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Robust Stability: Three Approaches for Discrete-Time Systems

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Abstract: The paper presents the results of estimating the robust stability bounds for discrete system in terms of three approaches based on scalar, vector and hierarchical Lyapunov functions. It is shown that the hierarchical Lyapunov function allows one to obtain the most wide bounds for the uncertain matrix in the investigation of discrete system. A numerical example is cited which illustrates the application of the proposed approach.

Keywords: *Discrete-time system; robust bounds; scalar; vector and hierarchical Lyapunov functions.*

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1 Introduction

In the present paper we consider an uncertain discrete-time system

$$x(\tau+1) = A\,x(\tau) + f(x(\tau),\alpha), \tag{1.1}$$

where $x \in R^n$, $\tau \in \mathcal{N} = \{t_0 + k, k = 0,1,2,\dots\}$, $t_0 \in R$, A is a constant $n \times n$ matrix, $f\colon R^n \times S \to R^n$, $\alpha \in S \subseteq R^d$, $d \geqslant 1$ is a compact set. Under specific conditions (we don't cite them here) dynamics of the system (1.1) are topologically equivalent with dynamics of the system

$$x(\tau+1) = (A+E)\,x(\tau), \tag{1.2}$$

where A is the same matrix, as in system (1.1), and E is an uncertain $n \times n$ matrix, about which it is known that it lies in some compact set $S_1 \subset R^{n\times n}$. Further we will investigate the system (1.2).

Our purpose is to compare the results of estimating the robust bounds of discrete system obtained in terms of three approaches involving scalar, vector and hierarchical

Lyapunov function. In the paper it is shown that the hierarchical Lyapunov function provides more wide bounds for estimation of the uncertain matrix.

2 Scalar Approach

We assume that for the matrix A the condition $|\sigma_i(A)| < 1$ is realized for all $i = 1, 2, \dots, n$. In this case the Lyapunov equation

$$A^{\mathrm{T}} P A - P = -G \tag{2.1}$$

has a unique solution $P \in R^{n \times n}$ for arbitrary symmetric and positive definite matrix $G \in R^{n \times n}$. Moreover the matrix P is symmetric and positive definite. According to the results of paper [6], we apply the function

$$v(x) = (x^{\mathrm{T}} P x)^{\frac{1}{2}}. \tag{2.2}$$

in robustness analysis of the system (1.2). Let us denote by $\sigma_m(P)$, $\sigma_M(P)$ the maximum and minimum eigenvalues of the matrix P.

Following the paper [6] we get the assertion.

Theorem 2.1 *Let the nominal system*

$$x(\tau + 1) = A\, x(\tau)$$

be asymptotically stable. If

$$\|E\| < \mu(G), \tag{2.3}$$

where

$$\mu(G) = \frac{\sigma_m(G)}{\sigma_M^{\frac{1}{2}}(P - G)\sigma_M^{\frac{1}{2}}(P) + \sigma_M(P)},$$

then the uncertain system (1.2) is asymptotically stable.

Here $\|E\| = \sup\limits_{\|x\| \leqslant 1} \|Ex\|$, $\|x\| = (x^{\mathrm{T}} x)^{\frac{1}{2}}$ is the Euclidean norm of vector x.

It is known [6], that $\mu(G)$ takes the largest value, if $G = I$ in (2.1). The expression (2.3) is a robust bound for the system (1.2), obtained in the framework of scalar approach with the function (2.2).

3 Vector Approach

We decompose system (1.2) into two interconnected subsystems

$$\hat{S}_i\colon \quad x_i(\tau + 1) = (A_i + E_i)\, x_i(\tau) + (B_j + U_j)\, x_j(\tau), \quad i, j = 1, 2 \quad \text{and} \quad i \neq j. \tag{3.1}$$

Here $x_i \in R^{n_i}$, A_i and B_i are submatrices of the known matrix

$$A = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix}, \tag{3.2}$$

E_i and U_i are submatrices of the uncertain matrix

$$E = \begin{pmatrix} E_1 & U_1 \\ U_2 & E_2 \end{pmatrix}, \tag{3.3}$$

where $B_1, U_1 \in R^{n_1 \times n_2}$, $B_2, U_2 \in R^{n_2 \times n_1}$, and $A_i, E_i \in R^{n_i \times n_i}$, $i = 1, 2$.

Assumption 3.1 *We assume that:*

(1) *the nominal subsystems*

$$x_i(\tau+1) = A_i x_i(\tau) \tag{3.4}$$

are asymptotically stable, i.e. there exist unique symmetric and positive definite matrices $P_i \in R^{n_i \times n_i}$, *which satisfy the Lyapunov matrix equations*

$$A_i^{\mathrm{T}} P_i A_i - P_i = -G_i, \quad i = 1, 2, \tag{3.5}$$

where G_i *are arbitrary symmetric and positive definite matrices;*

(2) *there exists a constant* $\gamma \in (0,1)$ *such that*

$$\|B_1\| \, \|B_2\| < \gamma^2 \mu_1 \mu_2$$

where $\mu_i = (\sigma_M^{\frac{1}{2}}(P_i - I_i)\sigma_M^{\frac{1}{2}}(P_i) + \sigma_M(P_i))^{-1}$, P_i *are solutions of the Lyapunov matrix equations (3.5) for the matrices* $G_i = I_{n_i}$, I_{n_i} *are* $n_i \times n_i$ *identity matrices,* $i = 1, 2$.

We define the constants

$$\begin{aligned}
&a = \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2), \quad b = \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2)(\|B_1\| + \|B_2\|),\\
&\mu_i = (\sigma_M^{\frac{1}{2}}(P_i - I_i)\sigma_M^{\frac{1}{2}}(P_i) + \sigma_M(P_i))^{-1}, \quad i = 1, 2,\\
&\alpha_i = \sigma_M^{\frac{1}{2}}(P_i)\mu_i = (\sigma_M^{\frac{1}{2}}(P_i - I_i) + \sigma_M^{\frac{1}{2}}(P_i))^{-1}, \quad i = 1, 2,\\
&c = \gamma^2 \alpha_1 \alpha_2 - \sigma_M^{\frac{1}{2}}(P_1)\sigma_M^{\frac{1}{2}}(P_2)\|B_1\| \, \|B_2\|,\\
&\epsilon = \frac{1}{2a}((b^2 + 4ac)^{\frac{1}{2}} - b),
\end{aligned}$$

where P_i are solutions of the Lyapunov matrix equations (3.5) for the matrices $G_i = I_{n_i}$, $i = 1, 2$.

Theorem 3.1 *Assume that for the uncertain system (1.2) the decomposition (3.1) – (3.3) takes place and all conditions of Assumption 3.1 are satisfied. If the submatrices* E_i *and* U_i *satisfy the inequalities*

$$\|E_i\| \leqslant (1-\gamma)\mu_i, \quad \|U_i\| < \epsilon, \quad i = 1, 2, \tag{3.6}$$

then the equilibrium $x = 0$ *of (1.2) is asymptotically stable.*

Proof For the nominal subsystems (3.4) by (3.5) we construct the normlike functions

$$v_i(x_i) = (x_i^{\mathrm{T}} P_i x_i)^{\frac{1}{2}}, \quad i = 1, 2, \tag{3.7}$$

and the scalar function

$$v(x) = d_1 v_1(x_1) + d_2 v_2(x_2), \tag{3.8}$$

where d_1, d_2 are some positive constants.

For the first forward differences $\Delta v_i(x_i)$ of the functions (3.7) with respect to τ along the solutions of (3.1) we have the estimates:

$$\Delta v_i(x_i)\big|_{\hat S_i} = v_i(A_ix_i) - v_i(x_i) + v_i((A_i+E_i)x_i) - v_i(A_ix_i) + v_i((A_i+E_i)x_i$$
$$+ (B_i+U_i)x_j) - v_i((A_i+E_i)x_i) \leqslant (x_i^{\mathrm T}A_i^{\mathrm T}P_iA_ix_i)^{\frac12} - (x_i^{\mathrm T}P_ix_i)^{\frac12} + \sigma_M^{\frac12}(P_i)\|E_ix_i\|$$
$$+ \sigma_M^{\frac12}(P_i)\|(B_i+U_i)x_j\| \leqslant \frac{x_i^{\mathrm T}A_i^{\mathrm T}P_iA_ix_i - x_i^{\mathrm T}P_ix_i}{(x_i^{\mathrm T}A_i^{\mathrm T}P_iA_ix_i)^{\frac12} + (x_i^{\mathrm T}P_ix_i)^{\frac12}} + \sigma_M^{\frac12}(P_i)\|E_i\|\,\|x_i\|$$
$$+ \sigma_M^{\frac12}(P_i)(\|B_i\| + \|U_i\|)\|x_j\| \leqslant -(\alpha_i - \sigma_M^{\frac12}(P_i)\|E_i\|)\|x_i\| + \sigma_M^{\frac12}(P_i)(\|B_i\| + \|U_i\|)\|x_j\|,$$

$i,\, j = 1,2,\ i \neq j$. Here we use the known inequality [6]

$$(p^{\mathrm T}Pp)^{\frac12} - (q^{\mathrm T}Pq)^{\frac12} \leqslant \sigma_M^{\frac12}(P)\|p-q\|$$

for all $p,\, q \in R^n,\ P \in R^{n\times n}$ is a symmetric and positive definite matrix. From here we arrive to the following inequality

$$\Delta v(x)\big|_{(\hat S_1,\hat S_2)} \leqslant d_1\Delta v_1(x_1)\big|_{\hat S_1} + d_2\Delta v_2(x_2)\big|_{\hat S_2} \leqslant -\tilde d^{\mathrm T}Wz, \tag{3.9}$$

where $\tilde d = (d_1,d_2)^{\mathrm T},\ z = (\|x_1\|,\|x_2\|)^{\mathrm T},\ W = (w_{ij})$ is a 2×2 matrix with the elements

$$w_{ij} = \begin{cases} \alpha_i - \sigma_M^{\frac12}(P_i)\,\|E_i\| & \text{if} \quad i = j, \\ -\sigma_M^{\frac12}(P_i)(\|B_i\| + \|U_i\|) & \text{if} \quad i \neq j. \end{cases}$$

As all conditions of Theorem 3.1 are satisfied, it is not difficult to verify that the matrix W is the M-matrix [8]. Really

$$w_{11}w_{22} - w_{12}w_{21} = [\alpha_1 - \sigma_M^{\frac12}(P_1)\|E_1\|][\alpha_2 - \sigma_M^{\frac12}(P_2)\|E_2\|] - \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)$$
$$\times (\|B_1\| + \|U_1\|)(\|B_2\| + \|U_2\|) > [\alpha_1 - \sigma_M^{\frac12}(P_1)(1-\gamma)\mu_1][\alpha_2 - \sigma_M^{\frac12}(P_2)(1-\gamma)\mu_2]$$
$$- \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)(\|B_1\| + \epsilon)(\|B_2\| + \epsilon)$$
$$= \gamma^2\alpha_1\alpha_2 - \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)(\|B_1\| + \epsilon)(\|B_2\| + \epsilon)$$
$$= -\sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)\epsilon^2 - \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)(\|B_1\| + \|B_2\|)\epsilon + \gamma^2\alpha_1\alpha_2 - \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}$$
$$\times (P_2)\|B_1\|\,\|B_2\| = -a\epsilon^2 - b\epsilon + c.$$

By condition (2) of Assumption 2.1

$$c = \gamma^2\alpha_1\alpha_2 - \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)\|B_1\|\,\|B_2\| = \sigma_M^{\frac12}(P_1)\sigma_M^{\frac12}(P_2)\big[\gamma^2\mu_1\mu_2 - \|B_1\|\,\|B_2\|\big] > 0$$

and therefore $-a\epsilon^2 - b\epsilon + c = 0$, and $w_{11}w_{22} - w_{12}w_{21} > 0$.

It is clear that the function (3.8) is positive definite and it's first forward difference (3.9) is negative definite. These conditions are sufficient [9] for the asymptotic stability of the equilibrium $x = 0$ of (1.2).

The proof of Theorem 3.1 is complete.

Thus the inequalities (3.6) are the robust bounds for the system (1.2), obtained in terms of the vector approach.

4 Hierarchical Approach

As is known [7], the essence of this method is as follows: beginning from the constructing an auxiliary Lyapunov function, we take into account a hierarchical structure of the system (1.2) or realize a multilevel decomposition of the initial system. Further the second approach is applied precisely.

We decompose each subsystems (3.1) into two interconnected components

$$\widetilde{C}_{ij}: \quad x_{ij}(\tau+1) = (A_{ij}+E_{ij})\,x_{ij}(\tau) + (B_{ij}+U_{ij})\,x_{ik}(\tau), \quad i,\,j,\,k = 1,2, \quad j \neq k, \tag{4.1}$$

where $x_{ij} \in R^{n_{ij}}$, $R^{n_i} = R^{n_{i1}} \times R^{n_{i2}}$, $A_{ij}, E_{ij} \in R^{n_{ij}\times n_{ij}}$, $B_{i1}, U_{i1} \in R^{n_{i1}\times n_{i2}}$, and $B_{i2}, U_{i2} \in R^{n_{i2}\times n_{i1}}$,

$$A_i = \begin{pmatrix} A_{i1} & B_{i1} \\ B_{i2} & A_{i2} \end{pmatrix}, \qquad E_i = \begin{pmatrix} E_{i1} & U_{i1} \\ U_{i2} & E_{i2} \end{pmatrix}.$$

Assume that the matrices B_i and U_i have a block structure:

$$B_i = \begin{pmatrix} M_{11}^{(i)} & M_{12}^{(i)} \\ M_{12}^{(i)} & M_{22}^{(i)} \end{pmatrix}, \qquad U_i = \begin{pmatrix} F_{11}^{(i)} & F_{12}^{(i)} \\ F_{12}^{(i)} & F_{22}^{(i)} \end{pmatrix},$$

where $M_{jk}^{(i)}, F_{jk}^{(i)} \in R^{n_{ij}\times n_{lk}}$, $i,\,j,\,k,\,l = 1,2,\ i \neq l$.

We extract from (4.1) the independent components

$$C_{ij}: \quad x_{ij}(\tau+1) = (A_{ij}+E_{ij})\,x_{ij}(\tau), \quad i,\,j = 1,2,$$

with the same designations of variables as in system (4.1).

In order to state the robust bounds we require the following assumptions.

Assumption 4.1 *The nominal components*

$$x_{ij}(\tau+1) = A_{ij}x_{ij}(\tau), \quad i,\,j = 1,2,$$

are asymptotically stable, i.e. there exist unique symmetric and positive definite matrices P_{ij}, which satisfy the Lyapunov matrix equations

$$A_{ij}^{\mathrm{T}}P_{ij}A_{ij} - P_{ij} = -G_{ij}, \quad i,\,j = 1,2, \tag{4.2}$$

where G_{ij} are arbitrary symmetric and positive definite matrices.

Let P_{ij} be solutions of the Lyapunov matrix equations (4.2) for the identity matrices $G_{ij} = I_{ij}$. We define the constants

$$\begin{aligned}
&\alpha_{ij} = \sigma_M^{\frac12}(P_{ij})\mu_{ij} = (\sigma_M^{\frac12}(P_{ij}-I_{ij}) + \sigma_M^{\frac12}(P_{ij}))^{-1},\\
&\mu_{ij} = (\sigma_M^{\frac12}(P_{ij}-I_{ij})\sigma_M^{\frac12}(P_{ij}) + \sigma_M(P_{ij}))^{-1},\\
&\epsilon_i = \frac{1}{2a_i}((b_i^2+4a_ic_i)^{\frac12} - b_i),\\
&a_i = \sigma_M^{\frac12}(P_{i1})\sigma_M^{\frac12}(P_{i2}),\\
&b_i = \sigma_M^{\frac12}(P_{i1})\sigma_M^{\frac12}(P_{i2})(\|B_{i1}\| + \|B_{i2}\|),\\
&c_i = \gamma_i^2\alpha_{i1}\alpha_{i2} - \sigma_M^{\frac12}(P_{i1})\sigma_M^{\frac12}(P_{i2})\|B_{i1}\|\,\|B_{i2}\|, \quad i,\,j = 1,2.
\end{aligned}$$

Assumption 4.2 *There exist constants $\gamma_i \in (0,1)$ such that*

$$\|B_{i1}\|\,\|B_{i2}\| < \gamma_i^2 \mu_{i1}\mu_{i2}, \quad i = 1,2.$$

Let us construct an auxiliary function on the base of the functions

$$v_{ij}(x_{ij}) = (x_{ij}^{\mathrm{T}} P_{ij} x_{ij})^{\frac{1}{2}},$$

by formular

$$v_i(x_i) = d_{i1}v_{i1}(x_{i1}) + d_{i2}v_{i2}(x_{i2}), \quad i = 1,2,$$

where d_{ij} are some positive constants. We introduce 2×2 matrices $W_i = (w_{jk}^{(i)})$ with the elements

$$w_{jk}^{(i)} = \begin{cases} \gamma_i \alpha_{ij} & \text{if} \quad j = k, \\ -\sigma_M^{\frac{1}{2}}(P_{ij})(\|B_{ij}\| + \bar{\epsilon}_i) & \text{if} \quad j \neq k. \end{cases}$$

Here $0 < \bar{\epsilon}_i < \epsilon_i$.

Further we need the following proposition.

Lemma 4.1 *We assume that*

(1) *discrete system (1.2) is decomposed on the first level to the system (3.1) and on the second level to the systems (4.1);*
(2) *all conditions of Assumptions 4.1 and 4.2 are satisfied;*
(3) *for the submatrices E_{ij}, U_{ij} of the matrices E_i, $i = 1,2$, the estimates*

$$\|E_{ij}\| \leqslant (1-\gamma_i)\mu_{ij}, \quad \|U_{ij}\| \leqslant \bar{\epsilon}_i, \quad i,\, j = 1,2.$$

are realized.

Then there exist vectors $\hat{d}_1$, $\hat{d}_2 \in R^2$ with positive components such that the first forward differences $\Delta v_i(x_i)\big|_{C_{ij}}$ for the functions $v_i(x_i)$ satisfy the estimates

$$\Delta v_i(x_i)\big|_{C_{ij}} \leqslant -\hat{d}_i^{\mathrm{T}} W_i z_i, \quad i = 1,2 \tag{4.3}$$

and the matrices W_i are the M-matrices.

Here $\hat{d}_i = (d_{i1}, d_{i2})^{\mathrm{T}}$ and $z_i = (\|x_{i1}\|, \|x_{i2}\|)^{\mathrm{T}}$.

The proof of Lemma 4.1 is analogous to that of Theorem 3.1.

Under the hypotheses of Lemma 4.1 the matrices W_i are the M-matrices and, according to [8], the vectors $\hat{d}_i^{\mathrm{T}} W_i = (d_{i1}w_{11}^{(i)} + d_{i2}w_{21}^{(i)}, d_{i1}w_{12}^{(i)} + d_{i2}w_{22}^{(i)})$ have positive components.

Let us denote

$$\pi_i = \min\{d_{i1}w_{11}^{(i)} + d_{i2}w_{21}^{(i)}; d_{i1}w_{12}^{(i)} + d_{i2}w_{22}^{(i)}\}, \quad i = 1,2,$$
$$m = \frac{1}{2}\left(\frac{\pi_1\pi_2}{(d_{11}\sigma_M^{\frac{1}{2}}(P_{11}) + d_{12}\sigma_M^{\frac{1}{2}}(P_{12}))\,(d_{21}\sigma_M^{\frac{1}{2}}(P_{21}) + d_{22}\sigma_M^{\frac{1}{2}}(P_{22}))}\right)^{\frac{1}{2}} \tag{4.4}$$

and give a method of optimal choice of the constants d_{i1}, d_{i2}, $i = 1,2$.

Lemma 4.2 *Let the matrices W_1 and W_2 be the M-matrices and $w_{12}^{(i)}, w_{21}^{(i)} < 0$, then*

$$\sup_{d\in D} m(d) = m(d_1^*, 1, d_2^*, 1) = \frac{1}{2}\left(\frac{w_{11}^{(1)}w_{22}^{(1)} - w_{12}^{(1)}w_{21}^{(1)}}{\sigma_M^{\frac{1}{2}}(P_{11})(w_{22}^{(1)} - w_{21}^{(1)}) + \sigma_M^{\frac{1}{2}}(P_{12})(w_{11}^{(1)} - w_{12}^{(1)})}\times \right.$$
$$\left. \times\, \frac{w_{11}^{(2)}w_{22}^{(2)} - w_{12}^{(2)}w_{21}^{(2)}}{\sigma_M^{\frac{1}{2}}(P_{21})(w_{22}^{(2)} - w_{21}^{(2)}) + \sigma_M^{\frac{1}{2}}(P_{22})(w_{11}^{(2)} - w_{12}^{(2)})}\right)^{\frac{1}{2}}, \tag{4.5}$$

where

$$D = \left\{d = (d_{11}, d_{12}, d_{21}, d_{22})^{\mathrm{T}} \in R^4 : -\frac{w_{21}^{(1)}}{w_{11}^{(1)}} < \frac{d_{11}}{d_{12}} < -\frac{w_{22}^{(1)}}{w_{12}^{(1)}},\ -\frac{w_{21}^{(2)}}{w_{11}^{(2)}} < \frac{d_{21}}{d_{22}} < -\frac{w_{22}^{(2)}}{w_{12}^{(2)}}\right\},$$
$$d_1^* = \frac{w_{22}^{(1)} - w_{21}^{(1)}}{w_{11}^{(1)} - w_{12}^{(1)}}, \qquad d_2^* = \frac{w_{22}^{(2)} - w_{21}^{(2)}}{w_{11}^{(2)} - w_{12}^{(2)}}.$$

Proof As the matrices W_1 and W_2 are the M-matrices, then $w_{11}^{(i)}, w_{22}^{(i)} > 0$, $w_{12}^{(i)}, w_{21}^{(i)} < 0$ and consequently, $-\frac{w_{22}^{(i)}}{w_{12}^{(i)}} > -\frac{w_{21}^{(i)}}{w_{11}^{(i)}} > 0$. On computing of the constant π_i and m we can set $d_{12} = d_{22} = 1$, $d_{11} = d_1$, $d_{21} = d_2$ and $d_i \in D_i = \left\{d_i \in R\colon -\frac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i < -\frac{w_{22}^{(i)}}{w_{12}^{(i)}}\right\}$, $i = 1, 2$. Let us denote

$$m_i(d_i) = \frac{\pi_i}{d_i\sigma_M^{\frac{1}{2}}(P_{i1}) + \sigma_M^{\frac{1}{2}}(P_{i2})} \quad i = 1, 2,$$

and note that

$$\sup_{d\in D} m(d) = \frac{1}{2}\Big(\sup_{d_1\in D_1} m_1(d_1) \sup_{d_2\in D_2} m_2(d_2)\Big). \tag{4.6}$$

By (4.4) for the function $m_i(d_i)$ we get the expressions

$$m_i(d_i) = \begin{cases} \frac{d_i w_{11}^{(i)} + w_{21}^{(i)}}{d_i\sigma_M^{\frac{1}{2}}(P_{i1}) + \sigma_M^{\frac{1}{2}}(P_{i2})}, & \text{if} \quad -\frac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i \leqslant d_i^*, \\ \frac{d_i w_{12}^{(i)} + w_{22}^{(i)}}{d_i\sigma_M^{\frac{1}{2}}(P_{i1}) + \sigma_M^{\frac{1}{2}}(P_{i2})}, & \text{if} \quad d_i^* \leqslant d_i < -\frac{w_{22}^{(i)}}{w_{21}^{(i)}}. \end{cases}$$

For the first derivatives $m_i'(d_i)$ we have

$$m_i'(d_i) = \begin{cases} \frac{w_{11}^{(i)}\sigma_M^{\frac{1}{2}}(P_{i2}) - w_{21}^{(i)}\sigma_M^{\frac{1}{2}}(P_{i1})}{\left(d_i\sigma_M^{\frac{1}{2}}(P_{i1}) + \sigma_M^{\frac{1}{2}}(P_{i2})\right)^2}, & \text{if} \quad -\frac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i < d_i^*, \\ \frac{w_{12}^{(i)}\sigma_M^{\frac{1}{2}}(P_{i2}) - w_{22}^{(i)}\sigma_M^{\frac{1}{2}}(P_{i1})}{\left(d_i\sigma_M^{\frac{1}{2}}(P_{i1}) + \sigma_M^{\frac{1}{2}}(P_{i2})\right)^2}, & \text{if} \quad d_i^* < d_i < -\frac{w_{22}^{(i)}}{w_{21}^{(i)}}, \end{cases}$$

therefore $m_i'(d_i) > 0$ for $-\frac{w_{21}^{(i)}}{w_{11}^{(i)}} < d_i < d_i^*$ and $m_i'(d_i) < 0$ for $d_i^* < d_i < -\frac{w_{22}^{(i)}}{w_{21}^{(i)}}$. From here it follows that

$$\sup_{d_i \in D_i} m_i(d_i) = m_i(d_i^*) = \frac{w_{11}^{(i)} w_{22}^{(i)} - w_{12}^{(i)} w_{21}^{(i)}}{\sigma_M^{\frac{1}{2}}(P_{i1})(w_{22}^{(i)} - w_{21}^{(i)}) + \sigma_M^{\frac{1}{2}}(P_{i2})(w_{11}^{(i)} - w_{12}^{(i)})}.$$

Substituting by the values of $m_i(d_i^*)$ into (4.6), we get the identity (4.5). Lemma 4.2 is proved.

Assumption 4.3 *Let for the submatrices $M_{jk}^{(i)}$ of the matrices B_i the inequalities*

$$\overline{m} = \max \|M_{jk}^{(i)}\| < m$$

be realized for all i, j, $k = 1, 2$.

The following proposition is basic in the method of hierarchical Lyapunov functions in the robust stability problem of the system (1.2).

Theorem 4.1 *We assume that for the uncertain system (1.2) the two-level decomposition (3.1), (4.1) is realized and all conditions of Assumptions 4.1–4.3 are satisfied. If the inequalities*

$$\|E_{ij}\| \leqslant (1 - \gamma_i)\mu_{ij}, \quad \|U_{ij}\| \leqslant \overline{\epsilon}_i, \quad \|F_{jk}^{(i)}\| < m - \overline{m}$$

are fulfilled for all i, j, $k = 1, 2$, then the equilibrium $x = 0$ of the system (1.2) is asymptotically stable.

Proof Under the hypotheses of Lemma 4.1 there exist constants $d_{ij} > 0$ for which $\hat{d}_i^{\mathrm{T}} W_i z_i > 0$. In view of designations (4.4), we get from estimate (4.3)

$$\Delta v_i(x_i)\big|_{S_i} \leqslant -\pi_i \left(\|x_{i1}\|^2 + \|x_{i2}\|^2\right)^{\frac{1}{2}} = -\pi_i \|x_i\|, \quad i = 1, 2.$$

Since for $i \neq k$ the estimates

$$\Delta v_{i1}(x_{i1})\big|_{\widehat{S}_i} \leqslant \Delta v_{i1}(x_{i1})\big|_{S_i} + \sigma_M^{\frac{1}{2}}(P_{i1})(2\overline{m} + \|F_{11}^{(i)}\| + \|F_{12}^{(i)}\|)\|x_k\|,$$
$$\Delta v_{i2}(x_{i2})\big|_{\widehat{S}_i} \leqslant \Delta v_{i2}(x_{i2})\big|_{S_i} + \sigma_M^{\frac{1}{2}}(P_{i2})(2\overline{m} + \|F_{21}^{(i)}\| + \|F_{22}^{(i)}\|)\|x_k\|,$$

are true, then

$$\begin{aligned}\Delta v_i(x_i)\big|_{\widehat{S}_i} &= d_{i1}\Delta v_{i1}(x_{i1})\big|_{S_i} + d_{i2}\Delta v_{i2}(x_{i2})\big|_{\widehat{S}_i} \leqslant -\pi_i\|x_i\| + \big[d_{i1}\sigma_M^{\frac{1}{2}}(P_{i1})\big(2\overline{m} + \\ &+ \|F_{11}^{(i)}\| + \|F_{12}^{(i)}\|\big) + d_{i2}\sigma_M^{\frac{1}{2}}(P_{i2})\big(2\overline{m} + \|F_{21}^{(i)}\| + \|F_{22}^{(i)}\|\big)\big]\|x_k\|.\end{aligned} \tag{4.7}$$

For the function

$$v(x) = d_1 v_1(x_1) + d_2 v_2(x_2)$$

in view of estimates (4.7) we get

$$\Delta v(x)\big|_{S} = d_1 \Delta v_1(x_1)\big|_{\widehat{S}_1} + d_2 \Delta v_2(x_2)\big|_{\widehat{S}_2} \leqslant -\hat{d}^{\mathrm{T}} W\, z, \tag{4.8}$$

where $\hat{d} = (d_1, d_2)^{\mathrm{T}}$, $z = (\|x_1\|, \|x_2\|)^{\mathrm{T}}$ and W is a 2×2-matrix with the elements

$$w_{jk} = \begin{cases} \pi_j & \text{for} \quad j = k, \\ -d_{j1}\sigma_M^{\frac{1}{2}}(P_{j1})(2\overline{m} + \|F_{11}^{(j)}\|) + \|F_{12}^{(j)}\|) - & \\ \qquad\qquad -d_{j2}\sigma_M^{\frac{1}{2}}(P_{j2})(2\overline{m} + \|F_{21}^{(j)}\|) + \|F_{22}^{(j)}\|) & \text{for} \quad j \neq k. \end{cases}$$

Under the hypotheses of Theorem 4.1 the matrix W in the estimate (4.8) is the M-matrix. Thus the matrices W_1, W_2, W are the M-matrices and it is sufficient [3] for asymptotic stability of the system (1.2).

5 Discussion and Some Applications

The hierarchical approach in robust stability problem permits a more complete allowance for the dynamic characteristics of the nominal system on each hierarchical level and thus a more exact definition of robust bounds for the system (1.2). We illustrate efficiency of the approach proposed in the paper by a simple example.

Let us assume that in the system (1.2) the matrix A has the form

$$A = \begin{pmatrix} 0.5 & 0.01 & 0.03 & 0 \\ 0.01 & 0.125 & 0 & 0.03 \\ 0.03 & 0 & 0.25 & 0.005 \\ 0 & 0.03 & 0.005 & 0.125 \end{pmatrix}. \tag{5.1}$$

5.1 Scalar approach

Let us compute the matrices and constants occurring in the framework of the scalar approach (see Theorem 2.1):

$$P = \begin{pmatrix} 1.336149 & 0.008512 & 0.032104 & 0.000737 \\ 0.008512 & 1.017019 & 0.000708 & 0.007761 \\ 0.032104 & 0.000708 & 1.068495 & 0.002057 \\ 0.000737 & 0.007761 & 0.002057 & 1.016891 \end{pmatrix};$$

$$\sigma(P) \approx 1.340176; \quad \sigma_M(P - I) \approx 0.340176; \quad \mu \approx 0.496185.$$

Here I is a 4×4-unit matrix. From here the robust bound for the system (1.2) with the matrix (5.1) is determined by the inequality

$$\|E\| < 0.496185 \tag{5.2}$$

for all matrices $E \in S_1$.

5.2 Vector approach

According to this approach we decomposed the matrix (5.1) and denote

$$A_1 = \begin{pmatrix} 0.5 & 0.01 \\ 0.01 & 0.125 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.25 & 0.005 \\ 0.005 & 0.125 \end{pmatrix}, \qquad B_1 = B_2 = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.03 \end{pmatrix}.$$

The uncertain matrix E is represented in the form (3.3). The matrices and constants occurring in the framework of vector the approach are:

$$P_1 \approx \begin{pmatrix} 1.333581 & 0.008469 \\ 0.008469 & 1.016029 \end{pmatrix}, \qquad P_2 \approx \begin{pmatrix} 1.066699 & 0.002031, \\ 0.002031 & 1.015902 \end{pmatrix},$$
$$\sigma_M(P_1) \approx 1.333807, \quad \sigma_M(P_2) \approx 1.066780, \quad \mu_1 \approx 0.449733, \quad \mu_2 \approx 0.749800.$$

Hence we have the estimates of submatrices norms in the form

$$\|E_1\| \leqslant 0.499733(1-\gamma), \quad \|E_2\| \leqslant 0.749800(1-\gamma), \quad \gamma \in (0,1). \tag{5.3}$$

Let $\gamma = 0.25$. Besides $\epsilon \approx 0.012303$.

Finally, for the matrix E represented in the form (3.3), we get the estimates:

$$\|E_1\| \leqslant 0.374800, \quad \|E_2\| \leqslant 0.562350, \quad \|U_i\| < 0.012303, \quad i = 1, 2. \tag{5.4}$$

For example the matrix

$$\widetilde{E} = \operatorname{diag}\{0.37,\, 0.37,\, 0.56,\, 0.56\}$$

satisfies the inequalities (5.4). But $\|\widetilde{E}\| = 0.56$, and consequently, the norm of uncertain matrix $\widetilde{E}$ does not satisfy the inequality (5.2).

5.3 Hierarchical approach

According to the proposed algorithm we accomplish the two-level decomposition of system (1.2) with the matrix (5.1) and as a result we get:

$$A_{11} = 0.5, \quad A_{12} = 0.125, \quad A_{21} = 0.25, \quad A_{22} = 0.125.$$

Let

$$\gamma_1 = 0.5, \qquad \gamma_2 = 0.125.$$

Numerical values of corresponding constants are:

$$\begin{aligned} &\sigma_M(P_{11}) \approx 1.333333, \qquad \sigma_M(P_{12}) \approx 1.015873, \qquad \mu_{11} = 0.5, \qquad \mu_{12} = 0.875, \\ &\sigma_M(P_{21}) \approx 1.066666, \qquad \sigma_M(P_{22}) \approx 1.015873, \qquad \mu_{21} = 0.75, \qquad \mu_{22} = 0.875, \\ &\qquad\qquad \epsilon_1 \approx 0.320718, \qquad \epsilon_2 \approx 0.096261. \end{aligned}$$

We shall set $\bar{\epsilon}_1 = 0.05$, and $\bar{\epsilon}_2 = 0.006$. In this case for the matrices W_1 and W_2 we get the expressions

$$W_1 \approx \begin{pmatrix} 0.288675 & -0.069282 \\ -0.060474 & 0.440958 \end{pmatrix}, \qquad W_2 \approx \begin{pmatrix} 0.096824 & -0.011360 \\ -0.011086 & 0.110239 \end{pmatrix}.$$

The matrices W_1 and W_2 are the M-matrices as their non-diagonal elements are negative and their principal minors are positive.

The constant m is computed by the formular (4.5): $m \approx 0.038392$. Thus, the following restrictions are imposed on submatrices of E:

$$\begin{gathered} \|E_{11}\| \leqslant 0.25, \quad \|E_{12}\| \leqslant 0.4375, \quad \|E_{21}\| \leqslant 0.65625, \quad \|E_{22}\| \leqslant 0.765625, \\ \|U_{1j}\| \leqslant 0.05, \quad \|U_{2j}\| \leqslant 0.006, \quad \|F_{jk}^{(i)}\| < 0.008392. \end{gathered} \tag{5.5}$$

For example the matrix

$$\overline{E} = \operatorname{diag}\{0.25,\, 0.43,\, 0.65,\, 0.76\}$$

satisfies the inequalities (5.5). Since $\|\overline{E}\| = 0.76$, the matrix $\overline{E}$ does not satisfy condition (5.2). Moreover $\|\operatorname{diag}\{0.65,\, 0.76\}\| = 0.76 > 0.75$ and it means that for the matrix $\overline{E}$ conditions (5.3) are not satisfied for any $\gamma \in (0,1)$.

Thus, the general conclusion from this example is: the hierarchical Lyapunov function allows a more complete use of the potential possibilities of direct Lyapunov method in robustness analysis of discrete system (1.2).

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