Nonlinear Dynamics and Systems Theory, **22** (3) (2022) 231–242

The Solution of the Second Part of the 16th Hilbert Problem for a Class of Piecewise Linear Hamiltonian Saddles Separated by Conics

R. Benterki *, L. Damene and L. Baymout

Department of Mathematics, University of Mohamed El Bachir El Ibrahimi of Bordj Bou Arŕeridj 34000, El Anasser, Algeria

Received: December 31, 2021; Revised: June 10, 2022

Abstract: In this paper, we study the existence of the maximum number of crossing limit cycles of planar piecewise differential systems formed by linear Hamiltonian saddles. Firstly, we prove that if we separate these systems by either a parabola or hyperbola or an ellipse, they can have at most three crossing limit cycles. Secondly, we provide an example of four crossing limit cycles when these systems have four zones separated by two intersecting straight lines $xy = 0$.

Keywords: *piecewise differential system, limit cycles, linear Hamiltonian saddles, conics.*

Mathematics Subject Classification (2010): 34A36, 34A07, 34C25.

1 Introduction

One of the important and difficult problems in the qualitative study of differential systems is the determination of the existence or non-existence of limit cycles and their position in the plane, the same problem arises for the piecewise linear differential systems separated by an algebraic curve. Planar discontinuous piecewise linear differential systems were firstly studied by Andronov, Vit and Khaikin [1].

Recently, these systems have been of great importance to the mathematical community due to their applicability to modeling and control of the environment, see for example the books [7, 14].

* Corresponding author: mailto:r.benterki@univ-bba.dz

Many authors studied the upper bound of crossing limit cycles that some families of discontinuous piecewise differential systems can have. In 2010, Han and Zhang [9] conjectured that we can have two crossing limit cycles when we separate planar discontinuous piecewise linear differential systems by a straight line, but in 2012, Huan and Yang [10] proved that the conjecture of Han is wrong by proving the existence of a numerical example with three limit cycles. Afterward, Llibre and Ponce in [12] proved analytically the existence of this example. In 2015, Llibre et al. [11] showed that the discontinuous piecewise linear differential centers sepatated by a straight line can not exhibit any limit cycle, while if we consider that the curve of discontinuity is different from a straight line, we can produce limit cycles, see for example the papers [3–5]. For another kind of discontinuous planar piecewise differential systems, Benterki and Llibre [2, 6] studied the existence of limit cycles of planar piecewise linear Hamiltonian systems without equilibrium points, where they solved the 16th Hilbert problem of these systems when the curve of separation are conics or irreducible cubic curves.

In [8], Damene and Benterki provided the maximum number of crossing limit cycles of two different families of discontinuous piecewise linear differential systems separated by cubic curves.

Our objective in this paper is to study the crossing limit cycles of planar piecewise differential systems with linear Hamiltonian saddles separated by conics.

We recognize that each conic occurs in nine canonical forms, but we omit some of them due to the fact that they do not separate the plane into connected regions such as $x^2 + 1 = 0$, $x^2 + y^2 = 0$, and $x^2 + y^2 + 1 = 0$.

In [13], the authors proved that the maximum number of limit cycles for discontinuous planar piecewise differential systems formed by linear Hamiltonian saddles and separated by two parallel straight lines is at most one.

The main goal of our work is to provide the upper bounds of crossing limit cycles of discontinuous planar piecewise differential linear Hamiltonian saddles (or simply **PHS**) separated by either an ellipse $x^2 + y^2 - 1 = 0$, or a parabola $y - x^2 = 0$, or a hyperbola $x^2 - y^2 = 1$ or by the two intersecting straight lines $xy = 0$. The main tool that we used to prove our results is the first integrals method.

A normal form for an arbitrary linear differential system with Hamiltonian saddles is given in the following proposition. For the proof, see for instance [13].

Proposition 1.1 *Differential systems with a linear Hamiltonian saddle can be written as*

$$\dot{x} = -bx - \delta y + d, \qquad \dot{y} = \alpha x + by + c, \tag{1}$$

where $\alpha \in \{0, 1\}$ *and* $b, \delta, c, d \in \mathbb{R}$. *Moreover, if* $\alpha = 0$, *then* $c = 0$, *and if* $\alpha = 1$, *then* $\delta = b^2 - \omega^2$ *with* $\omega \neq 0$. *The corresponding first integral of system* (1) *is*

$$H(x, y) = -(\alpha/2)x^2 - bxy - (\delta/2)y^2 - cx + dy.$$

2 Statements of the Main Results

In this section, and specialy in Theorem 2.1, we prove our results for discontinuous piecewise differential systems formed by linear Hamiltonian saddles intersecting the parabola, or hyperbola or ellipse at two points. While in Theorem 2.2 we are interested in studying the number of crossing limit cycles intersecting the straight lines $xy = 0$ at exactly four points. Our first main result is the following.

Theorem 2.1 *The following statements hold.*

(i) *The maximum number of crossing limit cycles of* **PHS** *intersecting the parabola at two points, is at most three. This maximum is reached in Figure 1.*

(ii) *The maximum number of crossing limit cycles of* **PHS** *intersecting the hyperbola at two points, is at most three. This maximum is reached in Figure 2.*

(iii) *The maximum number of crossing limit cycles of* **PHS** *intersecting the ellipse at two points, is at most three. This maximum is also reached in Figure 3.*

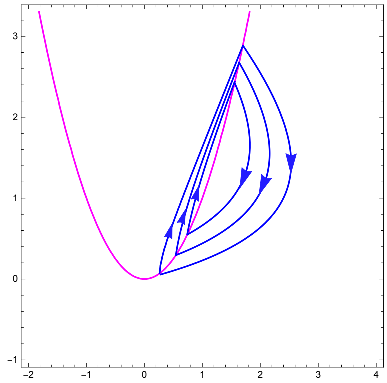

Figure 1: Three crossing limit cycles of piecewise differential system (7)–(8).

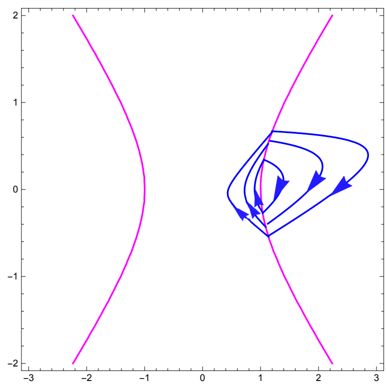

Figure 2: Three crossing limit cycles of piecewise differential system (10)–(11).

Theorem 2.2 *The maximum number of crossing limit cycles of piecewise linear differential systems formed by four linear Hamiltonian saddles and separated by the two intersecting straight lines* $xy = 0$, *is at most eight. There is an example of these systems with exactly four limit cycles.*

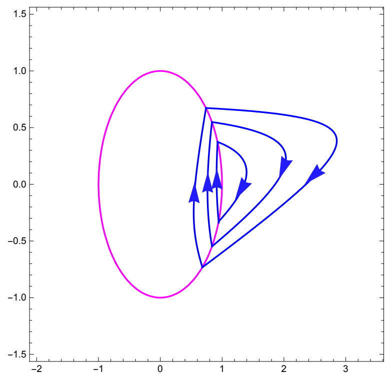

Figure 3: Three crossing limit cycles of piecewise differential system (13)–(14).

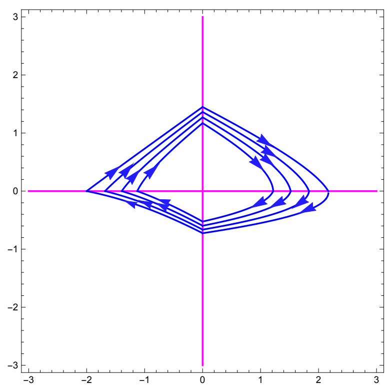

Figure 4: Four crossing limit cycles of piecewise differential system (21)–(24).

3 Proof of Theorem 2.1

Proof. In this part, we are going to prove the statement (i) of Theorem 2.1. Then in the first region $R_1 = \{(x,y) : y - x^2 \geq 0\}$, we consider the planar discontinuous piecewise Hamiltonian saddle

$$\dot{x} = -b_1 x - \delta_1 y + d_1, \quad \dot{y} = \alpha_1 x + b_1 y + c_1, \tag{2}$$

its corresponding Hamiltonian function is

$$H_1(x,y) = -\frac{\alpha_1}{2}x^2 - b_1 xy - \frac{\delta_1}{2}y^2 - c_1 x + d_1 y. \tag{3}$$

In the second region $R_2 = \{(x,y) : y - x^2 \leq 0\}$, we consider the **PHS** system

$$\dot{x} = -b_2 x - \delta_2 y + d_2, \quad \dot{y} = \alpha_2 x + b_2 y + c_2, \tag{4}$$

with its corresponding Hamiltonian function

$$H_2(x,y) = -\frac{\alpha_2}{2}x^2 - b_2 xy - \frac{\delta_2}{2}y^2 - c_2 x + d_2 y. \tag{5}$$

In order to have a crossing limit cycle that intersects the parabola $y - x^2 = 0$ at the points (x_i, y_i) and (x_k, y_k), with $i \neq k$, these points must satisfy the following system:

$$\begin{aligned} &H_1(x_i, y_i) - H_1(x_k, y_k)) = 0,\\ &H_2(x_i, y_i) - H_2(x_k, y_k) = 0,\\ &y_i - x_i^2 = 0,\ y_k - x_k^2 = 0. \end{aligned} \tag{6}$$

We suppose that the system (2)–(4) has four crossing limit cycles. Then, system (6) must have four pairs of points as solutions, namely, p_i and q_i taking the forms $p_i = (r_i, r_i^2)$ and $q_i = (s_i, s_i^2)$, with $i = 1, 2, 3, 4$. Due to the fact that these points satisfy system (6) and if we consider the points $p_1 = (r_1, r_1^2)$ and $q_1 = (s_1, s_1^2)$, then simple calculations give the following expressions of the parameters c_1 and c_2:

$$c_1 = \frac{1}{2}\Big(2d_1(r_1 + s_1) - 2b_1(r_1^2 + r_1 s_1 + s_1^2) - (r_1 + s_1)(\alpha_1 + (r_1^2 + s_1^2)\delta_1)\Big),$$

and c_2 has the same expression as c_1 with the change of $(d_1, \delta_1, b_1, \alpha_1)$ by $(d_2, \delta_2, b_2, \alpha_2)$.

If the two points $p_2 = (r_2, r_2^2)$ and $q_2 = (s_2, s_2^2)$ satisfy system (6), then by solving the two first equations of (6), we obtain the expressions of the two parameters d_1 and d_2

$$\begin{aligned} d_1 = \;& \frac{1}{2(r_1 - r_2 + s_1 - s_2)}\Big(2b_1(r_1^2 - r_2^2 + r_1 s_1 + s_1^2 - r_2 s_2 - s_2^2) - r_2\alpha_1 + s_1\alpha_1 \\ & -s_2\alpha_1 + r_1^3\delta_1 - r_2^3\delta_1 + r_1^2 s_1\delta_1 + s_1^3\delta_1 - r_2^2 s_2\delta_1 - r_2 s_2^2\delta_1 - s_2^3\delta_1 + r1(\alpha_1 + s_1^2\delta_1)\Big), \end{aligned}$$

and d_2 has the same expression as d_1 with the change of $(\delta_1, b_1, \alpha_1)$ by $(\delta_2, b_2, \alpha_2)$.

Now let us suppose that the points $p_3 = (r_3, r_3^2)$ and $q_3 = (s_3, s_3^2)$ satisfy system (6), then the parameters δ_1 and δ_2 must be $\delta_1 = A/B$, where

$$\begin{aligned} A = \;& -2b_1\Big((s_1 - s_2)(r_3^2 + (s_1 - s_3)(s_2 - s_3) - r_3(s_1 + s_2 - s_3)) + r_1^2(r_2 - r_3 + s_2 \\ & -s_3) + r_2^2(r_3 - s_1 + s_3) + r_1(-r_2^2 + r_3^2 - r_3 s_1 + r_2(s_1 - s_2) + s_1 s_2 - s_2^2 + r_3 s_3 \\ & -s_1 s_3 + s_3^2) - r_2(r_3^2 + r_3(-s_2 + s_3) - (s_1 - s_3)(s_1 - s_2 + s_3))\Big), \end{aligned}$$

$$\begin{aligned} B = \;& r_1^3(r_2 - r_3 + s_2 - s_3) + r_1^2 s_1(r_2 - r_3 + s_2 - s_3) + r_2^3(r_3 - s_1 + s_3) + r_2^2 s_2(r_3 \\ & -s_1 + s_3) + r_1(-r_2^3 + r_3^3 - r_3 s_1^2 - r_2^2 s_2 + s_1^2 s_2 - s_2^3 + r_2(s_1^2 - s_2^2) + r_3^2 s_3 + s_3^3 \\ & +r_3 s_3^2 - s_1^2 s_3) + (s_1 - s_2)(r_3^3 + r_3^2 s_3 + (s_1 - s_3)(s_2 - s_3)(s_1 + s_2 + s_3) - r_3(s_1^2 \\ & +s_1 s_2 + s_2^2 - s_3^2)) - r_2(r_3^3 - s_1^3 + s_1 s_2^2 + r_3^2 s3 - s_2^2 s3 + s_3^3 + r_3(-s_2^2 + s_3^2)), \end{aligned}$$

and we get the expression of δ_2 by changing (b_1, α_1) by (b_2, α_2) in the expression of δ_1. Finally, if we suppose that the points $p_4 = (r_4, r_4^2)$ and $q_4 = (s_4, s_4^2)$ satisfy system (6) and if $\alpha_i \in \{0, 1\}$ with $i = 1, 2$, then we obtain $b_1 = 0$ and $b_2 = 0$.

We replace $c_1, d_1, \delta_1, \alpha_1$ and b_1 in the expression of $H_1(x, y)$, and $c_2, d_2, \delta_2, \alpha_2$ and b_2 in the expression of $H_2(x, y)$, we known that the expression of the first integral $H_1(x, y)$ is the same as the expression of the first integral $H_2(x, y)$, i.e., $H_1(x, y) = H_2(x, y)$. Therefore, the piecewise linear differential system (2)–(4) becomes a linear differential system, which does not have limit cycles. Consequently, the maximum number of crossing limit cycles in this case is at most three.

Example with three limit cycles. Consider the planar discontinuous piecewise linear Hamiltonian saddle

$$\dot{x} = 75x + 250y - 550, \quad \dot{y} = -75y - 100, \tag{7}$$

in the region R_1 with its corresponding Hamiltonian function

$$H_1(x,y) = 125y^2 + 75xy + 100x - 550y.$$

In the region R_2, we consider the **PHS** system

$$\dot{x} = 0.07125x + 0.2375y - 0.0225, \quad \dot{y} = x - 0.07125y - 0.095, \tag{8}$$

its corresponding Hamiltonian function is

$$H_2(x,y) = -\frac{x^2}{2} + 0.0712..xy + 0.1187..y^2 - 0.0224..y + 0.095..x.$$

Now system (6) has the three solutions $(x_1^{(1)}, y_1^{(1)}, x_2^{(1)}, y_2^{(1)}) = (1.5569.., 2.4239.., 0.7417.., 0.5502..)$, $(x_1^{(2)}, y_1^{(2)}, x_2^{(2)}, y_2^{(2)}) = (1.6356.., 2.6753.., 0.543.., 0.2948..)$, and $(x_1^{(3)}, y_1^{(3)}, x_2^{(3)}, y_2^{(3)}) = (1.6977.., 2.8823.., 0.264.., 0.0698..)$, which provide the three limit cycles shown in Figure 1. This completes the proof of statement (i) of Theorem 2.1.

To prove the statement (ii) of Theorem 2.1, we consider in the region $R_1 = \{(x,y) : x^2 - y^2 - 1 \geq 0\}$ the **PHS** given in (2), with its corresponding Hamiltonian function given in (3).

In the region $R_2 = \{(x,y) : x^2 - y^2 - 1 \leq 0\}$, we consider the **PHS** given in (4), with its corresponding Hamiltonian function given in (5). In order to have a crossing limit cycle that intersects the hyperbola $x^2 - y^2 - 1 = 0$ at the points (x_i, y_i) and (x_k, y_k), with $i \neq k$, they must satisfy the system of equations

$$\begin{array}{l} H_1(x_i,y_i) - H_1(x_k,y_k) = 0, \\ H_2(x_i,y_i) - H_2(x_k,y_k) = 0, \\ x_i^2 - y_i^2 - 1 = 0, \ x_k^2 - y_k^2 - 1 = 0, \end{array} \tag{9}$$

we suppose that system (2)–(4) has four crossing limit cycles. So, system (9) must have four pairs of solutions which can be written as $p_i = (\cosh r_i, \sinh r_i)$ and $q_i = (\cosh s_i, \sinh s_i)$, for $i = 1, 2, 3, 4$.

Due to the fact that the two points $p_1 = (\cosh r_1, \sinh r_1)$ and $q_1 = (\cosh s_1, \sinh s_1)$ satisfy system (9), then by solving the two first equations in (9), we obtain the parameters c_1 and c_2 as follows:

$$\begin{aligned} c_1 = \ & \frac{1}{2(\cosh r_1 - \cosh s_1)}\Big(- \alpha_1 \cosh^2 r_1 + \alpha_1 \cosh^2 s_1 + 2d_1 \sinh r_1 - 2b_1 \cosh r_1 \\ & \sinh r_1 - \delta_1 \sinh^2 r_1 - 2d_1 \sinh s_1 + \delta_1 \sinh s_1^2 + b_1 \sinh 2s_1\Big). \end{aligned}$$

By changing $(\alpha_1, \delta_1, b_1, d_1)$ by $(\alpha_2, \delta_2, b_2, d_2)$ in the expression of c_1, we get the expression of c_2. We know that the two points $p_2 = (\cosh r_2, \sinh r_2)$ and $q_2 = (\cosh s_2, \sinh s_2)$ satisfy system (9), then from this system, we get the parameters d_1 and d_2, where

$$\begin{aligned} d_1 = \ & \frac{1}{4\big(\cosh\big(\frac{r_1 - 2r_2 + s_1}{2}\big) - \cosh\big(\frac{r_1 + s_1 - 2s_2}{2}\big)\big)}\Big(csch(\tfrac{r_1 - s_1}{2})\Big(\alpha_1 \cosh^2 r_2 \cosh s_1 \\ & + \alpha_1 \cosh^2 r_1 (\cosh r_2 - \cosh s_2) + \alpha_1 \cosh^2 s_1 \cosh s_2 - \alpha_1 \cosh s_1 \cosh^2 s_2 - \delta_1 \\ & \cosh s_2 \sinh^2 r_1 + \delta_1 \cosh s_1 \sinh^2 r_2 + b_1 \cosh s_1 \sinh(2r_2) + \delta_1 \cosh s_2 \sinh^2 s_1 \\ & + b_1 \cosh s_2 \sinh(2s_1) + \cosh r_2\big(- \alpha_1 \cosh^2 s_1 + \sinh(r_1 - s_1)\big(2b_1 \cosh(r_1 + s_1) \\ & + \delta_1 \sinh(r_1 + s_1)\big)\big) - 2b_1 \cosh s_1 \cosh s_2 \sinh s_2 - \delta_1 \cosh s_1 \sinh^2 s_2 \\ & + \cosh r_1\big(- \alpha_1 \cosh^2 r_2 + \alpha_1 \cosh^2 s_2 - 2b_1 \cosh s_2 \sinh r_1 + 2b_1 \cosh r_2 \sinh r_2 \\ & - \delta_1 \sinh^2 r_2 + \delta_1 \sinh^2 s_2 + b_1 \sinh(2s_2)\big)\Big)\Big), \end{aligned}$$

and by changing (α_1,δ_1,b_1) by (α_2,δ_2,b_2) in the expression of d_1, we obtain d_2.

We know that the points $p_3=(\cosh r_3,\sinh r_3)$ and $q_3=(\cosh s_3,\sinh s_3)$ satisfy system (9), then we obtain the values of δ_1 and δ_2. The value of δ_1 is given by $\delta_1=A/B$, where

$$\begin{aligned}
A = \;& -\alpha_1\Big(\cosh(\tfrac{r_1-r_2-r_3+s_1-s_2-3s_3}{2})+\cosh(\tfrac{r_1-r_2-r_3+s_1-3s_2-s_3}{2})\\
&-\cosh(\tfrac{r_1-r_2-3r_3+s_1-s_2-s_3}{2})+\cosh(\tfrac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}\\
&+\cosh(\tfrac{r_1-3r_2-r_3+s_1-s_2-s_3}{2})+\cosh(\tfrac{3r_1+r_2-r_3+s_1+s_2-s_3}{2})\\
&-\cosh(\tfrac{r_1+3r_2-r_3+s_1+s_2-s_3}{2})+\cosh(\tfrac{r_1+r_2-r_3+3s_1+s_2-s_3}{2})\\
&-\cosh(\tfrac{r_1+r_2-r_3+s_1+3s_2-s_3}{2})-\cosh(\tfrac{3r_1-r_2+r_3+s_1-s_2+s_3}{2})\\
&+\cosh(\tfrac{r_1-r_2+3r_3+s_1-s_2+s_3}{2})-\cosh(\tfrac{r_1-r_2+r_3+3s_1-s_2+s_3}{2})\\
&+\cosh(\tfrac{r_1-r_2+r_3+s_1-s_2+3s_3}{2})\Big)+2b_1\Big(\sinh(\tfrac{r_1-r_2-r_3+s_1-s_2-3s_3}{2})\\
&-\sinh(\tfrac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}+\sinh(\tfrac{r_1-r_2-3r_3+s_1-s_2-s_3}{2})\\
&-\sinh(\tfrac{r_1-3r_2-r_3+s_1-s_2-s_3}{2})+\sinh(\tfrac{3r_1+r_2-r_3+s_1+s_2-s_3}{2})\\
&-\sinh(\tfrac{r_1+3r_2-r_3+s_1+s_2-s_3}{2})+\sinh(\tfrac{r_1+r_2-r_3+3s_1+s_2-s_3}{2})\\
&-\sinh(\tfrac{r_1+r_2-r_3+s_1+3s_2-s_3}{2})-\sinh(\tfrac{3r_1-r_2+r_3+s_1-s_2+s_3}{2})\\
&+\sinh(\tfrac{r_1-r_2+3r_3+s_1-s_2+s_3}{2})-\sinh(\tfrac{r_1-r_2+r_3+3s_1-s_2+s_3}{2})\\
&+\sinh(\tfrac{r_1-r_2+r_3+s_1-s_2+3s_3}{2})\Big),
\end{aligned}$$

$$\begin{aligned}
B = \;& \cosh(\tfrac{r_1-r_2-r_3+s_1-s_2-3s_3}{2})-\cosh(\tfrac{r_1-r_2-r_3+s_1-3s_2-s_3}{2})\\
&+\cosh\tfrac{r_1-r_2-3r_3+s_1-s_2-s_3}{2})-\cosh(\tfrac{r_1-3r_2-r_3+s_1-s_2-s_3}{2})\\
&-\cosh(\tfrac{3r_1+r_2-r_3+s_1+s_2-s_3}{2})+\cosh(\tfrac{r_1+3r_2-r_3+s_1+s_2-s_3}{2})\\
&-\cosh\tfrac{r_1+r_2-r_3+3s_1+s_2-s_3}{2})+\cosh\tfrac{r_1+r_2-r_3+s_1+3s_2-s_3}{2})\\
&+\cosh(\tfrac{3r_1-r_2+r_3+s_1-s_2+s_3}{2})-\cosh(\tfrac{r_1-r_2+3r_3+s_1-s_2+s_3}{2})\\
&+\cosh(\tfrac{r_1-r_2+r_3+3s_1-s_2+s_3}{2})-\cosh(\tfrac{r_1-r_2+r_3+s_1-s_2+3s_3}{2}).
\end{aligned}$$

We get the expression of δ_2 by changing (α_1,b_1) by (α_2,b_2) in the expression of δ_1.

If $\alpha_1=\alpha_2=1$ or $(\alpha_1=\alpha_2=0)$, we assume that the points $p_4=(\cosh r_4,\sinh r_4)$ and $q_4=(\cosh s_4,\sinh s_4)$ satisfy system (9), then we obtain $b_1=0$ and $b_2=0$.

We replace c_1, d_1, δ_1, α_1 and b_1 in the expression of $H_1(x,y)$, and c_2, d_2, δ_2, α_2 and b_2 in the expression of $H_2(x,y)$ and we obtain $H_1(x,y)=H_2(x,y)$. Hence, in these cases, the piecewise linear differential system becomes a linear differential system, which does not have any limit cycle. Therefore, the maximum number of crossing limit cycles in the piecewise linear Hamiltonian saddles separated by a hyperbola is at most three.

Example with three limit cycles. We consider the **PHS** separated by the hyperbola

$$\dot{x}=-18x+95y+15,\quad \dot{y}=18y-14, \tag{10}$$

in the region $R_1 = \{(x,y) : x^2 - y^2 - 1 \geq 0\}$, which has the Hamiltonian function

$$H_1(x,y) = \frac{95}{2}y^2 - 18xy + 14x + 15y.$$

Now, in the region $R_2 = \{(x,y) : x^2 - y^2 - 1 \leq 0\}$, we consider the second **PHS**

$$\dot{x} = -0.2699..x + 2.425..y + 0.225.., \quad \dot{y} = x + 0.2699..y - 0.21, \tag{11}$$

its corresponding first integral is

$$H_2(x,y) = \frac{x^2}{2} - 0.2699..xy + 1.2124..y^2 + 0.21x + 0.2249..y.$$

The **PHS** (10)–(11) has exactly three crossing limit cycles because the system of equations (9) has three real solutions $(x_1^{(1)}, y_1^{(1)}, x_2^{(1)}, y_2^{(1)}) = (1.0571.., 0.3427.., 1.0362.., -0.2715..)$, $(x_1^{(2)}, y_1^{(2)}, x_2^{(2)}, y_2^{(2)}) = (1.1283.., 0.5227.., 1.0885.., -0.43..)$, and $(x_1^{(3)}, y_1^{(3)}, x_2^{(3)}, y_2^{(3)}) = (1.1969.., 0.6577.., 1.1385.., -0.5442)$, see Figure 2. This completes the proof of statement (ii).

Finally, to prove the statement (iii), we consider the **PHS** given in (2) in the region $R_1 = \{(x,y) : x^2 + y^2 - 1 \geq 0\}$, with its corresponding Hamiltonian function (3). We consider the **PHS** given in (4) in the region $R_2 = \{(x,y) : x^2 + y^2 - 1 \leq 0\}$, with its corresponding Hamiltonian function (5). In order that system (2)–(4) has crossing limit cycles intersecting the ellipse $y^2 + x^2 - 1 = 0$ at the points (x_i, y_i) and (x_k, y_k), with $i \neq k$, they must satisfy the system

$$\begin{aligned} &H_1(x_i,y_i) - H_1(x_k,y_k)) = 0,\\ &H_2(x_i,y_i) - H_2(x_k,y_k) = 0,\\ &y_i^2 + x_i^2 - 1 = 0, \;\; y_k^2 + x_k^2 - 1 = 0. \end{aligned} \tag{12}$$

Now we assume that system (2)–(4) has four crossing limit cycles. Consequently, system (12) must have four pairs of points $p_i = (\cos r_i, \sin r_i)$ and $q_i = (\cos s_i, \sin s_i)$ with $i = 1, \ldots, 4$ as solutions. So, if we consider the points $p_1 = (\cos r_1, \sin r_1)$ and $q_1 = (\cos s_1, \sin s_1)$ from (12), we obtain that the parameters c_1 and c_2 must be

$$\begin{aligned} c_1 = \; & \frac{1}{2(\cos r_1 - \cos s_1)}\Big(- \alpha_1 \cos^2 r_1 + \alpha_1 \cos^2 s_1 + 2d_1 \sin r_1 - 2b_1 \cos r_1 \sin r_1 - \\ & \delta_1 \sin^2 r_1 - 2d_1 \sin s_1 + \delta_1 \sin^2 s_1 + b_1 \sin(2s_1)\Big). \end{aligned}$$

Changing $(d_1, \delta_1, \alpha_1, b_1)$ by $(d_2, \delta_2, \alpha_2, b_2)$ in the expression of c_1, we get the expression of c_2. Due to the fact that the two points $p_2 = (\cos r_2, \sin r_2)$ and $q_2 = (\cos s_2, \sin s_2)$ satisfy system (12), then the parameters d_1 and d_2 have the expressions

$$\begin{aligned} d_1 = \; & \frac{\csc((r_1 - s_1)/2)}{4\big(\cos(2(r_1 - 2r_2 + s_1)/2) - \cos((r_1 + s_1 - 2s_2)/2)\big)}(\alpha_1 \cos^2 r_2 \cos s_1 \\ & + \alpha_1 \cos^2 r_1(\cos r_2 - \cos s_2) + \alpha_1 \cos^2 s_1 \cos s_2 - \alpha_1 \cos s_1 \cos^2 s_2 - \delta_1 \cos s_2 \sin^2 r_1 \\ & + \delta_1 \cos s_1 \sin^2 r_2 + b_1 \cos s_1 \sin(2r_2) + \delta_1 \cos s_2 \sin^2 s_1 + b_1 \\ & \cos s_2 \sin(2s_1) + \cos r_2(-\alpha_1 \cos^2 s_1 + \sin(r_1 - s_1)(2b_1 \cos(r_1 + s_1) + \delta_1 \sin(r_1 + s_1))) \\ & - 2b_1 \cos s_1 \cos s_2 \sin s_2 - \delta_1 \cos s_1 \sin^2 s_2 + \cos r_1(-\alpha_1 \cos^2 r_2 + \alpha_1 \cos^2 s_2 - 2b_1 \\ & \cos s_2 \sin r_1 - 2b_1 \cos r_2 \sin r_2 - \delta_1 \sin^2 r_2 + \delta_1 \sin^2 s_2 + b_1 \sin(2s_2))). \end{aligned}$$

We get the expression of d_2 by changing $(\delta_1, \alpha_1, b_1)$ by $(\delta_2, \alpha_2, b_2)$ in the expression of d_1.

Likewise, the points $p_3 = (\cos r_3, \sin r_3)$ and $q_3 = (\cos s_3, \sin s_3)$ satisfy system (12), then we obtain the expressions of δ_1 and δ_2 such that $\delta_1 = A/B$, where

$$
\begin{aligned}
A = \quad & \alpha_1\Big(\cos\Big(\frac{r_1 - r_2 - r_3 + s_1 - s_2 - 3s_3}{2}\Big) - \cos\Big(\frac{r_1 - r_2 - r_3 + s_1 - 3s_2 - s_3}{2}\Big) \\
& + \cos\Big(\frac{r_1 - r_2 - 3r_3 + s_1 - s_2 - s_3}{2}\Big) - \cos\Big(\frac{r_1 - 3r_2 - r_3 + s_1 - s_2 - s_3}{2}\Big) \\
& - \cos\Big(\frac{3r_1 + r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big) + \cos\Big(\frac{r_1 + 3r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big) \\
& - \cos\Big(\frac{r_1 + 3r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big) + \cos\Big(\frac{r_1 + r_2 - r_3 + s_1 + 3s_2 - s_3}{2}\Big) \\
& + \cos\Big(\frac{3r_1 - r_2 + r_3 + s_1 - s_2 + s_3}{2}\Big) - \cos\Big(\frac{3r_1 - r_2 + r_3 + s_1 - s_2 + s_3}{2}\Big) \\
& + \cos\Big(\frac{r1 - r_2 + r_3 + 3s_1 - s_2 + s_3}{2}\Big) - \cos\Big(\frac{r1 - r_2 + r_3 + s_1 - s_2 + 3s_3}{2}\Big)\Big) \\
& -2b_1\Big(\sin\Big(\frac{r_1 - r_2 - r_3 + s_1 - s_2 - 3s_3}{2}\Big) + \sin\Big(\frac{r_1 - r_2 - r_3 + s_1 - 3s_2 - s_3}{2}\Big) \\
& - \sin\Big(\frac{r_1 - r_2 - 3r_3 + s_1 - s_2 - s_3}{2}\Big) + \sin\Big(\frac{r_1 - 3r_2 - r_3 + s_1 - s_2 - s_3}{2}\Big) \\
& - \sin\Big(\frac{3r_1 + r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big) + \sin\Big(\frac{r_1 + 3r_2 - r_3 + s_1 + s2 - s3}{2}\Big) \\
& - \sin\Big(\frac{r_1 + r_2 - r_3 + 3s_1 + s2 - s3}{2}\Big) + \sin\Big(\frac{r_1 + r_2 - r_3 + s_1 + 3s2 - s3}{2}\Big) \\
& + \sin\Big(\frac{3r_1 - r_2 + r_3 + s_1 - s2 + s3}{2}\Big) - \sin\Big(\frac{r_1 - r_2 + 3r_3 + s_1 - s2 + s3}{2}\Big) \\
& + \sin\Big(\frac{r_1 - r_2 + r_3 + 3s_1 - s2 + s3}{2}\Big) - \sin\Big(\frac{r_1 - r_2 + r_3 + s_1 - s_2 + 3s_3}{2}\Big)\Big),
\end{aligned}
$$

and the expression of B is

$$
\begin{aligned}
B = \quad & \cos\Big(\frac{r_1 - r_2 - r_3 + s_1 - s_2 - 3s_3}{2}\Big) - \cos\Big(\frac{r_1 - r_2 - r_3 + s_1 - 3s_2 - s_3}{2}\Big) \\
& + \cos\Big(\frac{r_1 - r_2 - 3r_3 + s_1 - s_2 - s_3}{2}\Big) - \cos\Big(\frac{r_1 - 3r_2 - r_3 + s_1 - s_2 - s_3}{2}\Big) \\
& - \cos\Big(\frac{3r_1 + r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big) + \cos\Big(\frac{r_1 + 3r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big) \\
& - \cos\Big(\frac{r_1 + r_2 - r_3 + 3s_1 + s_2 - s_3}{2}\Big) + \cos\Big(\frac{r_1 + r_2 - r_3 + s_1 + 3s_2 - s_3}{2}\Big) \\
& + \cos\Big(\frac{3r_1 - r_2 + r_3 + s_1 - s_2 + s_3}{2}\Big) - \cos\Big(\frac{r_1 - r_2 + 3r_3 + s_1 - s_2 + s_3}{2}\Big) \\
& + \cos\Big(\frac{r_1 - r_2 + r_3 + 3s_1 - s_2 + s_3}{2}\Big) - \cos\Big(\frac{r_1 - r_2 + r_3 + s_1 - s_2 + 3s_3}{2}\Big).
\end{aligned}
$$

A simple change of (α_1, b_1) to (α_2, b_2) in the expression of δ_1, allows us to get the expression of δ_2.

Now, if we suppose that the points $p_4 = (\cos r_4, \sin r_4)$ and $q_4 = (\cos s_4, \sin s_4)$ satisfy equation (12) and if $\alpha_i \in \{0, 1\}$ with $i = 1, 2$, then we obtain $b_1 = 0$ and $b_2 = 0$.
We replace $c_1, d_1, \delta_1, \alpha_1$ and b_1 in the expression of $H_1(x, y)$, and $c_2, d_2, \delta_2, \alpha_2$ and b_2 in the expression of $H_2(x, y)$, we have $H_1(x, y) = H_2(x, y)$. Therefore, the piecewise linear differential system becomes a linear differential system, which does not have limit cycles. Therefore, the maximum number of crossing limit cycles in this case is at most three.

Example with three limit cycles. In the region $R_1 = \{(x, y) : x^2 + y^2 - 1 \geq 0\}$, we consider the linear **PHS**

$$\dot{x} = -18x + 95y + 15, \quad \dot{y} = 18y - 14, \tag{13}$$

with its Hamiltonian function $H_1(x,y) = 14x + 15y - 18xy + \frac{95}{2}y^2$. In the region $R_1 = \{(x,y) : x^2 + y^2 - 1 \leq 0\}$, we consider the linear **PHS**

$$\dot{x} = -0.2442..x + 0.2892..y + 0.203571, \quad \dot{y} = x + 0.2442..y - 0.19, \tag{14}$$

which has the Hamiltonian function

$$H_2(x,y) = -\frac{x^2}{2} - 0.2442..xy + 0.1446..y^2 + 0.19x + 0.20357..y.$$

The linear **PHS** (13)-(14) has exactly three crossing limit cycles because the system of equations (12) has exactly three real solutions $(x_1^{(1)}, y_1^{(1)}, x_2^{(1)}, y_2^{(1)}) = (0.9273.., 0.3741.., 0.9445.., -0.3282..)$, $(x_1^{(2)}, y_1^{(2)}, x_2^{(2)}, y_2^{(2)}) = (0.8357.., 0.5491.., 0.83658.., -0.5478..)$, and $(x_1^{(3)}, y_1^{(3)}, x_2^{(3)}, y_2^{(3)}) = (0.7397.., 0.6729.., 0.6809.., -0.732..)$, see Figure 3.

4 Proof of Theorem 2.2

In the quarter-plane $R_1 = \{(x,y) : x > 0, y < 0\}$, we consider the **PHS** given by (2). Its corresponding Hamiltonian function is given by equation (3).

In the quarter-plane $R_2 = \{(x,y) : x < 0, y < 0\}$, we consider the **PHS** given by (4), with its corresponding Hamiltonian function (5).

In the quarter-plane $R_3 = \{(x,y) : x < 0, y > 0\}$, we consider the **PHS**

$$\dot{x} = -b_3x - \delta_3y + d_3, \quad \dot{y} = \alpha_3x + b_3y + c_3, \tag{15}$$

its corresponding Hamiltonian function is

$$H_3(x,y) = -\frac{\alpha_3}{2}x^2 - b_3xy - \frac{\delta_3}{2}y^2 - c_3x + d_3y. \tag{16}$$

In the quarter-plane $R_4 = \{(x,y) : x > 0, y > 0\}$, we consider the **PHS**

$$\dot{x} = -b_4x - \delta_4y + d_4, \quad \dot{y} = \alpha_4x + b_4y + c_4. \tag{17}$$

Its corresponding Hamiltonian function is

$$H_4(x,y) = -\frac{\alpha_4}{2}x^2 - b_4xy - \frac{\delta_4}{2}y^2 - c_4x + d_4y. \tag{18}$$

In order to have a crossing limit cycle that intersects the two intersecting straight lines $xy = 0$ at the points $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$ and $(0, y_2)$, we must satisfy the following system:

$$\begin{aligned} P_1(x_1,y_1) &= H_1(x_1,0) - H_1(0,y_1) = 0, \\ P_2(x_2,y_1) &= H_2(0,y_1) - H_2(x_2,0) = 0, \\ P_3(x_2,y_2) &= H_3(x_2,0) - H_3(0,y_2) = 0, \\ P_4(x_1,y_2) &= H_4(0,y_2) - H_4(x_1,0) = 0, \end{aligned} \tag{19}$$

or, equivalently,

$$\begin{array}{l} P_1(x_1,y_1) = -2c_1x_1 - 2d_1y_1 - x_1^2\alpha_1 + y_1^2\delta_1 = 0, \\ P_2(x_2,y_1) = 2c_2x_2 + 2d_2y_1 + x_2^2\alpha_2 - y_1^2\delta_2 = 0, \\ P_3(x_2,y_2) = -2c_3x_2 - 2d_3y_2 - x_2^2\alpha_3 + y_2^2\delta_3 = 0, \\ P_4(x_1,y_2) = 2c_4x_1 + 2d_4y_2 + x_1^2\alpha_4 - y_2^2\delta_4 = 0. \end{array} \tag{20}$$

As $x_1 \neq x_2$ and $y_1 \neq y_2$, we know that the polynomials $P_1(x_1,y_1)$, $P_2(x_2,y_1)$, $P_3(x_2,y_2)$ and $P_4(x_1,y_2)$ are of degree 2. By using Bézout Theorem, we know that the number of solutions of the system (19) is bounded by the product of the degrees of the four polynomials $P_i(x_k,y_j)$, with $k,j = 1,2$, which is equal to 16. According to the symmetry of the solutions of this system, we know that the maximum number of solutions satisfying (20) is at most 8. Then the upper bound of limit cycles of system (2)–(17) is eight.

Because of the higher degree of these polynomials and the number of their parameters, we only can give an example with four limit cycles.

Example of four limit cycles for PHS separated by $xy = 0$. In the quarter-plane $R_1 = \{(x,y) : x > 0, y < 0\}$, we consider the **PHS**

$$\dot{x} = -4x - 35y + 8, \quad \dot{y} = -x + 4y + 8, \tag{21}$$

its Hamiltonian function is

$$H_1(x,y) = -\frac{1}{2}x^2 + 4xy + \frac{35}{2}y^2 + 8x - 8y.$$

In the quarter-plane $R_2 = \{(x,y) : x < 0, y < 0\}$, we consider the **PHS**

$$\dot{x} = -3x - 10.32..y + 5.17.., \quad \dot{y} = -x + 3y - 4.23.., \tag{22}$$

its Hamiltonian function is

$$H_2(x,y) = -\frac{1}{2}x^2 + 3xy + 5.16..y^2 - 4.23..x - 5.17..y.$$

In the quarter-plane $R_3 = \{(x,y) : x < 0, y > 0\}$, we consider the **PHS**

$$\dot{x} = +2.5x - 9y + 1, \quad \dot{y} = -x - 2.5y - 5, \tag{23}$$

where its Hamiltonian function is

$$H_3(x,y) = -\frac{1}{2}x^2 - 2.5xy + \frac{9}{2}y^2 - y - 5x.$$

In the quarter-plane $R_4 = \{(x,y) : x > 0, y > 0\}$, we consider the **PHS**

$$\dot{x} = -4x - 22.61..y + 4.46.., \quad \dot{y} = -x + 4y + 9.05.., \tag{24}$$

with the Hamiltonian function

$$H_4(x,y) = -\frac{1}{2}x^2 + 4xy + 11.3..y^2 - 4.46..y + 9.05..x.$$

The **PHS** (21)–(24) has exactly four crossing limit cycles because the system of equations (20) has four real solutions $(x_1^{(1)}, y_1^{(1)}, x_2^{(1)}, y_2^{(1)}) = (-0.524.., 1.2176.., -1.12.., 1.171..)$, $(x_1^{(2)}, y_1^{(2)}, x_2^{(2)}, y_2^{(2)}) = (-0.59.., 1.519.., -1.394.., 1.27..)$, $(x_1^{(3)}, y_1^{(3)}, x_2^{(3)}, y_2^{(3)}) = (-0.66.., 1.83.., -1.68.., 1.36..)$ and $(x_1^{(4)}, y_1^{(4)}, x_2^{(4)}, y_2^{(4)}) = (-0.72.., 2.16.., -2, 0, 1.44..)$, see Figure 4. This completes the proof of Theorem 2.2.

5 Conclusion

We have solved the extension of the second part of the 16th Hilbert problem for a family of discontinuous planar differential systems separated by conics. These piecewise differential systems are formed by planar linear Hamiltonian saddles. By using the fisrt integrals of these systems, we proved that the maximum number of crossing limit cycles of this family of systems is either three or eight depending on the curve of separation.

References

[1] A. Andronov, A. Vitt and S. Khaikin. *Theory of Oscillations.* Pergamon Press, Oxford, 1966.

[2] A. Belfar, R. Benterki and J. Llibre. Limit cycles of planar discontinuous piecewise linear Hamiltonian systems without equilibrium points and separated by irreducible cubics. *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications and Algorithms* **28** (2021) 399–421.

[3] R. Benterki and J. Llibre. The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves I. *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis* **28** (2021) 153-192.

[4] R. Benterki, L. Damene and J. Llibre. The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves II. *Differential Equations and Dynamical Systems* (2021).

[5] R. Benterki and J. Llibre. On the limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves III. Submitted (2021).

[6] R. Benterki and J. Llibre. Crossing limit cycles of planar piecewise linear Hamiltonian systems without Equilibrium Points. *Mathematics* **8** (5) 755 (2020).

[7] M. Di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk. *Piecewise-Smooth Dynamical Systems: Theory and Applications.* Appl. Math. Sci. Series, Springer-Verlag, London **163** (2008).

[8] L. Damene and R. Benterki. Limit cycles of discontinuous piecewise linear differential systems formed by centers or Hamiltonian without equilibria separated by irreducible cubics. *Moroccan Journal of Pure and Applied Analysis* **7** (2021) 248–276.

[9] M. Han and W. Zhang. On Hopf bifurcation in non-smooth planar systems. *Journal of Differential Equations* **248** (9) (2010) 2399–2416.

[10] S.M. Huan and X.S. Yang. On the number of limit cycles in general planar piecewise linear systems. *Discrete and Continuous Dynamical Systems* **32** (2012) 2147–2164.

[11] J. Llibre, D.D. Novaes and M.A. Teixeira. Maximum number of limit cycles for certain piecewise linear dynamical systems. *Nonlinear Dynamics* **82** (2015) 1159–1175.

[12] J. Llibre and E. Ponce. Three nested limit cycles in discontinuous piecewise linear differential systems with two zones. *Dyn. Cont. Disc. Impul. Syst. Ser. B* **19** (2012) 325–335.

[13] J. Llibre and C. Valls. Limit cycles of planar piecewise differential systems with linear Hamiltonian saddles. *Symmetry* **13** (7) (2021).

[14] D.J.W. Simpson. Bifurcations in piecewise-Smooth Continuous Systems. *World Scientific series on Nonlinear Science A* **70** (2010).