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A Mathematical Study of Wuhan Novel Coronavirus Epidemic Model

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Abstract: In this paper, we introduce a simplified model of the novel coronavirus pandemic (Covid-19), which appeared for the first time at Wuhan city in China. We compute the reproduction number $\mathcal{R}$, an epidemiologic index used to describe whether the disease spreads or ends. We study the model from a mathematical point of view, focusing on the local and global stability of the dynamical system by using Lyapunov functionals. We proof that for $\mathcal{R} < 1$, the disease dies and for $\mathcal{R} > 1$, the disease persists.

Keywords: *COVID-19; coronavirus pandemic; global stability; basic reproduction number; mathematical modeling; Lyapunov function.*

Mathematics Subject Classification (2010): 00A71, 34D23, 35N25, 37B25, 49K40, 60H10, 65C30, 91B70.

1 Introduction

Pandemics are large-scale outbreaks of infectious disease that can cause sudden, widespread morbidity and mortality over a wide geographic area and cause significant economic, social, and political disruption. Throughout history, there have been a lot of pandemics of diseases such as smallpox and tuberculosis. One of the most devastating pandemics was the Black Death, which killed an estimated $75-200$ million people in the 14th century. Other notable pandemics include the 1918 influenza pandemic (Spanish flu), the 2003 severe acute respiratory syndrome (SARS) pandemic, the 2009 influenza pandemic (H1N1), and the pandemic of human immunodeficiency virus/acquired immune deficiency syndrome, current HIV/AIDS. Over the past century, evidence suggests that the likelihood of pandemics has increased because of increased global travel, integration, urbanization and greater exploitation of the natural environment. These trends are likely to continue and intensify around the world with the appearance in 2019-2020

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of the current coronavirus pandemic. The origin of most diseases occurs through the "zoonotic" transmission of pathogens from animals to humans, on 31 December 2019, the World Health Organization (WHO), China Country Office, was informed of cases of the novel coronavirus (2019-nCoV) detected in Wuhan city. It is reported that the virus might have a bat origin, and the transmission of the virus might be related to the Huanan Seafood Wholesale Market in the same city [2]. Exported internationally via commercial and air travel, the virus reaches several countries around the world. There are now many times more cases outside of China than there were inside of it at the height of the outbreak. There are large outbreaks of the disease in multiple places, including Italy, Spain, France and the United States, which currently has the worst outbreak compared to any country in the world.

Populations, as with individuals, have unique patterns of disease. The science of epidemiology, which straddles biology, mathematical modeling, and dynamical systems, seeks to describe, understand, and utilize these patterns to improve population health. Therefore, several researches are focusing on mathematical modelling of Covid-19 to estimate the transmissibility and dynamic of the transmission of the virus [2]. These researches are focused on calculating the basic reproduction number $\mathcal{R}$.

In this study, we developed and analysed a mathematical model introduced in [2] and references therein, to describe the transmission of the virus from bats to people via the reservoir seafood market. We calculated the basic reproduction number $\mathcal{R}$. We study the basic and global properties of the model. By using Lyapunov functions and LaSalle's invariance principle, we have established the global stability of the equilibria of the model.

This paper is organized as follows. In Section 2, we propose the model and study its basic properties. In Section 3, the local stability of equilibria is established. Section 4 is offered to study the global stability of equilibria. In Section 5, we present some numerical examples to illustrate the obtained results.

2 Mathematical Model and Its Properties

We used a modelling framework similar to that by Chen et al. [2]. The variables of the model are introduced as follows: W denotes the SARS-CoV-2 in the reservoir (the seafood market). The population was divided into five compartments: susceptible individuals (S), exposed individuals (E), symptomatic infected individuals (I), asymptomatic infected individuals (A) and removed individuals (R) including recovered and dead individuals. The model parameters are given as follows: N represents the rate of the recruitment of susceptible (birth rate + rate of people travelling into Wuhan), c is the rate of individuals travelling out from the city. β_w is the transmission rate of the infection of individuals S from a sufficient contact with W, and β is the contagion rate due to the contact with infected people I. $1/\omega$ denotes the incubation period of human infection and $1/\gamma$ denotes the same infectious period of I and A. $1/\varepsilon$ describes the lifetime of the virus in W. The proportion of asymptomatic infection was defined as σ. θ denotes the multiple of the transmissibility of A to that of I (see Figure 1). The population is assumed constant, i.e., the births and natural deaths have the same value, due to the rapid disease spread. We assumed also that the transmissibility rate $\theta \in [0, 1]$.

The diagram (Figure 1) describes the dynamics of the reservoir-people (seafood market) transmission network model, and will be useful in the formulation of model equations. Based on the previous researches [1–4, 7–12, 15, 16] and using some assumptions,

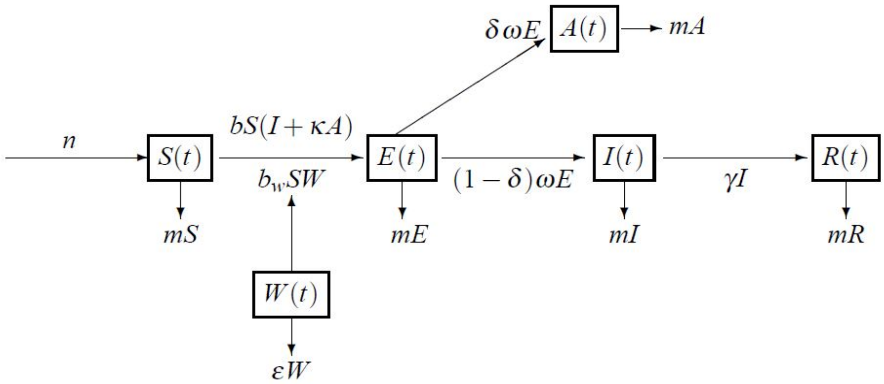

Figure 1: Flow diagram of the reservoir-people transmission network model.

the proposed mathematical model is given as follows:

$$\begin{aligned}
\dot{S} &= N - cS - \beta S(I+\theta A) - \beta_w SW,\\
\dot{E} &= \beta S(I+\theta A) + \beta_w SW - (\omega + c)E,\\
\dot{I} &= (1-\sigma)\omega E - (\gamma + c)I,\\
\dot{A} &= \sigma\omega E - (\gamma + c)A,\\
\dot{R} &= \gamma(I+A) - cR,\\
\dot{W} &= \varepsilon(I+\theta A - W).
\end{aligned} \tag{1}$$

It is subject to the conditions

$$S(0) > 0,\ E(0) > 0,\ I(0) \geq 0,\ A(0) \geq 0,\ R(0) \geq 0. \tag{2}$$

For epidemiological reasons, all model parameters are assumed to be positive. Next, we investigate the basic properties of model (1). Start by giving a result of boundedness and positivity of solutions.

Proposition 2.1 1. *All solutions of the model (1) with initial conditions (2) are bounded and non-negative.*
2. *The region* $\Omega = \{(S,E,I,A,R,W) \in \mathbb{R}^6_+ \ / \ S+E+I+A+R+W \leq \frac{N}{\bar{c}}\}$ *is a positively invariant attractor for system (1), where* $\bar{c} = \min(c-\varepsilon, \varepsilon)$.

Proof. 1. The solution is positive due to the fact below. Since $S=0$, one has $\dot{S} = N > 0$; if $E = 0$, then $\dot{E} = \beta S(I+\theta A) + \beta_w SW > 0$; once $I = 0$, then $\dot{I} = (1-\sigma)\omega E > 0$; if $A = 0$, then $\dot{A} = \sigma\omega E > 0$; if $R = 0$, then $\dot{R} = \gamma(I+A) > 0$; and if $W = 0$, then $\dot{W} = \varepsilon(I+\theta A) > 0$.

The boundedness of solutions of system (1) can be proved by summing up all equations of system (1), and denoting $T = S+E+I+A+R+W - \frac{N}{\bar{c}}$, then one obtains the following equation for the totality of individuals:

$$\begin{aligned}\dot{T} &= \dot{S}+\dot{E}+\dot{I}+\dot{A}+\dot{R}+\dot{W}\\ &= N-cS-cE-(c-\varepsilon)I-(c-\varepsilon\theta)A-cR-\varepsilon W\\ &\leq \bar{c}(\frac{N}{\bar{c}}-S-E-I-A-R-W)\\ &= -\bar{c}T.\end{aligned}$$

Then

$$S+E+I+A+R+W \leq \frac{N}{\bar{c}}+\Big(S(0)+E(0)+I(0)+A(0)+R(0)+W(0)-\frac{N}{\bar{c}}\Big)e^{-\bar{c}t}. \tag{3}$$

Then the boundedness of the solution of system (1) holds since all compartments of T are positive.

2. One can easily deduce from equality (3) that the set Ω is a positively invariant attractor for system (1).

3 Stability of the Equilibria of the System

The equilibria are obtained by putting all the equations of the system (1) to zero, as given below.

1. Disease-free equilibrium: $\mathcal{E}^0=(\frac{N}{c},0,0,0,0,0)$.

2. Endemic or positive equilibrium: $\mathcal{E}^\star=(S^\star,E^\star,I^\star,A^\star,W^\star)$.

To investigate the stability behavior of the equilibria, we need to compute the basic reproduction number $\mathcal{R}$ using the generation matrix method proposed by Diekmann, et al. [5] and elaborated by van den Driessche and Watmough [6] for an ODE compartmental model. Let

$$\dot{x}=F(x)-V(x),$$

where $x=(E,I,A,W)$, $F(x)$ is the matrix of new infection term, and $V(x)$ is the matrix of transfer terms into compartments and out of compartments. In our case, the Jacobian matrices of $F(x)$ and $V(x)$ at $\mathcal{E}^0$ are given by

$$F=\begin{pmatrix}0 & \frac{\beta N}{c} & \theta\frac{\beta N}{c} & \frac{\beta_w N}{c}\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}, \quad \text{and} \quad V=\begin{pmatrix}\omega+c & 0 & 0 & 0\\ -(1-\sigma)\omega & \gamma+c & 0 & 0\\ -\sigma\omega & 0 & \gamma+c & 0\\ 0 & -\varepsilon & -\varepsilon\theta & \varepsilon\end{pmatrix}.$$

Now,

$$V^{-1}=\begin{pmatrix}\frac{1}{\omega+c} & 0 & 0 & 0\\ A & \frac{1}{\gamma+c} & 0 & 0\\ B & 0 & \frac{1}{\gamma+c} & 0\\ D & E & G & \frac{1}{\varepsilon}\end{pmatrix},$$

where

$$\begin{cases} A &= \dfrac{(1-\sigma)\omega}{(\omega+c)(\gamma+c)}, \\ B &= \dfrac{\sigma\omega}{(\omega+c)(\gamma+c)}, \\ D &= \dfrac{(1-\sigma)\omega+\sigma\omega\theta}{(\omega+c)(\gamma+c)}, \\ E &= \dfrac{1}{\omega+c}, \\ G &= \dfrac{\theta}{\gamma+c}, \end{cases} \tag{4}$$

and then

$$FV^{-1} = \begin{pmatrix} 0 & \dfrac{\beta N}{c} & \theta\dfrac{\beta N}{c} & \beta_w\dfrac{N}{c} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{1}{\omega+c} & 0 & 0 & 0 \\ A & \dfrac{1}{\gamma+c} & 0 & 0 \\ B & 0 & \dfrac{1}{\gamma+c} & 0 \\ D & E & G & \dfrac{1}{\varepsilon} \end{pmatrix}$$

$$= \begin{pmatrix} A\dfrac{\beta N}{c}+B\theta\dfrac{\beta N}{c}+D\beta_w\dfrac{N}{c} & \dfrac{\beta N}{c(\omega+c)}+E\beta_w\dfrac{N}{c} & \theta\dfrac{\beta N}{c(\gamma+c)}+G\beta_w\dfrac{N}{c} & \beta_w\dfrac{N}{c\varepsilon} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the basic reproduction number for model (1) is given by $\mathcal{R} = \rho(FV^{-1})$, where ρ denotes the spectral radius of the next-generation matrix FV^{-1}. Therefore, the basic reproduction number $\mathcal{R}$ for our model is

$$\mathcal{R} = \frac{N(\beta+\beta_w)\big((1-\sigma)\omega+\sigma\omega\theta\big)}{c(\omega+c)(\gamma+c)} = R_1 + R_2, \tag{5}$$

where $R_1 = \dfrac{N\beta\big((1-\sigma)\omega+\sigma\omega\theta\big)}{c(\omega+c)(\gamma+c)}$ and $R_2 = \dfrac{N\beta_w\big((1-\sigma)\omega+\sigma\omega\theta\big)}{c(\omega+c)(\gamma+c)}$.

Next, the local stability of equilibria was discussed with respect to the basic reproduction number $\mathcal{R}$.

3.1 Analysis of the local stability for $\mathcal{E}^0$

The local stability of the disease-free equilibrium of the system (1) is given in the following theorem.

Theorem 3.1 *The disease-free equilibrium $\mathcal{E}^0$ is locally asymptotically stable when the basic reproduction number $\mathcal{R}$ is less than one and unstable when $\mathcal{R}$ is greater than one.*

Proof. The Jacobian matrix of (1) evaluated at $\mathcal{E}^0 = (\frac{N}{c}, 0, 0, 0, 0, 0)$ is given by

$$J^0 = \begin{pmatrix} -c & 0 & -\frac{\beta N}{c} & -\theta\frac{\beta N}{c} & 0 & -\beta_w\frac{N}{c} \\ 0 & -(\omega+c) & \frac{\beta N}{c} & \theta\frac{\beta N}{c} & 0 & \beta_w\frac{N}{c} \\ 0 & (1-\sigma)\omega & -(\gamma+c) & 0 & 0 & 0 \\ 0 & \sigma\omega & 0 & -(\gamma+c) & 0 & 0 \\ 0 & 0 & \gamma & \gamma & -c & 0 \\ 0 & 0 & \varepsilon & \varepsilon\theta & 0 & -\varepsilon \end{pmatrix}.$$

The characteristic equation of the matrix J^0 is

$$\begin{aligned} P^0(\lambda) =&(\lambda+\gamma+c)(\lambda+c)^2\big((\lambda+\omega+c)(\lambda+\gamma+c)(\lambda+\varepsilon)-(\omega+c)(\gamma+m)R_1\lambda \\ &-\varepsilon(\omega+c)(\gamma+c)\mathcal{R}\big). \end{aligned}$$

Obviously, $-c$ and $-\gamma-c$ are eigenvalues of J^0. To determine the other eigenvalues of J^0, let $p(\lambda) = (\lambda+\gamma+c)(\lambda+c)^2 p_3(\lambda)$, therefore

$$\begin{aligned} P_3(\lambda) =&\lambda^3 + (\omega+c+\gamma+c+\varepsilon)\lambda^2 + \Big(\varepsilon(\omega+c+\gamma+c)+(\omega+c)(\gamma+c)(1-R_1)\Big)\lambda \\ &+\varepsilon(\omega+c)(\gamma+c)(1-\mathcal{R}). \end{aligned}$$

We rewrite $p_3(\lambda)$ as $p_3(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$. The Routh-Hurwitz stability criterion ensures that $Re(\lambda) < 0$ under the conditions $A_1,\ A_3 > 0$ and $A_1A_2 - A_3 > 0$ for a monic polynomial of degree 3, then we have

$$\begin{aligned} A_1 =&\omega+c+\gamma+c+\varepsilon > 0, \\ A_2 =&\varepsilon(\omega+c+\gamma+c)+(\omega+c)(\gamma+c)(1-R_1) > 0 \quad \text{for } R_1 < \mathcal{R} < 1, \\ A_3 =&\varepsilon(\omega+c)(\gamma+c)(1-\mathcal{R}) > 0 \quad \text{for } \mathcal{R} < 1. \end{aligned}$$

Now we compute the term $A_1A_2 - A_3$:

$$\begin{aligned} A_1A_2 - A_3 =&(\omega+c+\gamma+c+\varepsilon)\big(\varepsilon(\omega+c+\gamma+c)+(\omega+c)(\gamma+c)(1-R_1)\big) \\ &-\varepsilon(\omega+c)(\gamma+c)(1-\mathcal{R}) \\ =&(\omega+c+\gamma+c)\big(\varepsilon(\omega+c+\gamma+c)+(\omega+c)(\gamma+c)(1-R_1)\big) \\ &+\varepsilon^2(\omega+c+\gamma+c)+\varepsilon(\omega+c)(\gamma+c)(\mathcal{R}-R_1) > 0 \quad \text{for } \mathcal{R} < 1. \end{aligned}$$

This completes the proof.

3.2 Existence and analysis of the local stability for $\mathcal{E}^\star$

In this section, the conditions for the existence of the endemic equilibrium $\mathcal{E}^\star = (S^\star, E^\star, I^\star, A^\star, R^\star, W^\star)$ are investigated and the local stability of the endemic equilibrium $\mathcal{E}^\star$ is discussed. The endemic equilibrium $\mathcal{E}^\star$ is obtained by putting all equations of the system (1) to zero as given below:

$$\begin{cases} N = cS^\star + \beta S^\star(I^\star + \theta A^\star) + \beta_w S^\star W^\star, \\ \beta S^\star(I^\star + \theta A^\star) + \beta_w S^\star W^\star = (\omega+c)E^\star, \\ (1-\sigma)\omega E^\star = (\gamma+c)I^\star, \\ \sigma\omega E^\star = (\gamma+c)A^\star, \\ \gamma(I^\star + A^\star) = cR^\star, \\ I^\star + \theta A^\star = W^\star. \end{cases} \tag{6}$$

Now we get

$$\mathcal{E}^\star = \Big(\frac{(\omega+c)(\gamma+c)}{(\beta+\beta_w)\big((1-\sigma)\omega+\sigma\omega\theta\big)}, \frac{N-cS^\star}{\omega+c}, \frac{(1-\sigma)\omega}{\gamma+c}E^\star, \frac{\sigma\omega}{\gamma+c}E^\star, \frac{1}{c(\gamma+c)}\gamma E^\star, \frac{(1-\sigma)\omega+\sigma\omega\theta}{\gamma+c}E^\star\Big).$$

Using the definition of reproduction number $\mathcal{R}$ in (5), we obtain

$$\begin{cases} S^\star &= \dfrac{N}{c\mathcal{R}}, \\ E^\star &= \dfrac{N}{\omega+c}(1-\frac{1}{\mathcal{R}}), \\ I^\star &= \dfrac{(1-\sigma)\omega N}{(\omega+c)(\gamma+c)}(1-\frac{1}{\mathcal{R}}), \\ A^\star &= \dfrac{\sigma\omega N}{(\omega+c)(\gamma+c)}(1-\frac{1}{\mathcal{R}}), \\ R^\star &= \dfrac{1}{c(\omega+c)(\gamma+c)}(1-\frac{1}{\mathcal{R}}), \\ W^\star &= \dfrac{(1-\sigma)\omega+\sigma\omega\theta}{(\omega+c)(\gamma+c)}N(1-\frac{1}{\mathcal{R}}). \end{cases} \tag{7}$$

Next, we study the local stability of system (1) around the endemic equilibrium $\mathcal{E}^\star$.

Theorem 3.2 *The endemic equilibrium $\mathcal{E}^\star$ exists and is locally asymptotically stable when the basic reproduction number $\mathcal{R}$ is less than one.*

Proof. The matrix J is evaluated at $\mathcal{E}^\star = (S^\star, E^\star, I^\star, A^\star, R^\star, W^\star)$ and is given by

$$J^\star = \begin{pmatrix} -c-\beta(I^\star+\theta A^\star)-\beta_w W^\star & 0 & -\beta S^\star & -\beta\theta S^\star & 0 & -\beta_w S^\star \\ \beta(I^\star+\theta A^\star)+\beta_w W^\star & -\omega-c & \beta S^\star & \beta\theta S^\star & 0 & \beta_w S^\star \\ 0 & (1-\sigma)\omega & -(\gamma+c) & 0 & 0 & 0 \\ 0 & \sigma\omega & 0 & -(\gamma+c) & 0 & 0 \\ 0 & 0 & \gamma & \gamma & -c & 0 \\ 0 & 0 & \varepsilon & \varepsilon\theta & 0 & -\varepsilon \end{pmatrix}.$$

Note that, by using (7), we have $-c-\beta(I^\star+\theta A^\star)-\beta_w W^\star = -c\mathcal{R}$ and $\beta(I^\star+\theta A^\star)+\beta_w W^\star = c(\mathcal{R}-1)$. The characteristic polynomial of the Jacobian matrix $J^\star$ is given by

$$p^\star(\lambda) = (\lambda+c)(\lambda+\gamma+c)\Bigg[(\lambda+c\mathcal{R})(\lambda+\varepsilon)(\lambda+\gamma+c)(\lambda+\omega+c) - (\lambda+c)S^\star\Bigg((1-\sigma)\omega\Big(\varepsilon\beta_w+\beta(\lambda+\varepsilon)\Big)+\sigma\omega\Big(\beta_w\varepsilon\theta+\beta\theta(\lambda+\varepsilon)\Big)\Bigg)\Bigg].$$

Clearly, the two roots of $p^\star$, $\lambda_1 = -c$ and $\lambda_2 = -\gamma-c$ are negative. The remaining roots can be determined by setting $p^\star(\lambda) = (\lambda+c)(\lambda+\gamma+c)p_4(\lambda)$, with $p_4(\lambda) = \lambda^4+B_1\lambda^3+B_2\lambda^2+B_3\lambda+B_4$.

We get

$$
\begin{aligned}
p_4(\lambda) =& \lambda^4 + \Big(\omega + c + \gamma + c + \varepsilon + c\mathcal{R}\Big)\lambda^3 \\
&+ \Big(\varepsilon c\mathcal{R} + (\omega + c)(\gamma + c) + (\varepsilon + c\mathcal{R})(\omega + c + \gamma + c) - \beta S^\star\big((1-\sigma)\omega + \sigma\omega\theta\big)\Big)\lambda^2 \\
&+ \Big(\varepsilon c\mathcal{R}(\omega + c + \gamma + c) + (\varepsilon + c\mathcal{R})(\omega + c)(\gamma + c) - S^\star\big((1-\sigma)\omega + \sigma\omega\theta\big)\times \\
&\big(\beta c + \varepsilon(\beta + \beta_w)\big)\Big)\lambda + \Big(\varepsilon c\mathcal{R}(\omega + c)(\gamma + c) - \varepsilon c S^\star\big((1-\sigma)\omega + \sigma\omega\theta\big)(\beta + \beta_w)\Big).
\end{aligned}
$$

Now we show, by a direct calculation, that all coefficients $B_i,\ i = 1, \cdots, 4$, of the polynomial p_4 are nonnegative, more precisely,

$$
\begin{aligned}
B_1 =& \omega + c + \gamma + c + \varepsilon + c\mathcal{R} > 0, \\
B_2 =& \varepsilon c\mathcal{R} + (\omega + c)(\gamma + c) + (\varepsilon + c\mathcal{R})(\omega + c + \gamma + c) - \beta S^\star\big((1-\sigma)\omega + \sigma\omega\theta\big) \\
=& \varepsilon c\mathcal{R} + (\omega + c)(\gamma + c) + (\varepsilon + c\mathcal{R})(\omega + c + \gamma + c) - (\omega + c)(\gamma + c)\frac{R_1}{\mathcal{R}} \\
=& \varepsilon c\mathcal{R} + (\varepsilon + c\mathcal{R})(\omega + c + \gamma + c) + (\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}}) > 0, \\
B_3 =& \varepsilon c\mathcal{R}(\omega + c + \gamma + c) + (\varepsilon + c\mathcal{R})(\omega + c)(\gamma + c) \\
&- S^\star\big((1-\sigma)\omega + \sigma\omega\theta\big)\big(\beta c + \varepsilon(\beta + \beta_w)\big) \\
=& \varepsilon c\mathcal{R}(\omega + c + \gamma + c) + (\varepsilon + c\mathcal{R})(\omega + c)(\gamma + c) - c(\omega + c)(\gamma + c)\frac{R_1}{\mathcal{R}} - \varepsilon(\omega + c)(\gamma + c) \\
=& \varepsilon c\mathcal{R}(\omega + c + \gamma + c) + c\mathcal{R}(\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}^2}) > 0 \quad \text{for } \mathcal{R} > 1,
\end{aligned}
$$

and

$$
\begin{aligned}
B_4 =& \varepsilon c\mathcal{R}(\omega + c)(\gamma + c) - \varepsilon c S^\star\big((1-\sigma)\omega + \sigma\omega\theta\big)(\beta + \beta_w) \\
=& \varepsilon c\mathcal{R}(\omega + c)(\gamma + c) - \varepsilon c(\omega + c)(\gamma + c) \\
=& \varepsilon c(\omega + c)(\gamma + c)(\mathcal{R} - 1) > 0.
\end{aligned}
$$

It follows, by using the Routh-Hurwitz criteria, that all the eigenvalues associated to $J^\star$ have negative real parts iff $B_i > 0,\ i = 1, 3, 4$, and $B_1(B_2B_3 - B_1B_4) - B_3^2 > 0$.

Now, calculating $B := B_2B_3 - B_1B_4$, and after simplifying negative terms, we get

$$
\begin{aligned}
B =& \Big(c\mathcal{R}(\omega + c + \gamma + c) + (\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}})\Big)\times \\
&\Big(\varepsilon c\mathcal{R}(\omega + c + \gamma + c) + c\mathcal{R}(\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}^2})\Big) \\
&+ \varepsilon c(c\mathcal{R} + \omega + c)\Big(\varepsilon\mathcal{R}(\omega + c + \gamma + c) + (\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}})\Big) \\
&+ \varepsilon c(\gamma + c)^2\Big(\varepsilon\mathcal{R} + (\omega + c)(1 - \frac{R_1}{\mathcal{R}})\Big) + \varepsilon^2 c(\gamma + c)(\omega + c).
\end{aligned}
$$

Let $B' = B_1(B_2B_3 - B_1B_4) - B_3^2 = B_1B - B_3^2$, after simplifying, we obtain

$$\begin{aligned}
B' =&(\gamma + c + c\mathcal{R})\Big(c\mathcal{R}(\omega + c) + c\mathcal{R}(\gamma + c) + (\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}})\Big)\times \\
&\Big(\varepsilon c\mathcal{R}(\omega + c + \gamma + c) + c\mathcal{R}(\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}^2})\Big) \\
&+ (\omega + c)\Big(c\mathcal{R}(\omega + c) + (\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}})\Big)\times \\
&\Big(\varepsilon c\mathcal{R}(\omega + c + \gamma + c) + c\mathcal{R}(\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}^2})\Big) \\
&+ \varepsilon(\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}})\Big(\varepsilon c\mathcal{R}(\omega + c + \gamma + c) + c\mathcal{R}(\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}^2})\Big) \\
+ &\varepsilon c(c\mathcal{R} + \omega + c)(\omega + c + \gamma + c + \varepsilon + c\mathcal{R})\Big(\varepsilon\mathcal{R}(\omega + c + \gamma + c) + (\omega + c)(\gamma + c)(1 - \frac{R_1}{\mathcal{R}})\Big) \\
&+ \varepsilon c(\gamma + c)^2(\omega + c + \gamma + c + \varepsilon + c\mathcal{R})\Big(\varepsilon\mathcal{R} + (\omega + c)(1 - \frac{R_1}{\mathcal{R}})\Big) \\
&+ \varepsilon^2 c(\gamma + c)(\omega + c)(\omega + c + \gamma + c + \varepsilon + c\mathcal{R}) + c^2\mathcal{R}^2(\omega + c)^2(\gamma + c)^2(1 - \frac{R_1}{\mathcal{R}^2})\frac{R_1}{\mathcal{R}^2} \\
&+ \varepsilon c^2\mathcal{R}^2(\omega + c + \gamma + c)(\omega + c)(\gamma + c)\frac{R_1}{\mathcal{R}^2}.
\end{aligned}$$

Since the compartments $1 - \frac{R_1}{\mathcal{R}}$ and $1 - \frac{R_1}{\mathcal{R}^2}$ are nonnegative for $\mathcal{R} > 1$, we get $B' > 0$. This ends the proof.

4 Global Stability Analysis of Both Equilibria of the System

In what follows, we investigate the global attractivity of both disease-free equilibria $\mathcal{E}^0$ and $\mathcal{E}^\star$.

Lemma 4.1 *The set* $\Omega_2 = \{(S, E, I, A, R, W) \in \mathbb{R}_+^6 \ / \ S + E + I + A + R + W \leq \frac{N}{\bar{c}}; S \leq \frac{N}{c}, W \leq I + \theta A\}$ *is a positively invariant attractor for system (1), where* $\bar{c} = \min(c - \varepsilon, \varepsilon)$.

Proof. It is proved in Proposition 2.1 that Ω_1 is a positive invariant attractor set of all solution of system (1). Now, since $\dot{S}(t) < 0$ for $S(t) > \frac{N}{c}$, one has $\liminf S(t) \leq \frac{N}{c}$. Similarly, since $\dot{W}(t) < 0$ for $W(t) > I(t) + \theta A(t)$, one has $\liminf W(t) \leq I(t) + \theta A(t)$. This completes the proof.

Theorem 4.1 *If* $\mathcal{R} \leq 1$, *then the disease-free equilibrium* $\mathcal{E}^0$ *is globally asymptotically stable (GAS). If* $\mathcal{R} > 1$, *then the disease-free equilibrium* $\mathcal{E}^0$ *is unstable.*

Proof. Construct the following Lyapunov function $\mathcal{L}(S, E, I, A, R, W)$ as:

$$\mathcal{L} = \omega(1 - \sigma + \sigma\theta)E + (\omega + c)(I + \theta A).$$

Along the trajectory of the solution of system (1), we have

$$\begin{aligned}
\dot{\mathcal{L}} =&\omega(1-\sigma+\sigma\theta)\dot{E}+(\omega+c)(\dot{I}+\theta\dot{A})\\
=&\omega(1-\sigma+\sigma\theta)\Big(\beta S(I+\theta A)+\beta_w SW-(\omega+c)E\Big)\\
&+(\omega+c)\Big((1-\sigma)\omega E-(\gamma+c)I+\sigma\theta\omega E-(\gamma+c)\theta A\Big)\\
=&\omega(1-\sigma+\sigma\theta)\Big(\beta S(I+\theta A)+\beta_w SW-(\omega+c)E\Big)\\
&+(\omega+c)\Big((1-\sigma+\sigma\theta)\omega E-(\gamma+c)I-(\gamma+c)\theta A\Big)\\
=&\omega(1-\sigma+\sigma\theta)\Big(\beta S(I+\theta A)+\beta_w SW\Big)-(\omega+c)(\gamma+c)(I+\theta A)\\
\leq&(1-\sigma+\sigma\theta)\frac{\omega N}{c}\Big(\beta(I+\theta A)+\beta_w(I+\theta A)\Big)-(\omega+c)(\gamma+c)(I+\theta A)\\
&\qquad\qquad\qquad\qquad\qquad\qquad(\text{since } S\leq\frac{N}{c}, W\leq I+\theta A)\\
=&\Big[(1-\sigma+\sigma\theta)\frac{\omega N}{c}(\beta+\beta_w)-(\omega+c)(\gamma+c)\Big](I+\theta A)\\
=&(\omega+c)(\gamma+c)\Big[\frac{(1-\sigma+\sigma\theta)}{(\omega+c)(\gamma+c)}\frac{\omega N}{c}(\beta+\beta_w)-1\Big](I+\theta A)\\
=&(\omega+c)(\gamma+c)(\mathcal{R}-1)(I+\theta A),\ \forall(S,E,I,A,R,W)\in\Omega_2.
\end{aligned}$$

Since all parameters of the model are non-negative, it follows that $\dot{\mathcal{L}}\leq 0$ for $\mathcal{R}\leq 1$ with $\dot{\mathcal{L}}=0$ only if $I=A=0$. Hence, $\mathcal{L}$ is a Lyapunov function on Ω_2. Further, by Lemma 4.1, Ω_2 is a compact, absorbing subset of $\mathbb{R}^6_+$, and the largest compact invariant set in $\{(S,E,I,A,R,W)\in\Omega_2 : \dot{\mathcal{L}}=0\}$ is the singleton $\{\mathcal{E}^0\}$. Therefore, by Lasalle's invariance principle (see, for instance, [13, Theorem 3.1]), every solution of system (1) with initial conditions in $\mathbb{R}^6_+$ converges to $\mathcal{E}^0$ as $t\to+\infty$.

The global stability of the disease-persistence (endemic) equilibrium $\mathcal{E}^\star$ is given in the following theorem.

Theorem 4.2 *If $\mathcal{R}>1$, then the disease-persistence equilibrium $\mathcal{E}^\star=(S^\star,E^\star,I^\star,A^\star,R^\star,W^\star)$ is GAS. If $\mathcal{R}\leq 1$, then the disease-persistence equilibrium $\mathcal{E}^\star$ is unstable.*

Proof. Introduce the following Lyapunov function:

$$\begin{aligned}
\mathcal{H}=&\Big(S-S^\star\ln(\frac{S}{S^\star})\Big)+\Big(E-E^\star\ln(\frac{E}{E^\star})\Big)+\frac{\omega+c}{(1-\sigma)\omega+\sigma\omega\theta}\Big(I+\theta A-(I^\star+\theta A^\star)\times\\
&\ln(\frac{I+\theta A}{I^\star+\theta A^\star})\Big)+\frac{\beta_w S^\star}{\varepsilon}\Big(W-W^\star\ln(\frac{W}{W^\star})\Big).
\end{aligned}$$

The equilibrium $\mathcal{E}^\star$ is the only internal stationary point of system (1). The function $\mathcal{H}(t)$ admits its minimum value $\mathcal{H}_{min}=S^\star+E^\star+\dfrac{\omega+c}{(1-\sigma)\omega+\sigma\omega\theta}(I^\star+\theta A^\star)+\dfrac{\beta_w}{\varepsilon}S^\star W^\star$ when $S=S^\star, E=E^\star, I=I^\star, A=A^\star, W=W^\star$, and $\mathcal{H}(t)\to+\infty$ at the boundary of the positive quadrant. Therefore, $\mathcal{E}^\star$ is the global minimum point, and the function is bounded from below.

Now we compute the derivative of $\mathcal{H}(t)$ along the solutions of system (1) as follows:

$$\begin{aligned}\dot{\mathcal{H}} =&(1-\frac{S^\star}{S})\dot{S}+(1-\frac{E^\star}{E})\dot{E}+\frac{\omega+c}{(1-\sigma)\omega+\sigma\omega\theta}(1-\frac{I^\star+\theta A^\star}{I+\theta A})(\dot{I}+\theta\dot{A})\\&+\frac{\beta_w}{\varepsilon}S^\star(1-\frac{W^\star}{W})\dot{W}\\=&(1-\frac{S^\star}{S})\Big(N-cS-\beta S(I+\theta A)-\beta_w SW\Big)\\&+(1-\frac{E^\star}{E})\Big(\beta S(I+\theta A)+\beta_w SW-(\omega+c)E\Big)\\&+\frac{\omega+c}{(1-\sigma)\omega+\sigma\omega\theta}\Big(1-\frac{I^\star+\theta A^\star}{I+\theta A}\Big)\Big(((1-\sigma)\omega+\theta\sigma\omega)E-(\gamma+c)(I+\theta A)\Big)\\&+\beta_w S^\star(1-\frac{W^\star}{W})(I+\theta A-W)\\=&(1-\frac{S^\star}{S})\Big(c(S^\star-S)+\beta S^\star(I^\star+\theta A^\star)-\beta S(I+\theta A)+\beta_w S^\star W^\star-\beta_w SW\Big)\\&+\beta S(I+\theta A)+\beta_w SW-(\omega+c)E-\frac{E^\star}{E}\beta S(I+\theta A)-\frac{E^\star}{E}\beta_w SW+(\omega+c)E^\star\\&+\Big(1-\frac{I^\star+\theta A^\star}{I+\theta A}\Big)\Big((\omega+c)E-\frac{(\omega+c)(\gamma+c)}{(1-\sigma)\omega+\sigma\omega\theta}(I+\theta A)\Big)\\&+\beta_w S^\star(I+\theta A)-\beta_w S^\star W-\beta_w S^\star\frac{W^\star}{W}(I+\theta A)+\beta_w S^\star W^\star.\end{aligned}$$

Using the fact that $(S^\star, E^\star, I^\star, R^\star, W^\star)$ is a solution of system (6), (7) and (5), we get

$$\begin{aligned}W^\star =&(I^\star+\theta A^\star), N=cS^\star+S^\star(\beta+\beta_w)(I^\star+\theta A^\star),\ (\omega+c)E^\star=\beta S^\star W^\star+\beta_w S^\star W^\star\\(\omega+c)E=&\frac{E}{E^\star}\beta S^\star W^\star+\frac{E}{E^\star}\beta_w S^\star W^\star \text{ and } \frac{(\omega+c)(\gamma+c)}{(1-\sigma)\omega+\sigma\omega\theta}=S^\star\beta+S^\star\beta_w.\end{aligned}$$

We obtain

$$\begin{aligned}\dot{\mathcal{H}}=&-c\frac{(S-S^\star)^2}{S}+\beta S^\star(I^\star+\theta A^\star)-\beta S(I+\theta A)+\beta_w S^\star W^\star-\beta_w SW\\&-\beta S^\star(I^\star+\theta A^\star)\frac{S^\star}{S}+\beta S^\star(I+\theta A)-\beta_w\frac{(S^\star)^2}{S}W^\star+\beta_w S^\star W+\beta S(I+\theta A)\\&+\beta_w SW-(\omega+c)E-\frac{E^\star}{E}\beta S(I+\theta A)-\frac{E^\star}{E}\beta_w SW+(\omega+c)E^\star+(\omega+c)E\\&-S^\star(\beta+\beta_w)(I+\theta A)-\frac{I^\star+\theta A^\star}{I+\theta A}(\omega+c)E+S^\star(\beta+\beta_w)W^\star+\beta_w S^\star(I+\theta A)\\&-\beta_w S^\star W-\beta_w S^\star\frac{W^\star}{W}(I+\theta A)+\beta_w S^\star W^\star.\end{aligned}$$

Therefore the expression of $\dot{\mathcal{H}}$ reduces to

$$\begin{aligned}\dot{\mathcal{H}} =& -c\frac{(S-S^\star)^2}{S} + \beta S^\star W^\star + \beta_w S^\star W^\star - \beta S^\star W^\star \frac{S^\star}{S} - \beta_w \frac{(S^\star)^2}{S} W^\star \\ &- \frac{E^\star}{E}\beta S(I+\theta A) - \frac{E^\star}{E}\beta_w SW + \beta S^\star W^\star + \beta_w S^\star W^\star \\ &- \frac{I^\star+\theta A^\star}{I+\theta A}\big(\frac{E}{E^\star}\beta S^\star W^\star + \frac{E}{E^\star}\beta_w S^\star W^\star\big) + S^\star \beta W^\star + S^\star \beta_w W^\star \\ &- \beta_w S^\star \frac{W^\star}{W}(I+\theta A) + \beta_w S^\star W^\star.\end{aligned}$$

More simply,

$$\begin{aligned}\dot{\mathcal{H}} =& -c\frac{(S-S^\star)^2}{S} + \beta S^\star W^\star \bigg(3 - \frac{S^\star}{S} - \frac{E^\star}{E}\frac{S}{S^\star}\frac{I+\theta A}{I^\star+\theta A^\star} - \frac{I^\star+\theta A^\star}{I+\theta A}\frac{E}{E^\star}\bigg) \\ &+ \beta_w S^\star W^\star \bigg(4 - \frac{S^\star}{S} - \frac{E^\star}{E}\frac{S}{S^\star}\frac{W}{W^\star} - \frac{I^\star+\theta A^\star}{I+\theta A}\frac{E}{E^\star} - \frac{I+\theta A}{W}\bigg).\end{aligned}$$

Note that

$$\frac{S^\star}{S}\frac{E^\star}{E}\frac{S}{S^\star}\frac{I+\theta A}{I^\star+\theta A^\star}\frac{I^\star+\theta A^\star}{I+\theta A}\frac{E}{E^\star} = 1,$$

and

$$\frac{S^\star}{S}\frac{E^\star}{E}\frac{S}{S^\star}\frac{W}{W^\star}\frac{I^\star+\theta A^\star}{I+\theta A}\frac{E}{E^\star}\frac{I+\theta A}{W} = 1.$$

We recall also the following inequality:

$$\sqrt[n]{x_1x_2x_3\cdots x_n} \le \frac{x_1+x_2+x_3+\cdots+x_n}{n}, \quad x_1,x_2,x_3,\cdots,x_n \ge 0. \tag{8}$$

Since the geometric mean of nonnegative real numbers is less than the arithmetical one, we obtain the inequalities

$$\begin{aligned}&3 - \frac{S^\star}{S} - \frac{E^\star}{E}\frac{S}{S^\star}\frac{I+\theta A}{I^\star+\theta A^\star} - \frac{I^\star+\theta A^\star}{I+\theta A}\frac{E}{E^\star} \le 0, \\ &4 - \frac{S^\star}{S} - \frac{E^\star}{E}\frac{S}{S^\star}\frac{W}{W^\star} - \frac{I^\star+\theta A^\star}{I+\theta A}\frac{E}{E^\star} - \frac{I+\theta A}{W} \le 0.\end{aligned}$$

Therefore $\dot{\mathcal{H}} \le 0$, and one deduces that $\mathcal{E}^\star = (S^\star, E^\star, I^\star, A^\star, W^\star)$ is stable in the sense of Lyapunov.

Now, to show the asymptotic stability of $\mathcal{E}^\star = (S^\star, E^\star, I^\star, A^\star, W^\star)$, we will use the Lasalle invariance principle cited, for instance, in Theorem 3.1 in [13]. To do this, let us define

$$\begin{aligned}s_2 =& -c\frac{(S-S^\star)^2}{S}, \\ s_3 =& 3 - \frac{S^\star}{S} - \frac{E^\star}{E}\frac{S}{S^\star}\frac{I+\theta A}{I^\star+\theta A^\star} - \frac{I^\star+\theta A^\star}{I+\theta A}\frac{E}{E^\star}, \\ s_4 =& 4 - \frac{S^\star}{S} - \frac{E^\star}{E}\frac{S}{S^\star}\frac{W}{W^\star} - \frac{I^\star+\theta A^\star}{I+\theta A}\frac{E}{E^\star} - \frac{I+\theta A}{W}.\end{aligned}$$

Then one has

$$\dot{\mathcal{H}}(S,E,I,A,W) = 0 \Longleftrightarrow s_2 = s_3 = s_4 = 0.$$

Using the above relations, we obtain the following implications:

$$\begin{array}{l} s_2 = 0 \Longrightarrow S = S^\star, \\ (S = S^\star, s_3 = 0) \Longrightarrow E^\star(I + \theta A) = E(I^\star + \theta A^\star), \\ (S = S^\star, E^\star(I + \theta A) = E(I^\star + \theta A^\star), s_4 = 0) \Longrightarrow E^\star W = EW^\star. \end{array}$$

Finally, we obtain

$$\dot{\mathcal{H}}(S, E, I, A, W) = 0 \Longleftrightarrow S = S^\star, E^\star(I + \theta A) = E(I^\star + \theta A^\star), E^\star W = EW^\star. \tag{9}$$

Let $r = \dfrac{E}{E^\star} = \dfrac{I + \theta A}{I^\star + \theta A^\star} = \dfrac{W}{W^\star}$, then $E = rE^\star$, $W = rW^\star$ and $I + \theta A = r(I^\star + \theta A^\star) = rW^\star$.

For $S = S^\star$, the first equation of system (1) gives

$$\dot{S} = \dot{S}^\star = N - cS^\star - \beta S^\star(I + \theta A) - \beta_w S^\star W = 0.$$

Replacing $I + \theta A, W$ in the above equation by their values given by (9) yields

$$N - cS^\star - r\beta S^\star(I^\star + \theta A^\star) - r\beta_w S^\star W^\star = 0.$$

By comparing with the first equation of system (6), we deduce that $r = 1$ and therefore $E = E^\star, W = W^\star$ and $I + \theta A = I^\star + \theta A^\star$ $\forall \theta > 0$. Finally,

$$\dot{\mathcal{H}}(S, E, I, A, W) = 0 \Longleftrightarrow (S = S^\star, E = E^\star, I = I^\star, A = A^\star, W = W^\star).$$

Thus $\{\mathcal{E}^\star = (S^\star, E^\star, I^\star, A^\star, W^\star)\}$ is the largest invariant set contained in $\{(S, E, I, A, W) | \dot{\mathcal{H}} = 0\}$. Then the global stability of the equilibrium $\mathcal{E}^\star = (S^\star, E^\star, I^\star, A^\star, W^\star)$ holds according to the Lasalle invariance principle [14].

5 Numerical Examples

The parameters used in the implementation of the model (1) are given by $c = 1$, $\beta = 0.5$, $\beta_w = 0.3$, $\omega = 3$, $\gamma = 5$, $\varepsilon = 0.3$, $\sigma = 0.75$, $\theta = 0.25$.

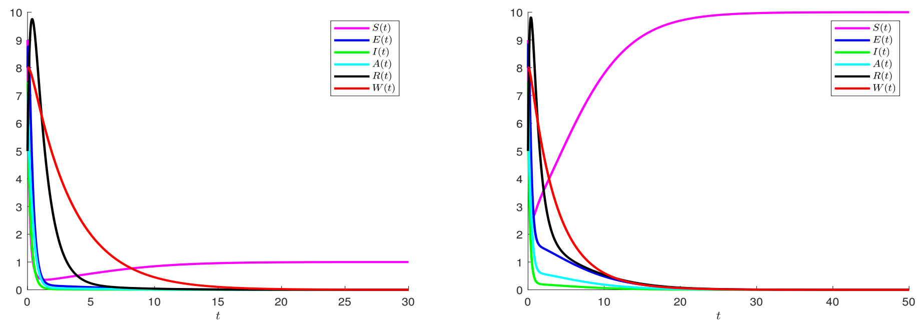

Figure 2: $(S(t), E(t), I(t), A(t), R(t), W(t))$ behaviours for $N = 1$(left), then $\mathcal{R} = 0.044 \leq 1$, and for $N = 10$(right), then $\mathcal{R} = 0.438 \leq 1$.

Four tests were considered. Two of them (Figure 2) confirming the global stability of the disease-free equilibrium $\mathcal{E}^0$ when $\mathcal{R} \leq 1$. We note that the solution of system (1) converges asymptotically to $\mathcal{E}^0$ and only susceptible compartment persists and the other compartments vanish.

The other two tests (Figure 3) confirm the global stability of the disease-persistence equilibrium $\mathcal{E}^\star$ when $\mathcal{R} > 1$. We observe that the solution of system (1) converges asymptotically to $\mathcal{E}^\star$ and all compartments persist.

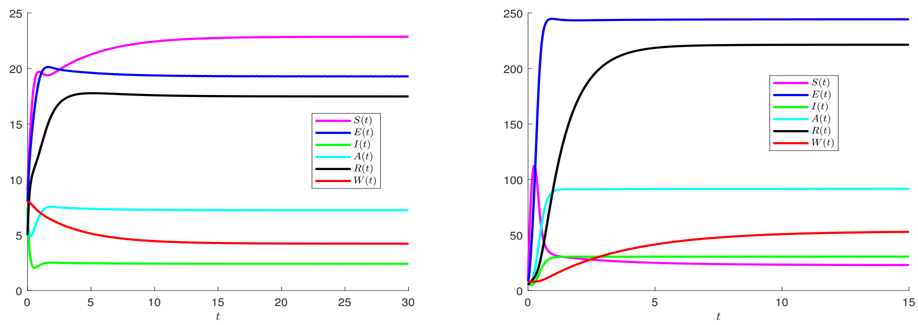

Figure 3: $(S(t), E(t), I(t), A(t), R(t), W(t))$ behaviours for $N = 100$(left), then $\mathcal{R} = 4.375 > 1$, and for $N = 1000$(right), then $\mathcal{R} = 43.75 > 1$.

6 Concluding Remarks

In this paper, we have considered an epidemic model for the Covid-19 coronavirus, in which we have divided the total population into five compartments, namely, susceptible, exposed, symptomatic infected, asymptomatic infected and recovered, and we investigated the dynamical behavior of this model. Here, we have found that

$$\mathcal{R} = \frac{N(\beta + \beta_w)\big((1-\sigma)\omega + \sigma\omega\theta\big)}{c(\omega + c)(\gamma + c)}$$

is the basic reproduction number of system (1), which helps us to determine the dynamical behavior of the system. We showed, for system (1), that the disease-free equilibrium $\mathcal{E}^0$ is globally asymptotically stable when $\mathcal{R} < 1$. However, when $\mathcal{R} > 1$, the endemic equilibrium $\mathcal{E}^\star$ is both locally and globally stable. These results have been verified numerically for some parameters of the model.

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