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Some Generalizations of Lyapunov's Approach to Stability and Control

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Abstract: In the paper presents a brief survey of some new developments in Lyapunov's approach including the generalized perturbation equation and its applications; the use of nonanalytic Lyapunov functions; an extension of the Barbashin-Krasovskii theorem related to asymptotic stability assured by a Lyapunov function with nonpositive derivative; the consistency condition for a time-space mosaic that constitutes a discontinuous Lyapunov function valid for investigation of stability; the introduction of non sign-definite functions for use in control (carrying surfaces); the extremal set construction for control, stabilization, and nonlinear asymptotic observer design.

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1 Introduction

After the seminal work of Lyapunov [1], stability theory was recognized as an independent and important field of knowledge. Since that time of 1892, it counted spectacular achievements such as Chetaev's instability theorem, Malkin's reduction principle, Krasovskii-Lyapunov functionals for delay differential equations, stability with respect to a part of variables, absolute stability of control systems, vector Lyapunov functions, matrix Lyapunov functions, to name just a few. These fundamental developments and some other important results can be found in [1 − 24], see also references therein.

This survey of some relatively recent developments concentrates on directions where the author personally participated.

The bibliography of the survey is limited to the topics considered which are presented in the order that relates to the areas of application and seems convenient for the reader. Efforts have been made to make the paper self-contained.

2 Generalized Perturbation Equation

This concept was proposed in the joint work [25] with V.V.Rumyantsev. In the classical stability theory, for a given nonlinear system

$$x' = \frac{dx}{dt} = f(x,t), \quad x \in R^n, \quad t \geq a \geq 0, \quad x(a) = b \tag{2.1}$$

under standard conditions ensuring the existence, uniqueness and extendibility of solutions in some region of initial data, the Lyapunov methods [1] can be applied to investigate stability of a certain particular solution of interest that corresponds to

$$x^*(t) = x(a,b,t), \quad x^*(a) = b. \tag{2.2}$$

Stability of solutions (2.2) is studied with the use of the perturbation equation which is obtained from (2.1), (2.2) by the transformation

$$x(t) = x^*(t) + w(t). \tag{2.3}$$

Substituting (2.3) into (2.1) and assuming the function f in (2.1) to be analytic with respect to x , one can use the expansion

$$\frac{dx^*}{dt} + \frac{dw}{dt} = f(x^* + w, t) = f(x^*, t) + \nabla f(x^*, t)w + g(w,t) \tag{2.4}$$

yielding, after cancellation of the first terms, the perturbation equation

$$w' = A(t)w + g(w,t), \quad g(0,t) = 0, \quad t \geq a. \tag{2.5}$$

Here $A(t)$ is the Jacobian matrix of $f(x,t)$, (2.1), calculated on the solution $x^*(t)$, (2.2), and $g(w,t)$ are higher order terms with all partial derivatives calculated on the same solution (2.2).

According to (2.3), the unperturbed motion $x^*(t)$ of (2.2) corresponds to the trivial solution $w(t) = 0$ of the perturbation equation (2.5). This allows us to substitute the problem of stability of the motion $x^*(t)$, (2.2), of the nominal equation (2.1) by the problem of stability of trivial solution $w = 0$ of the perturbation equation (2.5). This approach led to the powerful and elegant methods that constitute the classical stability theory, see, e.g. [1 – 16] and further references therein.

Consideration of perturbation equation (2.5) with all its comfort of using linear approximation $dw/dt = A(t)w$ and then, if necessary, successive higher order terms (in critical cases) has, however, some specific qualities.

First, if a particular solution $x^*(t)$, (2.2), is not given as an explicit function of a, b, t (i.e. as a formula), then perturbation equation (2.5) cannot be determined.

Second, if the solution (2.2) and, thus, the perturbation equation (2.5) are known, then the results of stability on that basis are applicable to that particular solution only.

To bypass these difficulties, let us not fix $x(a)$ in (2.1) and consider $x^*(t)$ of (2.2) as unknown parameter-function. Then the deviation $w(t)$ is governed by the equation

$$w' = \frac{dw}{dt} = f(x^* + w, t) - f(x^*, t) = q(w, x^*, t), \quad t \geq a \tag{2.6}$$

that follows from the first equality of (2.4). In contrast with equations (2.4), (2.5), the composite function q in (2.6) contains an unknown solution $x^*(t)$ as its argument. On the other hand, $q(0, x^*, t) = 0$ for all $x^*(t)$, t, thus $w(t) = 0$ is the solution of (2.6) for any $x^*(t)$. It means that trivial solution $w = 0$ can be put in correspondence to any particular solution $x^*(t)$, serving therefore, the whole region of possible initial data.

If $f(.)$ in (2.1) is analytic with respect to x, then $q(.)$ of (2.6) is analytic with respect to w, yielding the generalized perturbation equation

$$w' = A(x^*(t), t)w + g(w, x^*(t), t), \qquad g(0, x^*(t), t) = 0, \quad t \geq a. \tag{2.7}$$

For some particular $x^*(t)$, it is, of course, identical to (2.5) with corresponding $A(t)$, $g(w, t)$, where we use the same notation A, g for different functions. However, without fixing $x^*(t)$, it represents a bundle of equations given on a continuum of different particular solutions. With this meaning, we shall drop sometimes the indication of a particular solution, writing simply

$$w' = A(x, t)w + g(w, x, t), \quad t \geq a \tag{2.8}$$

with the understanding that (2.8) is a corresponding perturbation equation for every solution $x(t)$ of (2.1). It means that the form (2.8) is conserved while the terms are different for different $x(t)$.

Example 2.1 To illustrate the point, consider an example from [3, Sections 4, 44]:

$$x' = x(\alpha^2 - x^2), \quad \alpha > 0, \quad t \geq a. \tag{2.9}$$

According to (2.6), we have

$$w' = (\alpha^2 - 3x^2)w - 3xw^2 - w^3, \quad t \geq a \tag{2.10}$$

which is the generalized perturbation equation (2.8) in our case of (2.9).

With (2.10) we can do the standard stability analysis for (2.9) as follows. Equation (2.9) has three stationary solutions $x_1 = 0$, $x_{2,3} = \pm\alpha$. Substituting those solutions in (2.10), we immediately obtain instability for $x_1 = 0$ and asymptotic stability for $x_{2,3} = \pm\alpha$, all by the first approximation in (2.10). These results can also be established by considering the Lyapunov function $V = w^2/2$ which has the following derivative on trajectories of (2.10)

$$V' = w^2(\alpha^2 - 3x^2 - 3xw - w^2). \tag{2.11}$$

For $x_1 = 0$, we have from (2.11) that $V' > 0$ if $\alpha^2 - w^2 > 0$, asserting instability and yielding domain of repulsion $w \in (-\alpha, \alpha)$ with respect to nominal solution $x_1(t) = 0$.

For $x_{2,3} = \pm\alpha$, we have from (2.11)

$$V'_{2,3} = w^2(-2\alpha^2 \mp 3\alpha w - w^2), \tag{2.12}$$

asserting asymptotic stability of both solutions for small w. To find domain of attraction for $x_2 = \alpha$, we take the upper sign in (2.12) and solve the inequality $w^2 + 3\alpha w + 2\alpha^2 > 0$, yielding $w > -\alpha$ or $w < -2\alpha$, which in coordinates $t0x$ corresponds to $x > 0$ or $x < -\alpha$ since in this case $w = x - x_2 = x - \alpha$. However, in the region $x < -\alpha$ there is another attractor, namely, $x_3 = -\alpha$; hence, domain of attraction for $x_2 = \alpha$ is

$x \in (0, \infty)$. For $x_3 = -\alpha$, the same arguments with the lower sign in (2.12) yield domain of attraction $x \in (-\infty, 0)$; details are left to the reader.

We see that generalized perturbation equation can be used for all known solutions of the nominal equation. Moreover, it can be used for stability analysis of solutions that cannot be expressed as explicit integrals and for which one cannot write the specific perturbation equation (2.5) corresponding to a particular solution $x^*(t)$ (see (2.2) – (2.3)), *not given as a formula*. In such cases, the generalized perturbation equation represents a new and important tool for stability analysis.

Example 2.2 Use of bundles of first integrals [25].

Chetaev's method of construction of Lyapunov functions in the form of bundles of first integrals [2] (see also [5, Section 10] and further references therein) can be used with the generalized perturbation equation, that is, for stability analysis of sets of solutions. Consider the classical example of the Euler case in the motion of a rigid body around its fixed center of mass without external forces. Equations of such motion are usually written in the form

$$Ap' + (C - B)qr = 0, \tag{2.13}$$

$$Bq' + (A - C)rp = 0, \tag{2.14}$$

$$Cr' + (B - A)pq = 0, \tag{2.15}$$

where $t \geq a$ and p, q, r are projections of the vector of angular velocity on coordinate axes taken as principal axes of the ellipsoid of inertia, and A, B, C are principal moments of inertia of the rigid body.

Suppose that $p^*(t)$, $q^*(t)$, $r^*(t)$ is some particular solution of (2.13) – (2.15). Substituting $p = p^* + \xi$, $q = q^* + \eta$, $r = r^* + \zeta$ into (2.13) – (2.15), eliminating terms that are cancelled by virtue of nominal equations (2.13) – (2.15) and dropping the superscript, we obtain the generalized perturbation equations

$$A\xi' = (B - C)(r\eta + q\zeta + \eta\zeta), \tag{2.16}$$

$$B\eta' = (C - A)(p\zeta + r\xi + \zeta\xi), \tag{2.17}$$

$$C\zeta' = (A - B)(q\xi + p\eta + \xi\eta). \tag{2.18}$$

Here the prime ($\prime$) denotes time derivative, and p, q, r are fixed particular solutions of (2.13) – (2.15) defined by certain initial conditions $p(a) = p_0$, $q(a) = q_0$, $r(a) = r_0$.

By inspection, one can see that equations (2.13) – (2.15) have the following first integrals

$$T = Ap^2 + Bq^2 + Cr^2 = \text{const}, \tag{2.19}$$

$$M = A^2p^2 + B^2q^2 + C^2r^2 = \text{const}. \tag{2.20}$$

Case 1 $A = B = C$. In this case all solutions are stationary, $p = p_0$, $q = q_0$, $r = r_0$, and all are stable.

Case 2 $p = q = r = 0$. Equations (2.16) – (2.18) coincide with (2.13) – (2.15). Therefore, integrals T, M with ξ, η, ζ instead of p, q, r are also first integrals of perturbed motions. Being positive definite, they can be used as Lyapunov functions to conclude about stability of this trivial solution (at rest).

Case 3 $A \neq B \neq C \neq A$, and $p_0^2 + q_0^2 + r_0^2 > 0$. In this case, and taking into account (2.13) – (2.15), generalized perturbation equations (2.16) – (2.18) have the following first integrals

$$T^* = A(p+\xi)^2 + B(q+\eta)^2 + C(r+\zeta)^2 = \text{const}, \tag{2.21}$$

$$M^* = A^2(p+\xi)^2 + B^2(q+\eta)^2 + C^2(r+\zeta)^2 = \text{const}, \tag{2.22}$$

where constants T^*, M^* are defined by initial data p_0, q_0, r_0 and initial perturbations ξ_0, η_0, ζ_0. Since T^*, M^* do not vanish at $\xi = \eta = \zeta = 0$, they cannot be taken as Lyapunov functions.

Consider the function

$$V = (T^* - T)^2 + (M^* - M)^2. \tag{2.23}$$

This function is nonnegative, $V \geq 0$; vanishes if $\xi = \eta = \zeta = 0$, and its total derivative on trajectories of perturbed motions (2.16) – (2.18) of the system (2.13) – (2.15) is zero, $V' = 0$, since V is a bundle of integrals. If V were positive definite, one would conclude about stability of all motions. Unfortunately, this is not the case.

If ξ, η, ζ are not all zero, $|\xi| + |\eta| + |\zeta| > 0$, then $V = 0$ if and only if $T^* = T$ and $M^* = M$. To find the manifold on which $V = 0$, we can write, by virtue of (2.19) – (2.22)

$$T^* - T = A(2p\xi + \xi^2) + B(2q\eta + \eta^2) + C(2r\zeta + \zeta^2) = 0, \tag{2.24}$$

$$M^* - M = A^2(2p\xi + \xi^2) + B^2(2q\eta + \eta^2) + C^2(2r\zeta + \zeta^2) = 0. \tag{2.25}$$

Denoting parentheses in (2.24), (2.25) as x, y, z, we obtain for the case $A \neq B \neq C \neq A$ the integral-invariant manifold in the $\xi\eta\zeta$-space

$$\frac{x}{BC(B-C)} = \frac{y}{CA(C-A)} = \frac{z}{AB(A-B)} = \lambda(t). \tag{2.26}$$

Physically, it means that ξ, η, ζ satisfying (2.26) do not affect the energy nor the angular momentum of the body.

From conservation property of integrals at the left-hand side of (2.24), (2.25), it follows that perturbed trajectories either lie entirely on the manifold (2.26) or do not intersect it at all. If for nominal motions $p(t)$, $q(t)$, $r(t)$ there are no perturbed trajectories that lie on the manifold (2.26), then those motions are stable by Lyapunov's theorem on stability [1, Section 16] with the function V of (2.23) which is positive definite if (2.26) does not contain perturbed trajectories. Referring the reader to [25] for details, the conclusion is as follows.

Summary The rest $p = q = r = 0$ and all motions in trivial case $A = B = C$ are stable. The motion $p = q = 0$, $r(t) = r_0 = \text{const}$ in cases $A \leq B < C$ or $A \geq B > C$ (i.e. constant rotation around extreme axis C, including circular ellipsoids of inertia) is also stable. From the above analysis, we see that all other motions in the case $A \neq B \neq C \neq A$ are unstable. In the case of circular ellipsoid of inertia (ellipsoid of revolution, $A \neq B = C$), constant rotation around an equatorial axis is unstable and all other motions are stable.

Remark 2.1 The term "stability of sets of solutions" may sometimes be misinterpreted and confused with the notion usually referred to as "stability of sets", see, e.g., [26, 27] and references therein. The term "globally asymptotically stable set" means the existence of a globally contracting Lyapunov function acting outside of the set and bringing every trajectory from the exterior of the set onto that set. Such "stability sets" are also "viability sets", i.e., sets from which a trajectory cannot escape (the last term does not imply the global attraction of outside trajectories).

The stability of a set in this sense does not mean stability of solutions within that set. The use of Lyapunov functions to establish the global attraction of trajectories to some set has nothing to do with stability in the sense of Lyapunov. It means, in fact, a control application, proving certain quality referred to as ultimate boundedness, viability, practical stability, with some variations in terminology and definitions used by different authors. The level sets $V(x) \leq c$ can be used for construction of so-called overvaluing or comparison systems $dz/dt = h(t, z)$ with the property $z(t, t_0, z_0) \geq x(t, t_0, x_0)$ if $z_0 \geq x_0$.

In contrast, the generalized perturbation equation serves to establish stability of solutions in the sense of Lyapunov that start in some region of initial conditions.

3 Nonanalytic Lyapunov Functions

When N.N.Krasovskii (then my Ph.D. thesis supervisor) suggested the use of nonanalytic regulators for stabilization of nonlinear systems [28, 29], this naturally led to the introduction of nonanalytic Lyapunov functions.

Since the right-hand sides of perturbation equation are represented as convergent Maclaurin series around the trivial solution $x(t) = 0$, so the nonanalytic Lyapunov functions are also taken as finite sums of special power terms, containing absolute values and sign-functions of critical variables see [28 − 30]. Those sums are finite since asymptotic stability and instability are usually decided by terms up to a certain finite order.

Clearly, nonanalytic Lyapunov functions can be used also for other purposes. For example let us find the stability (viability) set in Example 1 of [27, p.248] for the system:

$$\frac{dx}{dt} = x(1 - x^2 - y^2) + yf(t, x, y), \qquad |f(.)| \leq 1;$$
$$\frac{dy}{dt} = xg(t, x, y) + y(1 - x^2 - y^2), \qquad |g(.)| \leq 1.$$

Taking $V = |x| + |y|$, we obtain on trajectories of the system

$$\begin{aligned}\frac{dV}{dt} &= (|x| + |y|)(1 - x^2 - y^2) + yf(.)\,\mathrm{sign}\,x + xg(.)\,\mathrm{sign}\,y \\ &\leq (|x| + |y|)(1 - x^2 - y^2) + |y| + |x| = V(2 - x^2 - y^2) \leq 0,\end{aligned}$$

if $x^2 + y^2 \geq 2$ which yields the circle of radius $\sqrt{2}$ as the global asymptotic stability set for the above system.

4 Extension of the Barbashin-Krasovskii Theorem unto Nonperiodic Systems

This theorem presents a sufficient condition for establishing asymptotic stability making use of a Lyapunov function $V(x) > 0,\ x \neq 0;\ V(0) = 0$ with nonpositive derivative $dV/dt \leq 0$ on the trajectories of the perturbation equation in a neighborhood of the origin. Such Lyapunov functions are usually constructed in practical cases of nonlinear systems. We reproduce the theorem in a simple formulation given by Barbashin [14, p.25].

Theorem 4.1 *If there is a positive definite function $V(x)$ such that $dV/dt < 0$ outside of a set M and $dV/dt \leq 0$ on M, where M is a set not containing entire trajectories (except for the origin), then the solution $x = 0$ is asymptotically stable.*

Note that it is easy to verify that M does not contain entire semitrajectories of a differential equation. Indeed, if a system is of the form

$$x' = g(x,t), \quad x(t_0) = x_0, \quad x \in R^n, \quad t \geq t_0 \tag{4.1}$$

and a surface M is given by $F(x) = 0$, then M does not contain entire trajectories if for some $t > T \geq t_0$ we have

$$\frac{dF}{dt} = \nabla F g(x,t) \neq 0.$$

For stationary systems $x' = g(x)$, not depending explicitly on t, the theorem (for the case of stability in the large) has been proved in [31] and is known as the Barbashin-Krasovskii theorem. For systems (4.1) where $g(x,t)$ is periodic in t, this theorem is proved in [4, Section 14] and is known as Krasovskii's theorem.

Further extension of this theorem follows from Theorem 4.1 for systems of class A, see [32, pp.21−27], as described below.

Definition 4.1 *System* (4.1) is said to be *of class A* if and only if the function $g(x,t)$ is such that for every solution $x(.,x_0,t_0)$ of the equation (4.1) and for any fixed $\bar{t} > t_0$ there is a sequence

$$\alpha_s > 0, \qquad \lim \alpha_s = 0, \tag{4.2}$$

such that there exists a sequence

$$\tau_s = \tau_s(x_0,t_0,\bar{t},\alpha_s) > 0, \quad \tau_{s+1} > \tau_s, \quad s = 1,2,\dots, \quad \lim \tau_s = \infty \tag{4.3}$$

for which

$$\|x(\bar{t},x_s,t_0) - x(\bar{t}+\tau_s,x_0,t_0)\| \leq \alpha_s, \qquad s = 1,2,\dots, \tag{4.4}$$

where

$$x_s = x(t_0+\tau_s,x_0,t_0), \qquad s = 1,2,\dots. \tag{4.5}$$

Remark 4.1 If one makes a drawing to illustrate conditions (4.2) to (4.5), it can be seen that those conditions, in application to solutions of differential equations, resemble the Cauchy criterion: a sequence $x_m \in R^n$ has a finite limit $x_0 = \lim x_m$ if and only if for every $\varepsilon > 0$ there is a number $N(\varepsilon)$ such that $\|x_p - x_q\| < \varepsilon$ whenever $p > N(\varepsilon)$ and $q > N(\varepsilon)$. In the above conditions, the role of ε is played by α_s of (4.2), the role of $N(\varepsilon)$ is played by τ_s of (4.3), and p, q, are played by $t_0 + \tau_s$ and $\bar{t} + \tau_s$ of (4.5), (4.4). Thus, class A contains systems with asymptotically contracting translations of every trajectory in some region, and if that region is a neighborhood of the origin, the Barbashin-Krasovskii Theorem follows. Conversely, if the Barbashin-Krasovskii Theorem is valid for some systems, those systems must be of the class A defined by (4.2) to (4.5).

Definition 4.2 If in the context of Definition 4.1 we can take $\alpha_s = 0$, $s = 1, 2, \ldots$, in (4, 6), then the *system* (4.1) is said to be *of class* A_0.

Definition 4.3 If in the context of Definition 4.1 we can take $\alpha_s = 0$ and $\tau_s = s\omega$, $s = 1, 2, \ldots$, with $\omega = \text{const} > 0$ defined by the function $g(x,t)$ in (4.1) but independent of x_0, t_0, $\bar{t}$, then the *system* (4.1) is said to be *of class* A^*.

It is clear that

$$A^* \subseteq A_0 \subseteq A. \tag{4.6}$$

Lemma 4.1 *The class A^* is nonempty and contains, in particular, all stationary systems and all systems where $g(x,t)$ is periodic in t.*

It is interesting and important that, in fact, classes A^*, A_0, A do not coincide: $A^* \neq A_0 \neq A$. Let us denote by G the general class of systems in (4.1) such that $g(x,t)$ satisfies only standard conditions of existence, uniqueness and extendibility.

Lemma 4.2 *Strictly: $A^* \subset A_0 \subset A \subset G$.*

Proof It is sufficient to provide examples, which are given in [32].

In the theorem that follows, notation θ denotes a closed neighborhood containing the origin, the sets Ω^-, Ω^+ are closed neighborhoods such that $\Omega^- \subseteq \theta \subset \Omega^+$, the closed set $C\theta = \Omega^+ - \Omega^-$, where Ω^- is open, other sets are closed and the set $\Omega_0(t) \subseteq C\theta$ plays the role of M as in the Barbashin-Krasovskii Theorem above.

Theorem 4.2* *If the system (4.1) is of class A and there is a function $V(x)$ such that for all $(x,t) \in C\theta \times [t_0, \infty)$ we have:*

$$\nabla V \cdot g(x,t) \leq 0, \tag{4.7}$$

where the equality is valid only at points of a set $\Omega_0(t) \subseteq C\theta$, $t \in [t_0, \infty)$, that contains no semitrajectories of (4.1), then there exists $T(x_0, t_0) > 0$ such that

$$\begin{aligned} x(t, x_0, t_0) \in \Omega^- \subseteq \theta \quad &\textit{for all} \quad x_0 \in \Omega^+ - \theta \\ &\textit{and all} \quad t \in [t_0 + T(x_0, t_0), \infty). \end{aligned} \tag{4.8}$$

The *proof* of this theorem which is cast in the context of differential games can be found in [32, pp.25–27]. Considering $V(x) > 0$, $x \neq 0$, $V(0) = 0$ in the case $g(0,t) = 0$, $\{0\} \in \theta$, and letting $\theta \to \{0\}$, we obtain the case of asymptotic stability for systems of class A of which stationary and periodic systems present particular cases of the smaller class A^*, $A^* \subset A_0 \subset A$. Thus, the Barbashin-Krasovskii Theorem is valid for far more general systems than stationary and periodic ones.

Example 4.1 Let

$$S\colon x' = -xt(1 + \sin 2t), \quad x(0) = x_0, \quad t \geq 0. \tag{4.9}$$

Consider $V = x^2$, then on trajectories of (4.9) we have

$$V' = 2xx' = -2x^2 t(1 + \sin 2t) \leq 0, \quad t \geq 0. \tag{4.10}$$

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Except for $x = 0$, which point is excluded from the complement $C\theta$ for any $\theta \to \{0\}$, derivative $V' = 0$ only at isolated points $t = 0$ and $t_n = \pi/2 + \pi n$, $n = 0, 1, \dots$, thus, trivial solution $x = 0$ is asymptotically stable. Equation (4.9) has separable variables, and it can be verified directly by Definitions 4.1, 4.2 that $S \in A$, $S \notin A_0$.

5 Lyapunov's Approach in Use for Control and Identification

Lyapunov's methods have been applied to control problems of different nature, see, e.g. [5, 9, 10, 18, 19, 21, 26−30, 32, 33, 35−45] and references therein. An interesting generalization for control of motion is developed in the joint work with J.M.Skowronski [33].

Consider a non-linear differential equation with controls:

$$x' = \frac{dx}{dt} = F(x, t, u), \quad x \in R^N, \quad u \in U \subset R^m, \quad t \in [0, t_f], \tag{5.1}$$

$$x(t_0) = x_0 \in \Delta_1 \subseteq \Delta \subset R^N, \quad t_0 \in [0, t_f], \tag{5.2}$$

$$u = u(x, t) \in U \subset R^m, \quad t \in [0, t_f]. \tag{5.3}$$

Equation (5.1) with control (5.3) takes the form

$$x' = \frac{dx}{dt} = f(x, t), \quad f(x, t) = F(x, t, u(x, t)), \quad t \in [0, t_f]. \tag{5.4}$$

We assume that the functions F, u and the sets U, Δ in (5.1)−(5.3) are such that the function f in (5.4) satisfies standard conditions for the existence and uniqueness of a solution $x(t)$ with values in Δ, given initial condition (5.2) and a control function $u(\cdot)$ with values in U. The sets U, Δ, Δ_1 are open connected sets (domains) and the set of control functions $\{u(\cdot)\}$ contains the function $u(\cdot) = 0$. We allow $t_f = \infty$.

With these hypotheses, the above relations are well defined and may be regarded in two ways:

(a) as nominal equations of a dynamical system with the motion $x(t) \in \Delta$, in phase coordinates;
(b) as perturbation equations of certain dynamical system, whereby $f(0, t) = 0$ and $x(t) \in \Delta$ represents a deviation from some unperturbed nominal motion which is not explicitly given; the nominal equations of the system are not written, but $x(t) = 0$ designates precisely its nominal motion.

In his doctoral dissertation [1] A.M.Lyapunov gave a thorough study of the problem (b). The principal idea of the approach is decomposition of motion $x(t)$ into two motions: a motion along a certain surface V and a motion of the surface V itself. This idea is not related to the kind of equation (the nominal or perturbation one), nor to certain assumptions of the Lyapunov theory. This allows us to generalize the approach in different directions.

The generalization for use in control is as follows.

(1) Equation (5.1) is regarded as a nominal equation and not as a perturbation one. The condition $f(0, t) = 0$ is dropped.
(2) The sets U, Δ, Δ_1 are not assumed to be small, on the contrary:

$$d(\Delta) \geq d(\Delta_1) = \sup \|x_1 - x_2\| \geq l > 0, \tag{5.5}$$

where $\|\cdot\|$ is a norm in R^n.

(3) The aim is to determine whether or not the motion $x(t)$ tends to a certain given domain $M \subset \Delta$ which is not a neighbourhood of the origin. In control applications the function $u(x,t)$ is to be chosen so as to make $x(t)$ enter M in finite time and remain there. We shall concentrate on sufficient conditions for the convergence $x(t) \to M$, and not on how to choose $u(\cdot)$. Consequently, the control function is assumed to have been chosen, so that we start with (5.4). More on how to choose $u(\cdot)$ can be found in [32, 35].

(4) Regarding the Lyapunov second method, the conditions $V(x) > 0$, $x \neq 0$, $V(0) = 0$ are dropped, the condition $dV/dt \leq 0$ modified, and certain other conditions are imposed. The functions $V(x)$ thus constructed are no longer Lyapunov functions and, to avoid confusion, they are called simply V-functions. We demonstrate, however, that stationary Lyapunov functions represent a subset in the set of general stationary V-functions.

(5) The sets Δ, Δ_1, M are explicitly introduced into the method, allowing us to obtain quantitative results.

Such are the major changes that aim at the two-fold objective:

(a) to facilitate direct control applications of Lyapunov's approach;

(b) to provide the means for investigation of nominal equations of a system and a tool for quantitative design of desired motions.

5.1 Geometry of V-functions

5.1.1 V-surfaces. We consider real C^1-functions $V(x)\colon R^N \to R$ such that for each constant $\nu_0 \in B \subset R$, B open, satisfy the following conditions:

(1*) There exists a surface $V(x) = \nu_0$ which is unique (single-sheeted) and of a finite measure.

(2*) There exist x_0 such that $V(x_0) < \nu_0$ and x_1 such that $V(x_1) > \nu_0$.

(3*) The set

$$\Omega(\nu_0) = \{x \mid V(x) < \nu_0\} \tag{5.6}$$

is bounded in R^N.

We consider the closure of Ω, or the level set

$$\mathrm{cl}\,\Omega(\nu_0) = \{x \mid V(x) \leq \nu_0\}, \tag{5.7}$$

its boundary

$$\partial\Omega(\nu_0) = \{x \mid V(x) = \nu_0\} \tag{5.8}$$

and the open complement or the exterior of Ω:

$$C\,\mathrm{cl}\,\Omega(\nu_0) = \{x \mid V(x) > \nu_0\} = \mathrm{ext}\ \mathrm{cl}\,\Omega. \tag{5.9}$$

The condition (2*) means that the interior and exterior of $\mathrm{cl}\,\Omega$ are not empty. If $V(x)$ is defined everywhere in R^N, then by (5.6), (5.8) (5.9) we have $\Omega + \partial\Omega + C\,\mathrm{cl}\,\Omega = R^N$. Also $\mathrm{ext}\,\Omega = \partial\Omega + C\,\mathrm{cl}\,\Omega \supset \mathrm{ext}\ \mathrm{cl}\,\Omega = C\,\mathrm{cl}\,\Omega$.

Lemma 5.1 *The boundary $\partial\Omega$ separates R^N into disjoint open sets:*

$$\Omega = \mathrm{int}\ \mathrm{cl}\,\Omega \quad \textit{and} \quad C\,\mathrm{cl}\,\Omega = \mathrm{ext}\ \mathrm{cl}\,\Omega, \quad \Omega \cap C\mathrm{cl}\,\Omega = \emptyset.$$

Lemma 5.2 *Any continuous curve L in R^N, joining $x_0 \in \Omega$ and $x_1 \in \operatorname{ext}\Omega$, intersects the boundary $\partial\Omega = \{x \mid V(x) = \nu_0\}$.*

Lemma 5.3 *If $v_0' < v_0$, then for the same $V(x)$ the surfaces $\partial\Omega(v_0')$ and $\partial\Omega(v_0)$ are strictly enclosed:*

$$\operatorname{cl}\Omega(\nu_0') \subset \Omega = \Omega(\nu_0). \tag{5.10}$$

Remark 5.1 The requirements of uniqueness and a finite measure of a V-surface are imposed in (1^*) to avoid unnecessary complications. Such pathological cases do exist, for example, the function $V = (x_1^2 + x_2^2)\sin^2(x_1^2 + x_2^2)$ with nice properties: $V(x) = 0$ for $\|x\|^2 = x_1^2 + x_2^2 = \pi n,\ n = 0, 1, \dots,$ otherwise $V(x) > 0$, presents for each $\nu_0 > 0$ a countable (denumerable) set of surfaces $V = \nu_0$ in R^2 which can be constructed by the equation $\sin^2(x_1^2 + x_2^2) = \nu_0/(x_1^2 + x_2^2)$. Such functions are not allowed by the condition (1^*).

The set of C^1-functions satisfying $(1^*)-(2^*)-(3^*)$ is not empty. Any real ellipsoid centered at the origin

$$V = \sum a_i x_i^2 = \nu_0, \quad a_i > 0, \quad i = 1 \div n$$

presents such a V-function for $v_0 > 0$, that is, $v_0 \in B = R_+$, thereby with additional properties: $V(x) > 0$ for all $x \neq 0$, $V(0) = 0$, that are not required in this research. The property $V(0) = 0$ disappears for ellipsoids centered not at the origin.

Non-sign-definite functions of the type:

$$V_k = \sum a_i (x_i - \alpha_i)^{2k} + \beta, \quad k = 1, 2, \dots, a_i > 0, \quad i = 1 \div n,$$

where β, α_i are real constants, are also allowed. Planes, cylinders, cones, paraboloids are not allowed since Ω is unbounded. Functions of the type

$$V_k = \sum a_i |x_i - \alpha_i|^{2k} + \beta, \quad k = 1, 2, \dots, a_i > 0, \quad i = 1 \div n$$

satisfy (1^*), (2^*), (3^*) for an appropriate interval $B \subset R$ but are not differentiable at $x_i = \alpha_i$. If however special care is taken at those corners, such functions can be allowed and were actually used for nonlinear stabilization in [28 – 30].

In some problems one might be interested in a bounded open region $\Delta \subset R^N$ only. In this case one can consider those $B \subset R$ and $\nu_0 \in B = B(\Delta)$ for which the conditions (1^*), (2^*), with x, x_0, x_1 all in $\Delta \subset R^N$ are satisfied and define the sets Ω_Δ, $C\operatorname{cl}\Omega_\Delta$ by the relations:

$$\Omega_\Delta = \{x \mid V(x) < \nu_0,\ x \in \Delta\} = \Omega(\nu_0, \Delta),$$
$$C\operatorname{cl}\Omega_\Delta = \{x \mid V(x) > \nu_0,\ x \in \Delta\}.$$

Clearly, $\Omega_\Delta \subset \Delta$ and is, therefore, always bounded so that (3^*) is automatically satisfied. To preserve the separation property in this case, we have to introduce it either directly by the condition:

(4^*a) The sets Ω_Δ and $C\Omega_\Delta$, nonempty by (2^*), are disjoint, that is

$$\Omega_\Delta \cap C\operatorname{cl}\Omega_\Delta = \emptyset;$$

or indirectly, by the condition:

(4*b) There exists $x_1 \in C\mathrm{cl}\,\Omega_\Delta$ such that $x_1 \notin \mathrm{cl}\,\Omega_\Delta$.

Condition (4*a) replaces Lemma 5.1 and it follows from (4*b) by Lemma 5.2. Geometrically it is clear that one of these conditions is necessary to exclude spiral and other surfaces that do not partition Δ into two disjoint subsets. Now planes, cylinders, paraboloids are allowed. We shall see, however, that this vast collection of V-functions is restricted by further considerations.

5.1.2 Moving V-surfaces. Suppose $x = x(t)$ is a C^1-function of time on $[t_0, \infty)$. Using one and the same $V(x)$, we can define the level function

$$\nu_0(t) = V(x(t)). \tag{5.11}$$

If this function is considered in (5.8) instead of a constant v_0, then we obtain a moving boundary

$$\partial\Omega(t) = \{x \mid V(x) = \nu_0(t)\} \tag{5.12}$$

and so in (5.12) $x \in R^N$ is any point on the surface and not the same as $x(t)$ in (5.11).

Take any $t_1 \in [t_0, t_f]$ and let the total derivative be negative:

$$\frac{d\nu_0}{dt} = \frac{dV}{dt} = \nabla V x' < 0, \qquad t = t_1. \tag{5.13}$$

Since $V(x) \in C^1$, then by continuity there exists $\delta > 0$ such that

$$t_2 = t_1 + \delta < t_f \quad \text{and} \quad \frac{d\nu_0}{dt} < 0 \quad \text{for all} \quad t \in [t_1, t_1 + \delta]. \tag{5.14}$$

The continuous function dv_0/dt is uniformly continuous on a closed segment $[t_1, t_2]$ and attains there its maximum:

$$\max \frac{d\nu_0}{dt} = c, \quad c < 0, \quad t \in [t_1, t_2]. \tag{5.15}$$

Now, (5.14) can be strengthened:

$$\frac{d\nu_0}{dt} \le -|c| < 0 \quad \text{for all} \quad t \in [t_1, t_2]. \tag{5.16}$$

Integrating (5.16) over $[t_1, t_2]$ yields

$$\nu_0(t_2) \le \nu_0(t_1) - |c|(t_2 - t_1) < \nu_0(t_1). \tag{5.17}$$

Thus, the new (moved) boundary $\delta\Omega(t_2)$ lies entirely in the interior of the old $\Omega(t_1)$, cf. Lemma 5.3:

$$\delta\Omega(t_2) \in \Omega(t_1) \tag{5.18}$$

and is separated from $\delta\Omega(t_1)$ by a band of the width (in terms of V-levels)

$$\Delta\nu_0 = \nu_0(t_1) - \nu_0(t_2) \ge |c|(t_2 - t_1). \tag{5.19}$$

Of course, here $c = c(\delta)$. Suppose now that (5.16) holds over the entire closed segment $[t_0, t_f]$. Then $c = \text{const} < 0$ and by the same argument we obtain that the curve $x(t)$ in finite time $\Delta t = t_f - t_0$ crosses the band between $\delta\Omega(t_0)$ and $\delta\Omega(t_f)$ at the moment $t = t_f + 0$ and stays there for a sufficiently small interval $(t_f, t_f + \varepsilon)$, $\varepsilon > 0$. If in addition

$$\frac{d\nu_0}{dt} = \nabla V x' < 0 \quad \text{for} \quad x(t) \in \delta\Omega(t_f), \;\; t \ge t_f, \tag{5.20}$$

where (5.20) is understood to hold every moment $t \ge t_f$ when the curve touches the boundary $\partial\Omega(t_f)$, then the curve $x(t)$ is not leaving the closure $\Omega(t_f)$, $\forall\, t \ge t_f$.

5.1.3 Carrying V-surfaces (V-carriages). Suppose that a family of trajectories $x(x_0, t_0, \cdot)$ is given by a differential equation

$$x' = \frac{dx}{dt} = f(x,t), \quad x_0 = x(t_0), \quad t \ge t_0. \tag{5.21}$$

Then (5.20) takes the simple form

$$\nabla V f(x,t) = \sigma(x,t) < 0 \tag{5.22}$$

and can be evaluated at every point of a region in space and time, in our case in $\Omega(t_0) \times [t_0, t_f]$, $\Omega(t_0) \subset R^N$, without integration of the equation (5.21). If $x_0 \in \Omega(t_0)$ and (5.22) holds for the closed region:

$$x \in \Omega(t_0) - \Omega(t_f), \quad \Omega(t_f) \subset \Omega(t_0) \;\; (\text{closed band in } R^N)$$
$$t \in [t_0, t_\alpha], \quad t_\alpha \ge t_f \;\; (\text{closed segment in time})$$

then the same argument holds and the entire family of solutions of (5.21) once trapped in $\Omega(t_0)$ crosses the band $\Omega(t_0) - \Omega(t_f)$ in finite times (depending on x_0)

$$\Delta t(x_0) \le \frac{1}{|c|}\,[\nu_0(t_0) - \nu_0(t_f)], \tag{5.23}$$

where

$$c = \max \sigma(x,t) = \text{const} < 0, \quad x \in \Omega(t_0) - \Omega(t_f), \quad t_0 \le t \le t_f$$

and every solution stays in $\Omega(t_f)$ at least until $t = t_\alpha$.

The construction resembles the well-known Lyapunov design. However, we do not require that $V(x)$ be sign-definite, nor that $V(0) = 0$.

5.2 The control theorem

Consider the set of all V-functions. Given $\Delta \subset R^N$, $M \subset \Delta$ and a function $V(x)$, define the following constants and sets ($\partial\Delta$, ∂M denote the boundaries of Δ, M):

$$\nu^+ = \sup V(x) \mid x \in \partial\Delta, \tag{5.24}$$

$$\Omega^+ = \{x \mid V(x) < \nu^+\}, \tag{5.25}$$

$$\nu^- = \inf V(x) \mid x \in \partial M, \tag{5.26}$$

$$\Omega^- = \{x \mid V(x) < \nu^-\}. \tag{5.27}$$

Unless otherwise stated, Ω^- is assumed to be non-empty. We assume $f(x,t)$ of (5.4) to be defined and solutions to exist in the closure Ω^+. Suppose that Ω^+ and Ω^- are simply connected. Discard all V-functions for which either $\nu^- \geq \nu^+$ or $\Delta \not\subseteq \Omega^+$, or $\Omega^- \not\subseteq M$. The remaining subset Π which is assumed to be non-empty contains only those $V(x)$ for which the following inclusions hold:

$$\Omega^- \subseteq M \subset \Delta \subseteq \Omega^+. \tag{5.28}$$

Denote the closed complement

$$CM = \Omega^+ - \Omega^-, \tag{5.29}$$

non-empty since $M \neq \Delta$.

Theorem 5.1 *Given $M \subset \Delta$, $x_0 = x(t_0) \in \Delta - M$ and a constant T, $t_f - t_0 > T > 0$, the motion $x(x_0, t_0, t)$ enters M not later than at the moment $t^* = t_0 + T$ and stays there, if there is a function $V \in \Pi$ such that for all $(x,t) \in CM \times [t_0, t_f)$ we have*

$$\nabla V f(x,t) \leq -c, \tag{5.30}$$

where

$$c = \frac{\nu^+ - \nu^-}{T} = \text{const} > 0. \tag{5.31}$$

Proof follows from the above considerations, see [33].

Remark 5.2 One cannot substitute M for Ω^- in (5.29).

It is apparent that the above theorem is well in the spirit of Lyapunov, with the difference that it presents sufficient conditions for guaranteed transfer from a given point into a given domain in finite time specified beforehand. This theorem can be specified to include the limit operation as $t \to \infty$ for the case of the perturbation equation in (5.4) with $t_f = \infty$, $f(0,t) = 0$, and to deduce the well known classical results of Lyapunov [1] in stability theory, see [34]. This makes clear that the set Π of V-functions is non-empty and contains positive definite functions used by Lyapunov. It also opens a way to apply known methods of constructing Lyapunov functions to more general functions $V \in \Pi$.

In [32] this approach is applied for differential games, cf. Theorem 4.2 above where θ is the target set. In [35] it is applied for asymptotic observer design in differential games with incomplete information.

In control applications, usually a part of coordinates of the state vector $x \in R^N$ are directly measured, or a function thereof that constitute the information vector $y = g(x,t) + \gamma$, $y \in R^k$, $k < n$, containing measurement noise $\gamma(t)$. In this case, a controller is taken either in the form $u = u(y,t)$ for the output feedback control, or in the form $u = u(z,t)$, where $z(t)$ is the observer, that is, an approximation to $x(t)$ computed from a model

$$\frac{dz}{dt} = h(y,u,t), \quad z(t_0) = z_0, \quad t \geq t_0 \tag{5.32}$$

constructed in such a way that the error

$$\epsilon(t) = z(t) - x(t) \tag{5.33}$$

does not leave some neighborhood of the origin and is attracted to the origin sufficiently fast. This way of obtaining an acceptable estimate of $x(t)$ for use in control is called asymptotical observation or adaptive identification [35, 37–44].

For a linear stationary control system

$$\frac{dx}{dt} = Ax + Bu, \quad y(t) = Cx \tag{5.34}$$

the construction of a model (5.32) is very simple [38,39]:

$$\frac{dz}{dt} = Hz + Qy + Bu, \quad H = A - QC. \tag{5.35}$$

Subtracting (5.34) from (5.35), we get the error equation, cf. (5.33):

$$\frac{d\epsilon}{dt} = Hz - Ax + Qy = H(z - x) = H\epsilon(t). \tag{5.36}$$

The matrices A, B, C are known, and it remains to provide appropriate eigenvalues for the matrix H in (5.35), (5.36) by the choice of the matrix Q, see [40,44].

For a nonlinear control system, the construction of the model (5.32) is not so simple and Lyapunov's approach should be used for a proper asymptotic observer design [35,37,41–43].

6 Stability by Time-Space Mosaic with Discontinuous Lyapunov Function

By a theorem of Massera [46], if the trivial solution $x = 0$ of a perturbation equation with Lipschitzian right-hand side is uniformly asymptotically stable in the large, then there exists a Lyapunov function $V(x,t)$ that guarantees this type of stability.

In practical cases, a particular solution may be uniformly asymptotically stable but not in the large. Too, stability in the large as well as uniform stability, though comfortable, are not usually required in practice.

Even if the existence of a Lyapunov function is established, there is no universal method for constructing Lyapunov functions, and its construction is difficult in almost all nontrivial cases. These difficulties led to the development of vector [22,23] and matrix [24] Lyapunov functions which act on regions of the subdivided state space through which trajectories are passing.

The generalized perturbation equation described in Section 2 opens a way to use different contracting Lyapunov functions for different periods of time. The surfaces defined by such Lyapunov functions form a time-space mosaic, or in other words, a discontinuous Lyapunov function, which is easier to construct and which can serve for establishing stability of motion. This approach was developed in the joint work [25] with V.V.Rumyantsev.

In stability analysis, deviations $w(t)$ are studied in a neighborhood H of the origin and one is interested to determine whether or not for every $\eta > 0$ there exists $\delta(\eta) > 0$ such that if

$$\|w_0\| \le \delta(\eta), \tag{6.1}$$

then

$$\|w(t)\| < \eta \quad \text{for all} \quad t > t_0, \tag{6.2}$$

where $\|\cdot\|$ is the Euclidean norm. If the answer to this question is in the affirmative, then the motion $w(t) = 0$ is called stable, otherwise, unstable. It means that if there exists

$\eta_0 > 0$ such that, whatever small $\delta > 0$ may be, there is a moment $t_* > t_0$ at which $\|w(t_*)\| = \eta_0$, then the motion is unstable. If at some moments $t_i^* > t_0$, perturbations grow to a fraction of the magnitude of a nominal coordinate, $|w_j(t_i^*)| = \alpha_j |x_j(t_i^*)|$, $\alpha_j = \text{const} \geq 1,\ 1 \leq j \leq n$, then the motion is unstable.

A stable motion with the additional property

$$\lim \|w(t)\| = 0, \quad t \to \infty \tag{6.3}$$

is called asymptotically stable. These are the classical definitions of stability given by Lyapunov [1]. With the notation (2.3), it refers, of course, to the stability of the solution $x^0(t)$. Let us not fix the initial condition $x^0(t_0) = x_0 \in \Delta_0$, considering instead a collection of nominal solutions $\{x^0(t)\} = x^0(\{x_0\}, t_0, t)$ corresponding to a set $\{x_0\} \subseteq \Delta_0$ of initial conditions; the notation $\{x_0\}$ may mean a finite collection or a set, a continuum.

To study and solve the problem by Lyapunov's second (direct) method, C^1-functions $V(w,t)$, $W(w)$, $W^1(w)$, $W^*(w)$ are considered that vanish if $w = 0$,

$$V(0,t) = W(0) = W^1(0) = W^*(0) = 0, \quad t \geq t_0, \tag{6.4}$$

and have some additional properties.

Recall the basic theorems of Lyapunov's second method.

Theorem 6.1 (Lyapunov [1]) *If there exists a function $V(w,t)$ satisfying the conditions*

$$\text{(a)} \quad V(w,t) \geq W(w) > 0, \quad w \in H, \quad w \neq 0, \quad t \geq t_0; \tag{6.5}$$

$$\text{(b)} \quad \frac{dV}{dt} = \frac{\partial V}{\partial t} + \nabla V \cdot q(w, x^0, t) \leq 0, \quad w \in H, \quad t \geq t_0 \tag{6.6}$$

on the trajectories of the perturbation equation, then the solution $w(t) = 0$ is stable.

Theorem 6.2 (Lyapunov [1]) *If there is a function $V(w,t)$ satisfying condition (a) and the strengthened (cf. (b)) conditions:*

$$\text{(c)} \quad \frac{dV}{dt} = \frac{\partial V}{\partial t} + \nabla V \cdot q(w, x^0, t) \leq -W^1(w) < 0, \quad w \in H,\ w \neq 0,\ t \geq t_0; \tag{6.7}$$

$$\text{(d)} \quad W^*(w) \geq V(w,t), \quad w \in H, \quad t \geq t_0, \tag{6.8}$$

then the solution $w(t) = 0$ is asymptotically stable.

Theorem 6.3 (Chetaev [2]) *If there exists a function $V(w,t)$ satisfying the conditions:*

$$\text{(e)} \quad \textit{the set} \quad \Sigma_\eta^t = \{w \in H \mid V(w,t) > 0,\ t \geq t_0\} \cap \{\|w\| < \eta,\ \eta > 0\} \neq \emptyset \tag{6.9}$$

is nonempty for all $t \geq t_0$ and any small $\eta > 0$;

$$\text{(f)} \quad V(w,t) \quad \textit{is bounded within } \Sigma_\eta^t; \tag{6.10}$$

$$\text{(g)} \quad \frac{dV}{dt} = \frac{\partial V}{\partial t} + \nabla V \cdot q(w, x^0, t) > 0, \quad w \in \Sigma_\eta^t, \quad w \neq 0, \tag{6.11}$$

on the trajectories of the perturbation equation, meaning that dV/dt *is positive definite in* Σ^t_η*, in the sense that for every small* $\varepsilon > 0$ *there is* $\gamma > 0$ *such that if* $V(w,t) \geq \varepsilon$*, then*

$$\frac{dV}{dt} \geq \gamma \quad \text{for all} \quad t \geq t_0, \tag{6.12}$$

then the motion $w(t) = 0$ *is unstable.*

Geometrically, condition (6.11) together with (6.12) mean that if $w(t) \in \Sigma^t_\eta$ is uniformly separated from the boundary $\partial\Sigma^t_\eta$ for all $t \geq t_0$, then $dV/dt \geq \gamma > 0$ is uniformly separated from zero for all $t \geq t_0$, see [2, Section 13]. As distinct from (6.5), a function $V(w,t)$ in (6.9) need not be positive definite.

Of course, stability, asymptotic stability or instability of the solution $w(t) = 0$ implied by Theorems 6.1 – 6.3 means the same property of all nominal solutions $\{x^0(t)\}$ for which (6.6), or (6.7), or (6.11) – (6.12), respectively, are fulfilled.

Consider x^0 in (2.6), (2.7) and x in (2.8) not as a particular solution, but as a parameter. Then inequalities (6.6), (6.7), (6.11) become characteristics of a domain (simply connected open set)

$$E = \mathcal{D} \times (t', t' + T), \quad \mathcal{D} \subseteq \Delta \subseteq R^n, \quad t' \geq t_0 \text{ fixed}, \quad T > 0, \tag{6.13}$$

where $\mathcal{D}$ may vary with $t \in (t', t' + T)$.

With x, t considered as independent variables, the left-hand side of (6.6), (6.7), (6.11) becomes a function $F\colon R^n \times R^n \times R \to R$ of three arguments

$$F(w,x,t) = \frac{\partial V}{\partial t} + \nabla V \cdot q(w,x,t), \tag{6.14}$$

which coincides with the total derivative $V' = dV/dt$ of a chosen function $V(w,t)$ on trajectories $w(t)$ of the perturbation equation (2.6).

Consideration of such functions (6.14) and domains (6.13) is motivated by the need to evaluate the rate of attraction of perturbed motions to a nominal solution of (2.1) within a finite time interval, and for all nominal trajectories passing through domain E of (6.13). For processes evolving in a finite space-time region, such information may be useful irrespective of stability properties on $[t_0, \infty)$. In such considerations, perturbations w do not have to be small.

Definition 6.1 If for a chosen $V(w,t)$ satisfying (6.5) on an interval $(t', t' + T)$, the condition (6.6) or (6.7) holds for $(x,t) \in E$, then *domain* E is called *neutral* or *contractive*, respectively.

Definition 6.2 If for a chosen $V(w,t)$ satisfying (6.9), (6.10) on an interval $(t', t'+T)$, the condition (6.11) holds for $(x,t) \in E$, then *domain* E contains a repulsive sector Σ^t_η; such domain E is called *repulsive.*

The statement that a certain domain E is contractive, neutral or repulsive means that there is a function $V(w,t)$ mentioned in Definitions 6.1, 6.2 which renders the corresponding property of E. The availability of such a function defines the corresponding domains. For example, if $V(w,t)$ satisfies (6.5), (6.7) for all $t \geq t_0$, then our domain becomes a contractive band $E = \mathcal{D} \times [t_0, \infty)$ with one sole Lyapunov function which is the classical case.

Remark 6.1 The names *contractive* or *repulsive domain* relating to the (x,t)-space should not be confused with the names *domain of attraction* or *repulsion* relating to the w-space, as in Example 2.1.

To illustrate the geometry corresponding to Definitions 6.1, 6.2, we can use the standard argument of the Lyapunov stability theory [1, 2]. Consider, for example, a neutral domain $E_1 = \mathcal{D}_1 \times [t_0, t_1)$. For a given $\eta > 0$, let

$$\gamma_1 = \inf W_1, \quad \|w\| = \eta; \quad \text{due to (6.5)}, \; \gamma_1 > 0. \tag{6.15}$$

Since $V_1(w, t_0)$ does not depend on t, so due to (6.4) and to the continuity of V_1 there is $\delta > 0$ such that for $\|w\| \le \delta$ we have $V_1(w_0, t_0) < \gamma_1$. Choosing such initial conditions and due to the relation

$$V_1 - V_1(w_0, t_0) = \int_{t_0}^{t} V_1' \, dt, \quad V_1' \le 0 \quad \text{as of (6.6)}, \; t \in [t_0, t_1], \tag{6.16}$$

we obtain that $w(t)$ is such that the following conditions are satisfied

$$W_1 \le V_1(w,t) \le V_1(w_0, t_0) < \gamma_1, \qquad t \in [t_0, t_1] \tag{6.17}$$

implying $\|w(t)\| < \eta$ for $t \in [t_0, t_1]$.

It means that, over a neutral domain, perturbations within a ball $\|w\| < \eta$, where (6.4) – (6.6) are satisfied cannot escape this ball whatever $(x,t) \in E_1 = \mathcal{D}_1 \times [t_0, t_1)$. If $t_1 = \infty$, stability follows.

If we have strict inequality $V_1' < 0$ in (6.16), compare with (6.7), then domain E_1 is contractive. If $t_1 = \infty$ and we use the additional condition (6.8), then asymptotic stability follows by the standard argument [1, 2].

However, if we consider two adjacent domains with different functions V_1, V_2 (with one common function it would be one single domain), then neutrality or contractivity of the union does not follow from the same property for component domains. Indeed, continuing the argument (6.13) – (6.17) for $E_2 = \mathcal{D}_2 \times [t_1, t_2]$, we denote $\eta_1 = \|w(t_1)\|$. Clearly, $\eta \ge \eta_1 > 0$ since, otherwise, the value $w(t_1) = 0$ of the solution $w(t) \not\equiv 0$ would contradict the uniqueness of a solution emanating from the point $(t_1, 0)$ due to the existence of the trivial solution $w(t) \equiv 0$. Let

$$\gamma_2 = \inf_{\|w\| = \eta_1} W_2. \tag{6.18}$$

Since $V_2(w, t_1)$ does not depend on t so due to (6.4) and to the continuity of V_2, there is $\delta_2 > 0$ such that for $\|w_1\| \le \delta_2$ we have $V_2(w_1, t_1) < \gamma_2$. However, $w_1 = w(t_1) = w(w_0, t_0, t_1)$ comes from E_1 and cannot be chosen so as $\|w_1\| \le \delta_2$ for appropriate $\delta_2 > 0$. Hence, to continue the argument and to assure that finite or countable union of adjacent neutral (contractive) domains be also neutral (contractive), we have to impose the following condition.

Consistency condition. A sequence of adjacent or overlapping neutral (contractive) domains $E_1, E_2, \ldots$ with functions $V_1, V_2, \ldots$, acting on $[t_0, t_1)$, $[t_1, t_2), \ldots$, and satisfying (6.4) – (6.6), or (6.7) for contractive domains, is called consistent if the functions $V_1, V_2, \ldots$ are such that, with the initial condition $\|w\| \le \delta(\gamma_1)$ for a given $\eta > 0$

in (6.15), we have $V_2(w_1, t_1) < \gamma_2$ for $w_1 = w(w_0, t_0, t_1)$ and any $x \in \mathcal{D}_2(t_1)$, then $V_3(w_2, t_2) < \gamma_3$ for $w_2 = w(w_1, t_1, t_2) = w(w_0, t_0, t_2)$ and any $x \in \mathcal{D}_3(t_2)$, etc., for all $V_n,\ n = 2, 3, 4, \dots$ in the sequence. It simply means that the solution $w(w_0, t_0, t)$ at times $t = t_1, t_2, \dots, t_n, \dots$ is picked by the next function with the same properties as previous functions plus the property of no escape from the sphere (ball) already attained. Consistent domains do exist, for example, if $V_n = c_n\|w\|^2$ or if V_n are considered as pieces on $[t_{n-1}, t_n)$ of one single Lyapunov function $V(w,t),\ t \in [t_0, \infty)$, existing under certain conditions [4, 46].

Definition 6.3 If there is a band $E_0 = \mathcal{D}_0 \times (t_0, \infty)$, $\mathcal{D}_0 \subseteq \Delta$, that can be covered by a finite or countable chain of consistent neutral (respectively, contractive) domains, such a *band* is called *neutral* (*respectively, contractive*).

Theorem 6.4 *Every solution which is entirely in a neutral band is stable.*

Proof There is a sequence of functions $V_1, V_2, \dots$ acting on $[t_0, t_1), [t_1, t_2), \dots$ and satisfying (6.4) – (6.6) that corresponds to a cover by a finite or countable chain of consistent neutral domains. If the chain is finite, we prove the theorem after a number of repetitions of the above argument (6.13) – (6.17) since the last $t_k = \infty$. If the chain is countable, then $t_n \to \infty$, thus, for every $t \in [t_0, \infty)$ there is a subsegment to which it belongs, yielding $\|w(t)\| < \eta$ for all $t \geq t_0$.

A solution which is entirely in a contractive band may not be asymptotically stable though its stability follows from Theorem 6.4 since (6.6) is implied by (6.7). If the chain is finite and for the last function $V_k(w,t)$ acting on $[t_k, \infty)$ the condition (6.8) is satisfied, then asymptotic stability follows from the classical Lyapunov Theorem [1].

For a countable chain of consistent contractive domains, consider a sequence of corresponding functions

$$V_i(w,t), \quad t \in [t_{i-1}, t_i), \quad t_i \to \infty \text{ as } i \to \infty, \quad i = 1, 2, \dots, \tag{6.19}$$

each acting over corresponding domain E_i of finite time length $\Delta t_i = t_i - t_{i-1} \geq \tau > 0$. Functions (6.19) may be regarded as components of a piecewise continuous function $V(w,t)$ acting on $[t_0, \infty)$, which components should satisfy the consistency condition stated above.

Now, condition (6.8) can be extended onto the sequence (6.19) as follows. From (6.5), (6.8) we have

$$W^*(w) \geq V(w,t) \geq W(w) > 0, \quad w \in H, \quad w \neq 0, \tag{6.20}$$

where $V(w,t)$ represents $V_i(w,t)$ over each $[t_{i-1}, t_i)$ of (6.19). Since $W^*(w) \to 0$ as $\|w\| \to 0$, so for appropriate $\eta > \eta_* > 0$ the surface $V(w,t) = \gamma$ is enclosed in the spherical ring

$$\eta \geq \|w\| \geq \eta_*, \tag{6.21}$$

provided that $\eta > \gamma > \eta_*$ and the ring (6.21) is in the region H. Indeed, it is sufficient to take such η, η_* that the sphere $\|w\| = \eta$ is circumscribed around $W^*(w) = \eta_1 \leq \eta$, and the sphere $\|w\| = \eta_*$ is inscribed in $W(w) = \eta_2 \geq \eta_*,\ \eta_1 > \eta_2$. Since $\eta_1 = W^*(w) \to 0$, as $\|w\| \to 0$, we can take $\eta \to 0$. Vice versa, if (6.8) holds, then for any spherical ring (6.21) in the region H, by virtue of (6.20), (6.8), there exist functions of (6.19) acting over this ring (we say in such case that ring (6.21) is covered by consistent contractive domains).

Take a decreasing sequence $\eta = \eta_1 > \cdots > \eta_k > \eta_{k+1} > \ldots$, $\lim \eta_k = 0$, and consider rings $R_k = \{w \in H \mid \eta_k \geq \|w\| \geq \eta_{k+1}\}$, $k = 1, 2, \ldots$. Consider all functions $V_i(w,t)$ from (6.19) acting over the ring R_k. By (6.7) every $V_i' < 0$ which means that there exists $W_i^1(w)$ such that over the segment of definition of $V_i(w,t)$ we have definite negative and bounded from zero total derivatives

$$-V_i'(w,x,t) \geq W_i^1(w) > 0, \quad w \in H, \quad w \neq 0,$$
$$(x,t) \in E_i = \mathcal{D}_i \times [t_{i-1}, t_i). \tag{6.22}$$

Let

$$\gamma_{ik} = \inf W_i^1(w) \geq \gamma_k > 0, \quad w \in R_k. \tag{6.23}$$

The uniform bound $\gamma_k > 0$ exists for all V_i acting over R_k since otherwise $W_i^1(w)$ would not be separated from zero within closed R_k, not containing zero, in contradiction with definition of a positive definite function.

Now, integrating the piecewise continuous function $V(w,t)$ with components (6.19) along a trajectory (or a part thereof) lying entirely within R_k, we obtain by (6.22), (6.23)

$$V - V(w_*, t_*) = \int_{t^*}^{t} V'\, dt \leq -\sum_i \gamma_{ik}(t_i - t_{i-1}) \leq -\gamma_k(t - t_*), \tag{6.24}$$

where the sum covers all components $V_i(w,t)$ acting over R_k and t_* is the starting time of a perturbed trajectory. From (6.5), (6.24), we get

$$0 < V(w,t) \leq V(w_*, t_*) - \gamma_k(t - t_*), \quad \gamma_k > 0, \tag{6.25}$$

meaning that there is only finite time $(t - t_*) \leq T_k < \infty$ during which a trajectory can stay within R_k. Since the band is contractive, the perturbed trajectory $w(t)$ will leave R_k, approaching zero, so that for $t > t_* + T_k$ we have $\|w(t)\| < \eta_{k+1}$. By (6.8), for any ring R_k, $k = 1, 2, \ldots$, there are V_i from (6.19) that act over that ring, hence $\lim_{t\to\infty} \|w(t)\| = \lim_{k\to\infty} \eta_k = 0$. This proves the following theorem.

Theorem 6.5 *If a contractive band is such that for any $\eta > 0$ there is $N(\eta)$ such that for all $i \geq N(\eta)$ functions $V_i(w,t)$ of (6.19) satisfy the condition $\eta \geq V_i(w,t) > 0$, $w \in H$, $w \neq 0$, $t \in [t_{i-1}, t_i)$, $t_i \to \infty$, as $i \to \infty$, then every solution passing entirely within such a band is asymptotically stable.*

Remark 6.2 The above arguments resemble the analysis based on property (A) or (B) in [4, Sections 4, 5], under which there exists a Lyapunov function $V(w,t)$ acting on $[t_0, \infty)$ with sign definite derivative that renders asymptotic stability of certain nominal solution $x^0(t)$. However, it may be difficult to find such a function and, if found, it serves one particular solution only. Functions (6.19) may be easier to construct, and they serve all solutions passing through corresponding domains E_i. If considered as components of one function $V(w,t)$, this function, though generally discontinuous, renders, under certain conditions, the same conclusions about stability or asymptotic stability as a classical Lyapunov function.

Remark 6.3 In contrast and similarity with vector Lyapunov functions introduced, e.g. in [22, 23], that create a space mosaic based on the idea that each subsequent function (all acting on $[t_0, \infty)$) covers a manifold (or a part thereof) where preceding functions are inconclusive (e.g. where $V' = 0$), the functions (6.19) correspond to a time-space mosaic of consistent domains which domains, if forming a band extending over $[t_0, \infty)$, deliver the same stability properties as a conventional Lyapunov function.

Functions V_i of (6.19) corresponding to a chain of consistent contractive domains can be used to obtain quantitative results concerning the measure of contraction within every domain E_i, see [25].

7 Conclusions

Developments presented in this survey complement the classical stability theory in different directions. First, it seems important that investigation of stability should be possible without integration of equations of motion. This possibility is provided by the generalized perturbation equation which implicitly contains trajectories of the nominal equation passing through the x-space included as parameter-space in the generalized perturbation equation acting in the w-space of perturbations. As a by-product, such relaxation of a fixed particular solution around which the classical perturbation equation is constructed allows us to investigate stability of all nominal solutions passing through the x-space. Thus, the explicit integration of the nominal equation which is difficult if not impossible in many practical cases becomes unnecessary. This also opens the avenue for numerical investigation of stability.

Second, Lyapunov functions usually constructed as smooth functions do not have to be differentiable. They can be even discontinuous, if certain consistency condition is respected. This expansion of the class of possible Lyapunov functions is of much interest in view of difficulties encountered in attempts to construct a Lyapunov function for a more complicated practical system.

Further, the extension of the Barbashin-Krasovskii theorem onto nonperiodic systems has been long overdue. Indeed, it was puzzling that this important and much used theorem should be valid only for systems with such easy-to-see fashionable property as being stationary or with a periodic right-hand side. The result presented in Section 4 extends the validity of this theorem to systems of class A whose solutions satisfy a condition that resembles the Cauchy compactness criterion.

Another generalization was to apply the idea of decomposition of motion (embodied in Lyapunov's approach) to the controller and observer design for nominal systems. This development required the relaxation or modification of classical Lyapunov conditions, leading, in fact, to new functions and to a different framework. Well in the spirit of Lyapunov, this approach can be used for new classes of problems such as motion control, dynamic games and asymptotic observer design. Quite naturally, in application to stability and stabilization it brings us back to the classical Lyapunov results.

Using this framework and the generalized perturbation equation, it became possible to develop a time-space mosaic method, a sort of Lyapunov-like assembly line along the time axis, that allows us to substitute a single continuous Lyapunov function acting on $[t_0, \infty)$ by separate independent functions easier to construct, provided the consistency condition is satisfied. Apart from analytical advantages in stability analysis, it opens a way to "practical stability" evaluations (on a finite interval of time, cf. [17, 27]) through on-line computations of the rate of attraction. If combined with the space-splitting furnished by vector and matrix Lyapunov functions, see [22 – 24], this presents a complete time-space mosaic in $R^n \times R$ which could provide a powerful tool for solution of complicated practical problems.

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