then $F$ has the BKZ property.

**Proof** By [19, Theorem 10 (i)], the compact set $C = g^{-1}(0)$ is globally attractive for (4) and the result follows by Theorem 4.2.

In Theorem 4.2, in order to conclude that $F$ has the BKZ property, hypotheses of a global nature were imposed (global in the sense that, for each $x^0 \in \mathbb{R}^N$, every maximal solution was posited to approach $C$). The following theorem imposes hypotheses of a local nature under which the BKZ property again persists: in particular, if there exists a closed ball that is locally asymptotically stable for (4), then $F$ has the BKZ property.
**Theorem 4.3** If there exists a closed ball $B_r(z) = B$ which is both stable and attractive for (4), then $F$ has the BKZ property.

**Proof** Without loss of generality, we may assume $z = 0$ and so $B = B_r \equiv B_r(0)$. By stability and attractivity of compact $B$, there exist $\alpha, \beta \in \mathbb{R}^+$ such that, for all $x^0 \in \mathbb{R}^N$,

$$d_B(x^0) \leq \alpha \implies \begin{cases} d_B(x(t)) \leq \beta & \forall t \in \mathbb{R}^+ \\ d_B(x(t)) \to 0 & \text{as } t \to \infty \end{cases}$$

for every maximal solution of (4). Let $\gamma \in (0, \alpha)$ be arbitrary. By stability of $B$, there exists $\mu \in (0, \gamma)$ such that, for all $x^0$,

$$d_B(x^0) \leq \mu \implies d_B(x(t)) \leq \gamma \forall t \in \mathbb{R}^+$$

(6)

for every maximal solution of (4). By Proposition 4.3(b), there exists $T > 0$ such that, for all $x^0$,

$$d_B(x^0) \leq \alpha \implies d_B(x(t)) \leq \mu \text{ for some } t \in [0, T]$$

which, together with (6), yields

$$d_B(x^0) \leq \alpha \implies d_B(x(t)) \leq \gamma \forall t \geq T$$

for every maximal solution $x$ of (4). We may now conclude that the hypotheses of Theorem 4.1 hold (with $\tau = -r, \delta = \alpha + r$ and $\rho = \beta + r$) and the proof is complete.

5 Feedback Control

We now turn to the main concern of the paper, namely, the consequences of the above results in a context of feedback control systems.

Let $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ be continuous and consider the controlled system

$$\dot{x} = f(x, u).$$

(7)

Henceforth, we assume that $f$ has the property that, for every non-empty convex set $C \subset \mathbb{R}^M$, the set $f(x, C) \subset \mathbb{R}^N$ is convex for all $x \in \mathbb{R}^N$.

As admissible feedback controls for (7), we take the class $\mathcal{K}$ of upper semicontinuous maps $x \mapsto k(x) \subset \mathbb{R}^M$ on $\mathbb{R}^N$, with non-empty convex and compact values. Therefore, for every feedback $k \in \mathcal{K}$, the map $F_k : x \mapsto f(x, k(x))$ is of class $\mathcal{U}$.

5.1 Persistence of the BKZ property in feedback systems

For system (7), a feedback $k \in \mathcal{K}$ is said to render a compact set $C \subset \mathbb{R}^N$ stable (respectively, attractive) if $C$ is stable (respectively, attractive) for (4) with $F = F_k$.

The following theorem and corollary are immediate consequences of Theorem 4.2 and Corollary 4.1.

**Theorem 5.1** Let $k \in \mathcal{K}$ and let $C \subset \mathbb{R}^N$ be non-empty and compact. If either of the following holds, then $f$ has the BKZ property:

(i) $k$ renders $C$ globally attractive for (4);
(ii) $k$ renders some closed ball $B$ stable and attractive for (4).
Corollary 5.1 Let \( k \in K \) and let \( g : \mathbb{R}^N \to \mathbb{R}_+ \) be as in Corollary 4.1. If, for each \( x^0 \in \mathbb{R}^N \), every maximal solution of (4) with \( F = F_k \) has interval of existence \( \mathbb{R}_+ \) and \( g \circ x \in L^1(\mathbb{R}_+) \), then \( f \) has the BKZ property.

References


Appendix: Proof of Theorem 4.1

Let \( D := \overline{B}_\rho \) and let \( \hat{F} \in \mathcal{U}(D) \) denote the restriction of \( F \in \mathcal{U} \) to \( D \).

Observe that \( 0 \notin \hat{F}(\partial B_\delta) \) (otherwise, there exists a constant solution \( t \mapsto x^0 \) of (4) with \( \|x^0\| = \delta \), contradicting the hypotheses). Therefore \( \deg(\hat{F}, B_\delta, 0) \) is well-defined and, in view of Lemma 3.1, to complete the proof it suffices to show that \( \deg(\hat{F}, B_\delta, 0) \neq 0 \).
By Proposition 2.1(ii) and property P1 of degree, there exists a sequence \((f_n)\) of locally Lipschitz functions \(D \to \mathbb{R}^N\) such that:

\[
\deg(\hat{F}, \mathbb{B}_\delta, 0) = \deg_B(f_n, \mathbb{B}_\delta, 0) \quad \forall n;
\]

\[
d(\text{graph}(f_n), \text{graph}(\hat{F})) \to 0 \quad \text{as} \quad n \to \infty.
\]  

By compactness of \(\hat{F}(D)\), the functions \(f_n\) are bounded and so, for each \(n\), the equation \(\hat{x} = f_n(x)\) generates a semiflow \(\varphi_n : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N\).

Write \(I := [0, 2T]\) and \(X := C(I; \mathbb{R}^N)\) (with the uniform norm). On \(\mathbb{B}_\delta\) define

\[
\mathcal{F} : x^0 \mapsto \{x \in X \mid x \text{ an } \hat{F}\text{-arc with } x(0) = x^0\}
\]

with \(\text{graph}(\mathcal{F}) := \{(x^0, x) \mid x^0 \in \mathbb{B}_\delta, x \in \mathcal{F}(x^0)\}\). For each \(n\), define \(\phi_n : \mathbb{B}_\delta \to X\) by

\[
(\phi_n(x^0))(t) := \varphi_n(t, x^0) \quad \forall t \in I.
\]

Fix \(\epsilon\) such that \(0 < \epsilon < \delta - \tau\). We claim that

\[
d(\text{graph}(\phi_m), \text{graph}(\mathcal{F})) < \epsilon \quad \text{for some} \quad m \in \mathbb{N}.
\]

Suppose otherwise. Then there exists a sequence \((x^0_n) \subset \mathbb{B}_\delta\) such that

\[
d(\text{graph}(\phi_n), \text{graph}(\mathcal{F})) \geq \epsilon \quad \forall n.
\]

By Proposition 4.1, we may assume (without loss of generality) that \((\phi(x^0_n)) \subset X\) converges uniformly to an \(\hat{F}\)-arc \(x \in AC(I; \mathbb{R}^N)\) with \(x(0) \in \mathbb{B}_\delta\) (and so \((x(0), x) \in \text{graph}(\mathcal{F}))\), which contradicts (10). Therefore, (9) is true.

Let \(x^0 \in \mathbb{B}_\delta\) be arbitrary. By (9), there exists \(y^0 \in \mathbb{B}_\delta\), with \(\|x^0 - y^0\| < \epsilon\), and \(y \in \mathcal{F}(y^0)\) such that \(\|\varphi_m(t, x^0) - y(t)\| < \epsilon\) for all \(t \in I\). Since the set \(\{y(t) \mid y \in \mathcal{F}(\mathbb{B}_\delta)\}\) lies in the ball \(\mathbb{B}_\tau\) for all \(t \in [T, 2T]\), we may conclude:

\[
\text{for all} \quad x^0 \in \mathbb{B}_\delta, \quad \varphi_m(t, x^0) \in \mathbb{B}_\delta \quad \text{for all} \quad t \in [T, 2T].
\]  

Define continuous \(h : [0, 1] \times \mathbb{B}_\delta \to \mathbb{R}^N\) by

\[
h(s, x^0) := \begin{cases} f_m(x^0), & s = 0 \\ \frac{1}{T} ([\phi_m(x^0)](sT) - x^0), & 0 < s \leq 1. \end{cases}
\]

We conclude that \(h(s, x^0) \neq 0\) for all \((s, x^0) \in [0, 1] \times \partial \mathbb{B}_\delta\) by the following argument. Suppose \(h(0, x^0) = f_m(x^0) = 0\) for some \(x^0 \in \partial \mathbb{B}_\delta\). Then, \(\varphi_m(t, x^0) = x^0 \in \partial \mathbb{B}_\delta\) for all \(t \in I\), which contradicts (11). Now suppose \(h(s, x^0) = 0\) for some \((s, x^0) \in (0, 1] \times \partial \mathbb{B}_\delta\). Then \(\varphi_m(nsT, x^0) = x^0 \in \partial \mathbb{B}_\delta\) for all \(n \in \mathbb{N}\) with \(ns \leq 2\). In particular, there exists \(n \in \mathbb{N}\) such that \(1 \leq ns \leq 2\) and \(\varphi_m(nsT, x^0) = x^0 \in \partial \mathbb{B}_\delta\). This contradicts (11).

Therefore, by (8) and the homotopic invariance property of the Brouwer degree,

\[
\deg(\hat{F}, \mathbb{B}_\delta, 0) = \deg_B(f_m, \mathbb{B}_\delta, 0) = \deg_B(h(0, \cdot), \mathbb{B}_\delta, 0)
\]

\[
= \deg_B(h(1, \cdot), \mathbb{B}_\delta, 0) = \deg_B(g_m, \mathbb{B}_\delta, 0),
\]
where, for notational convenience, $g_m$ denotes the function

$$g_m: x^0 \mapsto [(\phi_m(x^0))(T) - x^0]/T.$$ 

Now consider the continuous map

$$h_0: [0, 1] \times \overline{B}_\delta, \quad (s, x^0) \mapsto (1 - s)g_m(x^0) - sx^0.$$ 

Noting that $h_0$ is a homotopic connection of the function $g_m$ and the odd map $o: x^0 \mapsto -x^0$ and $h_0(s, x^0) \neq 0$ for all $(s, x^0) \in [0, 1] \times \partial \overline{B}_\delta$ by properties of the Brouwer degree, we may now conclude that

$$\deg(\hat{F}, \overline{B}_\delta, 0) = \deg_B(o, \overline{B}_\delta, 0) \neq 0.$$ 

This completes the proof.