Nonlinear Dynamics and Systems Theory, **22** (5) (2022) 489–502

Controllability of Dynamic Equations with Memory

Martin Bohner [1*], Cosme Duque [2] and Hugo Leiva [3]

[1] *Missouri S&T, Rolla, Missouri 65409-0020, USA.*
[2] *Department of Mathematics, Faculty of Sciences, Los Andes University, Mérida, Venezuela.*
[3] *School of Mathematical and Computational Science, Yachay Tech. University, San Miguel de Urcuqui, Imbabura, Ecuador.*

Received: May 26, 2022 Revised: November 11, 2022

Abstract: In this work, we consider a control system governed by a dynamic equation with memory. We obtain conditions under which the system is approximately controllable and approximately controllable on free time. In order to do this, we use a technique developed by Bashirov et al. [4–6], where we can avoid fixed point theorems. But first of all, we prove the existence and uniqueness of solutions of the system and after that, we prove the prolongation of solutions under some additional condition. Finally, we present several examples to illustrate the applicability of our results.

Keywords: *controllability; semilinear dynamic equations; memory; time scales.*

Mathematics Subject Classification (2010): 93C10, 93C23, 34N05, 34K42.

1 Introduction

Control theory addresses how a system can be modified through feedback, in particular, how an arbitrary initial state can be directed either exactly or approximately close to a given final state using a control in a set of admissible controls. In the last decades, control theory of dynamic equations on time scales has attracted the attention of several researches, because this is a powerful tool that allows to study from a unified point of view controllability of continuous systems, discrete systems, systems in which the time variable can vary both continuously and discretely, as well as other types of time variables. Among the works made, we can cite Bartosiewicz [1] who explored linear positive control systems, Bartosiewicz and Pawłuszewicz [2, 3] reviewed linear systems, Janglajew and Pawłuszewicz [15] analyzed constrained local controllability of linear dynamic systems,

* Corresponding author: mailto:bohner@mst.edu

Bohner and Wintz [8] studied controllability and observability of linear systems, Grow and Wintz [13] proved existence and uniqueness of solutions to a bilinear state system with locally essentially bounded coefficients on an unbounded time scale. Approximate and exact controllability of semilinear systems on time scales was studied by Duque, Leiva and Uzcátegui in [10,11], Malik and Kumar in [18] established exact controllability for time-varying neutral differential equations with impulses. More works can be seen in [9,17,19] and references therein.

In this regard, in this paper, we will consider a control system governed by the dynamic equation with memory

$$\begin{cases} z^{\Delta}(t) = -A(t)z^{\sigma}(t) + B(t)u(t) + a\displaystyle\int_{t_0}^{t} M(t,s)g(s,z_\tau(s))\Delta s \\ \qquad\qquad + bf(t,z(t),u(t)), \quad t \geq t_0 \geq 0, \\ z(t) = \phi(t), \quad t \in [\tau(t_0),t_0]_{\mathbb{T}}, \end{cases} \tag{1}$$

where $z(t) \in \mathbb{R}^n$ is the state function, $z_\tau(t) = z(\tau(t))$, and $\tau : \mathbb{T} \to \mathbb{T}$ is the delay function which is increasing and unbounded on $\mathbb{T}$ such that $\tau(t) \leq t$ for $t \in \mathbb{T}$ (see [12]). $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, $B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times m})$, the control $u \in L^2_\Delta(\mathbb{T}, \mathbb{R}^m)$, $M : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ is a function that is locally essentially bounded on $\mathbb{T}\times\mathbb{T}$, the functions $f : \mathbb{T}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{T}\times\mathbb{R}^n \to \mathbb{R}^n$ are rd-continuous and there exist rd-continuous functions $L_f, L_g : \mathbb{T} \to \mathbb{R}^+$ such that

C1) $\|f(t,z,u) - f(t,\tilde{z},\tilde{u})\| \leq L_f(t)(\|z-\tilde{z}\| + \|u-\tilde{u}\|)$, with $f(t,0,0) = 0$,

C2) $\|g(t,z) - g(t,\tilde{z})\| \leq L_g(t)\,\|z-\tilde{z}\|$, with $g(t,0) = 0$.

The function ϕ lies in the space $C_{\mathrm{rd}}([\tau(t_0),t_0]_{\mathbb{T}}, \mathbb{R}^n)$, which is a Banach space endowed with the norm

$$\|\phi\|_0 = \sup\{\|\phi(t)\| : t \in [\tau(t_0),t_0]_{\mathbb{T}}\}.$$

In this paper, we suppose that the time scale $\mathbb{T}$ satisfies $-\infty < \tau(t_0) < \sup\mathbb{T} = \infty$.

The main goal of this work is to study controllability of system (1). Specifically, we shall show that under certain conditions, controllability of the associated linear system implies controllability of the semilinear dynamic equation with memory. In order to prove this assertion, we impose some conditions on the nonlinear terms presented in the system, and then apply a direct approach developed by A. E. Bashirov et al. (see [4–6]) to avoid fixed point theorems, and approximate controllability is achieved. But before that, we prove existence, uniqueness and continuation of solutions of the system. Finally, we consider some examples in which our results can be applied.

2 Preliminaries

Before studying system (1), we give a brief introduction to the calculus on time scales, especially to clarify notations and definitions, which will help for a better understanding of the reader. For more details about time scales theory, we recommend the excellent monograph [7].

Time scales theory was introduced by Stefan Hilger (see [14]). We define a time scale as any arbitrary nonempty closed subset of $\mathbb{R}$, this set is denoted by $\mathbb{T}$. For every $t \in \mathbb{T}$, the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined, respectively, as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

A point $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, right-scattered if $\sigma(t) > t$, left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, left-scattered if $\rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$. The function $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(t) := \sigma(t) - t$ is known as the graininess function. It is assumed that $\mathbb{T}$ has the topology inherited from standard topology on the real numbers. The time scale interval $[a, b]_{\mathbb{T}}$ is defined by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$, with $a, b \in \mathbb{T}$, and similarly we define open intervals and open neighborhoods.

Definition 2.1 (See [7]) A function $f : \mathbb{T} \to \mathbb{R}^n$ is said to be right-dense continuous, or just rd-continuous, if f is continuous at every right-dense point $t \in \mathbb{T}$ and $\lim_{s \to t^-} f(s)$ exists (finite) for every left-dense point $t \in \mathbb{T}$. The class of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$. We define $f^\sigma : \mathbb{T} \to \mathbb{R}^n$ by $f^\sigma = f \circ \sigma$. We define the set $\mathbb{T}^\kappa$ by $\mathbb{T}^\kappa = \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$ if $\mathbb{T}$ has a left-scattered maximum, and $\mathbb{T}^\kappa = \mathbb{T}$ otherwise.

Definition 2.2 (See [7]) A function $f : \mathbb{T} \to \mathbb{R}^n$ is called delta differentiable (or simply Δ-differentiable) at $t \in \mathbb{T}^\kappa$ provided there exists $f^\Delta(t)$ with the property that given $\varepsilon > 0$, there is a neighborhood $U = (t - \delta, t + \delta)_{\mathbb{T}}$ for some $\delta > 0$ such that

$$\left\| f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s) \right\| \le \varepsilon \left| \sigma(t) - s \right| \text{ for all } s \in U.$$

In this case, $f^\Delta(t)$ is called the Δ-derivative of f at t.

If f is Δ-differentiable at $t \in \mathbb{T}^\kappa$, then it is easy to show that (see [7, Theorem 1.16])

$$f^\Delta(t) = \begin{cases} \dfrac{f^\sigma(t) - f(t)}{\sigma(t) - t} & \text{if } \sigma(t) > t, \\ \\ \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s} & \text{if } \sigma(t) = t. \end{cases}$$

Definition 2.3 (See [7]) A function $F : \mathbb{T} \to \mathbb{R}^n$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}^n$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. The Cauchy integral is defined by

$$\int_s^t f(\tau) \Delta\tau = F(t) - F(s), \quad t, s \in \mathbb{T},$$

where F is an antiderivative of f.

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$, $t \in \mathbb{T}$, and positively regressive if $1 + \mu(t)p(t) > 0$, $t \in \mathbb{T}$. We will denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions, and by $\mathcal{R}^+$ the set of all positively regressive and rd-continuous functions.

Definition 2.4 [See [7]] If $p \in \mathcal{R}$, then the generalized exponential function is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau \right),$$

where

$$\xi_\mu(z) := \begin{cases} \frac{\mathrm{Log}(1 + \mu z)}{\mu} & \text{if } \mu > 0, \\ z & \text{if } \mu = 0, \end{cases}$$

where $z \in \mathbb{C}_\mu := \{z \in \mathbb{C} : z \neq 1/\mu\}$ and $\mathrm{Log} z = \log |z| + i \arg z$, $-\pi < \arg z \le \pi$.

Definition 2.5 (See [7]) Let A be an $n \times n$ matrix-valued function on $\mathbb{T}$. We say that A is rd-continuous on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$, and the class of all such rd-continuous $n \times n$ matrix-valued functions on $\mathbb{T}$ is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$. A is called regressive (with respect to $\mathbb{T}$) provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^{\kappa}$, and the class of all such regressive and rd-continuous functions is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

Let $t_0 \in \mathbb{T}$ and A be an $n \times n$ regressive matrix-valued function defined on $\mathbb{T}$. Then, the unique solution of the initial value problem

$$X^{\Delta} = A(t)X, \quad X(t_0) = I,$$

is called the matrix exponential function, denoted by $e_A(t, t_0)$, and satisfies the properties

a) $e_0(t, s) \equiv I$ and $e_A(t, t) \equiv I$,

b) $e_A(t, s)e_A(s, r) = e_A(t, r)$,

c) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$,

d) $e_A(t, s) = e_A^{-1}(s, t) = e^*_{\ominus A^*}(s, t)$,

e) $e_A(t, s)e_B(t, s) = e_{A\oplus B}(t, s)$ if $A(t)$ and $B(t)$ commute,

where for $A, B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$,

$$A \oplus B = A + B + \mu AB \quad \text{and} \quad \ominus A = -(I + \mu A)^{-1}A.$$

3 Existence and Uniqueness

In this section, we show existence and uniqueness of solutions for system (1). The next theorem is a consequence of straightforward computation.

Theorem 3.1 *Consider a control $u \in L^2_{\Delta}(\mathbb{T}, \mathbb{R}^n)$. Then z is a solution of system* (1) *if and only if z satisfies the integral equation*

$$z(t) = \begin{cases} \phi(t), \quad t \in [\tau(t_0), t_0]_{\mathbb{T}}, \\ e_{\ominus A}(t, t_0)\phi(t_0) + \displaystyle\int_{t_0}^{t} e_{\ominus A}(t, s)B(s)u(s)\Delta s \\ \qquad + a\displaystyle\int_{t_0}^{t} e_{\ominus A}(t, s)\left[\int_{t_0}^{s} M(s, \xi)g(\xi, z_\tau(\xi))\Delta\xi\right]\Delta s \\ \qquad + b\displaystyle\int_{t_0}^{t} e_{\ominus A}(t, s)f(s, z(s), u(s))\Delta s, \quad t \geq t_0. \end{cases} \tag{2}$$

For fixed $\eta > t_0$, we denote

$$M_e = \sup\{\|e_{\ominus A}(t, s)\| : t, s \in [t_0, \eta]_{\mathbb{T}}\}, \quad M = \sup\{\|M(t, s)\| : t, s \in [t_0, \eta]_{\mathbb{T}}\},$$
$$\bar{L}_f = \sup\{L_f(t) : t \in [t_0, \eta]_{\mathbb{T}}\}, \quad \bar{L}_g = \sup\{L_g(t) : t \in [t_0, \eta]_{\mathbb{T}}\}.$$

Theorem 3.2 *Suppose there exists $\eta > t_0$ such that*

$$M_e \left(|a|\, M\bar{L}_g\eta + |b|\, \bar{L}_f\right)\eta < 1. \tag{3}$$

Then, for any $\phi \in C_{\mathrm{rd}}([\tau(t_0), t_0]_{\mathbb{T}}, \mathbb{R}^n)$ and $u \in L^2_{\Delta}(\mathbb{T}, \mathbb{R}^m)$, system (1) *has a unique solution through (t_0, ϕ) defined on $[\tau(t_0), \eta]_{\mathbb{T}}$.*

Proof. Let $\eta > t_0$ be such that (3) holds and consider $\phi \in C_{\mathrm{rd}}([\tau(t_0), t_0]_{\mathbb{T}}, \mathbb{R}^n)$ and $u \in L^2_\Delta([t_0, \eta]_{\mathbb{T}}, \mathbb{R}^m)$. Now, finding a solution of system (1) through (t_0, ϕ) is equivalent to solving the integral equation (2). In order to do this, we consider the function space

$$C_{\mathrm{rd}_\phi}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n) = \{z \in C_{\mathrm{rd}}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n) : z(t) = \phi(t) \text{ for } t \in [\tau(t_0), t_0]_{\mathbb{T}}\},$$

which is a Banach space endowed with the norm $\|z\|_* = \sup\{\|z(t)\| : t \in [\tau(t_0), \eta]_{\mathbb{T}}\}$, and we show that the operator

$$\mathcal{T} : C_{\mathrm{rd}_\phi}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n) \longrightarrow C_{\mathrm{rd}_\phi}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n)$$

defined by

$$(\mathcal{T}z)(t) = \begin{cases} \phi(t), \quad t \in [\tau(t_0), t_0]_{\mathbb{T}}, \\ e_{\ominus A}(t, t_0)\phi(t_0) + \displaystyle\int_{t_0}^{t} e_{\ominus A}(t, s)B(s)u(s)\Delta s \\ \qquad + a\displaystyle\int_{t_0}^{t} e_{\ominus A}(t, s)\left[\int_{t_0}^{s} M(s, \xi)g(\xi, z_\tau(\xi))\Delta\xi\right]\Delta s \\ \qquad + b\displaystyle\int_{t_0}^{t} e_{\ominus A}(t, s)f(s, z(s), u(s))\Delta s, \quad t \in [t_0, \eta]_{\mathbb{T}} \end{cases} \tag{4}$$

has a unique fixed point. Indeed, if $t \in [\tau(t_0), t_0]_{\mathbb{T}}$, then $(\mathcal{T}z)(t) = \phi(t) = z(t)$. If $t \in [t_0, \eta]_{\mathbb{T}}$, then for $z, \tilde{z} \in C_{\mathrm{rd}_\phi}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n)$ with $z \neq \tilde{z}$, we have

$$\begin{aligned}
&\|(\mathcal{T}z)(t) - (\mathcal{T}\tilde{z})(t)\| \\
&\le |a| \int_{t_0}^{t} \|e_{\ominus A}(t, s)\| \left[\int_{t_0}^{s} \|M(s, \xi)\| \, \|g(\xi, z_\tau(\xi)) - g(\xi, \tilde{z}_\tau(\xi))\| \, \Delta\xi\right] \Delta s \\
&\quad + |b| \int_{t_0}^{t} \|e_{\ominus A}(t, s)\| \, \|f(s, z(s), u(s)) - f(s, \tilde{z}(s), u(s))\| \, \Delta s \\
&\le |a| \int_{t_0}^{t} \|e_{\ominus A}(t, s)\| \left[\int_{t_0}^{s} M L_g(\xi) \, \|z(\tau(\xi)) - \tilde{z}(\tau(\xi))\| \, \Delta\xi\right] \Delta s \\
&\quad + |b| \int_{t_0}^{t} \|e_{\ominus A}(t, s)\| \, L_f(s) \, \|z(s) - \tilde{z}(s)\| \, \Delta s \\
&\le |a| \int_{t_0}^{t} M_e \left[\int_{t_0}^{s} M\bar{L}_g \, \|z - \tilde{z}\|_* \, \Delta\xi\right] \Delta s + |b| \int_{t_0}^{t} M_e \bar{L}_f \, \|z - \tilde{z}\|_* \, \Delta s \\
&\le |a| \int_{t_0}^{t} M_e M \bar{L}_g \eta \, \|z - \tilde{z}\|_* \, \Delta s + |b| \, M_e \bar{L}_f \eta \, \|z - \tilde{z}\|_* \\
&\le M_e \left(|a| \, M\bar{L}_g\eta + |b| \, \bar{L}_f\right) \eta \, \|z - \tilde{z}\|_* .
\end{aligned}$$

Therefore, using (3), we have

$$\|\mathcal{T}z - \mathcal{T}\tilde{z}\|_* \le M_e \left(|a| \, M\bar{L}_g\eta + |b| \, \bar{L}_f\right) \eta \, \|z - \tilde{z}\|_* < \|z - \tilde{z}\|_* ,$$

so that $\mathcal{T}$ satisfies all assumptions of the Banach contraction theorem, and therefore, $\mathcal{T}$ has only one fixed point in the space $C_{\mathrm{rd}_\phi}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n)$, which is the solution of problem (1).

Definition 3.1 We shall say that $[\tau(t_0), \eta)_{\mathbb{T}}$ is the maximal interval of existence of the solution z of system (1) if there is no solution of (1) on $[\tau(t_0), \eta^*)_{\mathbb{T}}$ with $\eta^* > \eta$.

Theorem 3.3 *If z is a solution of system (1) on $[\tau(t_0), \eta)_{\mathbb{T}}$ and η is maximal, then either $\eta = \infty$ or $z(t)$ is not bounded on any neighborhood of η.*

Proof. Suppose that $\eta < \infty$ and there is a neighborhood U of η such that $\|z(t)\| \le R$ for $t \in U$. In this case, we can suppose that $\|z(t)\| \le R$ for all $t \in [\tau(t_0), \eta)_{\mathbb{T}}$. If η is left-dense, then there is an increasing sequence $\{\eta_k\}_{k\ge 1}$ such that $\lim_{k\to\infty} \eta_k = \eta$ and $\lim_{k\to\infty} z(\eta_k) = z^*$ for some $z^* \in \mathbb{R}^n$. We shall see that $\lim_{t\to\eta^-} z(t) = z^*$.

Let $\varepsilon > 0$ be small enough. Since $\lim_{k\to\infty} \eta_k = \eta$, we can take $\eta_N \in (\eta-\varepsilon, \eta)_{\mathbb{T}}$ such that $\|z(\eta_N) - z^*\| < \varepsilon$. For $t \in (\eta-\varepsilon, \eta)_{\mathbb{T}}$ with $t > \eta_N$, we have

$$\|z(t) - z^*\| \le \|z(t) - z(\eta_N)\| + \|z(\eta_N) - z^*\| .$$

Now,

$$\begin{aligned}
\|z(t) - z(\eta_N)\| \le\ & \|e_{\ominus A}(t,t_0) - e_{\ominus A}(\eta_N,t_0)\| \, \|\phi(t_0)\| \\
& + \int_{t_0}^{\eta_N} \|e_{\ominus A}(t,s) - e_{\ominus A}(\eta_N,s)\| \, \|B(s)\| \, \|u(s)\| \, \Delta s \\
& + |a| \int_{t_0}^{\eta_N} \|e_{\ominus A}(t,s) - e_{\ominus A}(\eta_N,s)\| \left[\int_{t_0}^{s} \|M(s,\xi)\| \, \|g(\xi, z_\tau(\xi))\| \, \Delta\xi\right] \Delta s \\
& + |b| \int_{t_0}^{\eta_N} \|e_{\ominus A}(t,s) - e_{\ominus A}(\eta_N,s)\| \, \|f(s,z(s),u(s))\| \, \Delta s \\
& + \int_{\eta_N}^{t} \|e_{\ominus A}(t,s)\| \, \|B(s)\| \, \|u(s)\| \, \Delta s \\
& + |a| \int_{\eta_N}^{t} \|e_{\ominus A}(t,s)\| \left[\int_{t_0}^{s} \|M(s,\xi)\| \, \|g(\xi, z_\tau(\xi))\| \, \Delta\xi\right] \Delta s \\
& + |b| \int_{\eta_N}^{t} \|e_{\ominus A}(t,s)\| \, \|f(s,z(s),u(s))\| \, \Delta s \\
\le\ & \|e_{\ominus A}(t,t_0) - e_{\ominus A}(\eta_N,t_0)\| \, \|\phi(t_0)\| \\
& + \int_{t_0}^{\eta_N} \|e_{\ominus A}(t,s) - e_{\ominus A}(\eta_N,s)\| \, \|B(s)\| \, \|u(s)\| \, \Delta s \\
& + |a| \int_{t_0}^{\eta_N} \|e_{\ominus A}(t,s) - e_{\ominus A}(\eta_N,s)\| \left[\int_{t_0}^{s} M\bar{L}_g \, \|z(\tau(\xi))\| \, \Delta\xi\right] \Delta s \\
& + |b| \int_{t_0}^{\eta_N} \|e_{\ominus A}(t,s) - e_{\ominus A}(\eta_N,s)\| \, \bar{L}_f(\|z(s)\| + \|u(s)\|) \Delta s \\
& + \int_{\eta_N}^{t} \|e_{\ominus A}(t,s)\| \, \|B(s)\| \, \|u(s)\| \, \Delta s \\
& + |a| \int_{\eta_N}^{t} \|e_{\ominus A}(t,s)\| \left[\int_{t_0}^{s} M\bar{L}_g \, \|z(\tau(\xi))\| \, \Delta\xi\right] \Delta s \\
& + |b| \int_{\eta_N}^{t} \|e_{\ominus A}(t,s)\| \, \bar{L}_f(\|z(s)\| + \|u(s)\|) \Delta s \\
\le\ & \|e_{\ominus A}(t,t_0) - e_{\ominus A}(\eta_N,t_0)\| \, \|\phi(t_0)\| \\
& + \int_{t_0}^{\eta} \|e_{\ominus A}(t,s) - e_{\ominus A}(\eta_N,s)\| \, \|B(s)\| \, \|u(s)\| \, \Delta s
\end{aligned}$$

$$
\begin{aligned}
&+|a| \int_{t_0}^{\eta} \left\| e_{\ominus A}(t,s) - e_{\ominus A}(\eta_{_N},s) \right\| M \bar{L}_g R s \Delta s \\
&+|b| \int_{t_0}^{\eta} \left\| e_{\ominus A}(t,s) - e_{\ominus A}(\eta_{_N},s) \right\| \bar{L}_f (R + \|u(s)\|) \Delta s \\
&+\int_{\eta_{_N}}^{\eta} M_e \|B(s)\| \|u(s)\| \Delta s + |a| \int_{\eta_{_N}}^{\eta} M_e M \bar{L}_g R s \Delta s \\
&+|b| \int_{\eta_{_N}}^{\eta} M_e \bar{L}_f (R + \|u(s)\|) \Delta s.
\end{aligned}
$$

Hence, we get that, if $\eta_{_N} \to \eta$, then $\|z(t) - z(\eta_N)\| \to 0$, so $\lim\limits_{t\to\eta^-} z(t) = z^*$, and therefore, $z(t)$ can be continued beyond of η, contradicting our assumption.

If η is left-scattered, then $\rho(\eta) \in (t_0, \eta)_{\mathbb{T}}$ so that the solution z exists also at η, namely, by putting

$$
\begin{aligned}
z(\eta) =& [I + \mu(\rho(\eta))A(\rho(\eta))]^{-1} \Big\{ z(\rho(\eta)) + \mu(\rho(\eta))B(\rho(\eta))u(\rho(\eta)) \\
&+ a\mu(\rho(\eta)) \int_{t_0}^{\rho(\eta)} M(\rho(\eta), s) g(s, z_\tau(s)) \Delta s + b\mu(\rho(\eta)) f(\rho(\eta), z(\rho(\eta)), u(\rho(\eta)) \Big\},
\end{aligned}
$$

which is a contradiction.

Theorem 3.4 *If there exists Δ-differentiable $\varphi : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ such that*

$$\|g(t,z)\| \le \varphi^\Delta(t), \tag{5}$$

then the solution of system (1) *is defined on $[\tau(t_0), \infty)_{\mathbb{T}}$.*

Proof. Suppose that $z(t)$ is defined on $[\tau(t_0), \eta)_{\mathbb{T}}$ with $\eta < \infty$. Then, for $t \in (t_0, \eta)_{\mathbb{T}}$, we have

$$
\begin{aligned}
\|z(t)\| \le & \|e_{\ominus A}(t,t_0)\| \|\phi(t_0)\| + \int_{t_0}^{t} \|e_{\ominus A}(t,s)\| \|B(s)\| \|u(s)\| \Delta s \\
&+ |a| \int_{t_0}^{t} \|e_{\ominus A}(t,s)\| \left[\int_{t_0}^{s} \|M(s,\xi)\| \|g(\xi, z_\tau(\xi))\| \Delta \xi \right] \Delta s \\
&+ |b| \int_{t_0}^{t} \|e_{\ominus A}(t,s)\| \|f(s, z(s), u(s))\| \Delta s \\
\le & M_e \|\phi(t_0)\| + \int_{t_0}^{t} M_e \|B(s)\| \|u(s)\| \Delta s + |a| \int_{t_0}^{t} M_e \left[\int_{t_0}^{s} M \varphi^\Delta(\xi) \Delta \xi \right] \Delta s \\
&+ |b| \int_{t_0}^{t} M L_f(s) (\|z(s)\| + \|u(s)\|) \Delta s \\
\le & M_e \|\phi(t_0)\| + \int_{t_0}^{\eta} \left(M_e \|B(s)\| + |b| M \bar{L}_f \right) \|u(s)\| \Delta s + |a| \int_{t_0}^{\eta} M_e M \varphi(s) \Delta s \\
&+ |b| \int_{t_0}^{t} M \bar{L}_f \|z(s)\| \Delta s.
\end{aligned}
$$

By using Gronwall's inequality (see [7, Corollary 6.8]), we obtain

$$\begin{aligned}\|z(t)\| \leq & \Big[M_e \|\phi(t_0)\| + \int_{t_0}^{\eta} \big(M_e \|B(s)\| + |b| \bar{L}_f M\big) \|u(s)\| \Delta s \\ & + |a| \int_{t_0}^{\eta} M_e M \varphi(s) \Delta s\Big] e_{|b|\bar{L}_f M}(t, t_0) \\ \leq & \Big[M_e \|\phi(t_0)\| + \int_{t_0}^{\eta} \big(M_e \|B(s)\| + |b| \bar{L}_f M\big) \|u(s)\| \Delta s \\ & + |a| \int_{t_0}^{\eta} M_e M \varphi(s) \Delta s\Big] e_{|b|\bar{L}_f M}(\eta, t_0).\end{aligned}$$

This implies that $\|z(t)\|$ stays bounded in any neighborhood of η. So, from Theorem 3.3, we get $\eta = \infty$. This completes the proof.

4 Controllability of the Linear Equation

In order to study controllability of system (1), in this section, we shall present some characterization of controllability of a linear system associated to (1), namely,

$$\begin{cases} z^{\Delta}(t) = -A(t) z^{\sigma}(t) + B(t) u(t), \quad t \in [\delta, \eta]_{\mathbb{T}}, \\ z(\delta) = z^0. \end{cases} \tag{6}$$

The results presented in this section can be seen in [11], of course, with obvious modifications.

Note that, for all $z^0 \in \mathbb{R}^n$ and $u \in L^2_{\Delta}([\delta, \eta]_{\mathbb{T}}, \mathbb{R}^m)$, the initial value problem (6) admits only one solution, which is given by

$$z(t) = e_{\ominus A}(t, \delta) z^0 + \int_{\delta}^{t} e_{\ominus A}(t, s) B(s) u(s) \Delta s. \tag{7}$$

Definition 4.1 We say that (6) is controllable on $[\delta, \eta]_{\mathbb{T}}$ if for every $z^0, z^1 \in \mathbb{R}^n$, there exists $u \in L^2_{\Delta}([\delta, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ such that the solution z of (6) corresponding to u satisfies $z(\eta) = z^1$.

Definition 4.2 For the linear system (6), we define the following concepts:

1) The controllability operator $\mathcal{B}^{\eta} : L^2_{\Delta}([\delta, \eta]_{\mathbb{T}}, \mathbb{R}^m) \to \mathbb{R}^n$ is defined by

$$\mathcal{B}^{\eta} u = \int_{\delta}^{\eta} e_{\ominus A}(\eta, s) B(s) u(s) \Delta s. \tag{8}$$

2) The Gramian map is defined by $\mathcal{L}_{\mathcal{B}^{\eta}} = \mathcal{B}^{\eta} \mathcal{B}^{\eta *}$.

Proposition 4.1 *The adjoint $\mathcal{B}^{\eta *} : \mathbb{R}^n \to L^2_{\Delta}([\delta, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ of the operator $\mathcal{B}^{\eta}$ is given by*

$$(\mathcal{B}^{\eta *} z)(t) = B^*(t) e^*_{\ominus A}(\eta, t) z$$

and

$$\mathcal{L}_{\mathcal{B}^{\eta}} z = \int_{\delta}^{\eta} e_{\ominus A}(\eta, s) B(s) B^*(s) e^*_{\ominus A}(\eta, s) z \Delta s.$$

Theorem 4.1 *System* (6) *is controllable on* $[\delta, \eta]_{\mathbb{T}}$ *if and only if one of the following statements holds:*

1) $\operatorname{Range}(\mathcal{B}^{\eta}) = \mathbb{R}^n$,

2) $\langle L_{\mathcal{B}^{\eta}} z, z \rangle > 0$ *for every* $z \in \mathbb{R}^n \setminus \{0\}$,

3) there exists $\gamma > 0$ *such that* $\|\mathcal{B}^{\eta *} z\|_{L^2_{\Delta}} \geq \gamma \|z\|$ *for every* $z \in \mathbb{R}^n$,

4) $\mathcal{L}_{\mathcal{B}^{\eta}}$ *is invertible. Moreover,* $\mathcal{G}_{\eta} = \mathcal{B}^{\eta *} \mathcal{L}_{\mathcal{B}^{\eta}}^{-1}$ *is a right inverse of* $\mathcal{B}^{\eta}$, *and the control* $u \in L^2_{\Delta}([\delta, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ *steering the system from the initial state* z^{δ} *to a final state* z^1 *is given by*
$$u = \mathcal{B}^{\eta *} \mathcal{L}_{\mathcal{B}^{\eta}}^{-1} \left(z^1 - e_{\ominus A}(\eta, \delta) z^0\right). \tag{9}$$

5 Approximate Controllability of the Nonlinear System

Definition 5.1 (Approximate Controllability) System (1) is said to be approximately controllable on $[t_0, \eta]_{\mathbb{T}}$ if for every $\phi \in C_{\mathrm{rd}}([\tau(t_0), t_0]_{\mathbb{T}}, \mathbb{R}^m)$, $z^1 \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists a control $u \in L^2_{\Delta}([t_0, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ such that the solution z of (1) corresponding to u satisfies
$$z(t_0) = \phi(t_0) \quad \text{and} \quad \left\|z(t) - z^1\right\| < \varepsilon.$$

Theorem 5.1 *Suppose the system* (1) *is defined on* $[t_0, \eta]_{\mathbb{T}}$, *where* η *is such that* (3) *is satisfied. Assume that*

i) η *is left-dense,*

ii) there exists Δ*-differentiable* $\varphi : [t_0, \eta]_{\mathbb{T}} \to \mathbb{R}^+$ *such that* $\|g(t, z)\| \leq \varphi^{\Delta}(t)$ *for all* $t \in [t_0, \eta]_{\mathbb{T}}$,

iii) there exists rd-continuous $\psi : [t_0, \eta]_{\mathbb{T}} \to \mathbb{R}^+$ *such that* $\|f(t, z, u)\| \leq \psi(t)$ *for all* $t \in [t_0, \eta]_{\mathbb{T}}$.

If the linear system (6) *is controllable on* $[\delta, \eta]_{\mathbb{T}}$, *with* $t_0 \leq \delta < \eta$, *then system* (1) *is approximately controllable on* $[t_0, \eta]_{\mathbb{T}}$.

Proof. Given $\phi \in C_{\mathrm{rd}}([\tau(t_0), t_0]_{\mathbb{T}}, \mathbb{R}^n)$, a final state z^1 and $\varepsilon > 0$, we want to find a control $u^{\varepsilon} \in L^2_{\Delta}([t_0, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ steering the solution of system (1) to an ε-neighborhood of z^1 at time η. Indeed, let $\varepsilon > 0$ and consider a control $u \in L^2_{\Delta}([t_0, \eta]_{\mathbb{T}}, \mathbb{R}^m)$, arbitrary but fixed, and the corresponding solution $z(t) = z(t, t_0, \phi, u)$ of system (1). Since η is left-dense, there exists $\delta_{\varepsilon} \in (t_0, \eta)_{\mathbb{T}}$ such that
$$\eta - \delta_{\varepsilon} < \frac{\varepsilon}{M_e(|a|\, M\bar{\varphi} + |b|\, \bar{\psi})},$$
where $\bar{\varphi} = \sup\{\varphi(t) : t \in [t_0, \eta]_{\mathbb{T}}\}$ and $\bar{\psi} = \sup\{\psi(t) : t \in [t_0, \eta]_{\mathbb{T}}\}$. We define the control $u^{\varepsilon} \in L^2_{\Delta}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^m)$ by
$$u^{\varepsilon}(t) = \begin{cases} u(t) & \text{if } t \in [t_0, \delta_{\varepsilon}]_{\mathbb{T}}, \\ \tilde{u}(t) & \text{if } t \in (\delta_{\varepsilon}, \eta]_{\mathbb{T}}, \end{cases} \tag{10}$$
where
$$\tilde{u}(t) = B^*(t) e^*_{\ominus A}(\eta, t) \mathcal{L}_{\mathcal{B}^{\eta}}^{-1} \left(z^1 - e_{\ominus A}(\eta, \delta_{\varepsilon}) z(\delta_{\varepsilon})\right)$$

is the control steering the solution of system (6) from the initial state $z(\delta_\varepsilon)$ to the final state z^1 on $[\delta_\varepsilon,\eta]_{\mathbb{T}}$. The corresponding solution $z^{\delta_\varepsilon}(\cdot)=z^{\delta_\varepsilon}(\cdot,t_0,\phi,u^\varepsilon)$ of problem (1) at time η can be expressed by

$$\begin{aligned}
z^{\delta_\varepsilon}(\eta) =& e_{\ominus A}(\eta,t_0)\phi(t_0)+\int_{t_0}^{\eta} e_{\ominus A}(\eta,s)B(s)u^\varepsilon(s)\Delta s\\
&+a\int_{t_0}^{\eta} e_{\ominus A}(\eta,s)\left[\int_{t_0}^{s} M(s,\xi)g(\xi,z_\tau^{\delta_\varepsilon}(\xi))\Delta\xi\right]\Delta s\\
&+b\int_{t_0}^{\eta} e_{\ominus A}(\eta,s)f(s,z^{\delta_\varepsilon}(s)u_\alpha^\varepsilon(s))\Delta s\\
=& e_{\ominus A}(\eta,\delta_\varepsilon)\Bigg\{e_{\ominus A}(\delta_\varepsilon,t_0)\phi(t_0)+\int_{t_0}^{\delta_\varepsilon} e_{\ominus A}(\delta_\varepsilon,s)B(s)u(s)\Delta s\\
&+a\int_{t_0}^{\delta_\varepsilon} e_{\ominus A}(\delta_\varepsilon,s)\left[\int_{t_0}^{s} M(s,\xi)g(\xi,z_\tau(\xi))\Delta\xi\right]\Delta s\\
&+b\int_{t_0}^{\delta_\varepsilon} e_{\ominus A}(\delta_\varepsilon,s)f(s,z(s),u(s))\Delta s\Bigg\}\\
&+\int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta,s)B(s)\tilde{u}(s)\Delta s+a\int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta,s)\left[\int_{t_0}^{s} M(s,\xi)g(\xi,z_\tau^{\delta_\varepsilon}(\xi))\Delta\xi\right]\Delta s\\
&+b\int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta,s)f(s,z^{\delta_\varepsilon}(s),\tilde{u}(s))\Delta s\\
=& e_{\ominus A}(\eta,\delta_\varepsilon)z(\delta_\varepsilon)+\int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta,s)B(s)\tilde{u}(s)\Delta s\\
&+a\int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta,s)\left[\int_{t_0}^{s} M(s,\xi)g(\xi,z_\tau^{\delta_\varepsilon}(\xi))\Delta\xi\right]\Delta s\\
&+b\int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta,s)f(s,z^{\delta_\varepsilon}(s),\tilde{u}(s))\Delta s.
\end{aligned}$$

On the other hand, the corresponding solution $y(\cdot)=y(\cdot,\delta_\varepsilon,y(\delta_\varepsilon),\tilde{u})$ of initial value problem (6) at time $t=\eta$ is given by

$$y(\eta)=e_{\ominus A}(\eta,\delta_\varepsilon)y(\delta_\varepsilon)+\int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta,s)B(s)\tilde{u}(s)\Delta s.$$

Since the linear system (6) is controllable on $[\delta_\varepsilon,\eta]_{\mathbb{T}}$, we have that $y(\eta)=z^1$. Taking $y(\delta_\varepsilon)=z(\delta_\varepsilon)$, we get

$$\begin{aligned}
\left\|z^{\delta_\varepsilon}(\eta)-z^1\right\| \le& |a|\int_{\delta_\varepsilon}^{\eta}\|e_{\ominus A}(\eta,s)\|\left[\int_{t_0}^{s}\|M(s,\xi)\|\left\|g(\xi,z_\tau^{\delta_\varepsilon}(\xi))\right\|\Delta\xi\right]\Delta s\\
&+|b|\int_{\delta_\varepsilon}^{\eta}\|e_{\ominus A}(\eta,s)\|\left\|f(s,z^{\delta_\varepsilon}(s),\tilde{u}(s))\right\|\Delta s\\
\le& |a|\int_{\delta}^{\eta} M_e\left[M\int_{t_0}^{s}\varphi^\Delta(\xi)\Delta\xi\right]\Delta s+|b|\int_{\delta_e}^{\eta} M_e\psi(s)\Delta s\\
\le& |a|\int_{\delta_\varepsilon}^{\eta} M_eM\varphi(s)\Delta s+|b|\int_{\delta_\varepsilon}^{\eta} M_e\psi(s)\Delta s
\end{aligned}$$

$$\leq M_e \left(|a| \, M\bar{\varphi} + |b| \, \bar{\psi} \right) (\eta - \delta_\varepsilon) < \varepsilon.$$

So we get that system (1) is approximately controllable.

6 Approximate Controllability on Free Time

In this section, we prove the approximate controllability on free time of the system

$$\begin{cases} z^{\Delta}(t) = -A(t)z^{\sigma}(t) + B(t)u(t) + bf(t, z(t), u(t)), \quad t \geq t_0 \geq 0, \\ z(t_0) = z^0, \end{cases} \tag{11}$$

which is the system (1) without memory (i.e., taking $a \equiv 0$).

Definition 6.1 (Approximate Controllability on Free Time) System (11) is said to be approximately controllable on free time if for every $z^0, z^1 \in \mathbb{R}^n$, and $\varepsilon > 0$, there exist $\eta \in \mathbb{T}$ and $u \in L^2_{\Delta}([t_0, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ such that the corresponding solution of (1) satisfies

$$\left\| z(\eta) - z^1 \right\| < \varepsilon.$$

Theorem 6.1 *Suppose that*

i) There exists $M_e > 0$ such that $\|e_{\ominus A}(t,s)\| \leq M_e$ for all $t, s \in \mathbb{T}$,

ii) there exists rd-continuous $\psi : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ such that

$$\|f(t, z, u)\| \leq \psi(t) \quad \textit{with} \quad \int_{t_0}^{\infty} \psi(s)\Delta s < \infty.$$

If the linear system (6) is controllable on each interval $[\delta, \eta]_{\mathbb{T}}$, then the system (11) is approximately controllable on free time.

Proof. For $\varepsilon > 0$, $z_0 \in \mathbb{R}^n$ and a final state z^1, we want to find $\eta > t_0$ and a control $u^{\varepsilon} \in L^2_{\Delta}([t_0, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ steering the solution of system (11) to an ε-neighborhood of z^1 at time η. Since $\displaystyle\int_{t_0}^{\infty} \psi(s)\Delta s < \infty$, we can choose $\delta_\varepsilon, \eta \in \mathbb{T}$ big enough with $t_0 < \delta_\varepsilon < \eta$ such that

$$\int_{\delta_\varepsilon}^{\eta} \psi(s)\Delta s < \frac{\varepsilon}{|b| \, M_e}.$$

Now, defining $u^{\varepsilon} \in L^2_{\Delta}([t_0, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ as in (10) and proceeding similarly as in the proof of Theorem 5.1, we have

$$\left\| z^{\delta_\varepsilon}(\eta) - z^1 \right\| \leq |b| \int_{\delta_\varepsilon}^{\eta} \|e_{\ominus A}(\eta, s)\| \left\| f(s, z^{\delta_\varepsilon}(s), \tilde{u}(s)) \right\| \Delta s < \varepsilon.$$

So we get that system (11) is approximately controllable on free time.

7 Examples

Example 7.1 Let us consider the time scale $\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty}[2k, 2k+1]$ and the control system

$$\begin{cases} z^{\Delta}(t) = -z^{\sigma}(t) + 2u(t) + \frac{1}{100}\displaystyle\int_1^t e_{\ominus 1}(t,s)\sin(s)\sin(z(s/5))\Delta s \\ \qquad\qquad + \frac{1}{20}\cos(t)\sin(z(t)+u(t)), \quad t \in [1,5]_{\mathbb{T}}, \\ z(t) = \phi(t), \quad t \in [\frac{1}{5}, 1]_{\mathbb{T}}, \end{cases} \tag{12}$$

where $t_0 = 1$, $\tau(t) = \frac{t}{5}$, $M(t,s) = e_{\ominus 1}(t,s)$, $g(t,z) = \sin(t)\sin(z)$, $f(t,z,u) = \cos(t)\sin(z(t)+u(t))$, $A(t) = 1$, $B(t) = 2$ and $e_{\ominus A}(t,s) = e_{\ominus 1}(t,s)$. Since

$$\|g(t,z) - g(t,\tilde{z})\| \le |\sin(t)| \, \|z - \tilde{z}\|, \quad g(t,0) = 0,$$
$$\|f(t,z,u) - f(t,\tilde{z},\tilde{u})\| \le |\cos(t)| \left(\|z-\tilde{z}\| + \|u - \tilde{u}\|\right), \quad f(t,0,0) = 0,$$

and $M_e \left(|a| \, M\bar{L}_g\eta + |b| \, \bar{L}_f\right)\eta < \frac{1}{2}$, Theorem 3.2 ensures existence and uniqueness of solutions for problem (12) on $[\frac{1}{5}, 5]_{\mathbb{T}}$. On the other hand,

$$\|g(t,z)\| \le \varphi^{\Delta}(t) \quad \text{for all} \quad t \in [1,5]_{\mathbb{T}} \quad \text{with} \quad \varphi(t) = t,$$
$$\|f(t,z,u)\| \le \psi(t) \quad \text{for all} \quad t \in [1,5]_{\mathbb{T}} \quad \text{with} \quad \psi(t) = 1.$$

Furthermore, $\mathcal{L}_{\mathcal{B}^5} = 4\displaystyle\int_{\delta_\varepsilon}^5 e_{\ominus(1\oplus 1)}(5,s)\Delta s > 0$, so this operator is invertible, and hence the linear system

$$\begin{cases} z^{\Delta}(t) = -z^{\sigma}(t) + 2u(t), \quad t \in [\delta_\varepsilon, 5]_{\mathbb{T}}, \\ z(\delta_\varepsilon) = z^0, \end{cases}$$

is controllable and, since $\eta = 5$ is left-dense, by Theorem 5.1, system (12) is approximately controllable on $[1,5]_{\mathbb{T}}$.

Example 7.2 Let us consider the time scale $\mathbb{T} = \{3^n : n \in \mathbb{N}_0\}$ and the control system

$$\begin{cases} z^{\Delta}(t) = -2z^{\sigma}(t) + u(t) + \dfrac{1}{3t^2}\left(\tanh(z(t)) + \dfrac{u(t)}{1+u^2(t)}\right), \quad t > 1, \\ z(1) = z_0, \end{cases} \tag{13}$$

where $f(t,z,u) = \frac{1}{3t^2}\left(\tanh(z) + \frac{u}{1+u^2}\right)$, $A(t) = 2$, $B(t) = 1$ and $e_{\ominus A}(t,s) = e_{\ominus 2}(t,s)$. It is easy to see that the solution of (13) is defined on $[1,\infty)_{\mathbb{T}}$. On the other hand, we have

$$\|f(t,z,u)\| \le \frac{1}{3t^2}\left\|\tanh(z) + \frac{u}{1+u^2}\right\| \le \psi(t) \quad \text{with} \quad \psi(t) = \frac{2}{3t^2} \quad \text{and} \quad \int_1^{\infty}\frac{\Delta t}{t^2} < \infty.$$

For $\eta > \delta_\varepsilon$, the linear system

$$\begin{cases} z^{\Delta}(t) = -2z^{\sigma}(t) + u(t), \quad t \in [\delta_\varepsilon, \eta]_{\mathbb{T}}, \\ z(\delta_\varepsilon) = z^0, \end{cases}$$

is controllable since the operator $\mathcal{L}_{\mathcal{B}^\eta} = \displaystyle\int_{\delta_\varepsilon}^{\eta} e_{\ominus(2\oplus 2)}(\eta,s)\Delta s$ is invertible. Hence, by Theorem 6.1, we have that system (13) is approximately controllable on free time.

8 Conclusion and Final Remark

In this paper, we study a control system governed by a dynamic equation with memory on time scales. Specifically, first of all, we prove existence and uniqueness of solutions, then under an additional condition, and by applying Gronwall's inequality on time scales, we prove the prolongation of solutions. After that, we prove approximate controllability of the system assuming that the associated linear control problem on time scales is exactly controllable on $[\delta, \eta]_{\mathbb{T}}$, for any $\delta \in (t_0, \eta)_T$ with η being a left-dense point. In the case where the time scale does not have left-dense points, we consider the system without memory and we prove, under additional conditions, controllability on free time, i.e., we prove the existence of a time η such that the system (1) is approximately controllable. For difference equations, approximate controllability on free time was introduced by Uzcategui and Leiva in [16]. Finally, two examples show that our results are feasible. Of course, this work can be extended to evolution equations with memory on time scales in infinite-dimensional Banach spaces using strongly continuous semigroups on time scales approach.

Acknowledgements

The authors are thankful to the referee for carefully reading the manuscript and for suggesting several essential improvements.

References

[1] Zbigniew Bartosiewicz. Linear positive control systems on time scales; controllability. *Math. Control Signals Systems* **25** (3) (2013) 327–343.

[2] Zbigniew Bartosiewicz and Ewa Pawłuszewicz. Linear control systems on time scale: unification of continuous and discrete. In: *Proceedings of the 10th IEEE International Conference on Methods and Models in Automation and Robotics (MMAR2004), 30 August – 2 September 2004, Międzyzdroje, Poland*, pages 263–266, 2004.

[3] Zbigniew Bartosiewicz and Ewa Pawłuszewicz. Unification of continuous-time and discrete-time systems: the linear case. In: *Proceedings of the Sixteenth International Symposium on Mathematical Theory of Networks and Systems (MTNS2004), Katholieke Universiteit Leuven, July 5–9, 2004, Leuven, Belgium*, pages 1–6, 2004.

[4] Agamirza E. Bashirov and Noushin Ghahramanlou. On partial approximate controllability of semilinear systems. *Cogent Eng.* **1** (1) (2014) 1–13, Art. ID 965947.

[5] Agamirza E. Bashirov and Maher Jneid. On partial complete controllability of semilinear systems. *Abstr. Appl. Anal.* (2013) 1–8, Art. ID 521052.

[6] Agamirza E. Bashirov, Nazim I. Mahmudov, Nidai Şemi, and Huseyin Etikan. Partial controllability concepts. *Internat. J. Control* **80** (1) (2007) 1–7.

[7] Martin Bohner and Allan Peterson. *Dynamic Equations on Time Scales.* Birkhäuser Boston, Inc., Boston, MA, 2001. An introduction with applications.

[8] Martin Bohner and Nick Wintz. Controllability and observability of time-invariant linear dynamic systems. *Math. Bohem.* **137** (2) (2012) 149–163.

[9] Martin Bohner and Nick Wintz. The Kalman filter for linear systems on time scales. *J. Math. Anal. Appl.* **406** (2) (2013) 419–436.

[10] Cosme Duque, Jahnett Uzcátegui, and Hugo Leiva. Controllability of semilinear dynamic systems on time scales. *J. Abstr. Differ. Equ. Appl.* **8** (2) (2017) 70–80.

[11] Cosme Duque, Jahnett Uzcátegui, and Hugo Leiva. Approximate controllability of semilinear dynamic equations on time scale. *Asian J. Control* **21** (5) (2019) 2301–2307.

[12] Svetlin G. Georgiev. *Functional Dynamic Equations on Time Scales.* Springer, Cham, 2019.

[13] David Grow and Nick Wintz. Bilinear state systems on an unbounded time scale. *Appl. Math. Comput.* **397** (2021), Paper No. 125917, 14.

[14] Stefan Hilger. *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten.* PhD thesis, Universität Würzburg, 1988.

[15] Klara Janglajew and Ewa Pawłuszewicz. Constrained local controllability of dynamic systems on time scales. *Adv. Difference Equ.* **2015** (89) (2015) 1–8.

[16] Hugo Leiva and Jahnett Uzcátegui. Approximate controllability of discrete semilinear systems and applications. *Discrete Contin. Dyn. Syst. Ser. B* **21** (6) (2016) 1803–1812.

[17] Muslim Malik and Vipin Kumar. Existence, stability and controllability results of a Volterra integro-dynamic system with non-instantaneous impulses on time scales. *IMA J. Math. Control Inform.* **37** (1) (2020) 276–299.

[18] Muslim Malik and Vipin Kumar. Controllability of neutral differential equation with impulses on time scales. *Differ. Equ. Dyn. Syst.* **29** (1) (2021) 211–225.

[19] Akbar Zada, Luqman Alam, Poom Kumam, Wiyada Kumam, Gohar Ali, and Jehad Alzabut. Controllability of impulsive non-linear delay dynamic systems on time scale. *IEEE Access* **8** (2020) 93830–93839.